



The detection of cheating in multiple choice examinations



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HIGHLIGHTS

- Identification of connected answers.
- Two methods are proposed.
- Simulations show role of parameters.

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ABSTRACT

Cheating in examinations is acknowledged by an increasing number of organizations to be widespread. We examine two different approaches to assess their effectiveness at detecting anomalous results, suggestive of collusion, using data taken from a number of multiple-choice examinations organized by the UK Radio Communication Foundation. Analysis of student pair overlaps of correct answers is shown to give results consistent with more orthodox statistical correlations for which confidence limits as opposed to the less familiar “Bonferroni method” can be used. A simulation approach is also developed which confirms the interpretation of the empirical approach.

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1. Background

Cheating in examinations is now acknowledged by an increasing number of academic institutions to be widespread. Wesolowsky [1] notes one 2005 US study [2] which reported that from 10,000 faculty, 44% were aware of student cheating in the previous 3 years yet never reported the fact. He also points to a second study by the Josephson Institute of Ethics also in USA. This surveyed 43,000 high school students in October 2010, which found that more than one half admitted cheating on test during past 12 months and one third admitted cheating more than twice. Cizek [3] as a result of a survey of studies prior to 1999 concluded that “cheating is rampant”. Most research since suggests cheating is more widespread than is usually believed. In the UK cheating has been reported to occur during medical examinations [4]. But other specialisms are not exempt. Nor is it confined to schools and universities. Many professionals, aspiring car drivers, nuclear missile control supervisors and amateur radio operators are all required to take examinations in order to gain an appropriate licence. It is also not unknown for the teacher to collude with examinees. This was illustrated by a recent article in Time Magazine [5] which reported the indictment of a college Principal and four teachers for conspiracy and provision of examination answers to students before the examination. Multiple-choice examinations form a significant element of the testing process in all these cases. Here we report the development and implementation of a statistical method aimed at detecting cheating in

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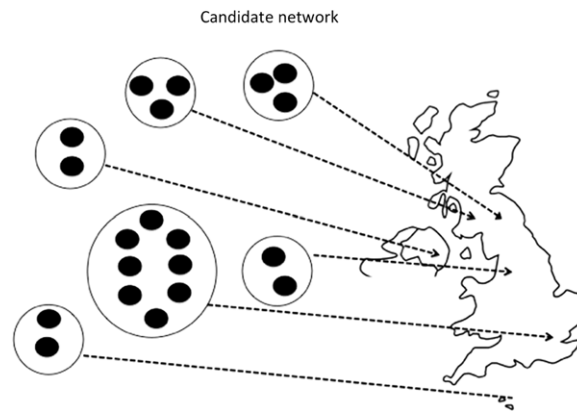


Fig. 1. Schematic illustration of the organization of examinations across the UK with relatively small numbers of students taking the examination at different centres simultaneously.

Source: Office of Communications Regulation (Ofcom) and Radio Communications Foundation (RCF).

multiple choice amateur radio examinations. Our empirical study is complemented by a set of simulations, which offer additional insight into the method and outcomes.

The next section outlines the way amateur radio examinations in the UK are administered and managed together with initial approaches to cheating detection. A statistical approach to detection is then described together with some output for three different examinations that illustrate the potential of the method. Finally we end with thoughts and conclusions.

Amateur radio has its origin in the early studies by Marconi and other physicists of electromagnetic wave propagation at the end of the 19th and beginning of the 20th centuries. As a hobby it began to grow substantially after world war one. It has been estimated that around two million people world wide are regularly involved with amateur radio. In the UK for the past few years new entrants number about 1000 annually. To participate, entrants across the world must obtain a licence. Most advanced countries issue this following success in a national examination. In the UK, the government office of communications regulation (Ofcom) confers responsibility for the examination to the Radio Communications Foundation (RCF) an independent charity established to support people and projects where radio communication, and amateur radio in particular, is the theme. The examination itself provides for three levels of licence: Foundation, Intermediate and Advanced. Here we are concerned with the Advanced examination which at the moment consists of 62 multiple choice questions that test knowledge of various aspects of radio including the licence conditions, basic electronics, radio and transmitter architecture, antennae and electromagnetic wave propagation, testing methods and safety. Each question is constructed with the correct answer and three distractors. The examination is held simultaneously at a number of approved centres located across the UK. The number of candidates at any centre varies depending on demand (see schematic shown in Fig. 1). As an example, in 2008 63 candidates sat for an advanced examination across 25 centres. The average number of candidates per centre varies from 1 to 6.

Locations are vetted and the conduct of the examination follows the traditional pattern found in most schools and academic establishments. Invigilators are required to be named and deemed suitable by the RCF prior to the examination. However unlike schools, the invigilators are volunteers and unpaid. Optical mark sheets record the answers and for each candidate one has therefore a 62 character string, viz: *ABDC . . . BCADC* plus two additional identifiers for the candidate and the particular centre in which the candidate sat the examination.

It can be noted that for confidentiality reasons all identification code numbers of candidates and examination centres have been removed from the results given in the paper.

2. Analysis of empirical data

In studying the data we follow a method similar to that proposed by McManus et al. [4] who examined the overlap, A_{ij} of correct answers for each and every pair of candidates in their cohort as a function of the geometric mean of the scores, A_i and A_j for the pair of students. For the definition of overlap and related notions, see Table 1. In our case it is useful to not only consider the total network of all student pairs but also the sub-network formed by pairs of students which sat the examination in each separate centre. A feature of the scatter plot such as shown in Fig. 2 is that each data point falls entirely inside the limited area defined by the dotted triangle. This may be understood when one realizes that the maximum value of the overlap occurs when the scores for each student are equal; the minimum value of the overlap depends on the values scored. If the students score less than 50%, the minimum possible value of the overlap is zero. For students scoring over 50% the minimum overlap increases linearly from zero to the maximum value when each student scores 100%.

Assuming no contact by phone, internet or other wireless means during the examination, we may expect the total overlap scores for the total student network to be completely uncorrelated since it is dominated by pairs from different examination

Table 1

Useful definitions for the analysis of multiple choice examinations.

Source: Radio Communications Foundation of the United Kingdom.

