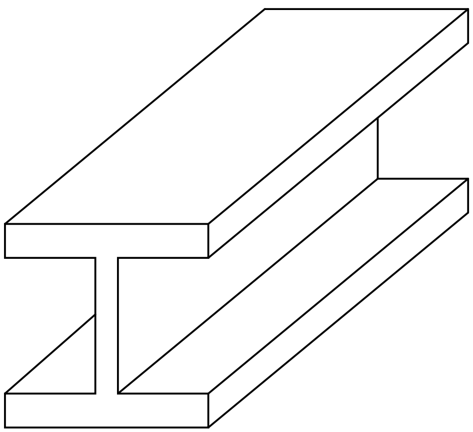
CEE384

Interpolation

Interpolation of Discrete Datasets



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1 Linear Interpolation

Linear interpolation is estimating the value between two data points given an input, x. The output y is given by solving for the point along a line between the two given data points in the data set. In figure 1, two data points are given (x_0, y_0) and (x_1, y_1).

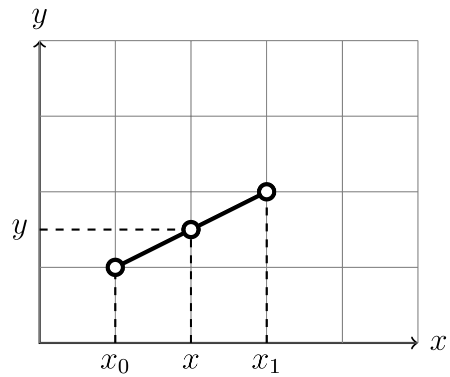

Figure 1: Example dataset plot with splines

From the figure, similar triangles are used to relate the three points.

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \tag{1}$$

Next, the output coordinate can be solved for:

$$y = y_0 + (x - x_0)\frac{y_1 - y_0}{x_1 - x_0} \tag{2}$$

2 Quadratic Interpolation

3 Quadratic Splines

A *quadratic spline* is a set of piecewise quadratic functions which intersect all points in a dataset. There are three conditions that mathematically govern these functions. The first condition is that the functions must be *continuous*. The second condition is that the functions must be *smooth*, meaning that the derivatives are continous.

There are $3n$ unknowns, where n is the number of data points in the dataset. This is because there is an a_i, b_i, and c_i, for each i data point. We will look at a particular example with 4 points (shown in table 1), and three connecting splines.

Table 1: Example dataset for quadratic spline

x	x_0	x_1	x_2	x_3
y	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

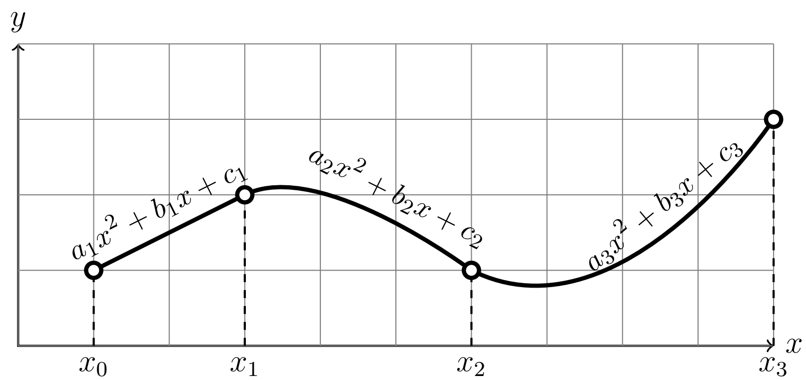

Figure 2: Example dataset plot with splines

3.1 Continuous Functions

The spline equations should be continuous and go through all the points in the dataset. This means that the

$$f(x) = \begin{cases} a_1x^2 + b_1x + c_1 & x_0 < x < x_1 \\ a_2x^2 + b_2x + c_2 & x_1 < x < x_2 \\ a_3x^2 + b_3x + c_3 & x_2 < x < x_3 \end{cases} \tag{3}$$

The first spline equation must be equal at the start of the interval, x_1, to the value of dataset $f(x_0)$. As an equation this is:

$$f(x_0) = a_1x_0^2 + b_1x_0 + c_1 \tag{4}$$

At the point, x_1, the first spline equation must be equal to the output value from the dataset, $f(x_1)$. The second equation must *also* be equal to this. Therefore this generates two additional equations:

$$f(x_1) = a_1x_1^2 + b_1x_1 + c_1 \tag{5}$$
$$f(x_1) = a_2x_1^2 + b_2x_1 + c_2 \tag{6}$$

At the second data point, x_2, the second and thrid spline must be equal to $f(x_2)$. This generates two additional equation.

$$f(x_2) = a_2x_2^2 + b_2x_2 + c_2 \tag{7}$$
$$f(x_2) = a_3x_2^2 + b_3x_2 + c_3 \tag{8}$$

Finally, the last spline must go through the last point, generating one more continuity equation:

$$f(x_3) = a_3x_3^2 + b_3x_3 + c_3 \tag{9}$$

This results in 6 equations total leaving 3 more unknowns that must be found. To generalize this pattern: for every n points the continuity condition generates $2(n-1)$ equations.

3.2 Smooth Functions

Next, take the derivative of the three spline equations:

$$f'(x) = \begin{cases} 2a_1x + b_1 \\ 2a_2x + b_2 \\ 2a_3x + b_3 \end{cases} \tag{10}$$

We know that the derivatives of the interior points must be continuous. We take the first two derivatives, evaluate them at the point x_1, and set them equal to each other. The same is done for spline 2 and 3. This results in two additional equations (11, 12).

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2 \tag{11}$$
$$2a_2x_2 + b_2 = 2a_3x_2 + b_3 \tag{12}$$

Rewriting the equations to move all the variables to the left side:

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0 \tag{13}$$
$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0 \tag{14}$$

To generalize this pattern, the smooth function condition produces $(n-2)$ equations. We now have 9 unknowns but only 8 equations. The last equation is the decision that the first spline will be linear. This is written in equation 15

$$a_1 = 0 \tag{15}$$

$$\begin{array}{c c} & \begin{array}{ccccccccc} a_1 & b_1 & c_1 & a_2 & b_2 & c_2 & a_3 & b_3 & c_3 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array} & \begin{bmatrix} x_0^2 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1^2 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^2 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2^2 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_3^2 & x_3 & 0 \end{bmatrix} \end{array} \tag{16}$$

4 Coding Quadratic Splines

Pseudocode

1. Define dataset or take dataset as input if programmed as a stand alone function
2. Initialize **A** as a matrix of zeros $(3n, 3n)$
3. Initialize **b** as a matrix of zeros $(30, 1)$
4. Loop through number of points and use continuous function information
   (a) Set values for first equation
5. Loop through number of points and use continuous derivatives information
   (a) Use interior points
6. Solve the system of equations for the constants
7. Plot and output values
   (a) Plot the piecewise quadratic function *only* in the intervals where they are valid.
   (b) Scatter plot points
   (c) Plot query point, if any