

Math 1301 Practice Final Exam Version 1

Vanderbilt University

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Name: Key

Please do not open the exam until instructed to do so.

You are allowed a one-page (double-sided) formula sheet.

Your instructor may ask to see your formula sheet.

No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. (20 points)

Q1 $\frac{d}{dx} \left[5^{\sqrt{x}} + \log_4(x^x) + \sin^{-1}(4x) \right] =$

$$5^{\sqrt{x}} \frac{\ln(5)}{2\sqrt{x}} + \log_4(x) + \frac{1}{\ln(4)x} + \frac{4}{\sqrt{1-16x^2}}$$

$$\frac{d}{dx} 5^{\sqrt{x}} = 5^{\sqrt{x}} \ln(5) \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} \log_4(x^x) = \frac{d}{dx} (x \log_4 x) = \log_4 x + \frac{x}{\ln(4)x}$$

$$\frac{d}{dx} \sin^{-1}(4x) = \frac{1}{\sqrt{1-16x^2}} \cdot 4$$

Q2 $\frac{d}{dx} \sec(x)^{\cos^{-1}(x)} =$

$$\sec(x)^{\cos^{-1}(x)} \left[\cos^{-1}(x) \tan(x) - \frac{\ln(\sec(x))}{\sqrt{1-x^2}} \right]$$

$$y = \sec(x)^{\cos^{-1}(x)}$$

$$\ln(y) = \cos^{-1}(x) \ln(\sec(x))$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{-\ln(\sec(x))}{\sqrt{1-x^2}} + \frac{\cos^{-1}(x) \sec(x) \tan(x)}{\sec(x)}$$

Q3 $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\tan^{-1}(x)} \right) =$

$$0$$

$$\lim_{x \rightarrow 0^+} \frac{\tan^{-1}(x) - x}{x \tan^{-1}(x)} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x^2} - 1}{\frac{1}{\tan^{-1}(x)} + \frac{x}{1+x^2}} = \lim_{x \rightarrow 0^+} \frac{1 - 1 - x^2}{(1+x^2)\tan^{-1}(x) + x} = \lim_{x \rightarrow 0^+} \frac{-x^2}{(1+x^2)\tan^{-1}(x) + x}$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{-2x}{1+2x\tan^{-1}(x)+1} = \frac{0}{2} = 0$$

Q4 The point of horizontal tangents for the curve $y = \ln^2(x+4)$ is

$$x = -3$$

$$y = \ln^2(x+4)$$

$$\frac{dy}{dx} = \frac{2 \ln(x+4)}{x+4}$$

horizontal tangent

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Leftrightarrow \ln(x+4) = 0$$

$$x = -3$$

radius of convergence

Part 2. (10 points) Find the interval of convergence for the following power series.

$$p(x) = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x+1)^n$$

$$\text{Let } a_n = \frac{(n!)^2}{(2n)!} (x+1)^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} |x+1|$$

$$= \frac{(n+1)^2}{4n^2 + 6n + 2} |x+1|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4} |x+1| < 1$$

$$\Rightarrow |x+1| < 4$$

$$-5 < x < 3$$

Part 3 (15 points) Evaluate the following integrals.

Q1 $\int \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \sin(e^x) (\ln(e^e))^x dx = \boxed{-e^x \cos(e^x) + \sin(e^x)}.$

$$\begin{aligned} & \int \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \sin(e^x) (\ln(e^e))^x dx = -t \cos(t) + \sin(t) \\ & = \int e^x \sin(e^x) e^x dx = -e^x \cos(e^x) + \sin(e^x). \\ & t = e^x \quad dt = e^x dx \\ & = \int t \sin(t) dt = -t \cos(t) + \int \cos(t) dt \\ & u = t \quad dv = \sin(t) dt \\ & du = dt \quad v = -\cos(t) \end{aligned}$$

