One-sample tests

Hypothesis testing

Testing the validity of various hypotheses about the population based on sampling results.

- **Tools:**

  Statistical tests

- **Results:**

  it is never a question of whether a hypothesis is true or not, but always of how plausible the hypothesis is in the light of the results of the sampling

Task

A sample is taken to check that the machine set for a 500-gram load is operating correctly in terms of average load weight. $(\alpha = 5\%)$

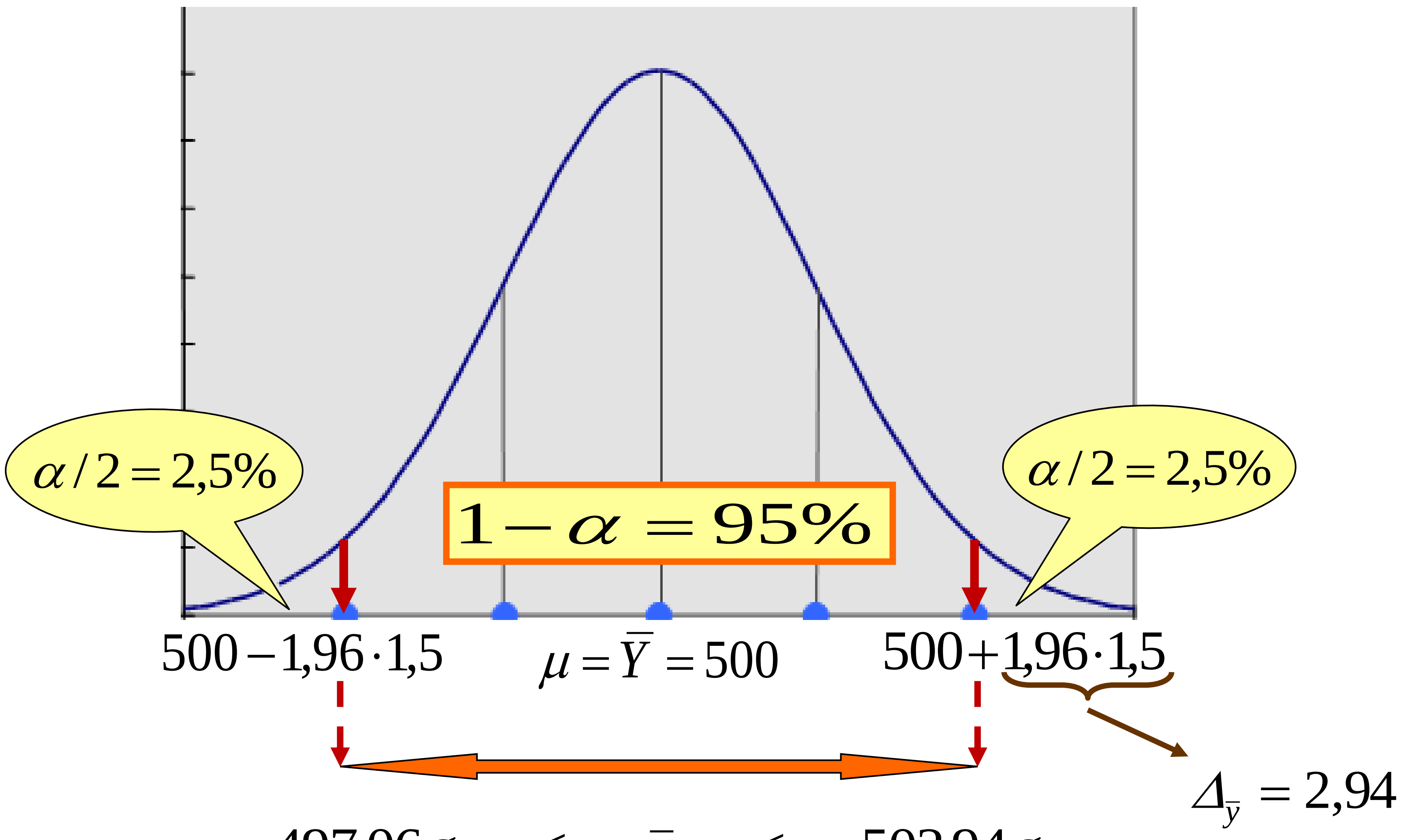
- **Population sd** $\sigma = 15\text{g}$
- ***n=100 - IID***
- **If the statement is true:**