Correct answers	4	1	4	4	3	3	3	4	1	4	
Answers of candidate 1 (R_1)	4	3	4	4	4	1	1	4	1	2	
Good (1)/wrong (0) answers of candidate 1 (S_1)	1	0	1	1	0	0	0	1	1	0	
Score of candidate 1 (A_1)											5
Answers of candidate 2 (R_2)	4	1	4	4	3	4	3	4	1	4	
Good (1)/wrong (0) answers of candidate 2 (S_2)	1	1	1	1	1	0	1	1	1	1	
Score of candidate 2 (A_2)											9
Overlap of correct answers (A_{12})											5
Overlap of all answers (T_{12})											5
$(A_1 + A_2)/2$ (arithmetic mean score of the pair 1, 2)											5.50
$\sqrt{A_1 A_2}$ (geometric mean score of the pair 1, 2)											5.48
Correlation between R_1, R_2											0.35
Correlation between S_1, S_2											0.33

Notes: For each question, the candidates had to select one of the 4 proposed answers. In the examination there were 62 questions but this table is limited to the first 10 answers. Candidates 1 and 2 were the first two candidates in the dataset of 2008. They took the examination in the same centre. Altogether for the 19 centres there were 110 candidates.

Initially, in the file of the UK Radio Communication Foundation, the answers were coded in the form A, B, C, D. Thus, the first step was to replace A, B, C, D by a numerical coding such as 1, 2, 3, 4, which can be done either with a text editor or with the Unix transliteration “tr” command. It can be seen that the arithmetic mean and the geometric mean (used by McManus 2005) are very much the same. In following sections their common value will be referred to as the average score of a pair of candidates.

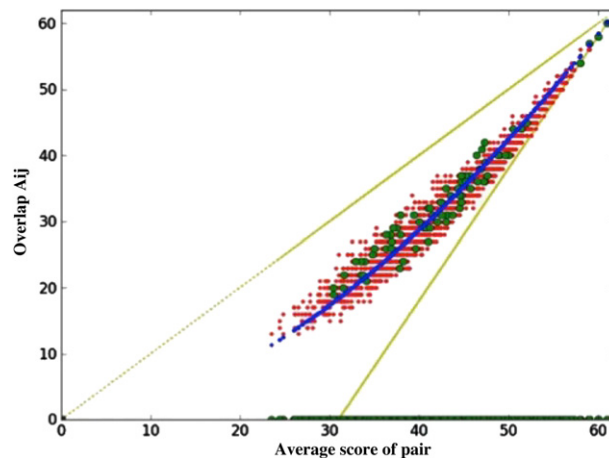


Fig. 2. The overlap A_{ij} as a function of the square root of the product of individual scores A_i and A_j . Student pairs from the same centre are shown in big green dots whereas student pairs from different centres are shown in small red dots. It can be observed that an electronic version of the paper with full colour figures is available on the arXiv data base of preprints at the following address: <http://xxx.lanl.gov/find/cond-mat>. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Source: Dataset from 2012.

centres. This is shown clearly using a dataset from an examination taken in 2012 and sat by a total of 62 students equating to 1953 student pairs.

Fig. 2 shows the outcome for the overlap A_{ij} plotted as a function of the square root of the product of individual scores A_i and A_j . The data points for the total network fall about a mean, $m = m(\sqrt{A_i A_j})$ that increases as the geometric mean itself increases. The solid line is a one parameter power law fit of this “mean” function. We deduce $m(\sqrt{A_i A_j}) = 62(\sqrt{A_i A_j})^\alpha$ where the exponent α is chosen such that the sum of the overlap fluctuations about the mean line is zero. In this instance $\alpha = 1.75$. The distribution of these overlap fluctuations about the mean line fits well a Gaussian distribution. Fig. 3 shows the cumulative distribution highlighting the final few points in the positive tail. No unusual data outside the Gaussian distribution is apparent.

We may now go further and examine the smaller sub-network of student overlaps for only the students within same centres. The big circles in Fig. 2 display the results for this sub-network. We observe no unusual clustering between these data. Each point appears to be distributed throughout the entire dataset. This suggests that the argument often put forward that students who work together will score similarly is not well founded.

However Figs. 4(a) and (b) show two further datasets from 2008 and 2011 respectively.

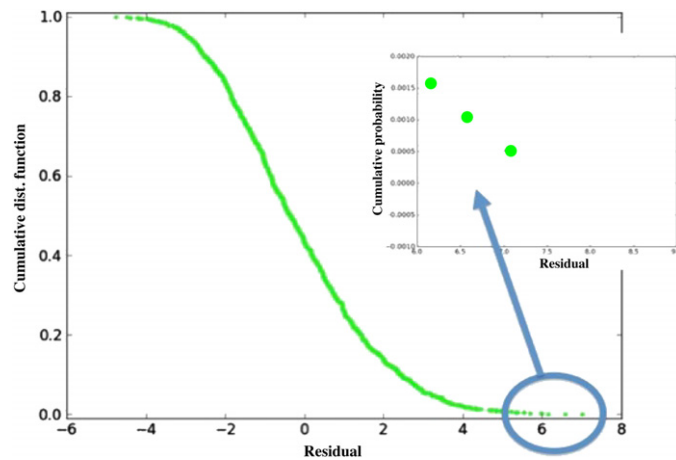


Fig. 3. Cumulative distribution for “overlap” fluctuations about the mean line (see text for definition) shown in Fig. 2. No untoward data points or outliers are evident from this dataset. The total amplitude on the log scale of the vertical axis of the inset is from -0.001 to $+0.002$.
Source: Dataset from 2012.

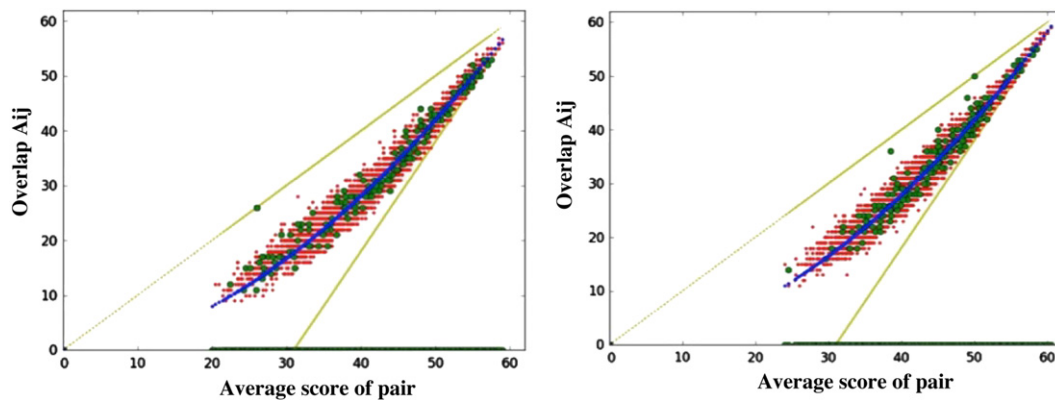


Fig. 4. (a) Data for an examination held in 2008. (b) Similar data for an examination held in 2011. Both exhibit outliers that correspond to student pairs from the same centre.
Source: Dataset from 2008 and 2011.

