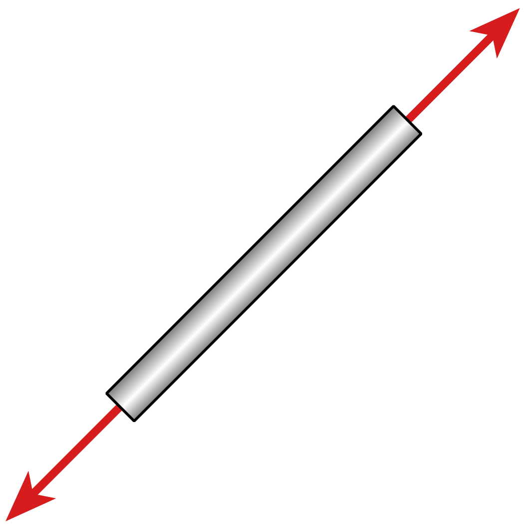
CEE321

Structural Analysis with the Direct Stiffness Method

Modeling Structural Systems as Springs



ARIZONA STATE UNIVERSITY

Brian Chevalier

Updated: June 29, 2018

Contents

1 Truss Direct Stiffness Method

The *Direct Stiffness Method* is a *Finite Element Method* of analysis that models structural elements as springs. We will begin with deriving the axial stiffness of a structural element. [?]

1.1 Axial Stiffness

Members in a truss are modelled as springs with only axial deformation. The axial force in the spring is modelled by Hooke's law:

$$N = ku \tag{1}$$

where N is the axial force, k is the axial stiffness, and u is the axial deformation. To define the axial stiffness we will look at a member of constant cross-secitonal area with only elastic deformation. The stress in such a member is defined by the stress-strain relationship:

$$\sigma = E\varepsilon \tag{2}$$

where σ is the stress, E is the modulus of elasticity, and ε is the strain. Multiplying both sides of the equation by the cross-secional area, A, will convert the stress into a force.

$$\sigma A = E\varepsilon A \tag{3}$$

$$N = E\varepsilon A \tag{4}$$

The engineering definition of strain will be used here. Strain is defined as:

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{u}{L} \tag{5}$$

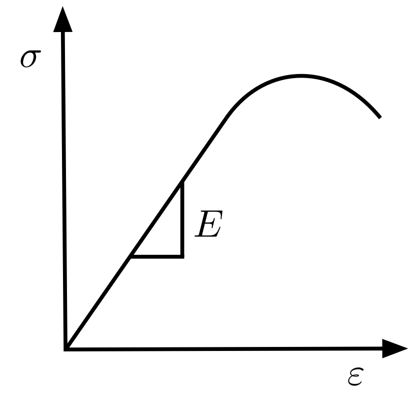


Figure 1: Stress-Strain Relationship

where L is the original length of the member and ΔL is the change in length of the member. Then substituting in the definition of engineering strain into the previous equation:

$$N = EA\frac{u}{L} \tag{6}$$

Therefore the axial stiffness from the equation is $\frac{AE}{L}$ for any structural member. The axial force of a structural element can be rewritten as:

$$N = \frac{AE}{L}u \tag{7}$$

1.2 Local Element Stiffness

Each element in a structural system will have its own local element stiffness equations. The goal of this section will be to develop the local element stiffness equations for a member in matrix form. Taking the nodal equilibrium from figure 2 becomes:

$$-N = ku_{1x'}$$
$$-N = ku_{2x'}$$

Next we will put these equations in matrix form in terms of the local coordinate system. The equation will take the form:

$$\mathbf{N} = \mathbf{ku} \tag{8}$$

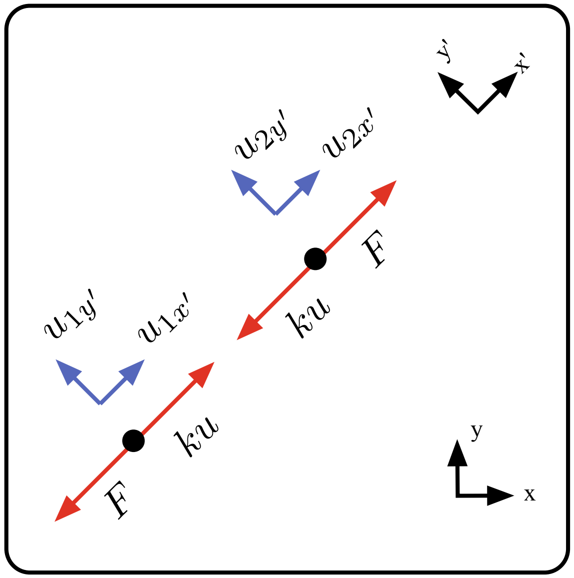


Figure 2: Local Member FBDs

$$\underbrace{\begin{Bmatrix} -N \\ 0 \\ \hline N \\ 0 \end{Bmatrix}}_{\mathbf{N}'} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{k}} \underbrace{\begin{Bmatrix} u_{1x'} \\ u_{1y'} \\ \hline u_{2x'} \\ u_{2y'} \end{Bmatrix}}_{\mathbf{u}'} \tag{9}$$

Next, we must take the local element stiffness matrix and derive the global element stiffness matrix.

