Probabilities and distributions

Special thanks to Rita Polt for the Hungarian raw material

Continuous distribution

If the probability variable is continuous, then the probability P(X=xk) is 0, because we are looking at the probability of a point relative to a continuum of infinities. This does not mean that the variable cannot take this value. (The probability of an impossible event is zero, but if the probability of an event is zero, it does not mean that it cannot occur at all.) In these cases, the probability of falling into a given interval is meaningful.

A continuous distribution is usually described by the distribution function. The distribution function always expresses the probability that the values of the probability variable take a value less than x. The distribution function can be used to calculate the probability of falling into an interval:

$$P(a < X < b) = F(b) - F(a)$$

Density function

$$F'(x) = f(x) \qquad F(x) = \int_{-\infty}^{x} f(t)\, dt$$

- **If the distribution F(x) is continuous and the function is derivable, then it has a density function**
- **The derivative of a distribution function is the density function**
- **Discrete distributions have no density function**
- **Not negative**

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

How to calculate probability

$$\int_a^b f(x)\,dx = P(a < X < b) = P(a \le X \le b) = F(b) - F(a)$$

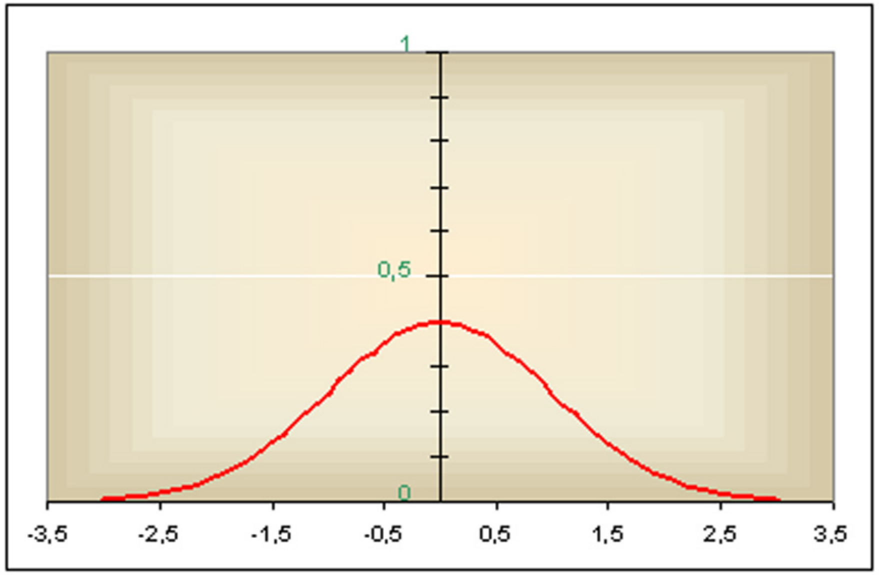
$$P(a < X < b)$$

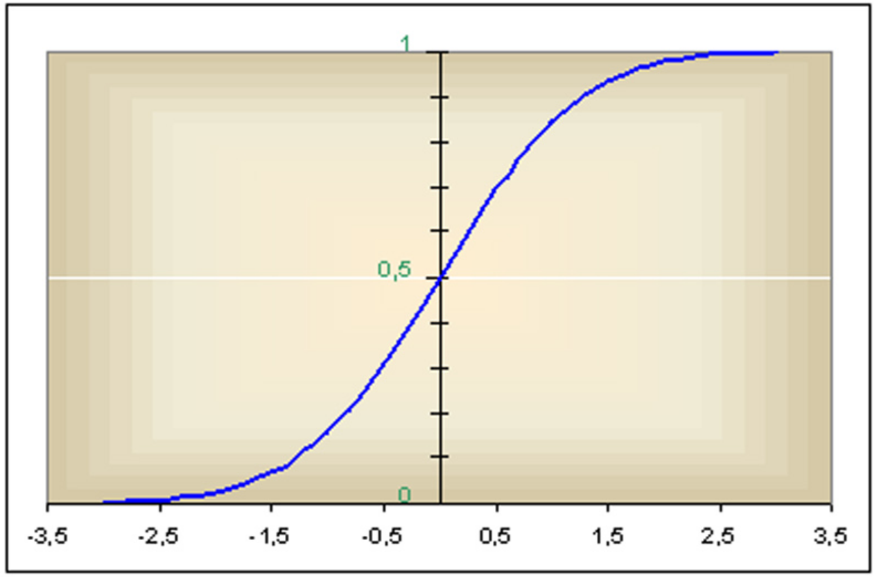
Probability is measured by the density function with the area under the function. The distribution function measures probability by the difference between the function values.

Continuous distributions

Type	Domain	CDF	Density function	Expected value	SD
Uniform	$[a\ ,\ b]$	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Normal	$(-\infty\ \text{-}\ \infty)$	**No closed form**	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$	m	σ
Exponential	$(0\ \text{-}\ \infty)$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$