Q2 $\int \ln(x^2 + x + 1) dx = \boxed{(x + \frac{1}{2}) \ln(x^2 + x + 1) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - 2x}$

$$\begin{aligned} & \int \ln(x^2 + x + 1) dx \\ & u = \ln(x^2 + x + 1) \quad dv = dx \\ & du = \frac{2x+1}{x^2+x+1} dx \quad v = x \\ & = x \ln(x^2 + x + 1) - \int \frac{2x^2+x}{x^2+x+1} dx \\ & \int \frac{2x^2+x}{x^2+x+1} dx = \int \frac{2x^2+x+x+2-x-2}{x^2+x+1} dx \\ & = \int \frac{2(x^2+x+1)}{x^2+x+1} dx - \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx - \frac{3}{2} \int \frac{dx}{x^2+x+1} \\ & = 2x - \frac{1}{2} \ln(x^2 + x + 1) - \frac{3}{2} \int \frac{dx}{x^2+x+1} \\ & x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} \rightarrow \frac{1}{x^2+x+1} = \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} \\ & \frac{3}{2} \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{3}{2} \int \frac{du}{u^2+\frac{3}{4}} = 2 \int \frac{du}{(\frac{2}{\sqrt{3}}u)^2+1} \\ & t = \frac{2}{\sqrt{3}}u \quad \sqrt{3} \int \frac{dt}{t^2+1} = \sqrt{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \end{aligned}$$

Q3 $\int \tan(x) \ln(\cos(x)) dx = \boxed{-\frac{1}{2} \ln(\cos(x))}$

$$\begin{aligned} & \int \tan(x) \ln(\cos(x)) dx = - \int s ds \\ & = -\frac{s^2}{2} \\ & = -\frac{1}{2} \ln(\cos(x)) \\ & = -\int \frac{\ln(u)}{u} du \\ & s = \ln(u) \\ & ds = \frac{du}{u} \end{aligned}$$

Part 4. (15 points)

Q1 The value of $\sum_{n=1}^{\infty} \frac{\sin^n(x) \cos(\pi n)}{3^n}$ is $\frac{-\sin(x)/3}{1 + \sin(x)/3}$

$$\sum_{n=1}^{\infty} \frac{\sin^2(x) \cos(\pi n)}{3^n} = \sum_{n=1}^{\infty} \left(-\frac{\sin(x)}{3} \right)^n = \frac{-\sin(x)/3}{1 + \sin(x)/3}$$

Q2 The values of p such that $\sum_{n=3}^{\infty} \frac{1}{n \ln(n) [\ln(\ln(n))]^p}$ converges are $p > 1$

- Let $f(x) = \frac{1}{x \ln(x) (\ln(\ln(x)))^p}$
- 1) $f(x)$ continuous
 - 2) $f(x)$ positive
 - 3) $f'(x) < 0$

$$\begin{aligned} \int_3^{\infty} f(x) dx &= \int_3^{\infty} \frac{dx}{x \ln(x) (\ln(\ln(x)))^p} \\ u &= \ln(x) \quad s = \ln(u) \\ &= \int_{\ln(3)}^{\infty} \frac{du}{u \ln^p(u)} = \int_{\ln(\ln(3))}^{\infty} \frac{ds}{s^p} \end{aligned}$$

converges if $p > 1$

Q3 The limit of the sequence defined by $a_n = \begin{cases} 2 & n = 1 \\ \frac{1}{3-a_n} & n > 1 \end{cases}$ is

Part 5. (15 points)

- Q1** Find a parametric equation for the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and find the slope of the tangent line at an arbitrary point using the parametric equations.

$$\begin{aligned} x &= a \cos(t) & 0 \leq t \leq 2\pi \\ y &= b \sin(t) \end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b \cos(t)}{-a \sin(t)} = -\frac{b}{a} \cot(t)$$

- Q2** Find the length of the polar curve of $r = e^{\theta/2}$ from $0 \leq \theta \leq \pi/2$

$$\begin{aligned} r &= e^{\theta/2} & r^2 + \left(\frac{dr}{d\theta}\right)^2 \\ \frac{dr}{d\theta} &= \frac{e^{\theta/2}}{2} & = e^\theta + \frac{1}{4} e^\theta = \frac{5}{4} e^\theta \end{aligned}$$