$$\bar{y} \quad \sim \quad N(\mu, \sigma/\sqrt{n})$$

$$\bar{y} \quad \sim \quad N(500\text{ g}, 15/\sqrt{100} = 1{,}5\text{ g})$$

Distribution of sample means

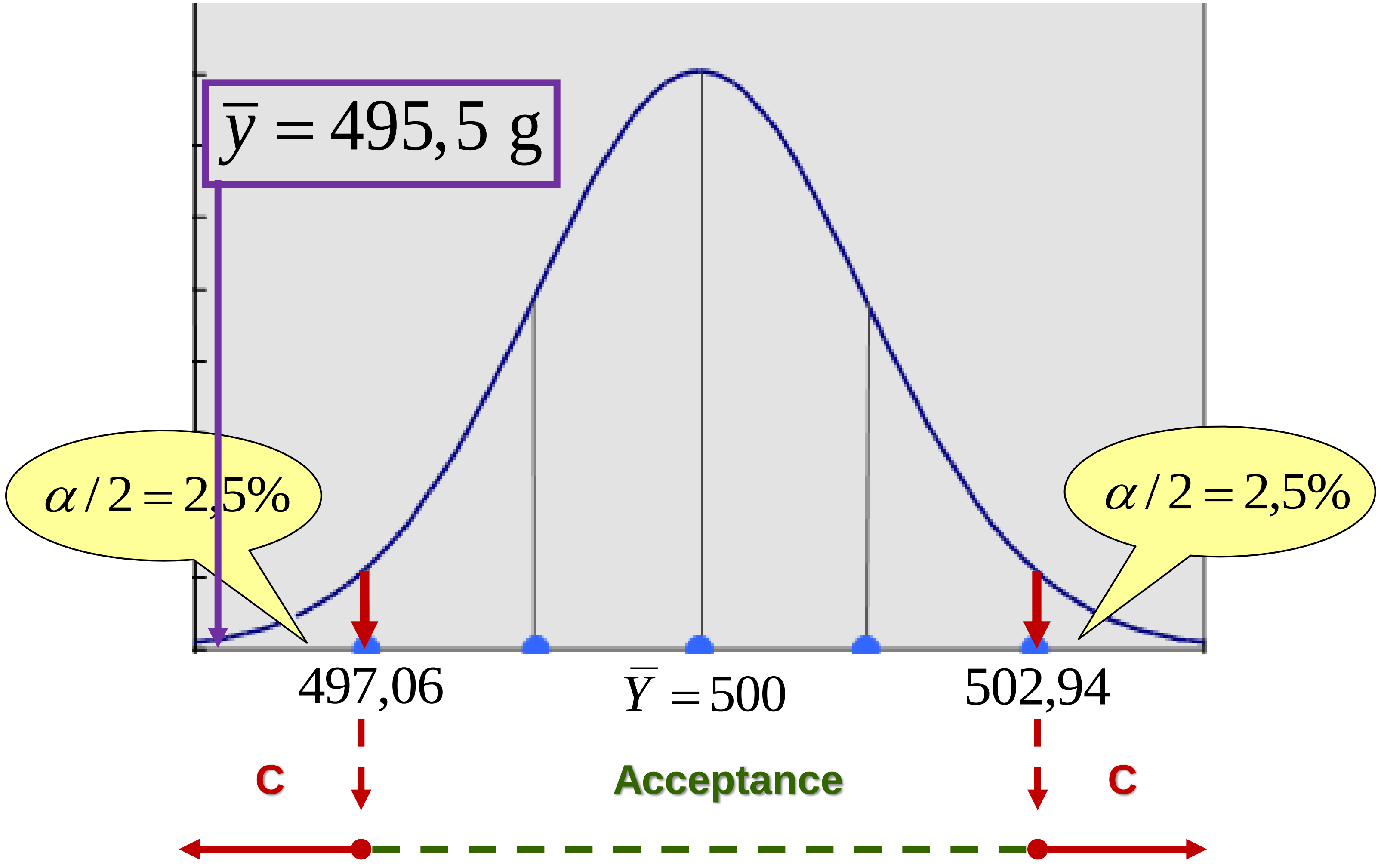
$$\bar{y} \sim N(500\,\text{g}\,,1{,}5\,\text{g}) \qquad \alpha = 5\%$$



