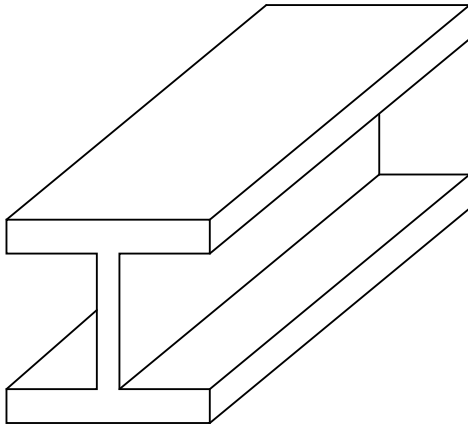


CEE384

Interpolation

Interpolation of Discrete Datasets



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1 Linear Interpolation

Linear interpolation is estimating the value between two data points given an input, x . The output y is given by solving for the point along a line between the two given data points in the data set. In figure 1, two data points are given (x_0, y_0) and (x_1, y_1) .

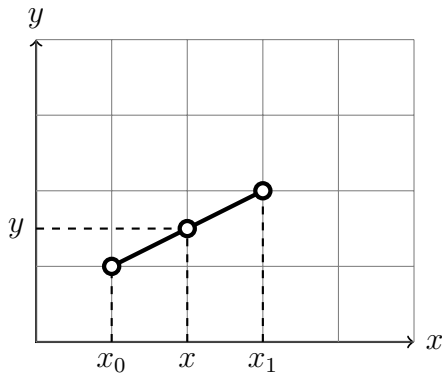


Figure 1: Example dataset plot with splines

From the figure, similar triangles are used to relate the three points.

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \quad (1)$$

Next, the output coordinate can be solved for:

2 Quadratic Interpolation

3 Quadratic Splines

A *quadratic spline* is a set of piecewise quadratic functions which intersect all points in a dataset. There are three conditions that mathematically govern these functions. The first condition is that the functions must be *continuous*. The second condition is that the functions must be *smooth*, meaning that the derivatives are continuous.

There are $3n$ unknowns, where n is the number of data points in the dataset. This is because there is an a_i , b_i , and c_i , for each i data point. We will look at a particular example with 4 points (shown in table 1), and three connecting splines.

Table 1: Example dataset for quadratic spline

x	x_0	x_1	x_2	x_3
y	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

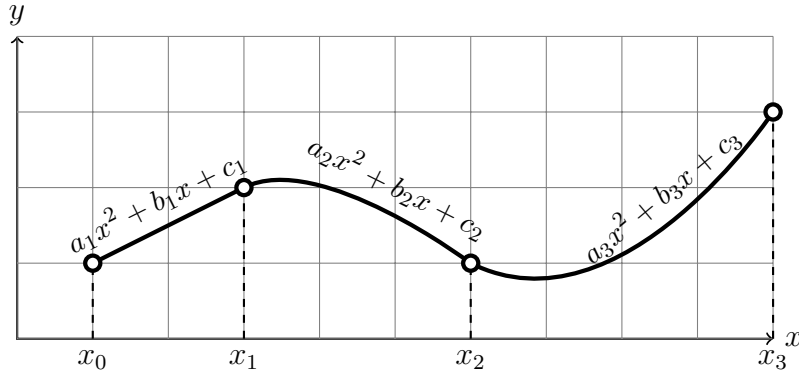


Figure 2: Example dataset plot with splines

3.1 Continuous Functions

The spline equations should be continuous and go through all the points in the dataset. This means that the

$$f(x) = \begin{cases} a_1x^2 + b_1x + c_1 & x_0 < x < x_1 \\ a_2x^2 + b_2x + c_2 & x_1 < x < x_2 \\ a_3x^2 + b_3x + c_3 & x_2 < x < x_3 \end{cases} \quad (3)$$

The first spline equation must be equal at the start of the interval, x_1 , to the value of dataset $f(x_0)$. As an equation this is:

$$f(x_0) = a_1x_0^2 + b_1x_0 + c_1 \quad (4)$$

At the point, x_1 , the first spline equation must be equal to the output value from the dataset, $f(x_1)$. The second equation must *also* be equal to this. Therefore this generates two additional equations:

$$f(x_1) = a_1x_1^2 + b_1x_1 + c_1 \quad (5)$$

$$f(x_1) = a_2x_1^2 + b_2x_1 + c_2 \quad (6)$$

At the second data point, x_2 , the second and third spline must be equal to $f(x_2)$. This generates two additional equations.

$$f(x_2) = a_2x_2^2 + b_2x_2 + c_2 \quad (7)$$

$$f(x_2) = a_3x_2^2 + b_3x_2 + c_3 \quad (8)$$

Finally, the last spline must go through the last point, generating one more continuity equation:

$$f(x_3) = a_3x_3^2 + b_3x_3 + c_3 \quad (9)$$

This results in 6 equations total leaving 3 more unknowns that must be found. To generalize this pattern: for every n points the continuity condition generates $2(n - 1)$ equations.

3.2 Smooth Functions

Next, take the derivative of the three spline equations:

$$f'(x) = \begin{cases} 2a_1x + b_1 \\ 2a_2x + b_2 \\ 2a_3x + b_3 \end{cases} \quad (10)$$

We know that the derivatives of the interior points must be continuous. We take the first two derivatives, evaluate them at the point x_1 , and set them equal to each other. The same is done for spline 2 and 3. This results in two additional equations (11, 12).

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2 \quad (11)$$

$$2a_2x_2 + b_2 = 2a_3x_2 + b_3 \quad (12)$$

Rewriting the equations to move all the variables to the left side:

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0 \quad (13)$$

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0 \quad (14)$$

To generalize this pattern, the smooth function condition produces $(n - 2)$ equations. We now have 9 unknowns but only 8 equations. The last equation is the decision that the first spline will be linear. This is written in equation 15

$$a_1 = 0 \quad (15)$$

$$\begin{array}{cccccccccc} & a_1 & b_1 & c_1 & a_2 & b_2 & c_2 & a_3 & b_3 & c_3 \\ f_1 & x_0^2 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_2 & x_1^2 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_3 & 0 & 0 & 0 & x_1^2 & x_1 & 0 & 0 & 0 & 0 \\ f_4 & 0 & 0 & 0 & x_2^2 & x_2 & 0 & 0 & 0 & 0 \\ f_5 & 0 & 0 & 0 & 0 & 0 & 0 & x_2^2 & x_2 & 0 \\ f_6 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2 & x_3 & 0 \end{array} \quad (16)$$

4 Coding Quadratic Splines

Pseudocode

1. Define dataset or take dataset as input if programmed as a stand alone function
2. Initialize \mathbf{A} as a matrix of zeros $(3n, 3n)$
3. Initialize \mathbf{b} as a matrix of zeros $(3n, 1)$
4. Loop through number of points and use continuous function information
 - (a) Set values for first equation
5. Loop through number of points and use continuous derivatives information
 - (a) Use interior points
6. Solve the system of equations for the constants
7. Plot and output values
 - (a) Plot the piecewise quadratic function *only* in the intervals where they are valid.
 - (b) Scatter plot points
 - (c) Plot query point, if any