Speckle in Ultrasound *B*-Mode Scans

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Abstract—**Ultrasound images obtained with a simple linear or sector scan show a granular appearance, called "speckle." This speckle is analyzed. The reduction in speckle that can be obtained with a compound scan with maximum amplitude writing is computed. The reduction in speckle is almost as large as can be obtained with averaging. It depends on the number of independent amplitude values that are measured. The condition for the independence of two amplitude values is derived, and thus a limit is given for the possible reduction in speckle.**

I. Introduction

AN ULTRASOUND *B*-mode scan is shown in Fig. 1 which was produced on a sector scanner described in a paper by the author and colleagues [1]. The echogram shows the placenta, fetal head, and amniotic fluid. The placenta and other parts show a granular structure, which will be called *speckle*. Similar speckle can be observed in scans of other authors if they were made with a simple scanning motion (i.e., either linear or sector scan). It is, however, much less pronounced in compound scans. Speckle is an undesirable property of the image as it masks small differences in grey level. This is the motivation for studying it in this paper.

As the paper is mainly theoretical, it seems useful to start with some experimental observations. When an object is scanned twice under exactly the same conditions, one obtains identical speckle patterns. Although of random appearance, speckle is therefore not random in the same sense as, e.g., electrical noise. If the same object is, however, scanned under different conditions, say different transducer aperture, pulse length, or transducer angulation, the speckle patterns are different. Usually the amplitude values at the same point are quite different in the two patterns. This suggests that the speckle pattern has only a tenuous relation to the actual object structure. It is also observed that the size of the speckle granules is about the same as the resolution of the scanner, both in the longitudinal as well as lateral direction. As already mentioned, speckle is much less pronounced in compound scans. This is most evident when "grey scale" compound scans are shown side by side with "grey scale" linear or sector scans (see, e.g., [2]).

In this paper speckle is investigated theoretically. It is argued that ultrasound speckle can be treated in a similar way as laser speckle. In the next section the treatment for a simple linear or sector scan is given. The tissue is modeled as a collection of scatterers which are so numerous that there are many within one resolution cell of the scanner. Since the scatterers occupy the same resolution cell, the wavelets scattered by them interfere and speckle is explained as an interference phenomenon. The question, how realistic such a model is, remains open. Limited experimental evidence presented in the next section shows reasonable agreement between theory and experiment. More experiments are certainly needed and it is hoped that the theory presented here will serve as a guide for further experiments.

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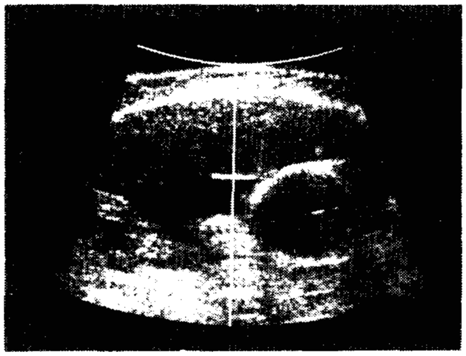

Fig. 1. Ultrasound *B*-mode scan of a placenta, fetal head, and amniotic fluid. The placenta and other parts show a granular appearance, called speckle.

Speckle in compound scans is discussed in Section III. It is shown there why and by how much speckle is reduced in compound scans. The speckle reduction is computed for a compound scan with *maximum amplitude writing*. It is shown that the reduction in speckle is almost as large as with amplitude averaging. The reduction in speckle depends on the number of different amplitude values that are measured. The question therefore arises under what conditions two amplitude values are independent. This condition is derived in Section IV and states that the transducer has to be translated by about half its width between measurements. This puts an upper limit to the possible reduction of speckle.

This paper discusses first-order statistics, i.e., probability density functions are computed for the amplitude at one point. The first-order statistics are independent of transducer aperture. We do not treat second-order statistics, i.e., the distribution of "sizes" of the speckle granules. As already mentioned, these do depend on resolution and therefore on the aperture of the transducer.

