Math 1301 Practice Final Exam Version 1

Vanderbilt University

8 December 2025

Name: Key

Please do not open the exam until instructed to do so.
You are allowed a one-page (double-sided) formula sheet.
Your instructor may ask to see your formula sheet.
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. (*20 points*)

Q1 $\frac{d}{dx}\left[5^{\sqrt{x}} + \log_4(x^x) + \sin^{-1}(4x)\right] =$ $\boxed{5^{\sqrt{x}}\frac{\ln(5)}{2\sqrt{x}} + \log_4(x) + \frac{1}{\ln(4)} + \frac{4}{\sqrt{1-16x^2}}}$

$$\frac{d}{dx}5^{\sqrt{x}} = 5^{\sqrt{x}}\ln(5)\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\log_4(x^x) = \frac{d}{dx}(x\log_4 x) = \log_4 x + \frac{x}{\ln(4)x}$$

$$\frac{d}{dx}\sin^{-1}(4x) = \frac{1}{\sqrt{1-16x^2}}\cdot 4$$

Q2 $\frac{d}{dx}\sec(x)^{\cos^{-1}(x)} =$ $\boxed{\sec(x)^{\cos^{-1}(x)}\left[\cos^{-1}(x)\tan(x) - \frac{\ln(\sec(x))}{\sqrt{1-x^2}}\right]}$

$$y = \sec(x)^{\cos^{-1}(x)}$$

$$\ln(y) = \cos^{-1}(x)\ln(\sec(x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-\ln(\sec(x))}{\sqrt{1-x^2}} + \frac{\cos^{-1}(x)\sec(x)\tan(x)}{\sec(x)}$$

Q3 $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\tan^{-1}(x)}\right) =$ $\boxed{0}$

$$\lim_{x\to0^+}\frac{\tan^{-1}(x)-x}{x\tan^{-1}(x)} \overset{L'H}{=} \lim_{x\to0^+}\frac{\frac{1}{1+x^2}-1}{\tan^{-1}(x)+\frac{x}{1+x^2}} = \lim_{x\to0^+}\frac{1-1-x^2}{(1+x^2)\tan^{-1}(x)+x} = \lim_{x\to0^+}\frac{-x^2}{(1+x^2)\tan^{-1}(x)+x}$$

$$\overset{L'H}{=} \lim_{x\to0^+}\frac{-2x}{1+2x\tan^{-1}(x)+1} = \frac{0}{2} = 0$$

Q4 The point of horizontal tangents for the curve $y = \ln^2(x+4)$ is $\boxed{x=-3}$

$$y = \ln^2(x+4)$$

$$\frac{dy}{dx} = \frac{2\ln(x+4)}{x+4}$$

horizontal tangent

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\leftrightarrow \ln(x+4) = 0$$

$$x = -3$$

Part 2. (*10 points*) Find the interval of convergence for the following power series.

$$p(x) = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(x+1)^n$$

Let $a_n = \frac{(n!)^2}{(2n)!}(x+1)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} \,(x+1)\,\Big|$$

$$= \frac{(n+1)^2}{4n^2+6n+2}|x+1|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}|x+1| < 1$$

$\Rightarrow |x+1| < 4$

$-5 < x < 3$

$$p(-5) = \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!} 4^n$$

let $b_n = (-1)^n \frac{(n!)^2}{(2n)!} 4^n$

$$\left|\frac{b_{n+1}}{b_n}\right| =$$

Part 3 (*15 points*) Evaluate the following integrals.