For these examples we see a number of outliers, which become very apparent on the corresponding cumulative distributions.

A similar situation prevails for the 2011 examination results. However now we see two extreme outliers, each representing a pair of students from the same centre. This approach is similar to that described by McManus [4] who used it for medical examinations. However we shall now turn to a simulation of the process, which as we shall see gives us much greater insight into the method.

3. Simulation model

It is possible to simulate situations which mimic the examinations. Such simulations have two main purposes.

- It allows us to study the influence of major parameters which define an examination, namely: the number of candidates, the number of questions, the number of possible answers, the average level of the candidates.
- By giving the opportunity to run many iterations, the simulation allows us to judge the effectiveness of different methods for the identification of cheating.

3.1. Principle of the simulation

Beyond the technical details (which are given in the Appendix) it is important to understand what is the nature of the probability problem at the core of the examination process.

In the examinations considered in this paper there are $a = 4$ answers proposed for each question. Let us for a moment suppose that there are only 2. Then, the whole examination process involving n candidates and q questions is equivalent

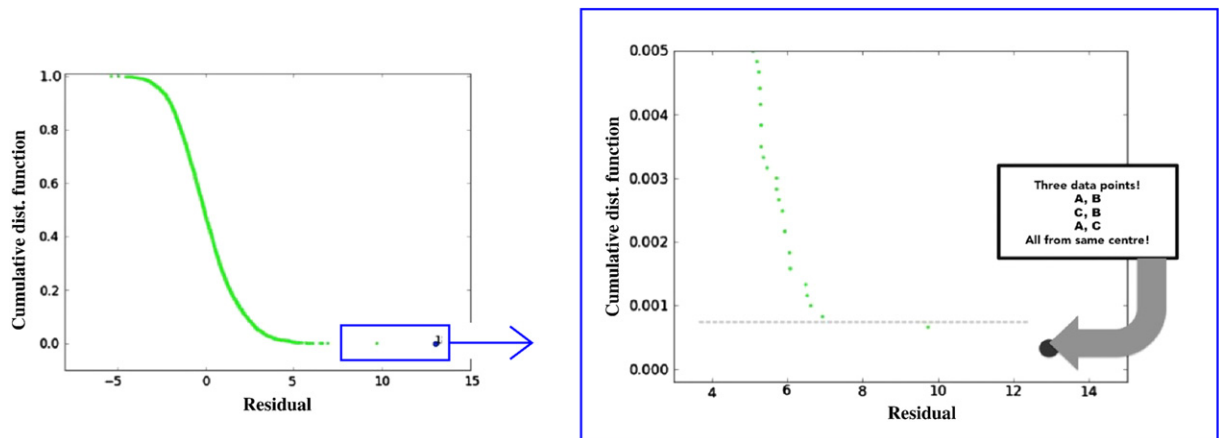


Fig. 5. The cumulative distribution for the 2008 examination (a). Detail from the tail of the distribution is in figure (b) from which it now becomes apparent that the outlier is actually three points corresponding to three students from one particular centre who have almost identical scores and identical overlaps.

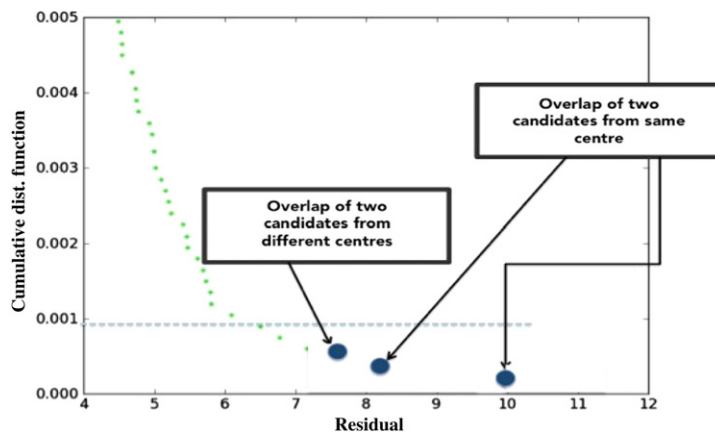


Fig. 6. The cumulative distribution for the 2011 examination showing two adjacent outliers from same centres. A small number of large random fluctuations for candidates in different centres are already visible in Figs. 2 and 4(a), (b). This is not surprising due to the large number of pairs: $\sim 100 \times 50 = 5000$.

to throwing q times n coins. The face-up sides in the throw number k give the answers to the question number k . In other words, the examination is a binomial random trial. Usually, in binomial trials the coins are supposed to be independent from one another. If, on the contrary, we suppose that two coins are in some way (electrically, magnetically or in any other way) in interaction, their results will become dependent in the same way as for the answers of two candidates who communicate with one another.

If we assume that $a = 6$ the examination is equivalent to rolling q times n dice. As before, the upper sides of the n dices in the trial number k will give the answers of the n candidates to the question number k . In other words, the examination is a multinomial random trial. When two candidates communicate, it means that two dice are in interaction.

In the real examination $a = 4$ but if we are only interested in whether the answers are right or wrong, the examination becomes again equivalent to a binomial trial.

In short, the detection of cheating refers to a fundamental problem of probability theory, namely how to determine whether or not two random variables are dependent.

