Assignment 1 - Image Deblurring

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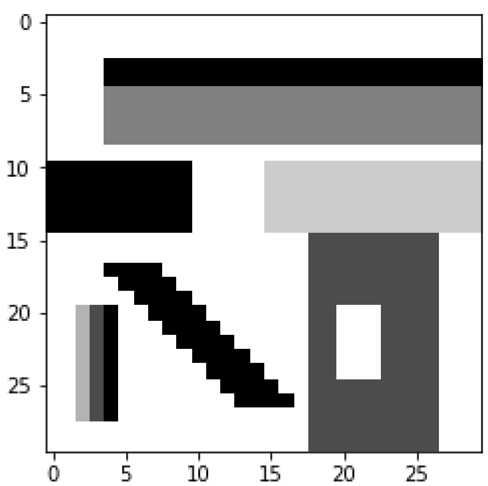
In [2]:
```python
import numpy as np
import scipy
import scipy.sparse
from scipy.sparse.linalg import inv
from PIL import Image
import matplotlib.pyplot as plt
from hessian_matrix import hessian_matrix
```

Test Image

You can use the following function to create a synthetic image:

In [3]:
```python
def create_random_binary_img_1():
    # RandInt returns an integer from 1 (inclusive) to 2 (exclusive), meaning it will al
    u = np.random.randint(1, 2, 30 * 30).reshape(30, 30).astype(float)
    u[5:9, 4:] = 0.5
    u[3:5, 4:] = 0
    u[10:15, :10] = 0
    u[10:15, 15:] = 0.8
    u[15:, 18:27] = 0.3

    for i in range(10):
        u[i + 17, i + 4:i + 8] = 0

    u[20:25:, 20:23] = 1
    u[20:28, 4] = 0
    u[20:28, 3] = 0.3
    u[20:28, 2] = 0.7

    return u


sample_image = create_random_binary_img_1()
plt.imshow(sample_image, cmap="gray")
plt.show()
```



hessian_matrix() function

We provide the implementation of the hessian matrix for $k = 0, 1, 2, 3$.

input variables:

- u: your image
- reg: regularization parameter
- k: one of the four kernel types 0-3

Example usage:

In [4]:
```
u_ex = np.zeros(9).reshape(3, 3)
reg_ex = 1
k_type_ex = 3
A = hessian_matrix(u_ex, reg_ex, k_type_ex)
A_n_t = A.toarray()
print(A_n_t)
```

```
[[ 4.  -2.   0.  -2.   0.   0.   0.   0.   0. ]
 [-2.   6.5 -2.   0.5 -2.   0.   0.   0.   0. ]
 [ 0.  -2.   4.5  0.   0.5 -2.   0.   0.   0. ]
 [-2.   0.5  0.   6.5 -2.   0.  -2.   0.   0. ]
 [ 0.  -2.   0.5 -2.   9.  -2.   0.5 -2.   0. ]
 [ 0.   0.  -2.   0.  -2.   6.5  0.   0.5 -2. ]
 [ 0.   0.   0.  -2.   0.5  0.   4.5 -2.   0. ]
 [ 0.   0.   0.   0.  -2.   0.5 -2.   6.5 -2. ]
 [ 0.   0.   0.   0.   0.  -2.   0.  -2.   4. ]]
```