Q1 $\displaystyle\int \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \sin(e^x)\,(\ln(e^e))^x \; dx =$ $\boxed{-e^x\cos(e^x)+\sin(e^x).}$

$$\int \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \sin(e^x)\,(\ln(e^e))^x\,dx$$

$$= \int e^x \sin(e^x) e^x\,dx$$

$t = e^x \quad dt = e^x\,dx$

$$= \int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt$$

$u = t \quad dv = \sin(t)\,dt$

$du = dt \quad v = -\cos(t)$

$$= -t\cos(t) + \sin(t)$$

$$= -e^x\cos(e^x) + \sin(e^x).$$

Q2 $\displaystyle\int \ln(x^2+x+1)\; dx =$ $\boxed{(x+\tfrac{1}{2})\ln(x^2+x+1) + \tan^{-1}\left(\tfrac{2x+1}{\sqrt{3}}\right) - 2x}$

$$\int \ln(x^2+x+1)\,dx$$

$u = \ln(x^2+x+1) \quad dv = dx$

$du = \dfrac{2x+1}{x^2+x+1}\,dx \quad v = x$

$$= x\ln(x^2+x+1) - \int \frac{2x^2+x}{x^2+x+1}\,dx$$

$$\int \frac{2x^2+x}{x^2+x+1}\,dx = \int \frac{2x^2+x+x+2-x-2}{x^2+x+1}\,dx$$

$$= \int \frac{2(x^2+x+1)}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$= 2x - \frac{1}{2}\ln(x^2+x+1) - 3/2 \int \frac{dx}{x^2+x+1}$$

$$x^2+x+1 = (x+1/2)^2 + 3/4 \rightarrow \frac{1}{x^2+x+1} = \frac{1}{(x+1/2)^2+3/4}$$

$$\frac{3}{2}\int \frac{dx}{(x+1/2)^2+3/4} \overset{u=x+1/2}{=} \frac{3}{2}\int \frac{du}{u^2+3/4} = 2\int \frac{du}{(2/\sqrt{3}\,u)^2+1}$$

$$\overset{t=\frac{2}{\sqrt{3}}u}{=} \sqrt{3}\int \frac{dt}{t^2+1} = \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Q3 $\displaystyle\int \tan(x)\ln(\cos(x))\; dx =$ $\boxed{-\tfrac{1}{2}\ln(\cos(x))}$

$$\int \tan(x)\ln(\cos(x))\,dx$$

$$= \int \frac{\sin(x)}{\cos(x)}\ln(\cos(x))\,dx$$

$u = \cos(x)\;, \quad du = -\sin(x)\,dx$

$$= -\int \frac{\ln(u)}{u}\,du$$

$s = \ln(u)$

$ds = \dfrac{du}{u}$

$$= -\int s\,ds$$

$$= -\frac{s^2}{2}$$

$$= -\frac{1}{2}\ln(\cos(x))$$

Part 4. (*15 points*)

Q1 The value of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^n(x)\sin(\pi n)}{3^n}$ is $\boxed{\dfrac{-\sin(x)/3}{1+\sin(x)/3}}$

$$\sum_{n=1}^{\infty} \frac{\sin^2(x)\sin(\pi n)}{3^n} = \sum_{n=1}^{\infty}\left(\frac{-\sin(x)}{3}\right)^n = \frac{-\sin(x)/3}{1+\sin(x)/3}$$

Q2 The values of p such that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)[\ln(\ln(n))]^p}$ converges are $\boxed{p>1}$

Let $f(x) = \dfrac{1}{x\ln(x)(\ln(\ln(x)))^p}$

1) $f(x)$ continuous
2) $f(x)$ positive
3) $f'(x) < 0$

$$\int_3^{\infty} f(x)dx = \int_3^{\infty} \frac{dx}{x\ln(x)(\ln(\ln(x)))^p}$$

$u = \ln(x)$ $s = \ln(u)$

$$= \int_{\ln(3)}^{\infty} \frac{du}{u\ln^p(u)} = \int_{\ln(\ln(3))}^{\infty} \frac{ds}{s^p}$$

converges if $p>1$

Q3 The limit of the sequence defined by $a_n = \begin{cases} 2 & n=1 \\ \frac{1}{3-a_n} & n>1 \end{cases}$ is $\boxed{}$

Part 5. (*15 points*)

Q1 Find a parametric equation for the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and find the slope of the tangent line at an arbitrary point using the parametric equations.