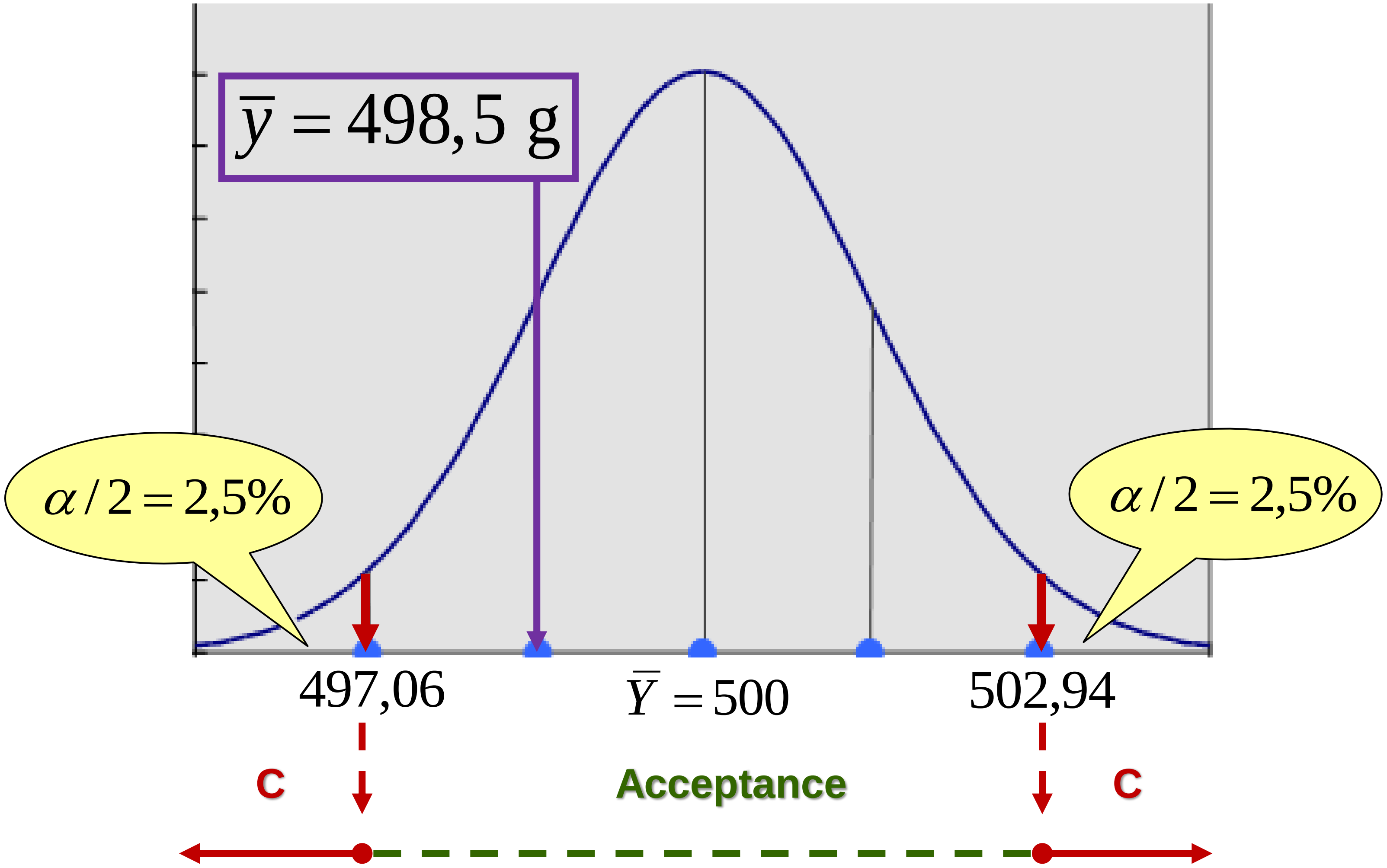
$$497{,}06\,\text{g} \quad < \quad \bar{y} \quad < \quad 502{,}94\,\text{g}$$

Decision:

Hypothesis is not acceptable
(mu=500)



Decision: The hypothesis is acceptable
(mu=500)



Steps of hypothesis testing

1. Define hypotheses
2. Choose the appropriate test
3. Significance level and critical value(s)
4. Sampling
5. Decision

Hypothesis

assumption(s) about the distribution of the population(s)
or one or more parameters of that distribution(s)

Null hypothesis (H_0)

Statistical tests are used to test the null hypothesis against alternative hypotheses

Alternative hipothesis (H_1)

H0 and H1 are mutually exclusive.

We make decision on it.

We make decision indirectly.

Task

1. One statement is that gout is a widespread disease, with more than 10% of the population over 50 suffering from the condition. A random sample of 500 observations used to test the statement.

$$H_0^T : P = 0,1 \qquad H_0 : \ P \leq 0,1 \qquad H_1 : \ P > 0,1$$

2. A canner will take delivery of tomatoes from a grower on condition that the given delivery contains at least 95% first class tomatoes. The hypothesis is tested by sampling.

$$H_0^T : P = 0,95 \qquad H_0 : \ P \geq 0,95 \qquad H_1 : \ P < 0,95$$

Task

3. To secure appropriate health condition, a person needs 40 minutes physical activite in a day. We test it from a sample.

$$H_0: \ \mu = 40 \qquad H_1: \mu \neq 40$$

4. We produce lasagne pasta in 250 gram packs. The maximum value of standard deviation is 10 grams. How would you test it?

$$H_0^T: \sigma = 10 \qquad H_0: \ \sigma \leq 10 \qquad H_1: \ \sigma > 10$$

Defining hypotheses

Left-tailed **Right-tailed**

$$H_0 : \mu = \mu_0 \qquad H_0 : \mu \geq \mu_0 \qquad H_0 : \mu \leq \mu_0$$

$$H_1 : \mu \neq \mu_0 \qquad H_1^b : \mu < \mu_0 \qquad H_1^j : \mu > \mu_0$$

Two-sided **One-sided**

Null hypothesis (H_0)

- **Simple** $$H_0:\ \mu = \mu_0$$

- **Complex:**

$$H_0:\ \mu \le \mu_0 \qquad H_0:\ \mu \ge \mu_0$$

Technical null hypothesis:

$$H_0^T:\ \mu = \mu_0$$

The simple H_0 hypothesis that least contradicts the one-sided alternative hypothesis.