Implementation

In [5]:
```python
def derivative_img_term(g, u):
    """
    Derivative of the image term
    g: Blurred image
    u: The image
    """
    result = np.zeros(u.shape)
    for x in range(u.shape[0]):
        for y in range(u.shape[1]):
            # Corners
            if x == 0 and y == 0:
                result[x, y] = 0
            elif x == 0 and y == u.shape[1] - 1:
                result[x, y] = -g[0, y - 1] + 0.5 * u[1, y] + 0.5 * u[0, y]
            elif x == u.shape[0] - 1 and y == 0:
                result[x, y] = 0
            elif x == u.shape[0] - 1 and y == u.shape[1] - 1:
                result[x, y] = - g[x - 1, y - 1] + 0.5 * u[x, y] + 0.5 * u[x - 1, y]
            # Edges
            elif x == 0:  # Upper edge
                result[x, y] = - g[0, y - 1] + 0.5 * u[0, y] + 0.5 * u[1, y]
            elif x == u.shape[0] - 1:  # Lower edge
                result[x, y] = - g[x - 1, y - 1] + 0.5 * u[x, y] + 0.5 * u[x - 1, y]
            elif y == 0:  # Left edge
                result[x, y] = 0
            elif y == u.shape[1] - 1:  # Right edge
                result[x, y] = - g[x, y - 1] - g[x - 1, y - 1] + u[x, y] + 0.5 * u[x + 1
            # Inside Image
            else:
                result[x, y] = u[x, y] + 0.5 * u[x + 1, y] + 0.5 * u[x - 1, y] - g[x, y
    return result


def derivative_gp_term(u):
    """
    Derivative of  Gaussian Prior Regularization Term
    u: The image
    """
    result = np.zeros(u.shape)
    for x in range(u.shape[0]):
        for y in range(u.shape[1]):
            # Corners
            if x == 0 and y == 0:
                result[x, y] = 4 * u[0, 0] - 2 * u[1, 0] - 2 * u[0, 1]
            elif x == 0 and y == u.shape[1] - 1:
                result[x, y] = 4 * u[0, y] - 2 * u[1, y] - 2 * u[0, y - 1]
            elif x == u.shape[0] - 1 and y == 0:
                result[x, y] = 4 * u[x, 0] - 2 * u[x - 1, 0] - 2 * u[x, 1]
            elif x == u.shape[0] - 1 and y == u.shape[1] - 1:
                result[x, y] = 4 * u[x, y] - 2 * u[x - 1, y] - 2 * u[x, y - 1]
            # Edges
            elif x == 0:  # Upper edge
                result[x, y] = 6 * u[0, y] - 2 * u[1, y] - 2 * u[0, y + 1] - 2 * u[0, y
            elif x == u.shape[0] - 1:  # Lower edge
                result[x, y] = 6 * u[x, y] - 2 * u[x - 1, y] - 2 * u[x, y + 1] - 2 * u[x
            elif y == 0:  # Left edge
                result[x, y] = 6 * u[x, 0] - 2 * u[x - 1, 0] - 2 * u[x + 1, 0] - 2 * u[x
            elif y == u.shape[1] - 1:  # Right edge
                result[x, y] = 6 * u[x, y] - 2 * u[x - 1, y] - 2 * u[x + 1, y] - 2 * u[x
            # Inside Image
            else:
                result[x, y] = 8 * u[x, y] - 2 * u[x - 1, y] - 2 * u[x + 1, y] - 2 * u[x
```

```python
    return result


def derivative_atv_term(u):
    """
    Derivative of Anisotropic Total Variation Regularization Term
    u: The image
    """
    result = np.zeros(u.shape)
    for x in range(u.shape[0]):
        for y in range(u.shape[1]):
            # Corners
            if x == 0 and y == 0:
                result[x, y] = - np.sign(u[1, 0] - u[0, 0]) - np.sign(u[0, 1] - u[0, 0]
            elif x == 0 and y == u.shape[1] - 1:
                result[x, y] = np.sign(u[0, y] - u[0, y - 1]) - np.sign(u[1, y] - u[0, y
            elif x == u.shape[0] - 1 and y == 0:
                result[x, y] = np.sign(u[x, 0] - u[x - 1, 0]) - np.sign(u[x, 1] - u[x, 0
            elif x == u.shape[0] - 1 and y == u.shape[1] - 1:
                result[x, y] = np.sign(u[x, y] - u[x, y - 1]) + np.sign(u[x, y] - u[x -
            # Edges
            elif x == 0:  # Upper edge
                result[x, y] = - np.sign(u[1, y] - u[0, y]) - np.sign(u[0, y + 1] - u[0,
                    u[0, y] - u[0, y - 1])
            elif x == u.shape[0] - 1:  # Lower edge
                result[x, y] = np.sign(u[x, y] - u[x - 1, y]) + np.sign(u[x, y] - u[x, y
                    u[x, y + 1] - u[x, y])
            elif y == 0:  # Left edge
                result[x, y] = - np.sign(u[x + 1, 0] - u[x, 0]) + np.sign(u[x, 0] - u[x
                    u[x, 1] - u[x, 0])
            elif y == u.shape[1] - 1:  # Right edge
                result[x, y] = np.sign(u[x, y] - u[x, y - 1]) + np.sign(u[x, y] - u[x -
                    u[x + 1, y] - u[x, y])
            # Inside Image
            else:
                result[x, y] = np.sign(u[x, y] - u[x - 1, y]) - np.sign(u[x + 1, y] - u[
                    u[x, y + 1] - u[x, y]) - np.sign(u[x, y] - u[x, y - 1])
    return result


def GD(g, reg_lambda, action=""):
    """
    Gradient Descent algorithm
    g: grayscale blurry image of size (M, N)
    reg_lambda: regularization parameter
    Optional 'action': Variant to be used.
    """
    u = np.zeros((g.shape[0] + 1, g.shape[1] + 1))
    epsilon = 0.1

    for x in range(g.shape[0]):
        for y in range(g.shape[1]):
            u[x, y] = g[x, y]

    if action == "gauss":  # Gaussian Regularization
        gradient = derivative_img_term(g, u) + reg_lambda * derivative_gp_term(u)
        # while (np.sum(np.power(gradient, 2)) > margin):
        for i in range(1000):
            u = u - epsilon * gradient
            gradient = derivative_img_term(g, u) + reg_lambda * derivative_gp_term(u)
    elif action == "aniso":  # Anisotropic Total Variation
        gradient = derivative_img_term(g, u) + reg_lambda * derivative_atv_term(u)
        for i in range(1000):
```

```
            u = u - epsilon * gradient
            gradient = derivative_img_term(g, u) + reg_lambda * derivative_atv_term(u)
    else:  # No Regularization
        gradient = derivative_img_term(g, u)
        while np.sum(np.power(gradient, 2)) > 0.000001:
            u = u - epsilon * gradient
            gradient = derivative_img_term(g, u)
    return u
```