II. Speckle in Images Obtained with a Simple Scanning Motion

Speckle has been treated extensively for the case of objects illuminated by laser light (see [3]). As mentioned, we will treat ultrasound speckle in a similar way. We consider one resolution cell of the scanner, i.e., an area which corresponds to the smallest resolvable detail. As was described in the last section, we assume that there are many scatterers within one resolution cell which scatter wavelets with random phases.

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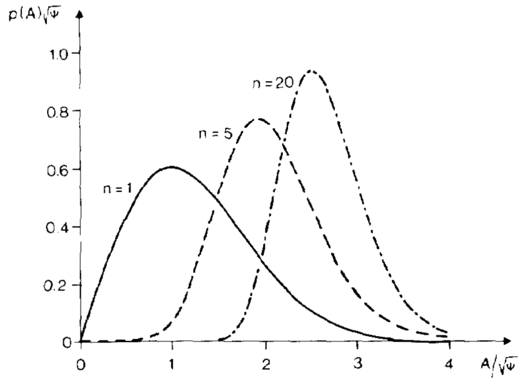

Fig. 2. Probability densities for the amplitude A. $n = 1$: Rayleigh probability density for the amplitude of a simple linear or sector scan. $n = 5, n = 20$: probability densities for the maximum amplitude of a maximum writing compound scan when the maximum of 5, respectively, 20 amplitudes is taken.

These wavelets are added up because the scatterers are all within one resolution cell. If constructive interference predominates the point in the image will be bright, if interference is mainly destructive, it will be dark.

Unlike laser light the ultrasound is pulsed. If the ultrasound pulse has a length of several cycles of the carrier wave and the scatterers are closer together than the spatial pulse length (i.e., are within one resolution cell) the phases between wavelets remain constant during the pulse. The waves can thus be treated like coherent waves. (The following treatment will have to be modified for pulses with a strong chirp.)

We assume that we add up many sinusoidal pulses with random phases. If we consider the phasors, this addition leads to the well-known random walk problem. If the number of scatterers within one resolution element is large and the phases are distributed uniformly between 0 and 2π, the amplitude A obeys a Rayleigh probability density function $p(A)$ [4, p. 401]

$$p(A) = \frac{A}{\psi} \exp(-A^2/2\psi) \tag{1}$$

where ψ is a parameter. $p(A)$ is shown as solid curve in Fig. 2 (labelled $n = 1$). From this curve it is evident that A has a rather large probable variation around its average. Equation (1) also applies to the light amplitude of laser speckle. In acoustics we investigate the statistics of A (rather than $|A|^2$ as in the laser speckle) because the transducer is pressure sensitive rather than intensity sensitive.

We will now determine the average amplitude $\bar{A}$ and the rms-deviation from the average which we will denote as noise amplitude N,

$$N = [\overline{(A - \bar{A})^2}]^{1/2} = [\overline{A^2} - \bar{A}^2]^{1/2}. \tag{2}$$

From [4, p. 402] we obtain

$$\bar{A} = (\pi\psi/2)^{1/2} \tag{3}$$

and $\overline{A^2} = 2\psi$. These formulas will be derived in Section III-B for a more general case (see (13) and (17)).

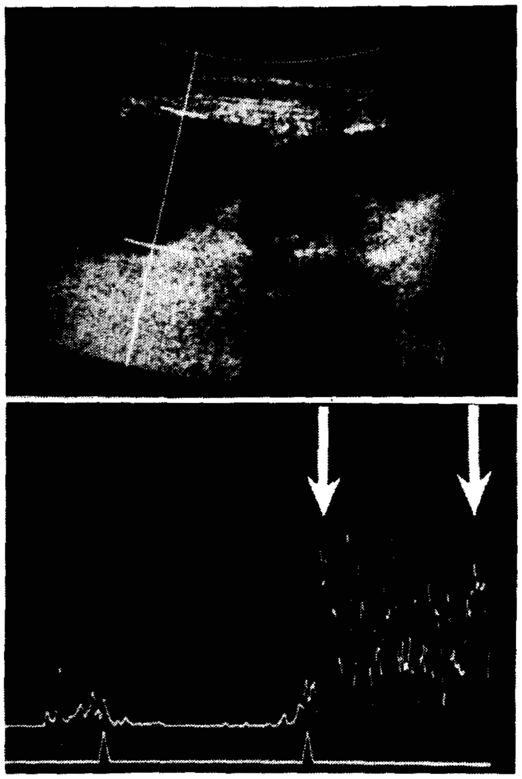

Fig. 3. B-scan and A-scan taken along the white radial line. The A-scan was evaluated between the white arrows and a SNR = 2.0 was obtained.