Standard normal distribution





$$f\left(x\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$m=0 \quad \text{és} \quad \sigma=1$$

https://www.desmos.com/calculator/6lk0lzjjpz

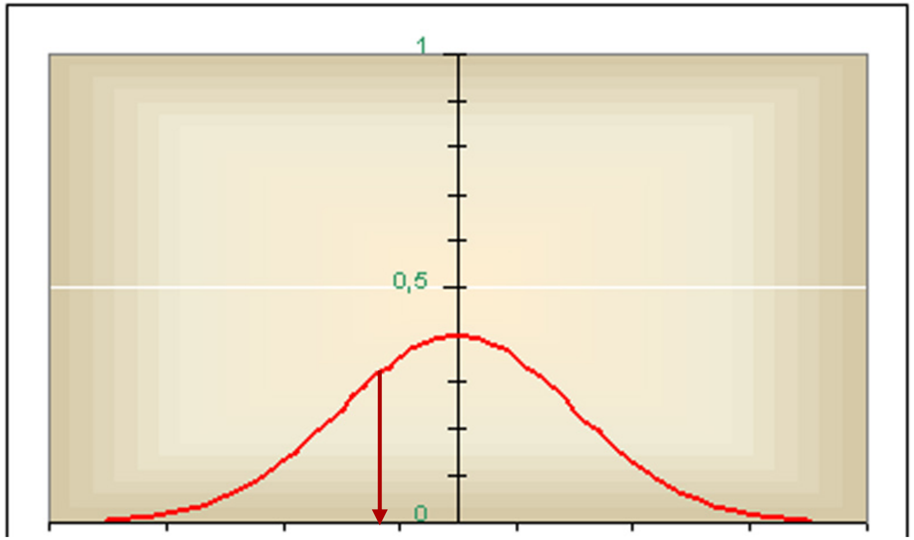
Normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

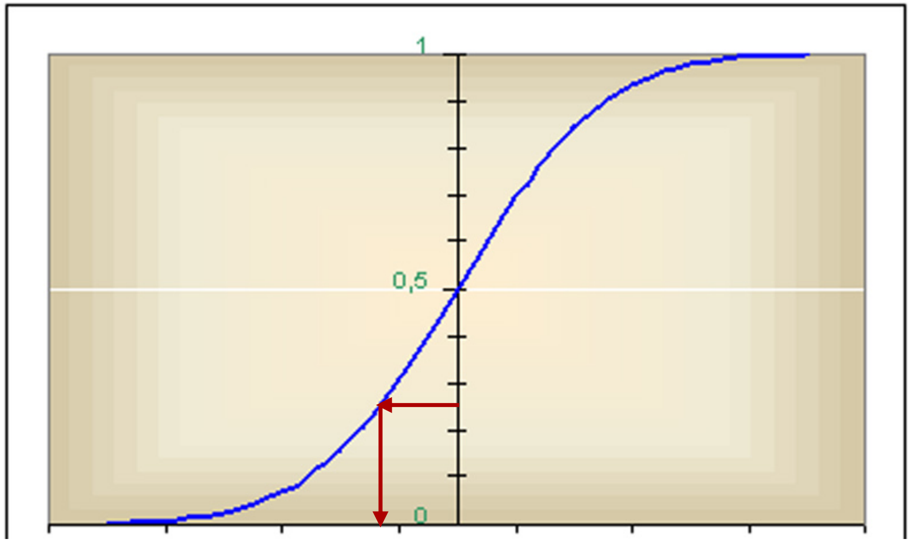
mu=3.6 million HUF and σ=0.9 million HUF

TASK

a) **What is the probability that payments will be below HUF 3 million?**



3,6



3,6

σ

Standard normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

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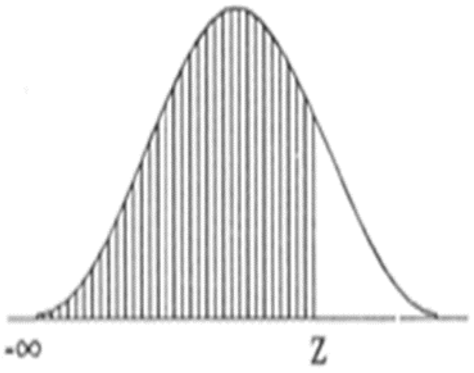
TASK

a) What is the probability that payments will be below HUF 3 million?