Linearization and Gauss-Seidel

In [6]:
```
def LGS(g, reg_lambda):
    """
    Creating the initial image u (1 unit larger than g in both dimensions), which is set
    Until the difference between the two different u's is smaller than 0.00001, the Line

    g: grayscale blurry image of size (M, N)
    reg_lambda: regularization parameter
    """
    margin = 0.000001
    u = np.zeros((g.shape[0] + 1, g.shape[1] + 1))
    old_u = np.zeros((g.shape[0] + 1, g.shape[1] + 1))

    for x in range(g.shape[0]):
        for y in range(g.shape[1]):
            u[x, y] = g[x, y]

    while np.sum(np.power(old_u - u, 2)) > margin:
        old_u = u
        u = iterative_linearization(g, u, reg_lambda)

    return u
```

In [7]:
```
def iterative_linearization(g, u, reg_lambda):
    """
    Gauss Seidel approach with linearized term using sparse matrices (speed-up!).
    g: Grayscale blurry image of size (m, n)
    u: Image passed from LSOR()
    reg_lambda: Regularization parameter
    """
    m, n = u.shape[0], u.shape[1]

    # Flatten image
    vector_u = u.reshape(m * n, 1)

    # Hessian Matrix, following theory from Lec03-04_energy minimization.pdf
    hess_mat = hessian_matrix(u, reg_lambda, k)
    # Compute b
    b = hess_mat * vector_u - (derivative_img_term(g, u) + reg_lambda * derivative_gp_te
    # Triangular lower sparse matrix
    lower_mat = scipy.sparse.tril(hess_mat, k=0)
    # Triangular upper sparse matrix
    upper_mat = scipy.sparse.triu(hess_mat, k=1)

    vector_u = inv(lower_mat) * (b - upper_mat * vector_u)

    # Reshape flat image to initial size
    u = vector_u.reshape(m, n)

    return u
```

In [8]:
```
def iterative_linearization_SOR(g, u, reg_lambda, w):
    """
    Successive Over-Relaxation Approach, using sparse matrices (speed-up!) for the linea
    g: Grayscale blurry image of size (m, n)
    u: Image passed from LSOR()
    reg_lambda: Regularization parameter
    w: SOR parameter
    """
    m, n = u.shape[0], u.shape[1]

    # Flatten image
    vector_u = u.reshape(m * n, 1)
    # Hessian Matrix
    hess_mat = hessian_matrix(u, reg_lambda, k)
    # Triangular lower sparse matrix
    lower_mat = scipy.sparse.tril(hess_mat, k=-1)
    # Triangular upper sparse matrix
    upper_mat = scipy.sparse.triu(hess_mat, k=1)
    # Diagonal sparse matrix, defined by the three matrices above
    diag_mat = hess_mat - lower_mat - upper_mat

    b = hess_mat * vector_u - (derivative_img_term(g, u) + reg_lambda * derivative_gp_te

    vector_u = inv(diag_mat + w * lower_mat) * (w * b - (w * upper_mat + (w - 1) * diag_

    # Reshape flat image to initial size
    u = vector_u.reshape(m, n)

    return u
```