Significance level and critical range

To check the correctness of the hypothesis, the entire range of possible values of the test function is split into two non-overlapping parts using suitable split points:

- **Acceptance range**

$$P\left(T\left(y_1,\ y_2,\ \ldots\ ,\ y_n\right)\in E\right)=1-\alpha$$

- **Rejection/critical range**

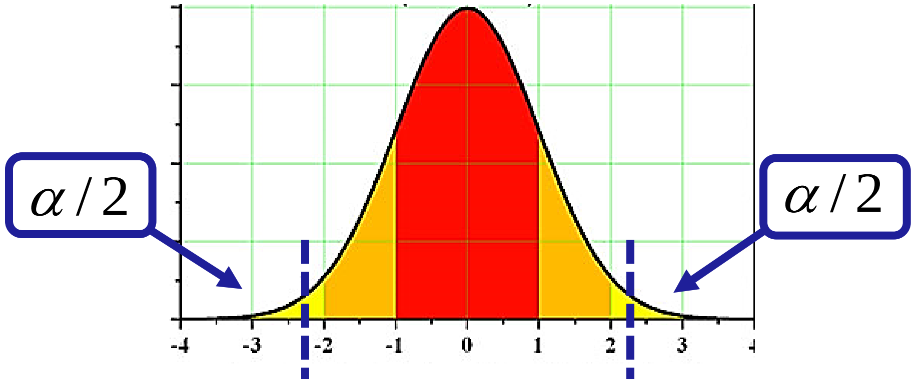
$$P\left(T\left(y_1,\ y_2,\ \ldots\ ,\ y_n\right)\in K\right)=\alpha$$

α : **significance level**

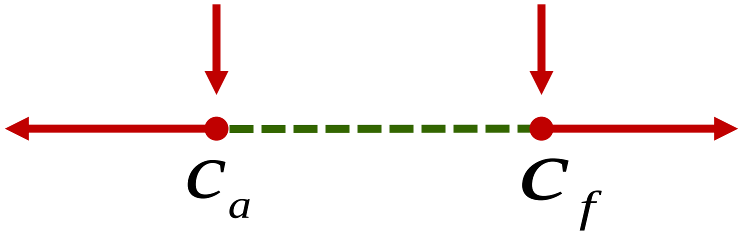
Simple case

1. $H_o : \mu = \mu_o$

$H_1 : \mu \neq \mu_o$



$$Z_{\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}}$$

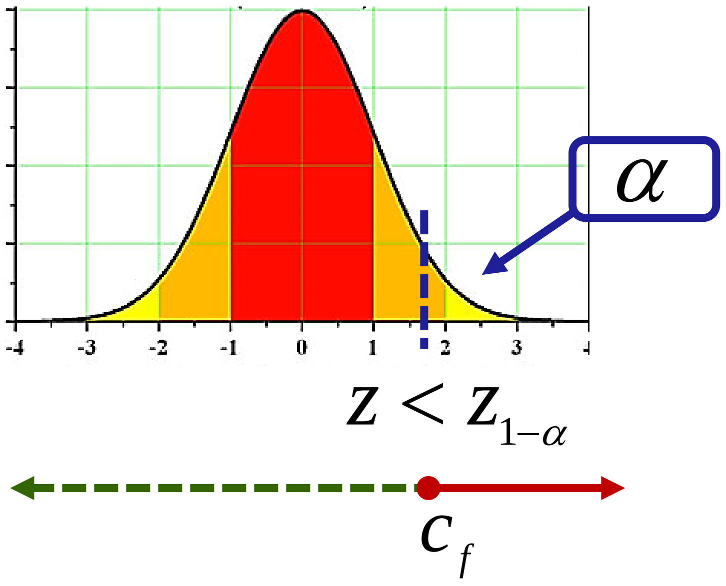


$$Z_{\frac{\alpha}{2}} = -Z_{1-\frac{\alpha}{2}}$$

$\alpha = 1\%$	$c_l = -2{,}575$	$c_u = 2{,}575$
$\alpha = 5\%$	$c_l = -1{,}96$	$c_u = 1{,}96$
$\alpha = 10\%$	$c_l = -1{,}645$	$c_u = 1{,}645$

Right-sided case

2. $$H_0^T : \mu = \mu_0$$
$$H_1^j : \mu > \mu_0$$



$$\alpha = 1\% \qquad c_u = z_{0,99} = 2{,}33$$

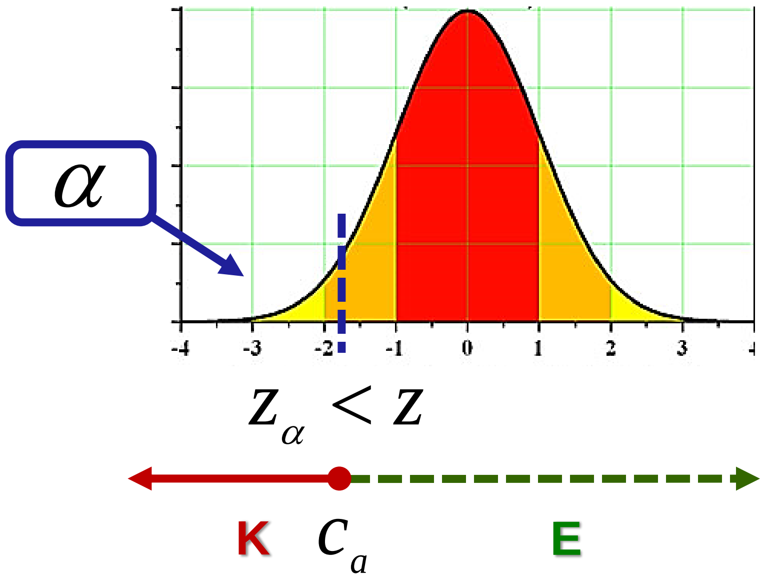
$$\alpha = 5\% \qquad c_u = z_{0,95} = 1{,}645$$