We can now define a signal-to-noise ratio (SNR)

$$\text{SNR} = \frac{\bar{A}}{N} = \frac{\bar{A}}{[\overline{A^2} - \bar{A}^2]^{1/2}} = \left(\frac{\pi}{4 - \pi}\right)^{1/2} = 1.91. \tag{4}$$

The SNR is larger than in laser speckle where it is only 1. This is a consequence of the fact that we measure amplitude and not intensity. All the same, it is much lower than in most pictures we are used to, e.g., television.

In order to check these computations, the SNR was determined on several A-scans which were judged to have a constant average, i.e., no discernible structure. Values for the SNR between about 1.5 and 2.5 were obtained. It is, however, sometimes hard to tell, whether the average is really constant. Fig. 3 shows an A-scan, together with the corresponding B-scan, where a SNR = 2.0 was obtained between the arrows shown, i.e., there is good agreement with the computation. The model we assumed thus seems to be reasonable.

We want to point out the following practical consideration. Most ultrasound systems use dynamic range compression because the dynamic range of the display is limited. Various arguments are given to justify the particular shape of the characteristic used for compression. Most characteristics are, however, or closely resemble a logarithmic curve. A logarithmic curve is very suitable because the SNR is independent of signal, i.e., large signals are associated with large noise levels. We have

$$\frac{\bar{A}}{N} = \text{SNR}$$

and $\log \bar{A} = \log N + \log \text{SNR}$.

This means that after being compressed logarithmically the signal is a constant level above the noise level, independent of signal level. Practically speaking, this means that we do not use up a lot of limited dynamic range to display the large random fluctuations associated with large signals.

III. Speckle in Images Obtained with a Compound Scanning Motion

The essence of compound scanning is that each object point is insonified in different directions and that the reflected amplitudes are obtained from different directions. For different directions the individual waves of the scatterers in a resolution cell add up with different random phases and the total amplitude is different for different directions. If the average of n different and independent amplitude values is computed, the rms deviation of the average is reduced by a factor $\sqrt{n}$ with respect to a single amplitude. (The reader might already be curious what values of n he can expect in a compound scan. As will be shown in the next section, two amplitude values are independent if the transducer is translated by about half its width between measurements. Therefore, an estimate of n is obtained by dividing the angle over which one point is "looked at" by half the angle that the transducer subtends at this point.) Such an averaging takes place in the so-called "open-shutter technique" where a time exposure of the display screen is taken while the compound scan is performed. More precisely, one does not perform an averaging operation but a summation of all the amplitude values. In practice one encounters the problem that the sound beam should dwell on each object point for the same time. It takes an experienced operator to meet this condition only approximately. Because of this difficulty the new compound scanning systems use the so-called *maximum amplitude writing* method. In this method the maximum amplitude obtained during the compound scanning motion is displayed for each point. All other amplitude values are ignored. In this mode of operation the image is not influenced by variations in scanning speed.

The question now is whether speckle is also reduced in a compound scan with maximum writing and, if so, by how much? In this section we will compute the SNR of a compound scan with maximum writing. We will see that the improvement in SNR is almost as large as for a compound scan with averaging.

A. Probability Density of the Amplitude for a Compound Scan with Maximum Amplitude Writing

We have n independent amplitudes and we will now compute the probability density $p_{\max}(A)$ of the maximum amplitude. From this we will then compute the average value and the rms deviation.