In [9]:
```
def LSOR(g, reg_lambda, w):
    """
    Creating the initial image u (1 unit larger than g in both dimensions), which is set
    Until the difference between the two different u's is smaller than 0.00001, the Line

    g: Grayscale blurry image of size (m, n)
    reg_lambda: Regularization parameter
    w: SOR parameter
    """
    u = np.zeros((g.shape[0] + 1, g.shape[1] + 1))
    prior_u = np.zeros((g.shape[0] + 1, g.shape[1] + 1))

    for x in range(g.shape[0]):
        for y in range(g.shape[1]):
            u[x, y] = g[x, y]

    while np.sum(np.power(prior_u - u, 2)) > 0.00001:
        prior_u = u
        u = iterative_linearization_SOR(g, u, reg_lambda, w)
    return u
```

Parameters

In [10]:
```

reg_lambda = 0.005  # lambda regularization parameter. you need to play with this
# 0.025 best for Gauss
# 0.005 best for anisotropic total variation

# k matrix. Look below how to choose it
k = 1
```

$$k = 0 \longrightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \qquad k = 1 \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \qquad k = 2 \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad k = 3 \longrightarrow \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

Blurring the image

In [11]:
```
def blurring(img, kernel_mat):
    """
    Blur image with convolution, could alternatively be done simply with scipy.signal.co
    img: The image passed
    kernel_mat: The kernel used
    """
    blurred_img = np.zeros((img.shape[0] - 1, img.shape[1] - 1))
    for row in range(img.shape[0] - 1):
        for column in range(img.shape[1] - 1):
            blur_pixel = 0
            for p in range(kernel_mat.shape[0]):
                for q in range(kernel_mat.shape[1]):
                    blur_pixel += kernel_mat[p, q] * img[row - p + 1, column - q + 1]
            blurred_img[row, column] = blur_pixel
    return blurred_img
```

Miscellanous functions

In [12]:
```python
def compute_loss(blurred_img, deblurred_img, kernel_mat):
    """
    Computes the data loss
    blurred_img: The blurred image
    deblurred_img: The deblurred image
    kernel_mat: The kernel used
    """
    loss = 0
    for x in range(blurred_img.shape[0]):
        for y in range(blurred_img.shape[1]):
            convolution = 0
            for p in range(kernel_mat.shape[0]):
                for q in range(kernel_mat.shape[1]):
                    convolution += kernel_mat[p, q] * deblurred_img[x - p + 1, y - q + 1
            loss += (blurred_img[x, y] - convolution) ** 2
    return loss


def compute_diff(deblurred_img, sample_img):
    """
    Calculates the sum of squared differences
    deblurred_img: The deblurred image
    sample_img: The unblurred image
    """
    return np.sum(np.power(deblurred_img - sample_img, 2))
```