$$\alpha = 10\% \qquad c_u = z_{0,9} = 1{,}28$$

Left-sided case

3. $H_0^T : \mu = \mu_0$

$H_1^b : \mu < \mu_0$



$$\alpha = 1\% \qquad c_l = -z_{0,99} = -2,33$$

$$\alpha = 5\% \qquad c_l = -z_{0,95} = -1,645$$

$$\alpha = 10\% \qquad c_l = -z_{0,9} = -1,28$$

Calculation of test function

$$\bar{y} = 495,5 \text{ g}$$

$$z = \frac{\bar{y} - \mu_0}{\sigma / \sqrt{n}}$$

$$z = \frac{495,5 - 500}{15/\sqrt{100}} = \frac{-4,5}{1,5} = -3$$

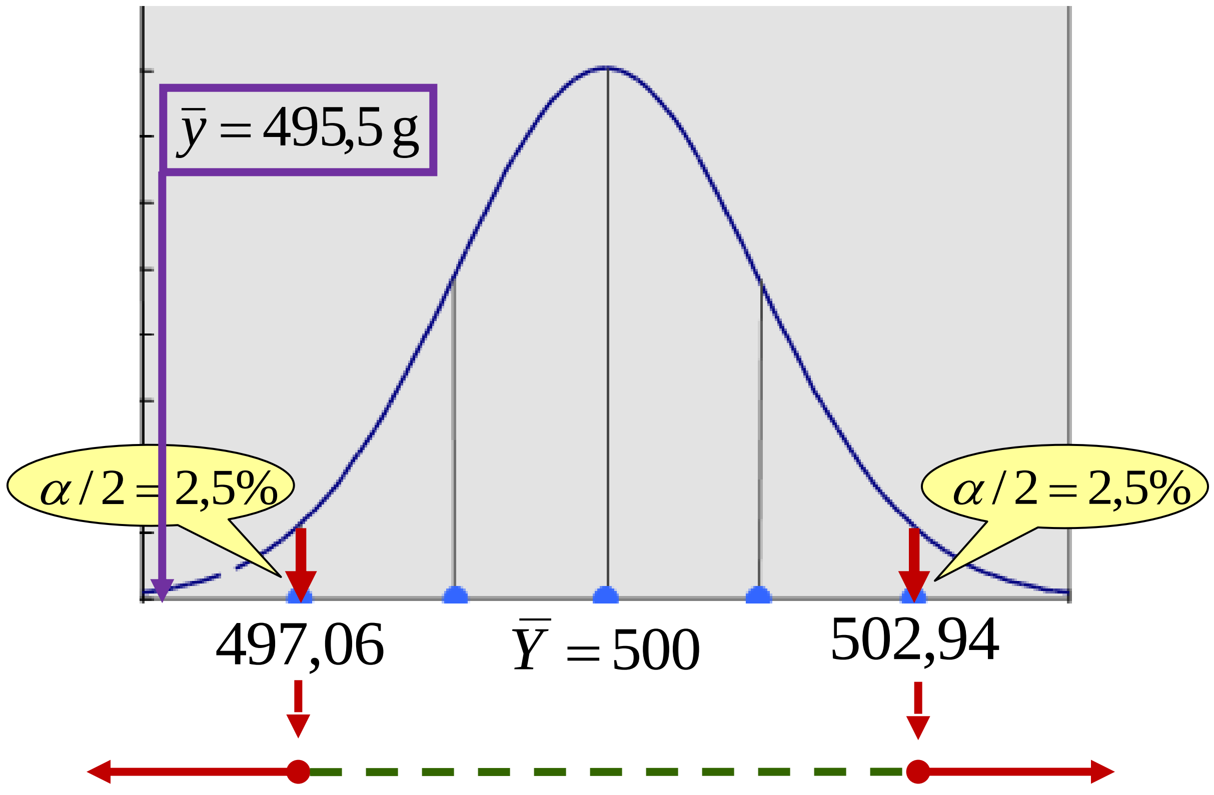
Decision

$$H_1: \quad \mu \neq 500$$

The value of the test function is -3.

$c_l = -1{,}96 \qquad c_u = 1{,}96$

Since this value is less than -1.96, i.e. the value of the test functionfalls within the rejection range, therefore H0 is rejected at the 5% significance level.



$\boxed{1.\ \ \bar{y} \le 497{,}06 \rightarrow z \le -1{,}96}$

$$\boxed{\begin{array}{c} 2.\ \ 497{,}06 < \bar{y} < 502{,}94 \\ \downarrow \\ -1{,}96 < z < 1{,}96 \end{array}}$$