$$F(y) = \Phi\left(\frac{y-\mu}{\sigma}\right) = \Phi(z) = P(Z < z)$$

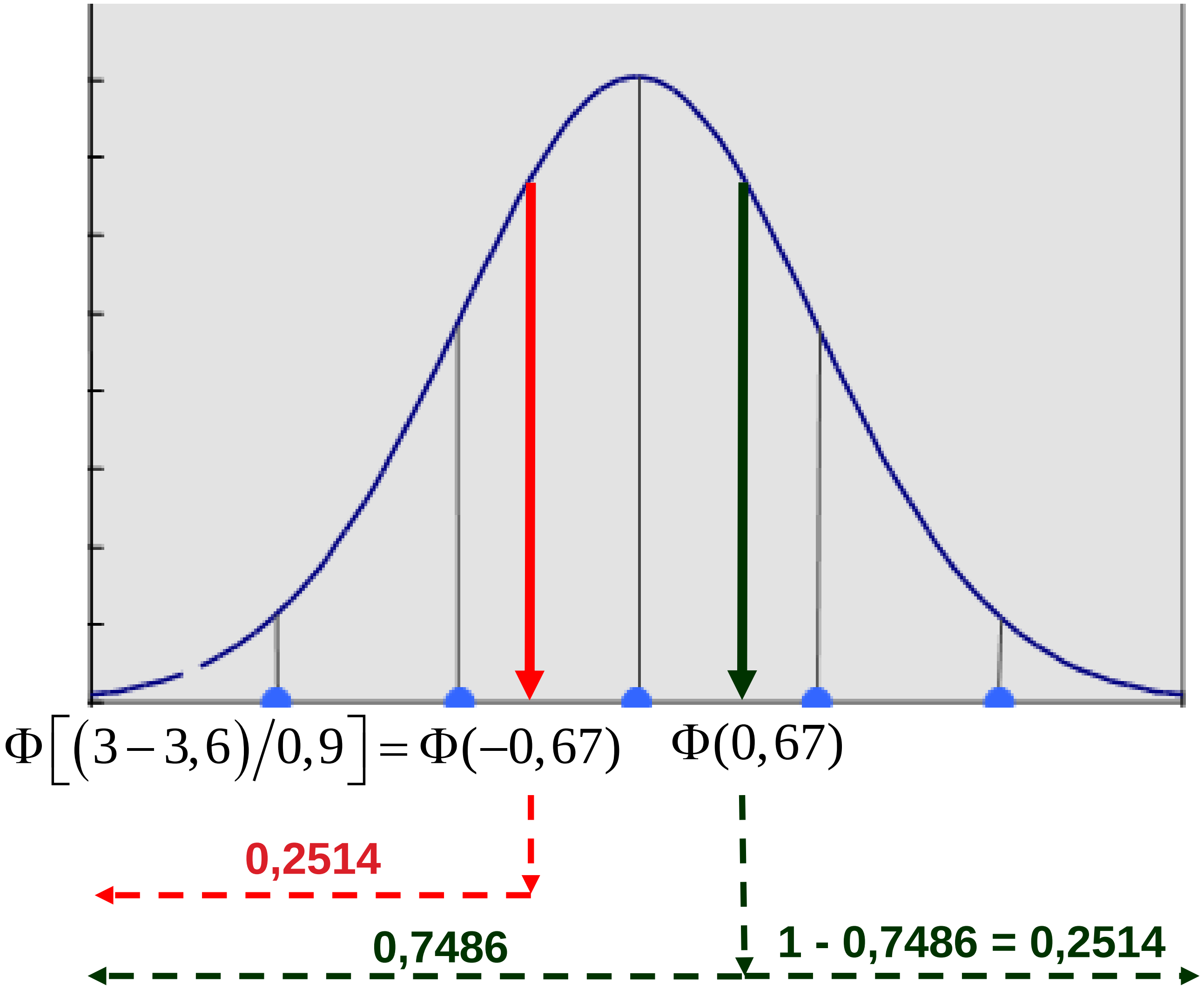


$$P(X < 3) = \Phi\left(\frac{3-3{,}6}{0{,}9}\right) = \Phi(-0{,}67) = 1 - \Phi(0{,}67) = 1 - 0{,}7486 = 0{,}2514$$

$$P(X < 3) = P(X > 4{,}2)$$

$$N(3{,}6\ ,\ 0{,}9)$$

$$\Phi(-z) = 1 - \Phi(z)$$



Standard normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

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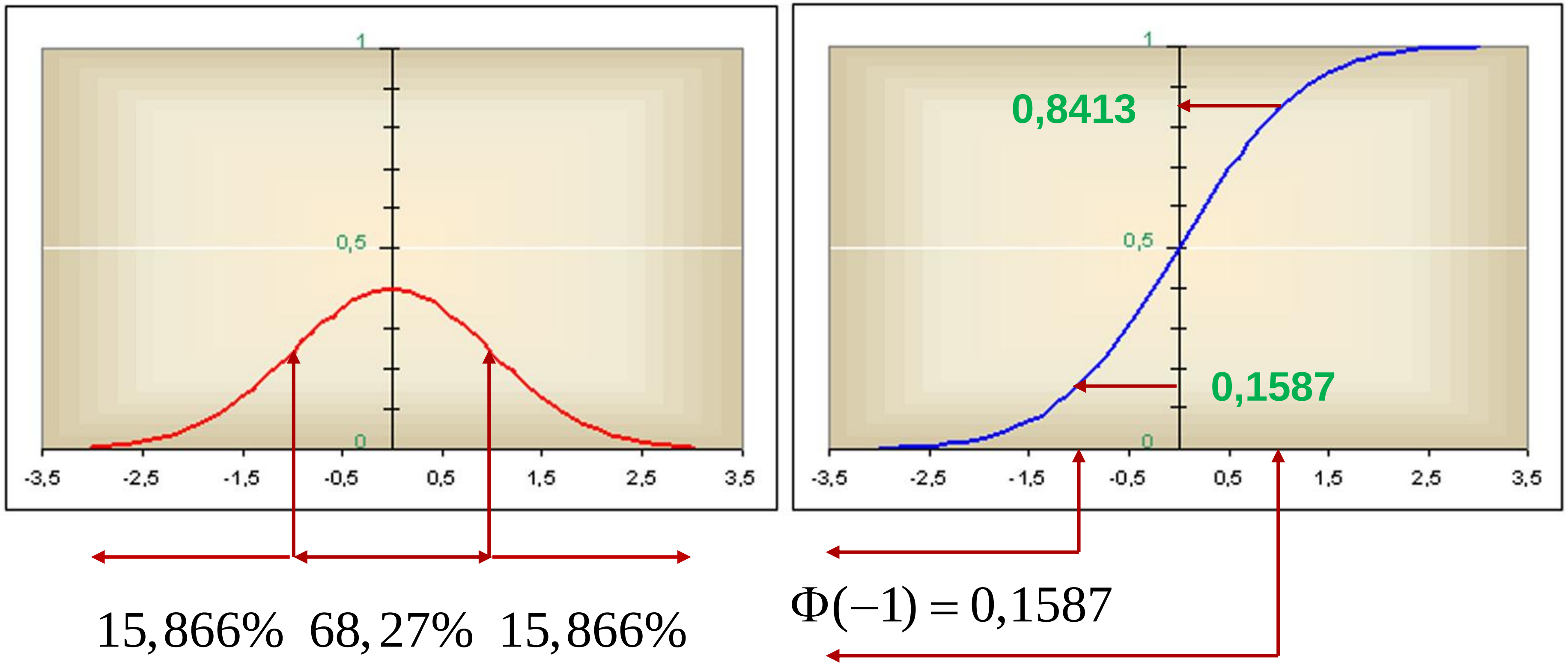
TASK

b) **What is the probability that the payments will be between HUF 2.7 and 4.5 million?**