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^{\pi/2} \frac{\sqrt{5}}{2} e^{\theta/2} d\theta = \sqrt{5} e^{\theta/2} \Big|_0^{\pi/2} = \sqrt{5} (e^{\pi/4} - 1) \end{aligned}$$

- Q3** Find the values of theta that the polar curve $r = 1 + \cos(\theta)$ has vertical and horizontal tangent lines.

$$r = 1 + \cos(\theta)$$

$$\frac{dr}{d\theta} = -\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}$$

$$\begin{aligned} \frac{dy}{d\theta} &= -\sin^2(\theta) + \cos^2(\theta) + \cos(\theta) \\ &= -(1 - \cos^2(\theta)) + \cos(\theta) \\ &= 2\cos^2(\theta) + \cos(\theta) - 1 \\ &= (2\cos(\theta) - 1)(\cos(\theta) + 1) \end{aligned}$$

$$\begin{aligned} \frac{dx}{d\theta} &= -\sin(\theta)\cos(\theta) - \sin(\theta) - \sin(\theta)\cos(\theta) \\ &= -2\sin(\theta)\cos(\theta) - \sin(\theta) \\ &= -\sin(\theta)(2\cos(\theta) + 1) \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\theta} = 0 &\Rightarrow 2\cos(\theta) - 1 = 0 & \cos(\theta) + 1 = 0 \\ \cos(\theta) &= \frac{1}{2} & \cos(\theta) = -1 \\ \theta &= \pi/3, 5\pi/3 & \theta = \pi \end{aligned}$$

$$\begin{aligned} \frac{dx}{d\theta} = 0 &\Rightarrow \sin(\theta) = 0, 2\cos(\theta) + 1 = 0 \\ \theta &= 0, \pi & \cos(\theta) = -\frac{1}{2} \\ \theta &= 2\pi/3, 4\pi/3 \end{aligned}$$

π is solution to $\frac{dx}{d\theta} = 0$ and

$\frac{dy}{d\theta} = 0$, so have to check

$$\lim_{\theta \rightarrow \pi} \frac{dy}{dx} .$$

Part 6. (10 points)

Q1 The general solution to $\cos(y) \frac{dy}{dx} = xe^{x^2 + \ln(1+\sin^2(y))}$ is

$$\tan^{-1}(\sin(y)) = \frac{1}{2}e^{x^2} + C$$

$$\cos(y) \frac{dy}{dx} = xe^{x^2 + \ln(1+\sin^2(y))}$$

$$\cos(y) \frac{dy}{dx} = xe^{x^2} e^{\ln(1+\sin^2(y))}$$

$$\frac{\cos(y)}{1+\sin^2(y)} dy = xe^{x^2} dx$$

$$\int \frac{\cos(y)}{1+\sin^2(y)} dy = \int xe^{x^2} dx$$

$$u_1 = \sin(y) \quad u_2 = x^2$$

$$du_1 = \cos(y) dy \quad du_2 = 2x dx$$

$$\int \frac{du_1}{1+u_1^2} = \frac{1}{2} \int e^{u_2} du_2$$

Q2 The integral $\int_1^\infty \frac{\cos(x)}{x} dx$ **CONVERGES** or DIVERGES. (explain your response)

$$\int_1^\infty \frac{\cos(x)}{x} dx$$

$$u = \frac{1}{x} \quad dv = \cos(x) dx$$

$$du = -\frac{dx}{x^2} \quad v = \sin(x)$$

$$\begin{aligned} &= \underbrace{\lim_{c \rightarrow \infty} \left[\frac{\sin(x)}{x} \right]_1^c}_{= \lim_{c \rightarrow \infty} \frac{\sin(c)}{c} - \sin(1)} + \underbrace{\int_1^\infty \frac{\sin(x)}{x^2} dx}_{\leq \int_1^\infty \frac{dx}{x^2}} \rightarrow \text{converges via } p\text{-integral test.} \\ &= -\sin(1) \end{aligned}$$

Part 7 (20 points) Determine if the following series converge or diverge. Show your work and state any necessary hypotheses.