The probability density of any of the amplitudes is given by (1). The probability $P(A_i < A)$ that the ith amplitude is smaller than the amplitude A of the kth amplitude is

$$P(A_i < A) = \int_0^A \frac{A}{\psi} \exp(-A^2/2\psi)\, dA = 1 - \exp(-A^2/2\psi). \tag{5}$$

The probability that *all* amplitudes except the kth amplitude are smaller than the kth amplitude is given by

$$P(A_1, A_2, \cdots, A_{k-1}, A_{k+1}, \cdots, A_n < A) = [P(A_i < A)]^{n-1} = (1 - \exp[-A^2/2\psi])^{n-1}. \tag{6}$$

Equation (6) gives the probability that the kth amplitude with value A is actually the largest one. The probability density $p_k(A)$ of the kth amplitude, given the condition that it be the largest amplitude, is obtained by multiplying the probability of (6) with the probability density of (1)

$$p_k(A)/_{A=A_{\max}} = \frac{A}{\psi} \exp(-A^2/2\psi)(1 - \exp[-A^2/2\psi])^{n-1}. \tag{7}$$

One of the n amplitudes A_1 or A_2 or A_3, etc., will now be maximum. Therefore, we obtain the probability density $p_{\max}(A)$ of the maximum amplitude by multiplying (7) by n,

$$p_{\max}(A) = n\frac{A}{\psi} \exp(-A^2/2\psi)(1 - \exp[-A^2/2\psi])^{n-1}. \tag{8}$$

Fig. 2 shows $p_{\max}(A)$ for $n = 1$ (Rayleigh), $n = 5$, and $n = 20$. With increasing n the average value increases and the curve becomes somewhat more narrow, i.e., it is to be expected that the SNR increases with increasing n.

In the Appendix we show that the integral over $p_{\max}(A)$ from zero to infinity is equal to one as required.

B. Computation of $\bar{A}$, $\overline{A^2}$, and SNR

The average amplitude $\bar{A}$ is computed as follows:

$$\bar{A} = \int_0^\infty A p_{\max}(A)\, dA = \int_0^\infty n \cdot \frac{A^2}{\psi} \exp(-A^2/2\psi) \cdot (1 - \exp[-A^2/2\psi])^{n-1}\, dA. \tag{9}$$

From the binomial theorem we obtain

$$\bar{A} = \int_0^\infty n\frac{A^2}{\psi}\left(\exp[-A^2/2\psi] - (n-1)\exp[-2A^2/2\psi] + \frac{(n-1)(n-2)}{2!}\exp[-3A^2/2\psi] - \cdots + \cdots\right) dA. \tag{10}$$

The kth term of this sum has the form

$$(-1)^{k-1} \int_0^\infty n\frac{A^2}{\psi} \cdot \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \cdot \exp(-kA^2/2\psi)\, dA. \tag{11}$$

By partial integration we obtain

$$\int_0^\infty \frac{A^2}{\psi} \exp(-kA^2/2\psi)\, dA = -\frac{A}{k}\exp(-kA^2/2\psi)\Big/_0^\infty + \int_0^\infty \frac{1}{k}\exp(-kA^2/2\psi)\, dA = \frac{1}{k}\left(\frac{\pi\psi}{2k}\right)^{1/2} \tag{12}$$

because the first term vanishes. The second term is the well-known error integral. From (10)-(12) we now obtain

$$\bar{A} = n\left(\frac{\pi\psi}{2}\right)^{1/2}\left(1 - \frac{n-1}{1}\frac{1}{2\sqrt{2}} + \frac{(n-1)(n-2)}{2!}\frac{1}{3\sqrt{3}} - \cdots + \cdots + (-1)^{k-1}\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \cdot \frac{1}{k\sqrt{k}} + \cdots\right). \tag{13}$$

We now want to compute the mean of the amplitude squared $\overline{A^2}$:

$$\overline{A^2} = \int_0^\infty A^2 p_{\max}(A)\,dA = \int_0^\infty n\frac{A^3}{\psi}\exp(-A^2/2\psi) \cdot (1 - \exp[-A^2/2\psi])^{n-1}\,dA. \tag{14}$$

From the binomial theorem we have

$$\overline{A^2} = \int_0^\infty n\frac{A^3}{\psi}\left(\exp[-A^2/2\psi] - (n-1)\exp[-2A^2/2\psi] + \frac{(n-1)(n-2)}{2!}\exp[-3A^2/2\psi] - \cdots + \cdots\right)dA. \tag{15}$$