1.3 Global Element Stiffness

Transforming local coordinates to global coordinates is described by the following equation:

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}} \begin{Bmatrix} u'_x \\ u'_y \end{Bmatrix} \tag{10}$$

where $\mathbf{R}$ is the general rotation matrix. The particular rotation matrix for the two degree of freedom system takes the form:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{11}$$

Note that the values of $\cos\theta$ and $\sin\theta$ can be determined by solving for the x and y component of the element unit vector, respectively. The unit vector can be calculated by:

$$\mathbf{n} = \frac{\mathbf{x_2} - \mathbf{x_1}}{||\mathbf{x_2} - \mathbf{x_1}||} \tag{12}$$

where $\mathbf{n}$ is the unit vector along the element, and $\mathbf{x}_2$ and $\mathbf{x}_1$ are the coordinates of node 1 and node 2, respectively. Now, we take equation 9, and multiply both sides of the equation by the local to global transformation matrix, then solve for the axial forces $\mathbf{N}'$. Note that the global deformation, $\mathbf{u}$ is equal to $\mathbf{T}^{\intercal}\mathbf{u}$

$$\mathbf{N}' = \mathbf{k}\mathbf{u}' \tag{13}$$

$$\mathbf{T}\mathbf{N}' = \mathbf{k}\mathbf{T}^{\intercal}\mathbf{u} \tag{14}$$

$$\mathbf{N}' = \underbrace{\mathbf{T}\mathbf{k}\mathbf{T}^{\intercal}}_{\mathbf{k}_{global}}\mathbf{u} \tag{15}$$

This gives us the relationship between the global deformation of nodes to the axial force. Enforcing the global nodal boundary conditions will allow solving the system of equations for the member forces. Expanding the equation for $\mathbf{k}_{global}$ looks like:

$$\mathbf{k}_{global} = \frac{AE}{L}\begin{bmatrix} l & -m & 0 & 0 \\ m & l & 0 & 0 \\ 0 & 0 & l & -m \\ 0 & 0 & m & l \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}$$

where l and m are $\cos\theta$ and $\sin\theta$, respectively. Therefore the global element stiffness matrix takes the form of:

$$\mathbf{k}_{global} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \tag{16}$$

Notice that the $\mathbf{k}_{global}$ matrix is symetric about the main diagonal.

1.4 Global Structural Stiffness

Deriving the global structural stiffness matrix for a truss is a matter of combining the global element stiffness matricies.

Brian Chevalier

1.5 Boundary Conditions

2 Beam Direct Stiffness Method

A beam element has four degrees of freedom. Two degrees of freedom at each node which includes the rotation, θ, and the deflection or transverse displacement, w.

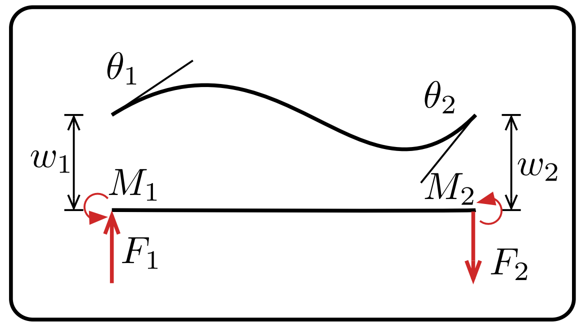


Figure 3: Degrees of Freedom

2.1 Bending Stiffness

We will isolate each degree of freedom by applying boundary conditions at each node, such that each beam will only have one unrestrained kinematic unknown. Figure 4 shows the various boundary conditions that will be used to determine the beam bending stiffness. This will allow us to derive the force-displacement relationship necessary to solve the beam.

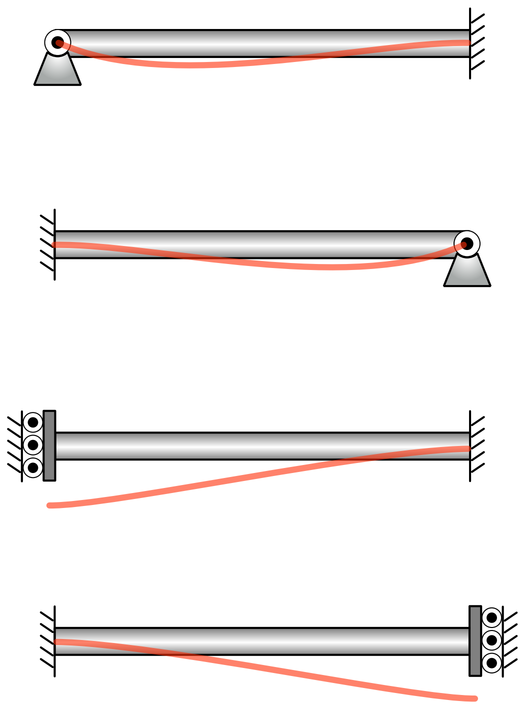


Figure 4: Beam Boundary Conditions

$$w_0 = 0 \;\; \theta_0 = ?$$
$$w_L = 0 \;\; \theta_L = 0$$

The following variables will be used for notational conveninece when defining the beam stiffness matrix.

