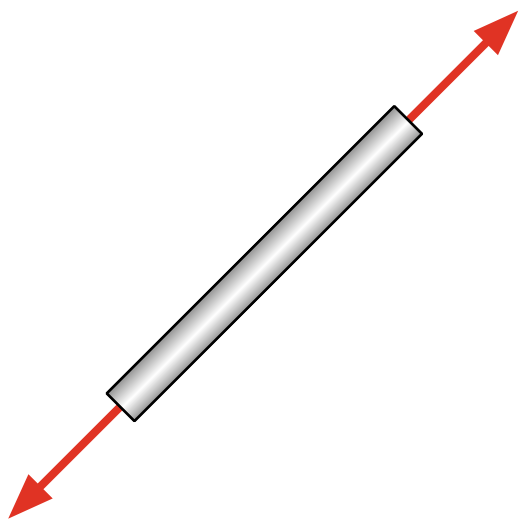
CEE321

Structural Analysis with the Direct Stiffness Method

Modeling Structural Systems as Springs



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1 Truss Direct Stiffness Method

The *Direct Stiffness Method* is a *Finite Element Method* of analysis that models structural elements as springs. We will begin with deriving the axial stiffness of a structural element. [Felippa, 2000]

1.1 Axial Stiffness

Members in a truss are modelled as springs with only axial deformation. The axial force in the spring is modelled by Hooke's law:

$$F = ku \tag{1}$$

where F is the axial force, k is the axial stiffness, and u is the axial deformation. To define the axial stiffness we will look at a member of constant cross-secitonal area with only elastic deformation. The stress in such a member is defined by the stress-strain relationship:

$$\sigma = E\varepsilon \tag{2}$$

where σ is the stress, E is the modulus of elasticity, and ε is the strain. Multiplying both sides of the equation by the cross-secional area, A, will convert the stress into a force.

$$\sigma A = E\varepsilon A \tag{3}$$

$$F = E\varepsilon A \tag{4}$$

The engineering definition of strain will be used here. Strain is defined as:

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{u}{L} \tag{5}$$

where L is the original length of the member and ΔL is the change in length of the member. Then substituting in the definition of engineering strain into the previous equation:

$$F = EA\frac{u}{L} \tag{6}$$

Therefore the axial stiffness from the equation is $\frac{AE}{L}$ for any structural member. The axial force of a structural element can be rewritten as:

$$F = \frac{AE}{L}u \tag{7}$$

1.2 Local Element Stiffness

Each element in a structural system will have its own local element stiffness equations. The goal of this section will be to develop the local element stiffness equations for a member in matrix form. Taking the nodal equilibrium from figure 1 becomes:

$$-F = ku_{1x'}$$
$$-F = ku_{2x'}$$

Next we will put these equations in matrix form in terms of the local coordinate system. The equation will take the form:

$$\mathbf{F} = \mathbf{ku} \tag{8}$$

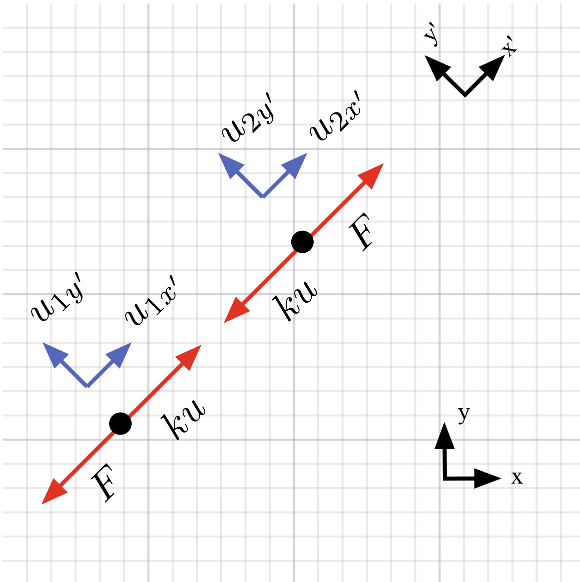

Figure 1: Local Member FBDs

$$\underbrace{\begin{bmatrix} -F \\ 0 \\ F \\ 0 \end{bmatrix}}_{\mathbf{F}'} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{k}} \underbrace{\begin{Bmatrix} u_{1x'} \\ u_{1y'} \\ u_{2x'} \\ u_{2y'} \end{Bmatrix}}_{\mathbf{u}'} \tag{9}$$

Next, we must take the local element stiffness matrix and derive the global element stiffness matrix.

1.3 Global Element Stiffness

Transforming local coordinates to global coordinates is described by the following equation:

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$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}} \begin{Bmatrix} u'_x \\ u'_y \end{Bmatrix} \tag{10}$$

where $\mathbf{R}$ is the general rotation matrix. The particular rotation matrix for the two degree of freedom system takes the form:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{11}$$

Now, we take equation 9, and multiply both sides of the equation by the local to global transformation matrix, then solve for the axial forces $\mathbf{F}'$. Note that the global deformation, $\mathbf{u}$ is equal to $\mathbf{T}^\intercal\mathbf{u}$

$$\mathbf{F}' = \mathbf{k}\mathbf{u}' \tag{12}$$
$$\mathbf{T}\mathbf{F}' = \mathbf{k}\mathbf{T}^\intercal\mathbf{u} \tag{13}$$
$$\mathbf{F}' = \mathbf{T}\mathbf{k}\mathbf{T}^\intercal\mathbf{u} \tag{14}$$

This gives us the relationship between the global deformation of nodes to the axial force.

1.4 Global Structural Stiffness

References

[Felippa, 2000] Felippa, C. A. (2000). A historical outline of matrix structural analysis: A play in three acts.