In Fig. 7 we show the results of 62 students completing 62 questions. In both figures it is assumed that the probability of choosing the correct answer is constant across all questions and all students. We do not expect this to be a realistic assumption but we shall see that it serves as a toy model that allows us to understand how variations in the various aspects of the matter affect the outcomes. A feature which is not possible using only the available empirical data. Fig. 7(a) assumes a fairly smart set of students who answer each question correctly with a probability of $p = 0.8$ but the student pair, 1 and 2, cheat with an overlap correlation of 0.99. Fig. 7(b) is similar but the students are assumed to be less smart and the probability of answering any question correctly is only $p = 0.5$. The cheating pair is immediately spotted as an outlier. The distribution of the bulk of the data points is more limited than those for the actual student cohort but clearly a real class will have students with different abilities so we can expect a distribution of values for p to be more appropriate. What is evident

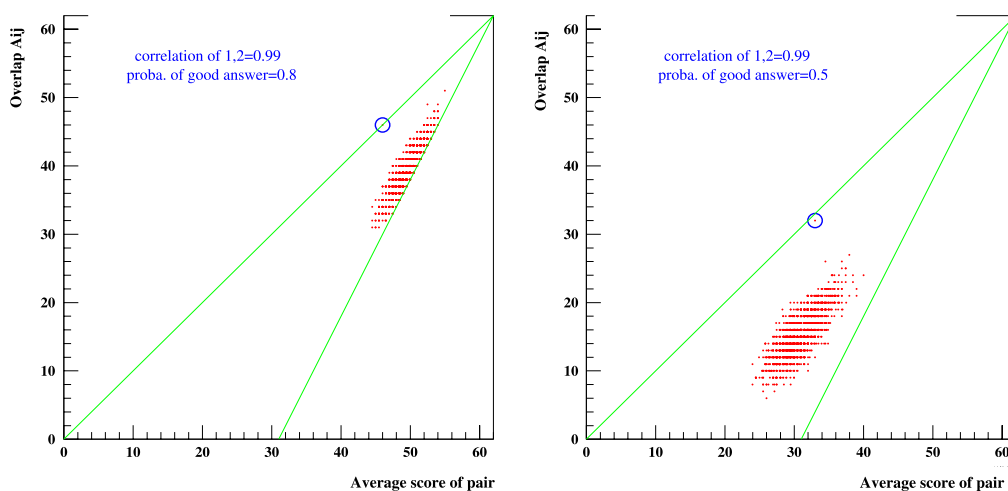


Fig. 7. Simulation for a group of 62 candidates. There are also 62 questions. Each dot corresponds to a pair of candidates. One pair, namely (1, 2) is supposed to cheat with probability 0.99 in the sense that the correlation between their sets of results is 0.99. Fig. 7(a) is for candidates who answer each question correctly with a probability 0.8. Fig. 7(b) is the same simulation but the probability of answering questions correctly is only 0.5. A_{ij} is the overlap of correct answers.

from the two figures is the deviation of the data from the right hand line as the value of p is reduced in line with actual data. However further calculations to explore this issue will be left for another study.

Fig. 8 shows the degree to which correlation between the cheating pair influences the position of the outlier. If the correlation parameter is less than 0.5, it would make no sense to assign it as an outlier. But what value ought to be chosen to eliminate false positives? We return to this point in the next section after looking at one further outcome from the simulations.

Fig. 9 shows the effect on the results of changing the number of students relative to the number of questions in the examination paper. We see that an outlier is more obvious when the number of students is less than the number of questions. As an aside we note here that the total number of student pairs remains constant across the graphs; that this may not appear so reflects the fact that many points are superimposed when integer values of the overlap coincide.

From Fig. 10, we see that an outlier is also more obvious for students who are not so smart than when they all score highly.

Our examination data is usually for a cohort which is greater in number than 62, the number of examination questions. But they are not all equally smart and moreover our network, as we noted at the outset is fragmented, with many centres hosting only a few examinees. Overlaying the sub-network on the complete network of overlaps as we showed earlier now helps overcome any potential masking of the outliers. Examples are shown in Fig. 4(a), (b).

The majority of data points for the sub-network are more confined forming a slightly tighter distribution within the total network leaving the single outlier in this example more clearly exposed. As for the 2012 data the majority of points within the sub-network are more or less randomly scattered about the mean relative to the totality of points, so again we cannot conclude that students who study together gain any additional advantage over those who do not and once more we see that the frequent assertion that students who study together will have common overlaps does not pass muster.

Wesolowsky stresses the importance of using a conservative method to identify false positives. He kindly agreed to process our data using his proprietary method and the specific cheaters he identified are the same as obtained here. He imposes a boundary value of 10^{-4} on the probabilities shown in the cumulative distributions in Figs. 5(b) and 6 as a light blue line. Points lying below this value are the outliers. We may understand this by plotting our data on a log–log scale and superimposing a Gaussian distribution chosen to fit the points in the main bulk of the distribution. Fig. 11 illustrates the detail. The green line is the cumulative distribution function for our data; the red line is a Gaussian fit to the bulk of the data. In this case the outlier has a probability 8 orders of magnitude greater than that which we would expect from the Gaussian distribution function. Simple enough to identify one would think. But at this point it is usual to apply a statistical test ascribed to Bonferroni in order to reduce the chance of false positive results.¹ It is argued that the distribution function computed from the data represents the outcome of a single experiment. However we have N candidates sitting the examination. The Bonferroni correction says that the true probabilities ought to be $N(N-1)/2$ greater than the single Gaussian values. It is not easy to understand why this should be so. Moreover in our case the application of this is a little ambiguous since candidates take the examination in small groups. Do we choose a value of the order of the group size typically 3 to 6 or do we choose $N = 63$ the total size of the cohort? We shall ignore this and select the most conservative approach which is to

¹ For more details see the Wikipedia article entitled “Bonferroni correction”; in particular see the references to the statistical papers cited in this article.