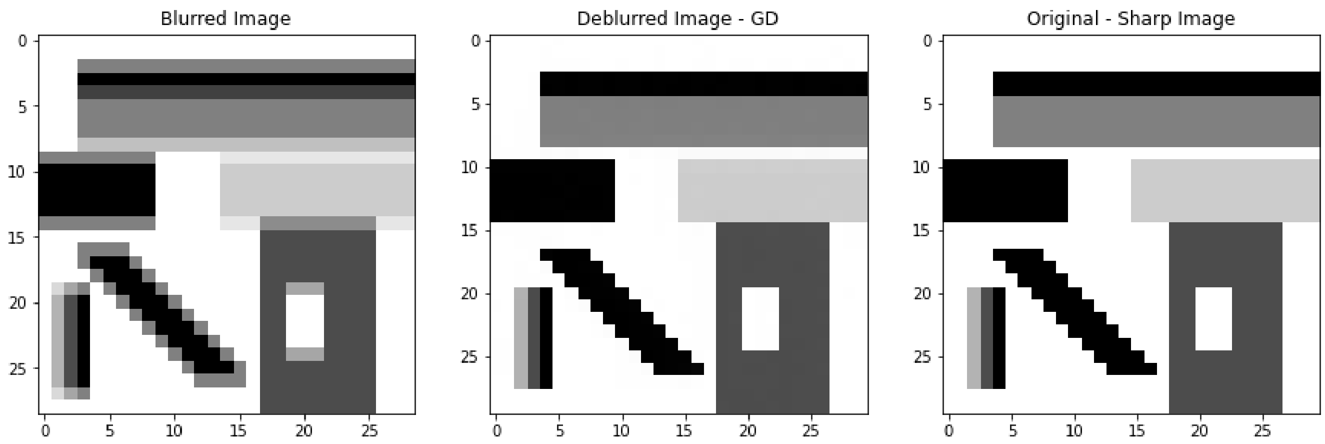
Reading image

You are provided with only the synthetic image. The purpose of using a synthetic image is to see possible changes in the deblurred image after applying your algorithm. You can convolve the original image using the [scipy library (https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve2d.html)](https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.convolve2d.html). Note that you must choose **valid** as the mode in your convolution.

In [13]:
```python
img_org = sample_image = create_random_binary_img_1()

# Kernel k = 1
kernel = np.array([[0.5, 0], [0.5, 0]])

img_blurred = blurring(sample_image, kernel)
```

GRADIENT DESCENT

In [14]:
```python
# run gradient descent algorithm
img_deblurred_gd = GD(img_blurred, reg_lambda, "aniso")

# for visualization purpose
# img_deblurred_gd = np.clip(u_final_gd, 0, 1)

plt.figure(figsize=(15, 5))
ax1 = plt.subplot(1, 3, 1)
plt.imshow(img_blurred, cmap='gray')
ax2 = plt.subplot(1, 3, 2)
plt.imshow(img_deblurred_gd, cmap='gray')
ax3 = plt.subplot(1, 3, 3)
plt.imshow(img_org, cmap='gray')

ax1.set_title("Blurred Image")
ax2.set_title("Deblurred Image - GD")
ax3.set_title("Original - Sharp Image")
plt.show()

print(f"For lambda {reg_lambda:.5f}\t \nk_type {k:1d}")
print(f"Loss Data Term = {compute_loss(img_blurred, img_deblurred_gd, kernel):.5f}")
print(f"Sum of Squared Distance: deblurred vs. original = {compute_diff(img_deblurred_g
```

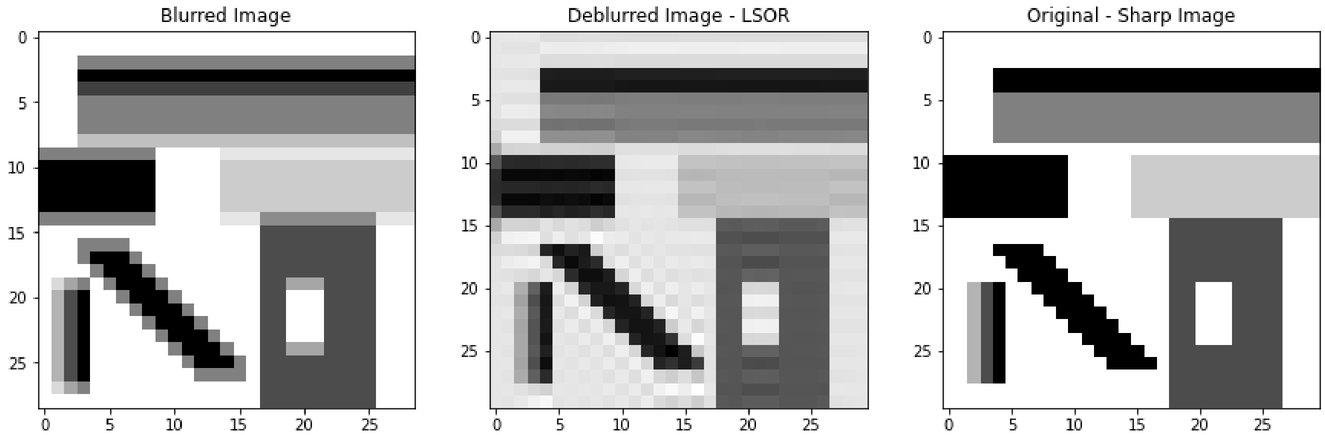



```
For lambda 0.00500
k_type 1
Loss Data Term = 0.00733
Sum of Squared Distance: deblurred vs. original = 0.01874
```