$\boxed{3.\ \ \bar{y} \ge 502{,}94 \rightarrow z \ge 1{,}96}$

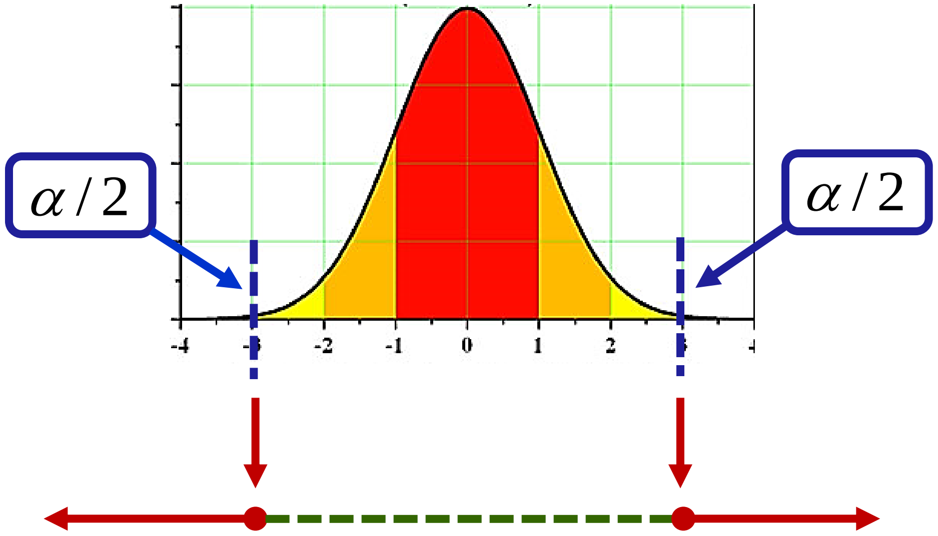
$$z = \frac{497{,}06 - 500}{15/\sqrt{100}} = \frac{-2{,}94}{1{,}5} = -1{,}96$$

$$z = \frac{502{,}94 - 500}{15/\sqrt{100}} = \frac{2{,}94}{1{,}5} = 1{,}96$$

P-value
(empirical significance level)

1. $H_o: \mu = 500$
$H_1: \mu \neq 500$

$z = -3$



$\alpha = 5\%$ $\quad c_l = -1{,}96$ $\quad c_u = 1{,}96$

$\alpha = 1\%$ $\quad c_l = -2{,}575$ $\quad c_u = 2{,}575$

$\alpha = 0{,}5\%$ $\quad c_l = -2{,}81$ $\quad c_u = 2{,}81$

$\alpha = 0{,}3\%$ $\quad c_l = -2{,}96$ $\quad c_u = 2{,}96$

P-value

Value of the test function: $z = -3$

$H_1:\ \mu \neq 500g$ **(Two-sided)**

$z_{p/2} = -3$ $z_{1-p/2} = 3$ $\Phi(3) = 0,9987$

$1 - p/2 = 0,9987$ $p = 2(1 - 0,9987) = 0,0026$

P-value

The p-value is the lowest significance level at which hypothesis H0 can be rejected

Decision based on p-value:

$p \le \alpha_0$ ***Reject H_0***

$p > \alpha_0$ **fail to reject H_0**

Errors that can be made while deciding about H0

- **Type I error:**

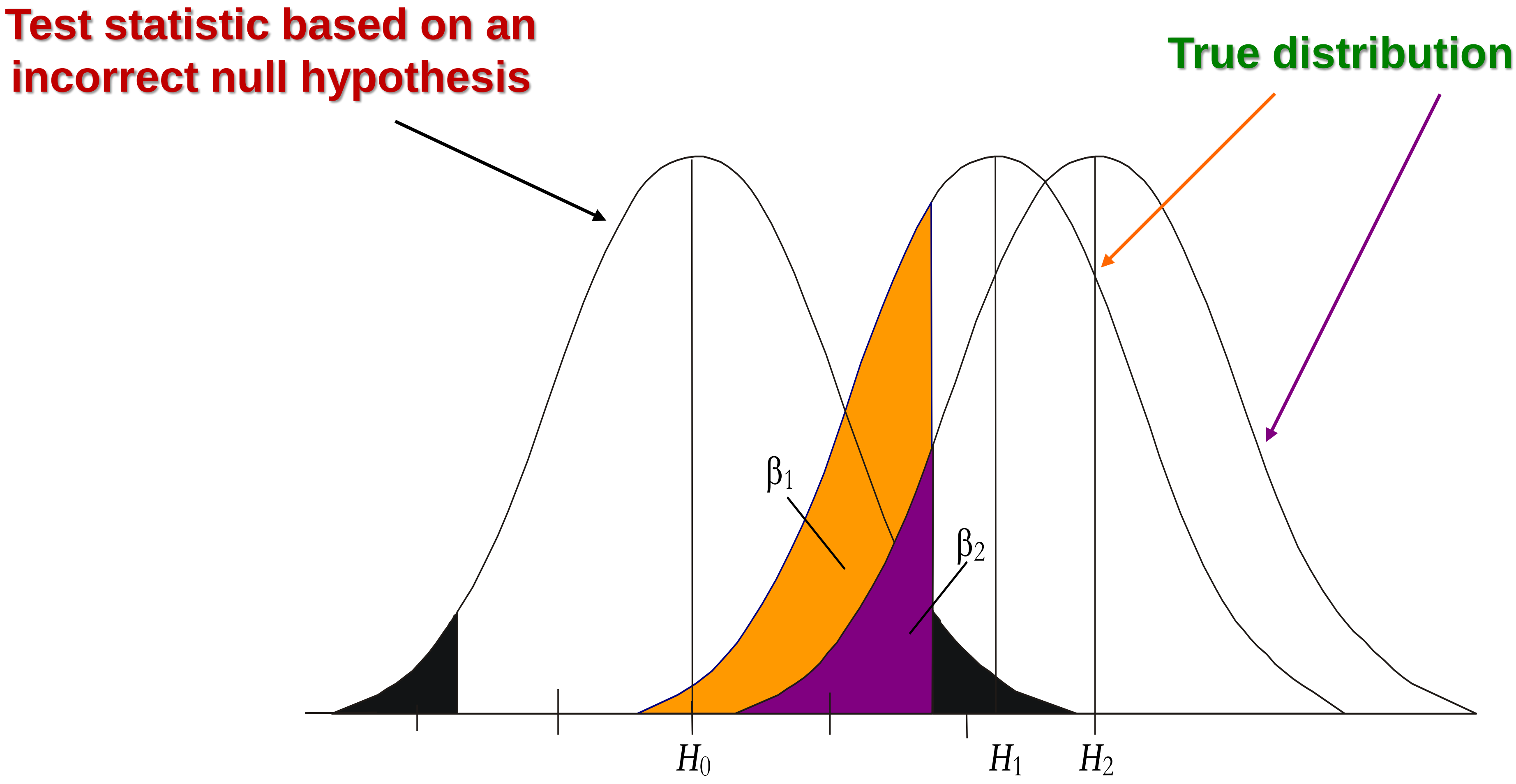
  We reject H_0, even though it is actually true (we reject H_0 incorrectly)