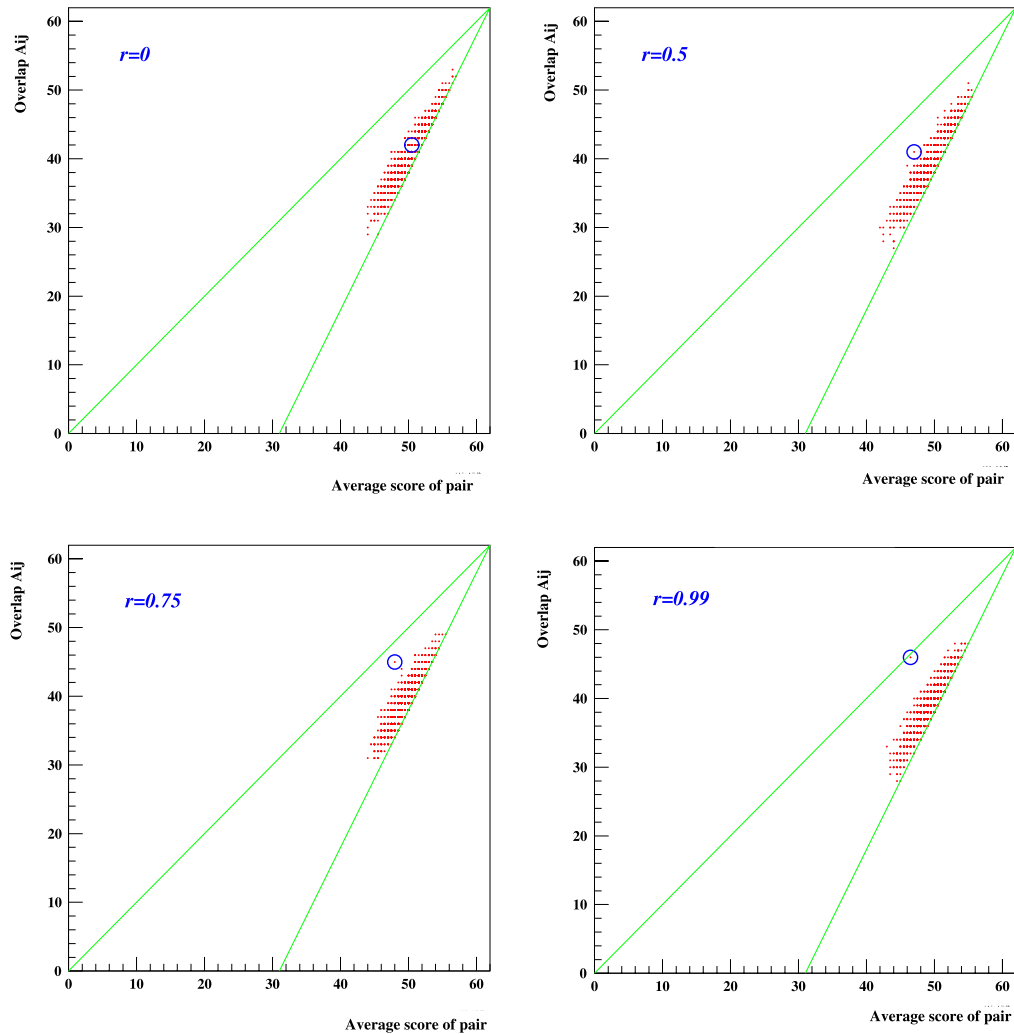


Fig. 8. The graphs show the emergence of the outlier as the correlation between the answers of candidates 1 and 2 increases from zero to one.

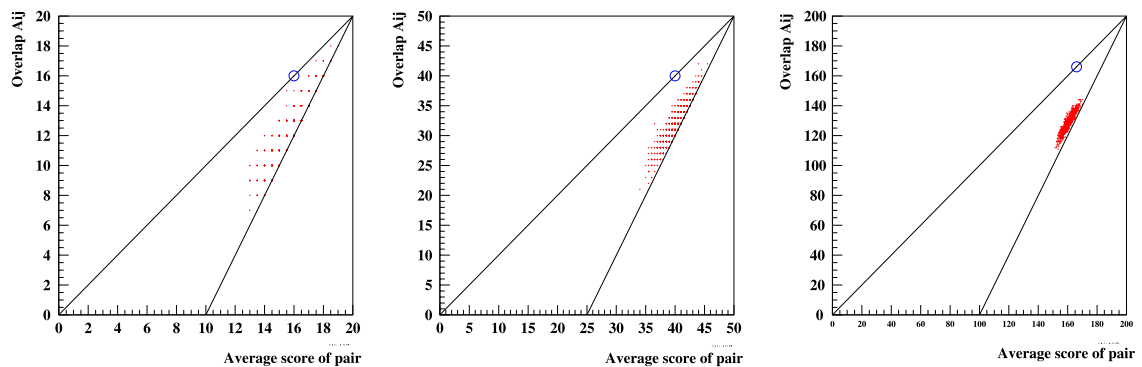


Fig. 9. For the 3 graphs there are 50 candidates but from left to right the number of questions increases from 20 to 50 and 200. As expected the outlier is more obvious when the number of questions becomes larger.

choose $N = 63$. Our correction factor is then of the order of 10^{-4} which means the corrected probability is still far less than that observed in this case so we can be quite sure that our outliers are true outliers and not just erroneous points associated with the Gaussian.