SOR

In [15]:
```python
w = 1.075  #play with this

# run LSOR algorithm
img_deblurred_sor = LSOR(img_blurred, reg_lambda, w)

plt.figure(figsize=(15, 5))
ax1 = plt.subplot(1, 3, 1)
plt.imshow(img_blurred, cmap='gray')
ax2 = plt.subplot(1, 3, 2)
plt.imshow(img_deblurred_sor, cmap='gray')
ax3 = plt.subplot(1, 3, 3)
plt.imshow(img_org, cmap='gray')

ax1.set_title("Blurred Image")
ax2.set_title("Deblurred Image - LSOR")
ax3.set_title("Original - Sharp Image")
# plt.savefig("results_" + str(k) + ".png")
plt.show()

print(f"For lambda {reg_lambda:.5f}")
print(f"Loss Data Term = {compute_loss(img_blurred, img_deblurred_sor, kernel):.5f}")
print(f"Sum of Squared Distance: deblurred vs. original = {compute_diff(img_deblurred_so
```

```
c:\users\brian\appdata\local\programs\python\python39\lib\site-packages\scipy\sparse\l
inalg\dsolve\linsolve.py:318: SparseEfficiencyWarning: splu requires CSC matrix format
  warn('splu requires CSC matrix format', SparseEfficiencyWarning)
c:\users\brian\appdata\local\programs\python\python39\lib\site-packages\scipy\sparse\l
inalg\dsolve\linsolve.py:215: SparseEfficiencyWarning: spsolve is more efficient when
sparse b is in the CSC matrix format
  warn('spsolve is more efficient when sparse b '
```

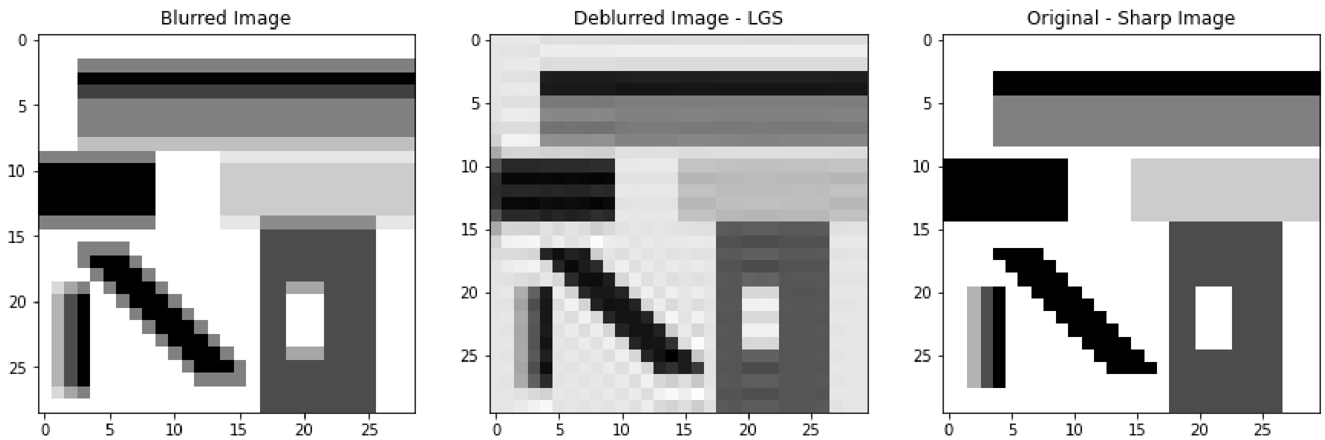



```
For lambda 0.00500
Loss Data Term = 0.03840
Sum of Squared Distance: deblurred vs. original = 1.93167
```