One sigma



$$15,866\% \quad 68,27\% \quad 15,866\%$$

$$\Phi(-1) = 0,1587$$

$$\Phi(1) = 0,8413$$

Standard normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

mu=3.6 million HUF and σ=0.9 million HUF

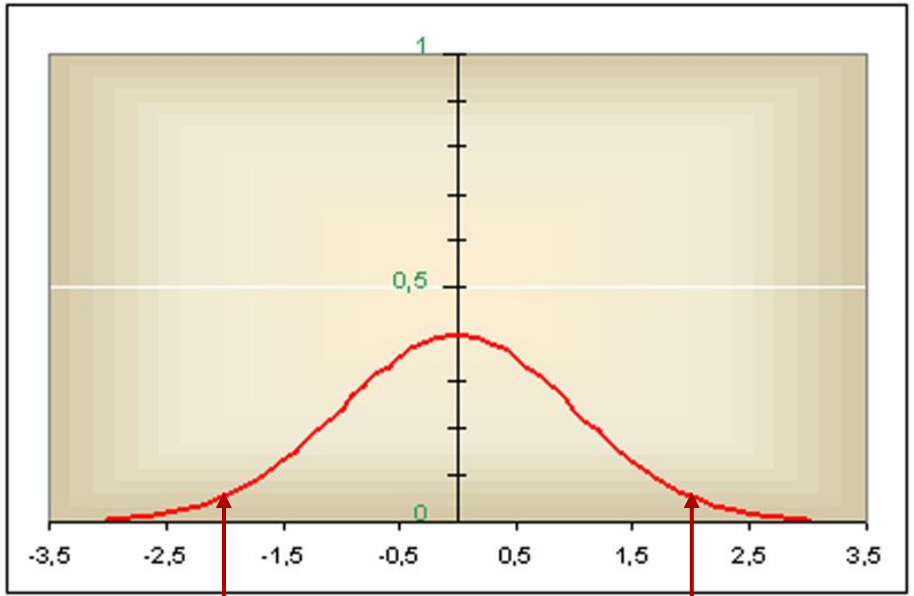
TASK

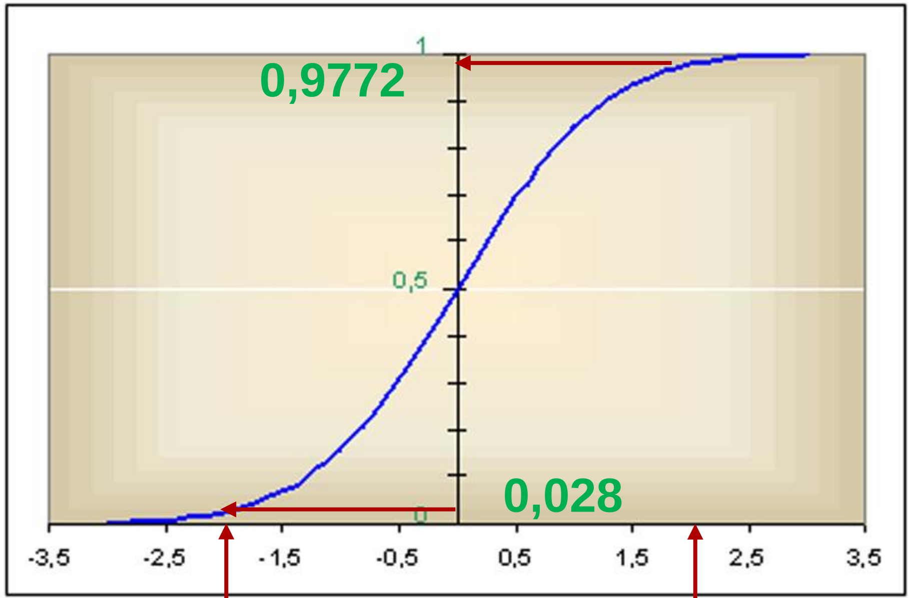
b) What is the probability that the payments will be between HUF 2.7 and 4.5 million?