$x = a\cos(t) \quad 0 \le t \le 2\pi$
$y = b\sin(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos(t)}{-a\sin(t)} = -\frac{b}{a}\cot(t) = -\frac{b}{a}\, y/x.$$

Q2 Find the length of the polar curve of $r = e^{\theta/2}$ from $0 \le \theta \le \pi/2$

$r = e^{\theta/2}$

$\frac{dr}{d\theta} = \frac{e^{\theta/2}}{2}$

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = e^{\theta} + \frac{1}{4}e^{\theta} = \frac{5}{4}e^{\theta}$

$$L = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta = \sqrt{5}\, e^{\theta/2}\Big|_0^{\pi/2} = \sqrt{5}\left(e^{\pi/4} - 1\right)$$

Q3 Find the values of theta that the polar curve $r = 1+\cos(\theta)$ has vertical and horizontal tangent lines.

$r = 1 + \cos(\theta)$

$\frac{dr}{d\theta} = -\sin(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

$\frac{dy}{d\theta} = -\sin^2(\theta) + \cos^2(\theta) + \cos(\theta)$

$= -(1-\cos^2(\theta)) + \cos(\theta)$

$= 2\cos^2(\theta) + \cos(\theta) - 1$

$= (2\cos(\theta) - 1)(\cos(\theta)+1)$

$\frac{dx}{d\theta} = -\sin(\theta)\cos(\theta) - \sin(\theta) - \sin(\theta)\cos(\theta)$

$= -2\sin(\theta)\cos(\theta) - \sin(\theta)$

$= -\sin(\theta)(2\cos(\theta)+1)$

$\frac{dy}{d\theta} = 0 \Rightarrow 2\cos(\theta) - 1 = 0 \qquad \cos(\theta)+1 = 0$

$\cos(\theta) = 1/2 \qquad \cos(\theta) = -1$

$\theta = \pi/3,\ 5\pi/3 \qquad \theta = \pi$

$\frac{dx}{d\theta} = 0 \Rightarrow \cos(\theta) = 0, \qquad 2\cos(\theta)+1 = 0$

$\theta = \pi/2,\ 3\pi/2 \qquad \cos(\theta) = -1/2$

$\theta = 2\pi/3,\ 4\pi/3$

Part 6. (*10 points*)

Q1 The general solution to $\cos(y)\dfrac{dy}{dx} = xe^{x^2+\ln(1+\sin^2(y))}$ is $\boxed{\tan^{-1}(\sin(y)) = \frac{1}{2}e^{x^2}+C}$

$$\cos(y)\frac{dy}{dx} = xe^{x^2+\ln(1+\sin^2(y))}$$

$$\cos(y)\frac{dy}{dx} = xe^{x^2}e^{\ln(1+\sin^2(y))}$$

$$\frac{\cos(y)}{1+\sin^2(y)}dy = xe^{x^2}dx$$

$$\int\frac{\cos(y)}{1+\sin^2(y)}dy = \int xe^{x^2}dx$$

$u_1 = \sin(y)$ $\quad$ $u_2 = x^2$

$du_1 = \cos(y)dy$ $\quad$ $du_2 = 2x\,dx$

$$\int\frac{du_1}{1+u_1^2} = \frac{1}{2}\int e^{u_2}du_2$$

$$\tan^{-1}(u_1) = \frac{1}{2}e^{u_2}+C$$

$$\tan^{-1}(\sin(y)) = \frac{1}{2}e^{x^2}+C.$$

Q2 The integral $\displaystyle\int_1^\infty \frac{\cos(x)}{x}\,dx$ $\boxed{\textbf{CONVERGES}}$ or **DIVERGES**. (explain your response)

$$\int_1^\infty \frac{\cos(x)}{x}dx$$

$u = \dfrac{1}{x}$ $\quad$ $dv = \cos(x)dx$

$du = -\dfrac{dx}{x^2}$ $\quad$ $v = \sin(x)$

$$= \underbrace{\lim_{c\to\infty}\frac{\sin(x)}{x}\Big|_1^c}_{} + \underbrace{\int_1^\infty\frac{\sin(x)}{x^2}dx}_{}$$

$$= \lim_{c\to\infty}\frac{\sin(c)}{c} - \sin(1) \qquad \le \int_1^\infty \frac{dx}{x^2} \longrightarrow \text{converges via p-integral test.}$$

$$= -\sin(1)$$

Part 7 (*20 points*) Determine if the following series converge or diverge. Show your work and state any necessary hypotheses.