By partial integration we obtain

$$\int_0^\infty \frac{A^3}{\psi}\exp(-kA^2/2\psi)\,dA = -\frac{A^2}{k}\exp(-kA^2/2\psi)\Big|_0^\infty + \frac{2}{k}\int_0^\infty A\exp(-kA^2/2\psi)\,dA = -\frac{2\psi}{k^2}\exp(-kA^2/2\psi)\Big|_0^\infty = 2\psi/k^2. \tag{16}$$

Therefore we have

$$\overline{A^2} = 2\psi n\left(1 - (n-1)\frac{1}{4} + \frac{(n-1)(n-2)}{2!}\cdot\frac{1}{9} - \cdots + (-1)^{k-1}\frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!}\frac{1}{k^2} + \cdots\right). \tag{17}$$

Equations (13) and (17) were evaluated on a programmable pocket calculator for various values of n. The signal to noise ratio $\bar{A}/N$ was then computed

$$\bar{A}/N = \bar{A}/(\overline{A^2} - \bar{A}^2)^{1/2}.$$

The results are summarized in the following table where $\bar{A}$ is given in multiples of $\sqrt{\psi}$ and $\overline{A^2}$ in multiples of ψ.

n	$\bar{A}$	$\overline{A^2}$	$\bar{A}/N$	Improvement	$\sqrt{n}$
1	1.2533	2.0000	1.91	—	—
2	1.6204	3.0000	2.65	1.39	1.41
3	1.8249	3.6667	3.15	1.65	1.73
4	1.9636	4.1667	3.52	1.84	2.00
5	2.0675	4.5666	3.83	2.01	2.24
7	2.2180	5.1857	4.30	2.25	2.65
10	2.3699	5.8579	4.82	2.52	3.16
15	2.5334	6.6365	5.42	2.84	3.87
20	2.6442	7.1955	5.86	3.07	4.47

The column "Improvement" gives the improvement in SNR with respect to $n = 1$, the column $\sqrt{n}$ gives the improvement for amplitude averaging. For small values of n the improvement with maximum amplitude writing is almost as large as that which is obtained with an averaging method. For large values of n the improvement becomes somewhat smaller. The disadvantage is small compared to the other advantages of maximum amplitude writing and therefore maximum amplitude writing should be the preferred mode of operation.

IV. Condition for the Independence of the Amplitudes

In the last section we computed the improvement in SNR when the maximum of n *independent* amplitudes is used. In this section we want to show under what conditions the amplitude values are independent from each other. Intuitively one would say that a second amplitude value is not independent from a first amplitude value, if between measurements the ultrasound transducer has been translated by a distance that is much smaller than its diameter.

We first treat a simplified case which does not correspond to reality. A realistic case will be treated afterwards. As shown in Fig. 4 a quadratic receiving transducer of width a is focused on a surface S. The surface S is made up of very many scatterers which reflect with random phase. Contrary to reality we assume that the surface is insonified by a second stationary transmitting ultrasound transducer. The whole problem can be treated in one dimension. Since the transducer is focused on the surface its point spread function on the surface, $s_1(x)$ is given by

$$s_1(x) = a(\sin \pi a\xi)/(\pi a\xi), \qquad \xi = x/(\lambda d) \tag{18}$$

where λ is the wavelength of the ultrasound, d the distance between transducer and surface, and the origin $x = 0$ is chosen at the focal point of the transducer. If we now translate the receiving transducer by a distance b as shown in Fig. 4 and focus again on the same point, its point spread function $s_2(x)$ will be

$$s_2(x) = a\frac{\sin \pi a\xi}{\pi a\xi}\exp(2\pi i\xi b). \tag{19}$$

Equation (19) is obtained from the shift theorem in Fourier transform theory. The amplitudes scattered can be written as phasors with two components X_n and Y_n. We only treat imaging of the component X_n, for Y_n the argument is the

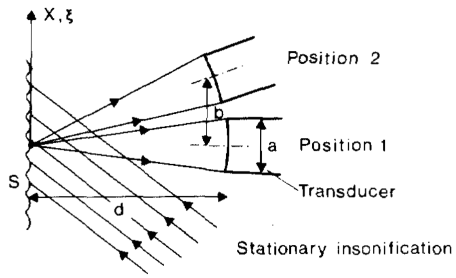

Fig. 4. A quadratic receiving transducer of width a is focused on the surface S. After having been translated by the distance b, it is again focused on the same point. In the first case treated, the surface is insonified by a second stationary transducer. In the second case treated, the transducer shown here is used for transmission and reception.