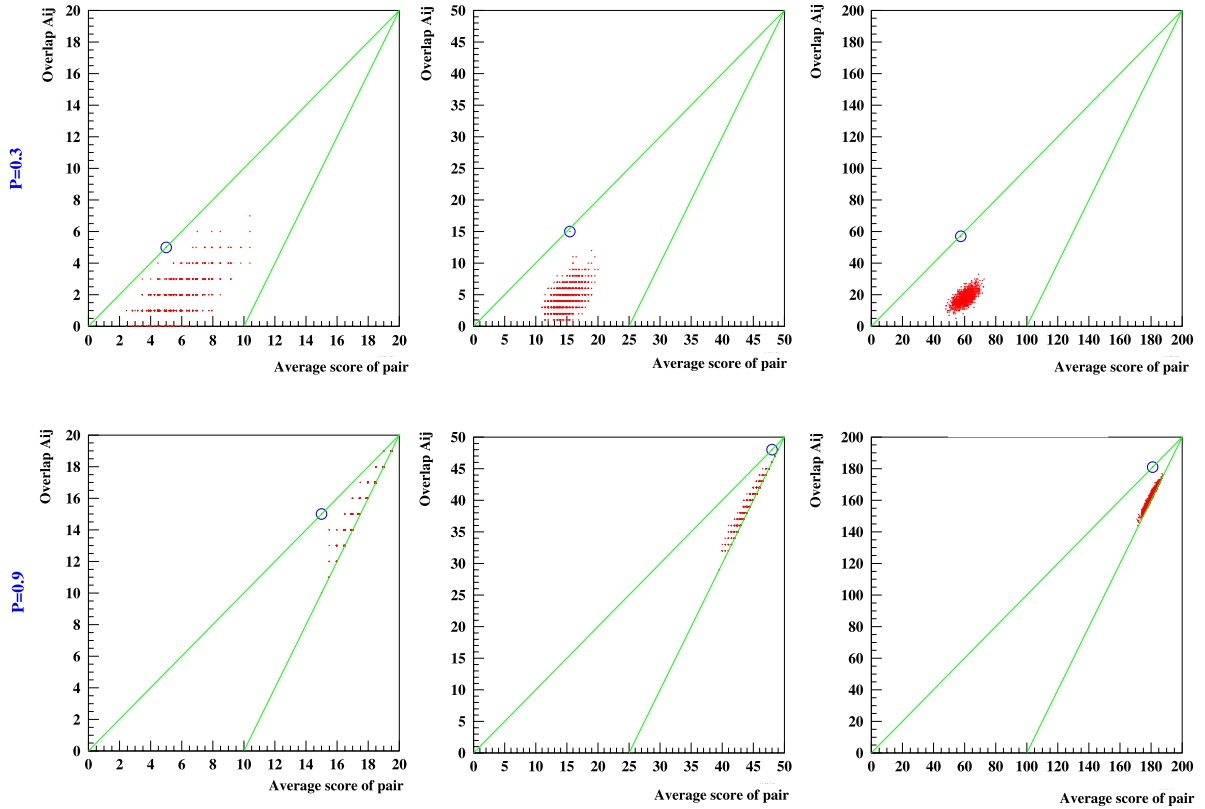


Fig. 10. These graphs combine two effects: change in the smartness of the students through the probability p and change in the number of questions. The two effects work in opposite directions. The bottom right graph shows that when $p = 0.9$ even with a number of questions as high as 200 (which from a practical perspective seems to be an upper limit) the outlier is hardly detectable.

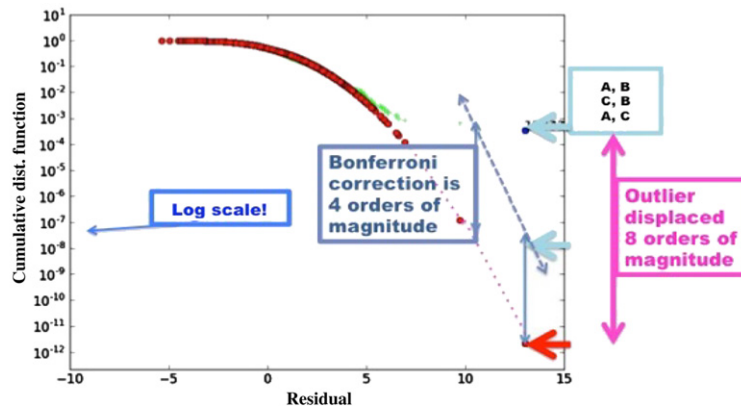


Fig. 11. The cumulative distribution of overlaps for the data of 2008 plotted on a log log scale. Now the deviation of the actual distribution (circles) from a Gaussian (points) is clear. The three data points from the same examination centre and suggestive of collusion are visible and well separated from the Gaussian curve. The Bonferroni correction offers only 4 orders of magnitude of hope leaving a further 4 orders of magnitude unaccounted for!

Analysis of the second data (Fig. 6) set suggests both outliers are true examples of anomalous pairs each being for students from the same examination centre. Although the penultimate point only just falls within this conservative cut-off!

4. Overlaps or correlations?

The method of looking at overlap of correct answers (as used by McManus [4]) does not use information relating to wrong answers. Moreover, and this point is probably more meaningful, it does not take into the specific answer numbers

selected by the candidates. In this section we propose an alternative method based on the correlations between the two results vectors of pairs of candidates. In the present case the results vector of a candidate is a set of 62 numbers comprised between 1 and 4.

Why should one use correlations when in this problem the natural variable is the overlap between the answers of different candidates? When a teacher begins to suspect some “abnormalities”, the first reaction is to assess the similarity in the answers of the candidates. That is why the overlap was used as a key-variable in the first part of this paper.

However, the overlap is not a standard statistical concept and does not easily lead to an assessment in terms of confidence intervals. That is why, instead of the overlap, we will here use the correlation. Overlap and correlation are two measures of the degree of similarity between two sets of results and it is easy to check that the two variables increase together when the results become more identical. However by using correlations we can use standard theorems that give confidence interval for a given level of confidence probability.

More specifically the confidence interval limits that we will use are obtained following Morice and Chartier [6]. Lying between -1 and 1 , the correlations can certainly not be considered as Gaussian variables. That is most unfortunate because the confidence intervals of Gaussian variables can be obtained most easily. Therefore it is natural to apply a change of variable which will transform the interval $(-1, 1)$ into an interval covering the whole line from minus infinity to plus infinity. The inverse hyperbolic tangent function does this. Then the correlations so transformed can be fairly well described by a Gaussian variable. Once we have performed the standard confidence analysis for Gaussian variables we return to the correlations by way of applying an hyperbolic tangent function.

Before using this method we need to decide the kind of results to which we want to apply the methodology. There are basically two possible options.

The first approach is to use the results in the form “good answer” versus “wrong answer”. In this framework, the set of results of each candidate will be a simplified vector whose elements are 0 (wrong answer) or 1 (correct answer). For the sake of simplicity, the previous simulations have been performed using this framework.

Alternatively we can use all the available information. As every teacher knows, common wrong answers are more suggestive of possible cheating than common good answers. For instance, if the correct answer is A, two D answers may mean either that the two candidates had the same misunderstanding or that there was some form of communication between them. In the present case where there are 4 possible answers the set of results of each candidate will be a vector whose elements can take the values 1, 2, 3, 4.² Because it uses more information it is of course natural to expect that this second approach will lead to more accurate identification of interdependent results. This is indeed confirmed by observation as shown in Figs. 12a and 12b.

As in previous graphs, green dots correspond to pairs of candidates who took their examination in the same centre. The black vertical lines shown for a number of outliers are the lower parts of the confidence intervals for a probability level of 0.95 (which means that if we repeat 100 times the same experiment about 95 of the correlations for the same pair of candidates will fall in the confidence interval). For the 3 candidates which correspond to the two squares at the top (almost at correlation 1) the confidence interval is so small that the confidence interval can hardly be seen. This is of course an extreme case in the sense that 99% of the answers selected by these 3 candidates were identical. In such a case cheating is hardly in doubt.

However, for the other cases the situation is less clear. If we wish to interpret them in terms of cheating or not cheating an additional assumption is necessary, namely one needs to assume that the bulk of the candidates did not *cheat*. This assumption has already been made in the Bonferroni method. In cases where all candidates would cheat the identification of outliers would only reveal the candidates who cheat more than the average.

With this assumption in mind, one can adopt the rule that when the confidence interval does not cross the staircase line that represents the average of all candidates, there is a significant “abnormal” similarity. Needless to say, such a statement depends on the confidence level that is selected. With a confidence level of 0.99 (instead of 0.95), the confidence intervals would be broader and the number of significant cases would be reduced.