- **Type II error:**

  We do not reject (we accept) H_0, even though it is not true in reality.

  (We incorrectly fail to reject H_0.)

Type II error



What does the probability of committing a Type II error depend on?

- What is the value of α?
- How „far” is the true null hypothesis?
- What is the sample size?
  - (The probabilities of Type I and Type II errors can only be reduced simultaneously by increasing the sample size)

Decision tavle

Strict „hard” decision

H_0 In reality	H_0 hypothesis	
	We accept	We reject
True	Correct decision $(1-\alpha)$	Type I error (α)
False	Type II error (β)	Correct decision $(1-\beta)$

Weak „soft” decision

One-sample parametric tests for the expected value

$$H_0^T : \mu = \mu_0$$

1. Z-test

Conditions for application:

- normally distributed population
- **known** population sd
- IID sample

Test statistic:

$$z = \frac{\bar{y} - \mu_0}{\sigma / \sqrt{n}}$$

Regardless of the sample size, the standardized sample mean follows a N(0,1) distribution.

One-sample parametric tests for the expected value

$$H_0^T : \mu = \mu_0$$

2. T-test

Conditions for application:

- normally distributed population
- **unknown** population sd
- IID sample

Test statistic:

$$t = \frac{\bar{y} - \mu_0}{s / \sqrt{n}}$$

The test statistic follows a t-distribution with n – 1 degrees of freedom.

One-sample parametric tests for the expected value

$$H_0^T : \mu = \mu_0$$

3. Asymptotic Z-test

Conditions for application:

- **large sample** from any distribution with finite variance (IID sample)

Test statistics:

$$z = \frac{\bar{y} - \mu_0}{s / \sqrt{n}}$$

The test statistic is asymptotically standard normally distributed due to CLT.

Large-sample test for a population proportion

$$H_0^T: \quad P = P_0$$

Condition for application:

- ***n*** **element IID sample**

Test statistic:

$$z = \frac{p - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}}$$

If the sample taken from the population is large enough such that

$$\min\{nP_0, n(1-P_0)\} \geq 10$$

,then the test statistic is asymptotically standard normally distributed.

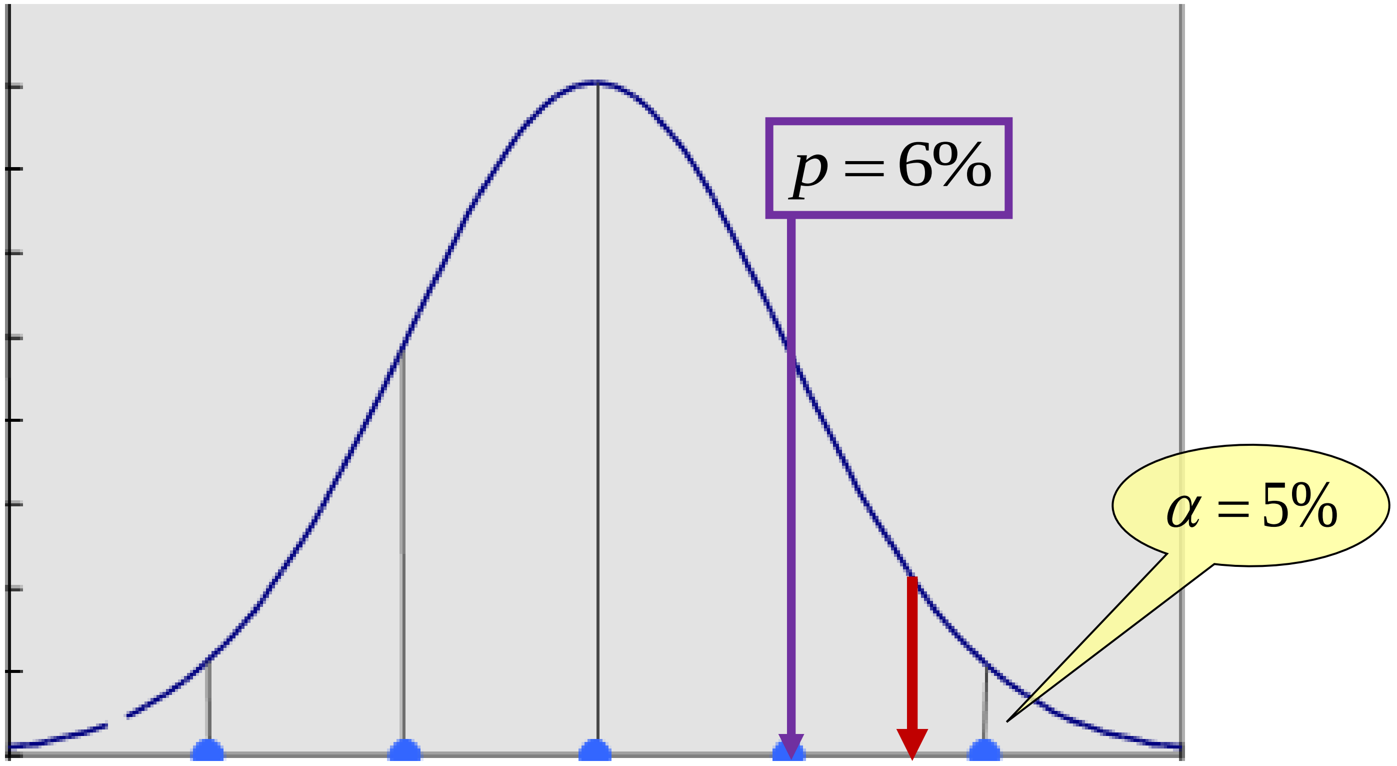
Example: proportion of defective products