$$P(2,7 < X < 4,5) = F(4,5) - F(2,7) = \Phi\left(\frac{4,5-3,6}{0,9}\right) - \Phi\left(\frac{2,7-3,6}{0,9}\right)$$

$$= \Phi(1) - \Phi(-1) = 0,8413 - (1 - 0,8413) = 0,6826$$

2 sigma rule



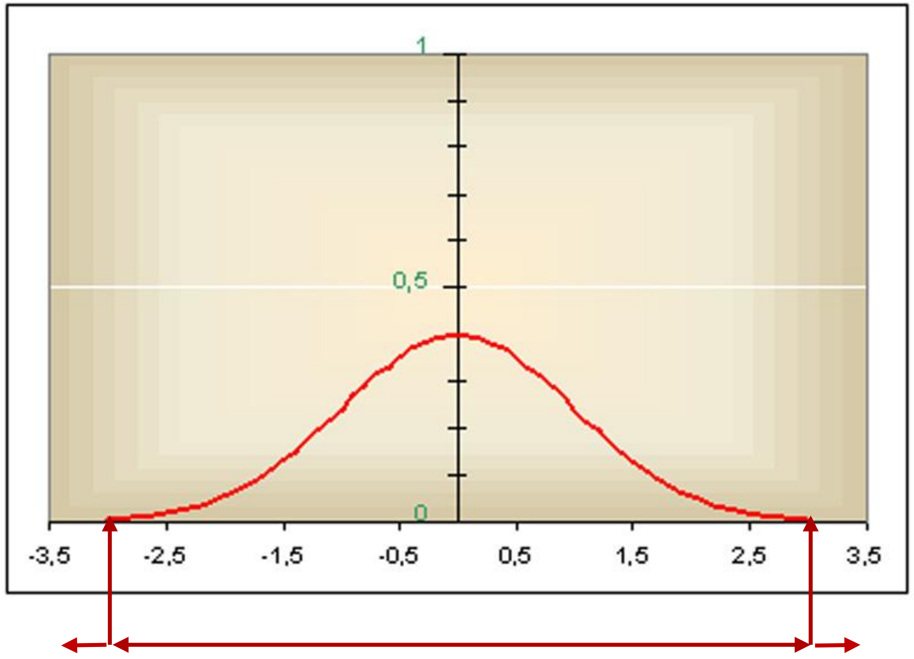


$2,275\%$ $95,45\%$ $2,275\%$

$$\Phi(-2)=0,028$$

$$\Phi(2)=0,9772$$

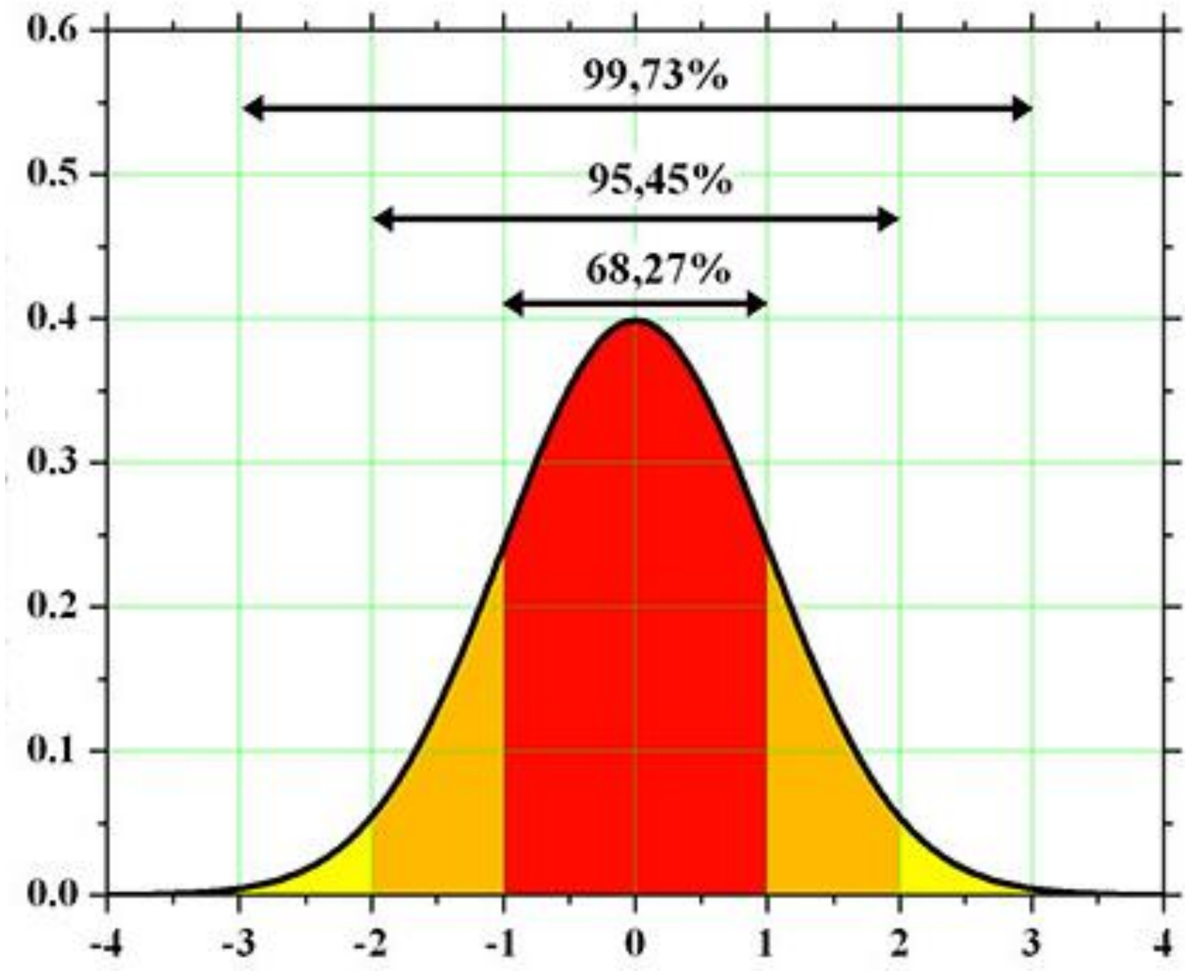
Three sigma rule