5. Conclusions

5.1. How can one optimize the detection of cheating?

The method of overlaps, that compares correct answers for pairs of candidates, is shown to be effective at identifying anomalous outcomes suggestive of cheating, in multiple-choice examinations. Simulation confirms the method and offers insight into parameters that might be controlled in future examinations to optimize such identification. The method can be criticized for focusing on overlap only of correct answers, taking no notice of overlap of incorrect answers. Intuitively it seems that overlap of incorrect answers could be a stronger pointer to collusion. The use of correlations allows the use of well-defined and more widely understood confidence limits for eliminating false positives, rather than the Bonferroni method which becomes rather ill-defined in our case where the examination cohort is split across numerous centres. Since

² As can easily be checked, the correlation between two sets of answers remains the same if instead of 1, 2, 3, 4 we use 11, 12, 13, 14 or 1, 3, 5, 7. Whereas the first case is obvious because it is merely a translation, the second is somewhat less obvious.

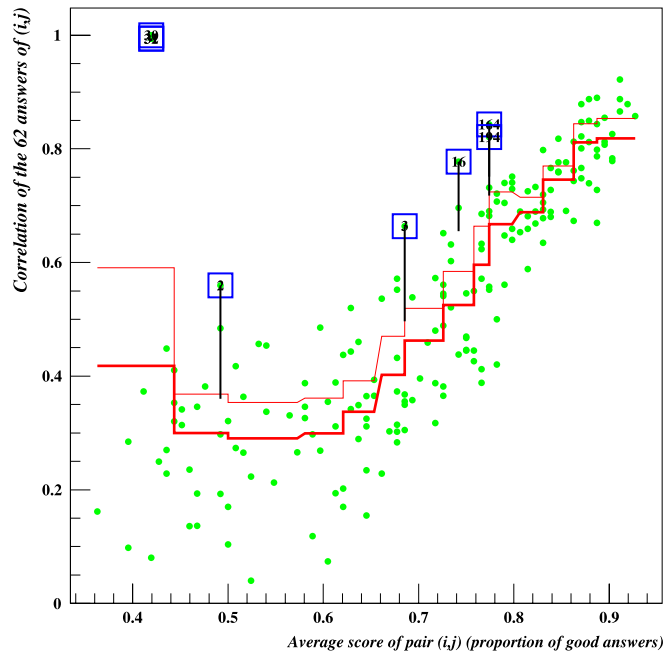


Fig. 12a. Analysis of 2008 data (110 candidates) in terms of correlation between the answers to the 62 questions. Answers are described as vectors whose elements are numbers comprised between 1 and 4 depending on which one of the 4 possible answers the candidate selected. The thick (red) staircase curve gives the averages for groups of 14 successive points. The thin line gives the confidence interval of the averages (at probability level of 0.95). It can be observed that the correlations C_{ij} of the answers used in this graph are closely related to their overlap in terms of right/wrong answers, that is to say the A_{ij} used in earlier graphs; the correlation between the C_{ij} and A_{ij} is $r = 0.97$. The outliers whose confidence intervals (also with probability level of 0.95 and represented by the black vertical lines) do not cross the staircase curve can be considered as having a significantly abnormal correlation. Altogether there are 8 such pairs: the most conspicuous correspond to the three squares at the top of the graph almost at probability 1 (its confidence interval is so small that it does not appear on the graph). They correspond to the three candidates who took the examination at the same centre and identified already in Fig. 5 by the Gaussian method. The 5 other squares, namely those numbered 2, 3, 16, 164 and 194, correspond to 5 pairs of candidates who took their examination in 4 centres C_1, C_2, C_3, C_4 . We can say that pair number 2 and pair number 3 took their examination in the same centre C_1 , whereas the 3 other centres were all different. The exact code numbers of the centres as well as the code numbers of the candidates have been removed for confidentiality reasons. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
Source: Dataset from the UK Radio Communication Foundation (2008).

it is conceivable that cases of cheating could be the subject of legal action, it is important that well understood methods for elimination of false positives are used. We show that the method of correlations appears to be equally applicable in our case giving identical outcomes.

A practical way to minimize the potential for cheating in multiple choice examinations such as those discussed here is to randomize the order of potential answers on candidate question sheets and this approach is now used by the RCF in the UK.

5.2. Extension of the present analysis to similar problems

This paper is about a specific problem, namely how to detect people who cheat at examinations. However, we believe that the identification methods proposed in the paper can also be used every time one wants to detect “abnormal” similarities in a selection process.

This can be illustrated by the following example. We consider a group of N customers. Every customer is shown 4 brands of various products (tea, milk, beer, rice, cookies and so on) and is asked which one he or she prefers.

The dispersion of the correlations (or overlaps) of the vectors of selected brands will provide a way to measure how similar or closely connected the customers are. The analog of cheating will correspond to an abnormally high correlation suggesting a strong connection between two customers.

One can also introduce the notion of right/wrong answers. For instance, “right” answers may be defined as the brands for which there had been a recent advertisement campaign. Then, one can define the score of a customer as the number of his (or her) correct answers. The customers with highest scores (the analogue of the smart students) are likely to be those who watch TV the most.

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We are grateful to the Radio Communications Foundation of the United Kingdom for permission to use their data. This collaboration began with a discussion at Ways of Seeing, a workshop sponsored by COST action TD1210 Analyzing the