$$a = \frac{12EI}{L^3} \quad b = \frac{6EI}{L^2} \quad c = \frac{4EI}{L} \quad d = \frac{2EI}{L}$$

Brian Chevalier

The local element stiffness matrix for a beam is therefore:

$$\begin{Bmatrix} V_0 \\ M_0 \\ \hline V_L \\ M_L \end{Bmatrix} = \left[\begin{array}{cc|cc} a & b & -a & b \\ b & c & -b & d \\ \hline -a & -b & a & -b \\ b & d & -b & c \end{array}\right] \begin{Bmatrix} w_0 \\ \theta_0 \\ \hline w_L \\ \theta_L \end{Bmatrix} \tag{17}$$

The global element stiffness matrix will not be discussed since a beam only has elements along one coordinate system.

3 Frame Direct Stiffness Method

The *Direct Stiffness Method* extends solving statically indeterminate frames. Each node in a frame has three degrees of freedom in comparison to two degrees of freedom for a truss. The additional degree of freedom is the rotation at each node.

3.1 Global Element Stiffness

The global transformation matrix is:

$$\begin{Bmatrix} u_x \\ u_y \\ \theta \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u'_x \\ u'_y \\ \theta' \end{Bmatrix} \tag{18}$$

Note that this transformation matrix has no effect on the rotation since rotating the whole coordinate system does not change the relative rotation at a node.

Combining the *local* element stiffness matricies for both a beam element and truss element yields:

$$\begin{Bmatrix} N_0 \\ V_0 \\ M_0 \\ \hline N_L \\ V_L \\ M_L \end{Bmatrix} = \left[\begin{array}{c|cc|c|cc} e & 0 & 0 & -e & 0 & 0 \\ \hline 0 & a & b & 0 & -a & b \\ 0 & b & c & 0 & -b & d \\ \hline -e & 0 & 0 & e & 0 & 0 \\ \hline 0 & -a & -b & 0 & a & -b \\ 0 & b & d & 0 & -b & c \end{array}\right] \begin{Bmatrix} u_0 \\ w_0 \\ \theta_0 \\ \hline u_L \\ w_L \\ \theta_L \end{Bmatrix} \quad (19)$$

The global element stiffness matrix for a frame element is then:

$$\begin{Bmatrix} N_0 \\ V_0 \\ M_0 \\ \hline N_L \\ V_L \\ M_L \end{Bmatrix} = \underbrace{\left[\begin{array}{c|cc|c|cc} am^2+el^2 & -alm+elm & -bm & -am^2-el^2 & alm-elm & -bm \\ \hline -alm+elm & al^2+em^2 & bl & alm-elm & -al^2-em^2 & bl \\ -bm & bl & c & bm & -bl & d \\ \hline -am^2-el^2 & alm-elm & bm & am^2+el^2 & -alm+elm & bm \\ \hline alm-elm & -al^2-em^2 & -bl & -alm+elm & al^2+em^2 & -bl \\ -bm & bl & d & bm & -bl & c \end{array}\right]}_{\mathbf{k}^{frame}_{global}} \begin{Bmatrix} u_0 \\ w_0 \\ \theta_0 \\ \hline u_L \\ w_L \\ \theta_L \end{Bmatrix} \quad (20)$$

3.2 Equivalent Nodal Loading

To solve the deformations with the governing equations, the equivalent nodal loading must be known (i.e. N_0, V_0, etc).

The equivalent nodal loading for a frame element with an applied point load

$$\mathbf{q}' = \left\{0, \frac{Pb^2(L+2a)}{L^3}, \frac{Pab^2}{L^2}, 0, \frac{Pa^2(L+2b)}{L^3}, \frac{-Pa^2b}{L^2}\right\} \quad (21)$$

$$\mathbf{q}' = \left\{0, \frac{L(7w_0+3w_L)}{20}, \frac{L^2(3w_0+2w_L)}{60}, 0, \frac{L(3w_0+7w_L)}{20}, -\frac{L^2(2w_0+3w_L)}{60}\right\} \quad (22)$$

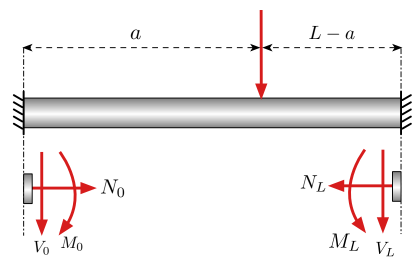


Figure 5: Equivalent Nodal Load

References

[Felippa, 2000] Felippa, C. A. (2000). A historical outline of matrix structural analysis: A play in three acts.