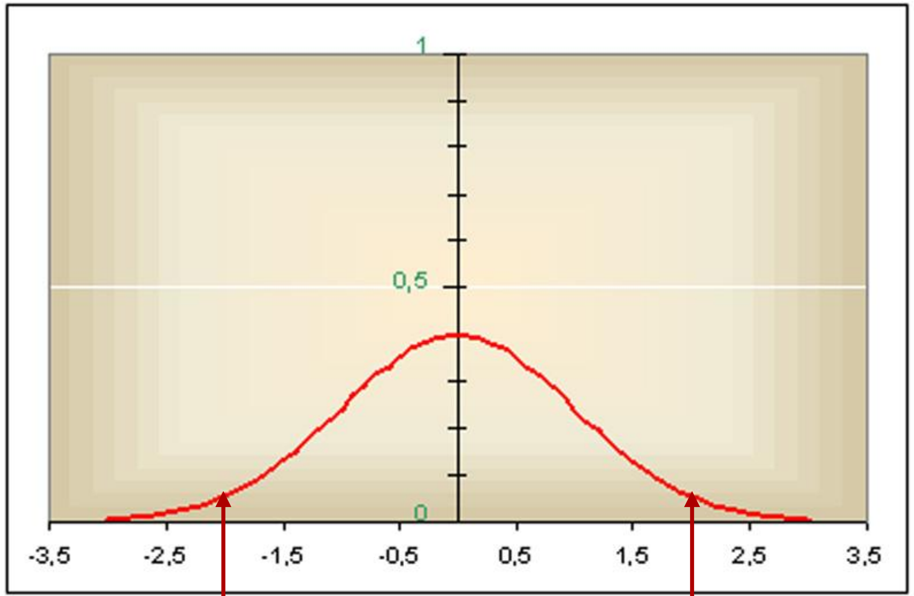
$$\Phi(3) = 0,9987$$

$0,135\%$ $99,73\%$ $0,135\%$

Density function of standard normal distribution



Inner 95%

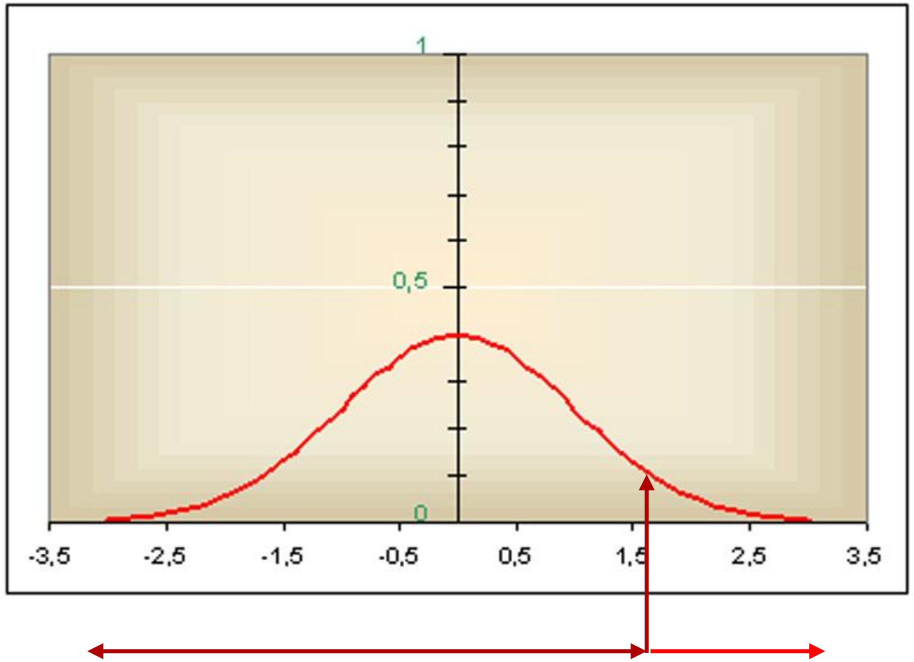


$$2,50\% \qquad 95,00\% \qquad 2,50\%$$

$$\Phi(z) = 0,975$$

$$z = 1,96$$

How many times should I add the standard deviation to the mean so that 95% of the cases fall below this value?