GAUSS - SEIDEL

In [16]:
```
nb_iter = 1

# run LGS algorithm
img_deblurred_gs = LGS(img_blurred, reg_lambda)

plt.figure(figsize=(15, 5))
ax1 = plt.subplot(1, 3, 1)
plt.imshow(img_blurred, cmap='gray')
ax2 = plt.subplot(1, 3, 2)
plt.imshow(img_deblurred_gs, cmap='gray')
ax3 = plt.subplot(1, 3, 3)
plt.imshow(img_org, cmap='gray')

ax1.set_title("Blurred Image")
ax2.set_title("Deblurred Image - LGS")
ax3.set_title("Original - Sharp Image")
#plt.savefig("figures/results_LSOR_"+str(k)+".png")
plt.show()

print(f"For lambda {reg_lambda:.5f}")
print(f"Loss Data Term = {compute_loss(img_blurred, img_deblurred_gs, kernel):.5f}")
print(f"Sum of Squared Distance: deblurred vs. original = {compute_diff(img_deblurred_gs
```

```
c:\users\brian\appdata\local\programs\python\python39\lib\site-packages\scipy\sparse\l
inalg\dsolve\linsolve.py:144: SparseEfficiencyWarning: spsolve requires A be CSC or CS
R matrix format
  warn('spsolve requires A be CSC or CSR matrix format',
```



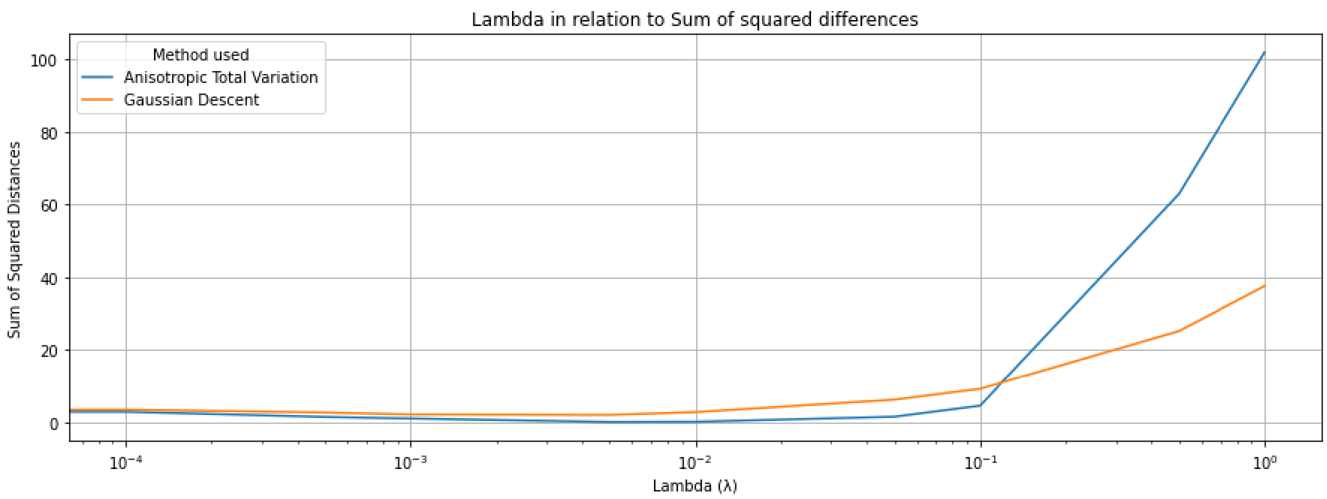

```
For lambda 0.00500
Loss Data Term = 0.03844
Sum of Squared Distance: deblurred vs. original = 1.92149
```

Sum of Squared Distances Calculation

In [17]:
```python
lambdas = [0, 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 1]
ssd_atv = []
ssd_gauss = []

for value in lambdas:
    img_anisotropic = GD(img_blurred, value, "aniso")
    ssd_atv.append(compute_diff(img_anisotropic, sample_image))

    img_gaussian = GD(img_blurred, value, "gauss")
    ssd_gauss.append(compute_diff(img_gaussian, sample_image))
```

Sum of Squared Distances Plotting

In [19]:
```python
plt.figure(figsize=(15, 5))

plt.title("Lambda in relation to Sum of squared differences")

plt.xlabel("Lambda (λ)")
plt.ylabel("Sum of Squared Distances")
plt.plot(lambdas, ssd_atv, label = "Anisotropic Total Variation")
plt.plot(lambdas, ssd_gauss, label = "Gaussian Descent")
plt.legend(title='Method used')
plt.xscale("log")
plt.grid()
plt.show()
```



Optimal values are between 10^-3 and 10^-1 as expected, thanks to the exercise description. This fits for both Anisotropic Total Variation and Gaussian Descent.

If we move above the optimal values, the reconstructed image is very noise and below the impact of the regularization term is negligible.