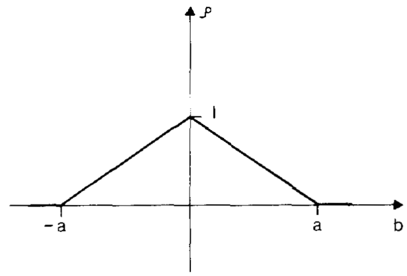

Fig. 5. The correlation coefficient ρ as a function of translation b of a quadratic transducer of width a. Insonification by a second stationary transducer is assumed.

same. In position 1 one obtains for the total x-component X_{tot} of the phasor received by the transducer

$$X_{tot} = \sum_n X_n a \frac{\sin \pi a \xi}{a \pi \xi}. \tag{20}$$

In position 2, translated by the distance b, we obtain

$$X'_{tot} = \sum_n X_n a \frac{\sin \pi a \xi}{a \pi \xi} \exp(2\pi i \xi b). \tag{21}$$

Since the phasors have random phase between 0 and 2π, we have $\overline{X_{tot}} = \overline{X'_{tot}} = 0$ and therefore we obtain for the correlation coefficient ρ [5, p. 572]:

$$\rho = C \overline{X_{tot} X'_{tot}} \tag{22}$$

where C is a constant of proportionality. Inserting (20) and (21) into (22) we obtain

$$\rho = C \left\langle \left(\sum_n X_n a \frac{\sin \pi a \xi}{\pi a \xi} \right) \left(\sum_m X_m a \frac{\sin \pi a \xi}{\pi a \xi} \cdot \exp[2\pi i \xi b] \right) \right\rangle \tag{23}$$

where $\langle \cdots \rangle$ means average. The average of the products for which $m \neq n$ vanishes and we are left with

$$\rho = C \left\langle \sum_n X_n^2 a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \exp(2\pi i \xi b) \right\rangle. \tag{24}$$

We now write X_n as a continuous function, $X(\xi)$ and integrate over ξ,

$$\rho = C \left\langle \int_{-\infty}^{\infty} X^2(\xi) a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \exp(2\pi i \xi b)\, d\xi \right\rangle. \tag{25}$$

Since $X(\xi)$ is real, $X^2(\xi) \geq 0$. We can write

$$X^2(\xi) = K + X_{AC}^2(\xi) \tag{26}$$

where K is a positive constant and

$$\overline{X_{AC}^2(\xi)} = 0.$$

We assume that $X(\xi)$ and therefore $X_{AC}^2(\xi)$ vary much more rapidly than the point spread function, (18). Therefore,

$$\left\langle \int_{-\infty}^{\infty} X_{AC}^2(\xi) a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \exp(2\pi i \xi b)\, d\xi \right\rangle = 0.$$

Equation (25) therefore becomes

$$\rho = K \cdot C \left\langle \int_{-\infty}^{\infty} a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \exp(2\pi i \xi b)\, d\xi \right\rangle. \tag{27}$$

This is recognized as Fourier transform of $(\sin \pi a \xi / \pi a \xi)^2$ and therefore

$$\rho = K \cdot C \left(1 - \left| \frac{b}{a} \right| \right), \quad \text{for } |b| \leq a$$

$$\rho = 0, \quad \text{for } |b| > a.$$

For $b = 0$ we must have $\rho = 1$, therefore $KC = 1$ and

$$\rho = \left(1 - \left| \frac{b}{a} \right| \right), \quad \text{for } |b| \leq a$$

$$\rho = 0, \quad \text{for } |b| > a. \tag{28}$$

Fig. 5 shows ρ as a function of the translation b. We have the intuitively plausible result that the correlation decreases linearly as a function of the translation b. The correlation is zero for translations equal or larger than the width of the transducer a. Therefore we obtain *independent* amplitude values if we translate the transducer by its width or more.