$$\Phi(z) = 0,95$$

$$z = 1,645$$

95,00%

5,00%

Standard normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

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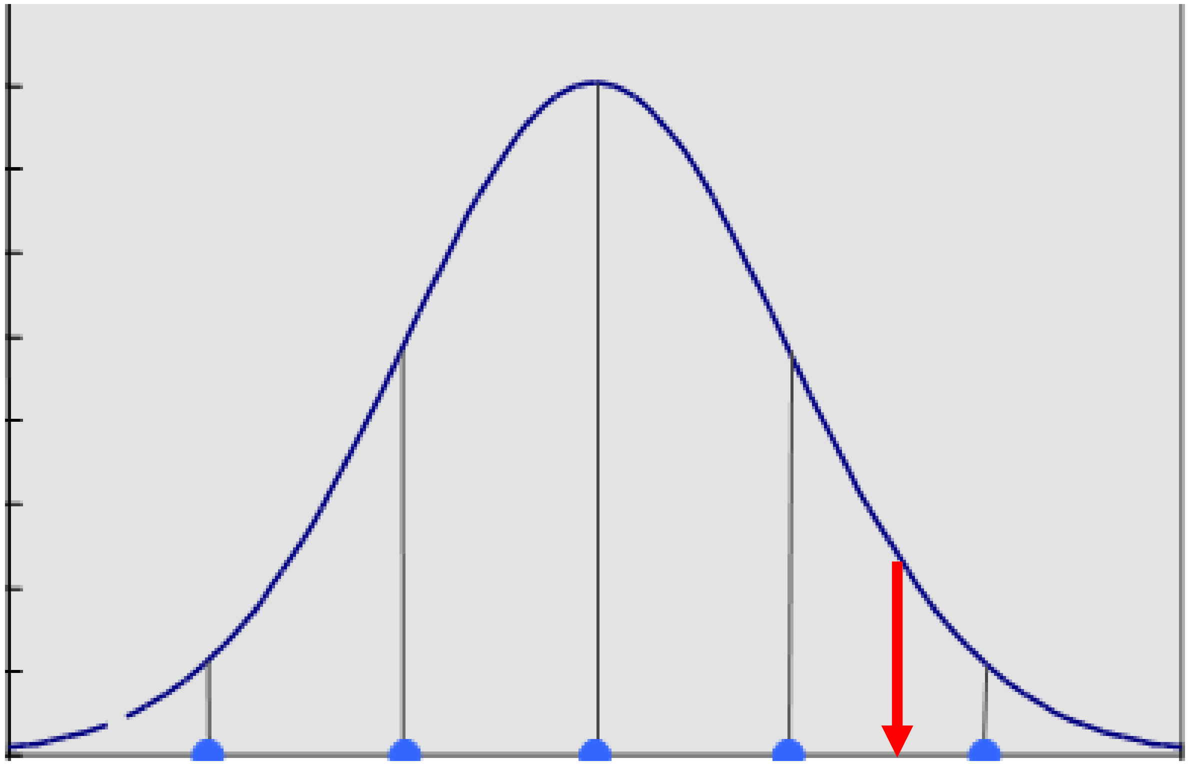
TASK

c) What is the probability that payments will be above 5 million HUF?

$$P(X > 5) \quad = 1 - P(X < 5) \quad = 1 - \Phi\left(\frac{5 - 3{,}6}{0{,}9}\right) = 1 - \Phi(1{,}56)$$

$$= 1 - 0{,}9406 = 0{,}0594$$

$$N(3{,}6\ ,\ 0{,}9)$$



$$\Phi\left[(5-3{,}6)/0{,}9\right]=\Phi(1{,}56)$$

0,9406 **1 - 9406 = 0,0594**

Standard normal distribution

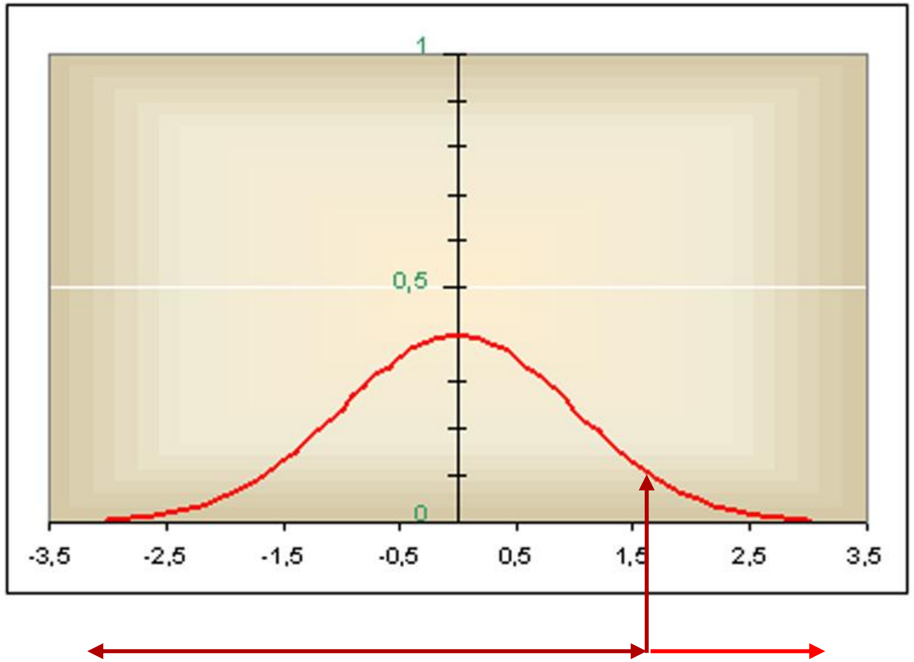
The amount of daily payments in a bank follows a normal distribution with these parameters:

mu=3.6 million HUF and σ=0.9 million HUF

TASK

d) How much money do you need to keep in your bank if you want to ensure a 95% probability of payment?

How many times should I add the standard deviation to the mean so that 95% of the cases fall below this value?