Q1 $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}$

Set $a_n = (-1)^n \dfrac{\pi^{2n}}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\pi^{2n+2}}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{\pi^{2n}}\right|$$

$$= \frac{\pi^2}{(2n+2)(2n+1)} \to 0 \text{ as } n\to\infty.$$

$\sum a_n$ converges via the ratio test

Q2 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sin(n)}$

Set $b_n = \dfrac{1}{2+\sin(n)}$. As $\frac{1}{3} \le b_n \le 1$

$\lim_{n\to\infty} b_n \neq 0$. Hence, $\sum b_n$ diverges via the divergence test.

Q3 $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{\pi n}\right)^{n^2}$

Set $c_n = \left(1+\frac{1}{\pi n}\right)^{n^2}$. Then, $\sqrt[n]{|c_n|} = \left(1+\frac{1}{\pi n}\right)^n$ and $\lim_{n\to\infty}\sqrt[n]{|c_n|} = e^{1/\pi}$. As $e^{1/\pi} > 1$, the root test yields divergence.

Q4 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+\cos^2(n)n^2+3^n n+4}$

We have $n^3 < n^3 + \cos^2(n)n^2 + 3^n n + 4$

$$\Rightarrow \frac{1}{n^3} > \frac{1}{n^3+\cos^2(n)n^2+3^n n+4}$$

$$\Rightarrow \frac{1}{n^2} > \frac{n}{n^3+\cos^2(n)n^2+3^n n+4}$$

As $\sum \frac{1}{n^2}$ converges via p-series test $(p=2)$, the direct comparison test yields convergence.

Part 8 (*20 points*) Define the function $f(x) = \arctan(\ln(x+1)) - \arccos(\ln(\sqrt[4]{x^2+2x+1}))$

Q1 The domain of $f(x)$ is $\boxed{e^{-2}-1 \le x \le e^{2}-1}$

$$f(x) = \underbrace{\tan^{-1}(\ln(x+1))}_{\substack{x+1>0 \\ x>-1}} - \underbrace{\cos^{-1}\left(\tfrac{1}{2}\ln(x+1)\right)}_{\substack{-1 \le \frac{1}{2}\ln(x+1) \le 1 \\ -2 \le \ln(x+1) \le 2 \\ e^{-2}-1 \le x \le e^{2}-1}}$$

Q2 $f'(x) = \boxed{\dfrac{1}{1+(\ln(x+1))^2}\cdot\dfrac{1}{1+x} + \dfrac{1}{\sqrt{1-\frac{1}{4}\ln^2(1+x)}}\cdot\dfrac{1}{2}\,\dfrac{1}{1+x}}$

Q3 $f(x)$ has an inverse on $\boxed{e^{-2}-1 \le x \le e^{2}-1}$ (justify your answer)!

$$f'(x) = \underbrace{\frac{1}{1+x}}_{<0}\left[\underbrace{\frac{1}{1+\ln^2(1+x)}}_{<0} + \underbrace{\frac{1/2}{\sqrt{1-\frac{1}{4}\ln^2(1+x)}}}_{<0}\right]$$

As $f'(x) > 0$ for all x in the domain of $f(x)$, f has an inverse on its entire domain.

Q4 $f\left(e^{\sqrt{3}}-1\right) = \boxed{\pi/6}$

$$f(e^{\sqrt{3}}-1) = \tan^{-1}(\ln(e^{\sqrt{3}}-1+1)) - \cos^{-1}\left(\tfrac{1}{2}\ln(e^{\sqrt{3}}-1+1)\right)$$

$$= \tan^{-1}(\sqrt{3}) - \cos^{-1}\left(\sqrt{3}/2\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \pi/6$$

Part 9 (*20 points*) Define the function $g(x) = a^x$.