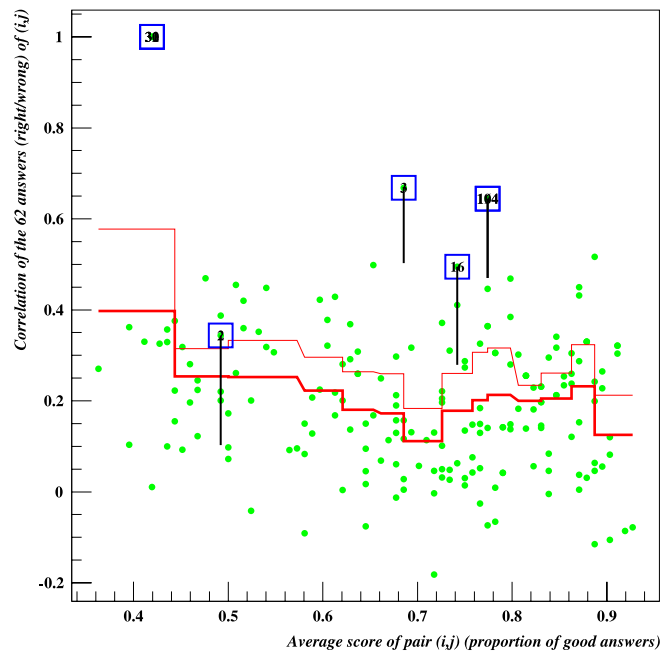


Fig. 12b. Analysis of 2008 data (110 candidates) in terms of correlation between the right/wrong characteristics of the 62 answers. Answers are described as vectors whose elements are 1 or 0 depending on whether the answer is right or wrong. The staircase curves have the same meaning as in Fig. 12a. It can be observed that the correlations C_{ij} of the answers used in this graph are less closely related to their overlaps A_{ij} ; the correlation is $r = 0.75$. This is mainly due to the fact that the C_{ij} are lower and hence more dispersed (due to larger confidence intervals) than in Fig. 12a. With the exception of the data point number 2, the outliers are the same as in Fig. 12a. In contrast with Fig. 12a, the correlation does not display an upward trend. This is due to the fact that for low scores the vectors of the answers contain many 0 in the same way as they have many 1 for high scores; this makes the situation fairly symmetrical; it can be noted that the high level of the first step of the staircase curve is due to the inclusion into the average of the 3 top data points with correlations almost equal to 1.

Source: Dataset from the UK Radio Communication Foundation (2008).

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Appendix. Simulating interdependent random variables

In this Appendix we wish to answer two related questions regarding the definition of a system X_i of symmetric binary interdependent random variables:

$$x_i \in \{0, 1\} \quad E(X_i) = p \quad \sigma^2(X_i) = p(1-p) \quad \text{corr}(X_i, X_j) = \rho_{ij} = r \quad 1 \leq i, j \leq n. \quad (1)$$

(1) In the simulation we used a system $\{X_i\}$ introduced by Lunn and Davis [7]. How is it defined?

(2) Apart from the previous one, are there other $\{X_i\}$ systems which share the properties defined in (1)?

It will be seen that it is only for $n = 2$ that the correlation matrix ρ_{ij} uniquely defines the probability distribution. In other words, the set of random variables introduced by Lunn and Davies is only one among many possible answers.

A.1. The set $\{X_i\}$ proposed by Lunn and Davies [7]

It is based on the following result. $(0, 1; p)$ denotes a binary variable whose expectation is p .

Proposition The Y_i , $(1 \leq i \leq n)$ are a set of independent binary random variables $(0, 1; p)$. Z is also a binary variable $(0, 1; p)$. The U_i are a set of binary variables $(0, 1; \sqrt{r})$. The Y_i, Z, U_i are all independent.

Then the variables $X_i = (1 - U_i)Y_i + U_iZ$ are a system of symmetric dependent binary variables $(0, 1; p)$ whose correlation matrix is $\rho_{ij} = r$.

The proof is easy and given in the paper. Let us add two remarks.

- We used the expression “symmetric variables” to reflect the fact that all X_i play the same role. The expression “exchangeable variables” is often used with the same meaning.

- The correlation matrix has only positive elements. This is of course imposed by the symmetry condition. $\rho_{12} < 0$ and $\rho_{23} < 0$ would imply $\rho_{13} > 0$, thus violating the symmetry requirement.

In the following subsections we will be concerned with the question of uniqueness of the set of X_i generated above. Needless to say, it is useful to know whether the proposition gives *the* answer or only one among many. More precisely, the problem can be stated as follows.

In order to specify the probability distribution of one X_i variable one needs h parameters. In addition, the correlation matrix contains one parameter. Are these $h_1 = h + 1$ parameters sufficient to determine the joint probability distribution of the whole set $\{X_i, i = 1, \dots, n\}$.

A.2. Uniqueness for two binary dependent variables

The definition (1) contains only two parameters, namely p and r . In this case $h_1 = 2$. It will be seen that many systems (indeed an infinite number) of variables can be defined which fulfil these conditions. The only exception is the case $n = 2$. In this case the correlation matrix defines completely the probability distribution.

For two variables X_1, X_2 the probability distribution function is completely defined by two joint probabilities:

$$p_{11} = P\{X_1 = 1 \text{ and } X_2 = 1\}, \quad p_{01} = p_{10} = P\{X_1 = 1 \text{ and } X_2 = 0\} \quad \text{then: } p_{00} = 1 - p_{11} - 2p_{01}.$$

Thus, through an argument based on degrees of freedom, one sees that there is a one-to-one correspondence between the parameters p, r and the probabilities p_{11}, p_{01} .

A.3. Non uniqueness for more than two binary variables

On the contrary, a system of 3 variables is defined by 3 probabilities, namely

$$p_{111}, p_{011}, p_{001}, \quad \text{with: } p_{000} = 1 - p_{111} - 3p_{011} - 3p_{001}.$$

Thus, one of these 3 numbers can be chosen arbitrarily. There is no longer a one-to-one correspondence with the parameters p, r .

The same property can be seen in a slightly different way.

Clearly, $E(X_1 X_2) = 1 \times 1 \times p_{11}$; similarly $p_{01} = E[(1 - X_1)X_2] = p - E(X_1 X_2)$. This shows that the probabilities which define the distribution are specified by $E(X_1 X_2)$ that is to say basically by the correlation matrix. As a matter of fact, an easy calculation gives the expressions of the probabilities ($q = 1 - p$):

$$p_{11} = r^2 pq + p^2, \quad p_{01} = p_{10} = pq(1 - r^2), \quad p_{00} = r^2 pq + q^2.$$

On the contrary, the expression of p_{111} , namely $p_{111} = E(X_1 X_2 X_3)$, shows that for 3 binary variables one needs a three-point moment. The two-point moment of the correlation matrix is no longer sufficient.

A.4. Non uniqueness for all non-binary variables

The uniqueness property for the two-variable case is special to binary variables.

If X_i can take 3 values, say 0, 1, 2, its probability function is defined by two numbers, for instance p_0, p_1 , with $p_2 = 1 - p_0 - p_1$. Thus, $h_1 = 3$. However, for two variables one needs 4 numbers to define the symmetric joint probability distribution namely:

$$p_{00}, p_{10}, p_{20}, p_{11}, \quad \text{with: } p_{22} = 1 - p_{00} - 2p_{10} - 2p_{20}.$$

Thus, there is no uniqueness.

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