$95,00\%$ $5,00\%$

$$\Phi(z) = 0,95$$

$$z = 1,645$$

$$N(3{,}6\ ,\ 0{,}9)$$

$$3,6 + 1,645 \cdot 0,9 = 3,6 + 1,4805 = 5,08$$

Standard normal distribution

The amount of daily payments in a bank follows a normal distribution with these parameters:

mu=3.6 million HUF and σ=0.9 million HUF

TASK

d) How much money do you need to keep in your bank if you want to ensure a 95% probability of payment?

$$P(X < \mathrm{A}) = 0{,}95$$

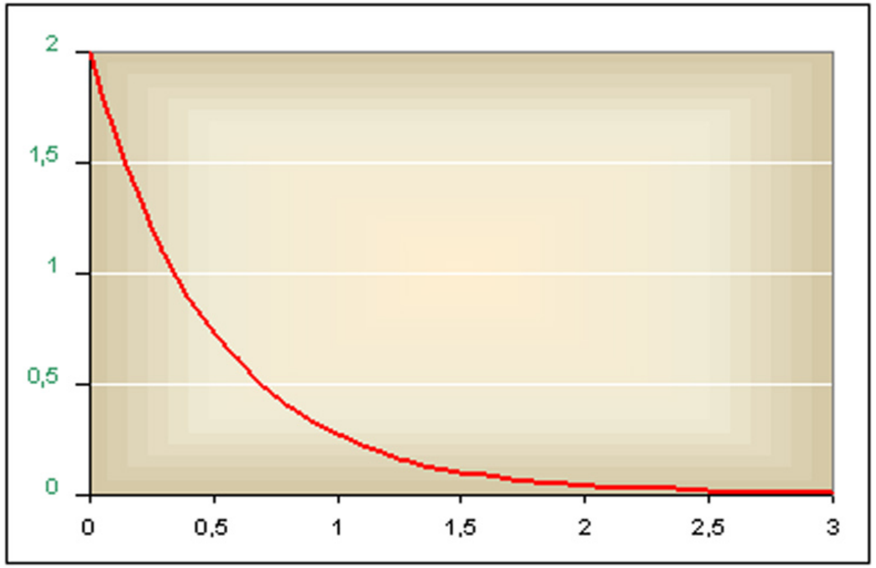
$$\Phi\left(\frac{A-3{,}6}{0{,}9}\right) = 0{,}95$$

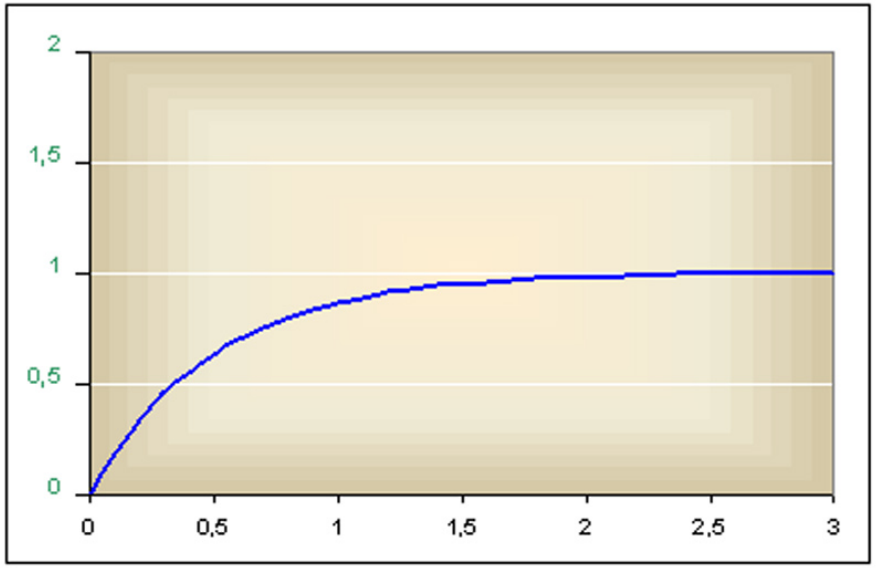
$$\Phi(1{,}645) = 0{,}95$$

$$\frac{A-3{,}6}{0{,}9} = 1{,}645$$

A=5,08 million HUF

Exponential distribution





$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & \text{ha} \quad x > 0 \\ 0, & \text{ha} \quad x \leq 0 \end{cases} \qquad F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{ha} \quad x > 0 \\ 0, & \text{ha} \quad x \leq 0 \end{cases}$$

$$\lambda > 0$$

Exponential distribution (forever young/memoryless property)

The exponential distribution is „forever young", which means that if the event I am waiting for has not yet happened, it is as if I am starting to wait for it now

$$P\left(T > a+b \middle| T > a\right) = \frac{P\left(T > a+b\right)}{P\left(T > a\right)} = \frac{1-P\left(T \leq a+b\right)}{1-P\left(T \leq a\right)} =$$

$$= \frac{1-\left(1-e^{-\lambda(a+b)}\right)}{1-\left(1-e^{-\lambda a}\right)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P\left(T > b\right)$$

Beringer 7.4.

The lifetime of a light bulb is exponentially distributed in hours, with an expected 1000 hours of use before it burns out.

a) What is the probability that the bulb you just bought will last longer than 100 hours but less than 500 hours?
b) What is the probability that the bulb you just bought will last at least 500 hours?
c) The bulb in the lamp has been burning for 1000 hours. What is the probability that it will last for at least 500 hours?

a)

$$P(100<\xi<500)=F(500)-F(100)=1-e^{-0,001\cdot 500}-\left(1-e^{-0,001\cdot 100}\right)=$$

$$=1-0,6065-(1-0,9048)=0,3935-0,0952=0,2983$$

b)

$$P(\xi\geq 500)=1-F(500)=1-\left(1-e^{-0,001\cdot 500}\right)=e^{-0,5}=1-0,3935=0,6065$$

c)

$$P(\xi\geq 1500|\xi\geq 1000)=P(\xi\geq 500)=e^{-0,5}=0,6065$$