Q1 Use logarithmic differentiation to prove $g'(x)$.

$y = a^x$

$\ln(y) = x \ln(a)$

$\frac{d}{dx} \ln(y) = \frac{d}{dx} x \ln(a)$

$\frac{1}{y} \frac{dy}{dx} = \ln(a)$

$\frac{dy}{dx} = y \ln(a)$

$\frac{dy}{dx} = \ln(a) a^x$

Q2 Find the first four terms of the Taylor series of $g(x)$ centered at $x = 3$.

$g(x) = a^x$, $g(3) = a^3$

$g'(x) = \ln(a) a^x$, $g'(3) = \ln(a) a^3$

$g''(x) = \ln^2(a) a^x$, $g''(3) = \ln^2(a) a^3$

$g^3(x) = \ln^3(a) a^x$, $g^3(3) = \ln^3(a) a^3$

$g(x) \approx a^3 + \ln(a) a^3 (x-3) + \ln^2(a) a^3 (x-3)^2 + \ln^3(a) a^3 (x-3)^3$

Q3 Using **Q1**, write down a differential equation that $g(x)$ satisfies. Is the differential equation separable?

$\frac{dy}{dx} = \ln(a) y$

Yes, this ODE is separable.

Q4 The values of a such that $\int_0^\infty g(x)\,dx$ converges are $\boxed{0 < a < 1}$.

$\int_0^\infty g(x)dx = \lim_{c \to \infty} \int_0^c a^x dx$

$= \lim_{c \to \infty} \frac{1}{\ln(a)} a^x \Big|_0^c = \frac{1}{\ln(a)} \underbrace{\lim_{c \to \infty} a^c - 1}_{}$

converges if and only if $0 < a < 1$

Part 10 (*25 points*) Consider the power series $h(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$.

Q1 The radius of convergence of $h(x)$ is $\boxed{\text{ROC} = 1}$.

Set $a_n = (-1)^{n+1}\frac{x^{2n}}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n}|x|^2 \to |x|^2 \text{ as } n \to \infty$$

Q2 The power series $h(x)$ is given by the function $\boxed{\ln(1+x^2)}$.

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$\Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$$

Q3 The 41 partial sum of $h(1/2)$ is an $\boxed{\textbf{OVERESTIMATE}}$ or **UNDERESTIMATE** of $h(1/2)$.

$$S_{41} = \sum_{n=1}^{41}(-1)^{n+1}\frac{(1/2)^{2n}}{n}$$

The 41^{st} term is $\frac{(1/2)^{82}}{41} > 0 \to$ overestimate

Q4 The number of terms required for to estimate $g(0.1)$ with an error less than $\frac{1}{100}$ is $\boxed{2}$.

$$|a_1| = \frac{(0.1)^2}{1} = \frac{1}{100}$$

$$|a_2| = \frac{(0.1)^4}{2} = \frac{1}{2 \cdot 10^4} < \frac{1}{100}$$

Q5 Using **Q1**, $\int_0^1 h(x)\,dx = \boxed{\ln(2) - 2 + \pi/2}$.

$$\int_0^1 \ln(1+x^2)\,dx$$

$u = \ln(1+x^2)$ $\quad dv = dx$

$du = \frac{2x}{1+x^2}dx$ $\quad v = x$

$$= \underbrace{x\ln(1+x^2)\Big|_0^1}_{=\ln(2)} - 2\int_0^1 \frac{x^2}{1+x^2}dx$$

$$\int_0^1 \frac{x^2}{1+x^2}dx = \int_0^1 1 - \frac{1}{1+x^2}dx$$

$$= x - \tan^{-1}(x)\Big|_0^1 = 1 - \tan^{-1}(1)$$

$$= 1 - \pi/4$$

Part 11 *Extra space for work*

End of Exam. Check your work!