We now want to treat the realistic case where the same transducer is used for transmission and reception. The geometry is the same as in Fig. 4 except that there is no separate stationary transducer for insonification. The point spread function $s_1(x)$ for transmission and reception then is

$$s_1(x) = a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \tag{29}$$

where the same symbols have been used as in (18). When the transducer is translated by the distance b we obtain for the spread function $s_2(x)$

$$s_2(x) = a^2 \left(\frac{\sin \pi a \xi}{\pi a \xi} \right)^2 \exp(4\pi i \xi b). \tag{30}$$

Going through the steps that lead to (27), we now obtain for the correlation coefficient

$$\rho = KC \int_{-\infty}^{\infty} a^4 \left(\frac{\sin \pi a\xi}{\pi a\xi} \right)^4 \exp (4\pi i\xi b)\, d\xi. \tag{31}$$

If we make the substitution $b' = 2b$ we again obtain a Fourier transform. The function to be transformed is the square of the function to be transformed in (27). We therefore obtain ρ as the autocorrelation of the triangle function shown in Fig. 5 but now written as function of b'. The autocorrelation is best performed graphically and we obtain after changing back from b' to b

$$\rho = \frac{(a-b)^2}{a^2}, \qquad \text{for } |b| \leqslant a$$

$$\rho = 0, \qquad \text{for } |b| > a. \tag{32}$$

Again $\rho = 0$ for the translation $b = a$, but we have $\rho = 1/4$ for $b = a/2$. The amplitude values are "almost" independent if we translate the transducer by half its width.

If a certain length is available over which we can compound, i.e., over which we can measure the reflected amplitude from a point, we therefore obtain more independent amplitude values for a transducer with a small aperture than with a transducer with a large aperture. Therefore, we can obtain a better SNR with a transducer with a small aperture than with a transducer with a large aperture. The resolution, however, increases with increasing aperture. It is possible to trade in resolution against SNR. The increase in SNR is then proportional to $\sqrt{m}$, where m is the loss in horizontal resolution. The reverse, trading in SNR against resolution, is not possible.

V. Conclusion

Speckle in ultrasound images has been analyzed for simple scans and for compound scans. It was especially shown that the improvement in SNR is almost as good for a "maximum amplitude writing" compound scan as for an "averaging" compound scan. In view of its other advantages the first mode of operation is to be preferred. It was then shown that the transducer has to be translated by about half its width when independent amplitude values are to be obtained. This places a definite limit on the possible improvement in SNR. The possible improvement is lower for high-resolution (i.e., large aperture) transducers than for low-resolution (i.e., small aperture) transducers.

Appendix

We want to show that the integral over the probability density $p_{\max}(A)$, (8) is indeed equal to one as required. We want to compute

$$\int_0^{\infty} p_{\max}(A)\, dA = \int_0^{\infty} n(A/\psi) \exp(-A^2/2\psi) \cdot (1 - \exp[-A^2/2\psi])^{n-1}\, dA.$$

For $n = 1$ (Rayleigh) the integration is straightforward. For $n \neq 1$ we integrate partially $(n - 1)$ times. We then obtain

$$\int_0^{\infty} p_{\max}(A)\, dA = \int_0^{\infty} \frac{n(n-1) \cdot \cdots \cdot 1}{1 \cdot 2 \cdot \cdots \cdot (n-1)} \frac{A}{\psi} \cdot \exp\left(-\frac{nA^2}{2\psi}\right) dA$$
$$= -\frac{n(n-1) \cdot \cdots \cdot 1}{1 \cdot 2 \cdot \cdots \cdot (n-1)} \cdot \frac{1}{n} \cdot \exp\left(-\frac{nA^2}{2\psi}\right) \Big|_0^{\infty} = 1. \qquad \text{Q.E.D.}$$

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