- **Can it be stated at a 5% significance level that the proportion of defective products in the population is above 5%?**
- ***n = 500, IID sample*,**
- ***k = 30* (number of defective products)**
- **Hypotheses:**

$$H_0^T : P = 0{,}05$$

$$H_0 : P \leq 0{,}05 \qquad H_1^R : P > 0{,}05$$

$$H_1^R : P > 0{,}05 \qquad s_p = \sqrt{\frac{0{,}95 \cdot 0{,}05}{500}} = 0{,}00975$$



$$P = 5\% \qquad 6{,}6\% = 5 + 1{,}645 \cdot 0{,}975$$

Acceptance region

c

Example: one-tailed (right-tailed) testing

$$H_0^T : P = 0{,}05 \qquad H_0 : P \le 0{,}05 \qquad H_1^R : P > 0{,}05$$

acceptance and rejection region of H0 (α = 5%)

$$c_u = z_{0,95} = 1{,}645$$

$$z = \frac{p - P_0}{\sqrt{\dfrac{P_0(1-P_0)}{n}}} = \frac{0{,}06 - 0{,}05}{\sqrt{\dfrac{0{,}05 \cdot 0{,}95}{500}}} = \frac{0{,}01}{0{,}00975} = 1{,}03$$

Decision: **We accept the technical null hypothesis and reject the alternative hypothesis at a 5% significance level.**

p-value in the example:

Value of the test statistic: $z = 1{,}03$

$H_1^R : P > 0{,}05$ **(one-tailed)**

$$\Phi(1{,}03) = 0{,}8485$$

$1 - p = 0{,}8485$ $\quad$ $p = 1 - 0{,}8485 = 0{,}1515$

$15{,}15\% \le \alpha_0$ ***We reject H_0***

$15{,}15\% > \alpha_0$ **We accept H_0**

Tests for Variance

$$H_0^T : \sigma^2 = \sigma_0^2$$

Conditions for application: **normal distribution, IID sample**

Test statistic:
$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

χ^2 **- distribution with n-1 degrees of freedom**

Critical values:

$$H_1^R : \quad \sigma^2 > \sigma_0^2 \qquad \rightarrow c_u = \chi^2_{1-\alpha}(\nu)$$

$$H_1^L : \quad \sigma^2 < \sigma_0^2 \qquad \rightarrow c_l = \chi^2_{\alpha}(\nu)$$

$$H_1 : \quad \sigma^2 \neq \sigma_0^2 \qquad \rightarrow c_l = \chi^2_{\alpha/2}(\nu) \text{ és } c_u = \chi^2_{1-\alpha/2}(\nu)$$

Example: standard deviation of filling weight

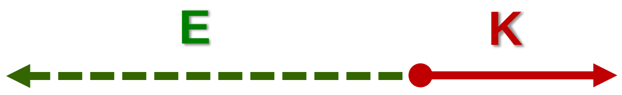
- *n* = *101,* IID sample; *s* = 12 g
- Allowed standard deviation: 10g;
- Filling weights are normally distributed
- Can it be stated at a 5% significance level that the requirement for the standard deviation is met?
- Hypotheses: $H_0^T : \sigma = 10$

$$H_0 : \sigma \leq 10 \qquad H_1^R : \sigma > 10$$

Example: one-tailed (right-tailed) testing

$$H_0^T : \sigma = 10 \qquad H_0 : \sigma \leq 10 \qquad H_1^R : \sigma > 10$$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{100 \cdot 12^2}{10^2} = 144$$



$$c_u = \chi^2_{1-\alpha}(\nu) = \chi^2_{0,95}(100) = 124{,}3$$

Decision: **We reject the technical null hypothesis and accept the alternative hypothesis at a 5% significance level.**