Q1 $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

Set $a_n = (-1)^n \frac{\pi^{2n}}{(2n)!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\pi^{2n+2}}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{\pi^{2n}} \right| \\ = \frac{\pi^2}{(2n+2)(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$\sum a_n$ converges via the ratio test

Q2 $\sum_{n=1}^{\infty} \frac{1}{2 + \sin(n)}$

Set $b_n = \frac{1}{2 + \sin(n)}$. As $\frac{1}{3} \leq b_n \leq 1$

$\lim_{n \rightarrow \infty} b_n \neq 0$. Hence, $\sum b_n$ diverges via the divergence test.

Q3 $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\pi n}\right)^{n^2}$

Set $c_n = \left(1 + \frac{1}{\pi n}\right)^{n^2}$. Then, $\sqrt[n]{|c_n|} = \left(1 + \frac{1}{\pi n}\right)$
 and $\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = e^{1/\pi}$. As $e^{1/\pi} > 1$, the root test yields divergence.

Q4 $\sum_{n=1}^{\infty} \frac{n}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$

We have $n^3 < n^3 + \cos^2(n)n^2 + 3^n n + 4$

$$\Rightarrow \frac{1}{n^3} > \frac{1}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$$

$$\Rightarrow \frac{1}{n^2} > \frac{n}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$$

As $\sum \frac{1}{n^2}$ converges via p-series test ($p=2$), the direct comparison test yields convergence.

Part 8 (20 points) Define the function $f(x) = \arctan(\ln(x+1)) - \arccos(\ln(\sqrt[4]{x^2 + 2x + 1}))$

Q1 The domain of $f(x)$ is

$$e^{-2}-1 \leq x \leq e^2-1$$

$$f(x) = \underbrace{\tan^{-1}(\ln(x+1))}_{\begin{array}{l} x+1 > 0 \\ x > -1 \end{array}} - \underbrace{\cos^{-1}\left(\frac{1}{2} \ln(x+1)\right)}_{\begin{array}{l} -1 \leq \frac{1}{2} \ln(x+1) \leq 1 \\ -2 \leq \ln(x+1) \leq 2 \end{array}}$$

$$e^{-2}-1 \leq x \leq e^2-1$$

Q2 $f'(x) =$

$$\frac{1}{1+(\ln(x+1))^2} \cdot \frac{1}{1+x} + \frac{1}{\sqrt{1-\frac{1}{4} \ln^2(1+x)}} \cdot \frac{1}{2} \cdot \frac{1}{1+x}$$

Q3 $f(x)$ has an inverse on

$$e^{-2}-1 \leq x \leq e^2-1$$

(justify your answer)!

$$f'(x) = \underbrace{\frac{1}{1+x}}_{<0} \left[\underbrace{\frac{1}{1+\ln^2(1+x)}}_{<0} + \underbrace{\frac{1/2}{\sqrt{1-\frac{1}{4} \ln^2(1+x)}}}_{<0} \right]$$

As $f'(x) > 0$ for all x in the domain of $f(x)$,
 f has an inverse on its entire domain.

Q4 $f(e^{\sqrt{3}} - 1) =$

$$\pi/6$$

$$\begin{aligned} f(e^{\sqrt{3}} - 1) &= \tan^{-1}(\ln(e^{\sqrt{3}} - 1 + 1)) - \cos^{-1}\left(\frac{1}{2} \ln(e^{\sqrt{3}} - 1 + 1)\right) \\ &= \tan^{-1}(\sqrt{3}) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{3} - \frac{\pi}{6} = \pi/6 \end{aligned}$$

Part 9 (20 points) Define the function $g(x) = a^x$.

Q1 Use logarithmic differentiation to prove $g'(x)$.

$$\begin{aligned} y &= a^x & \frac{dy}{dx} &= y \ln(a) \\ \ln(y) &= x \ln(a) \\ \frac{d}{dx} \ln(y) &= \frac{d}{dx} x \ln(a) & \frac{dy}{dx} &= \ln(a) a^x \\ \frac{1}{y} \frac{dy}{dx} &= \ln(a) \end{aligned}$$

Q2 Find the first four terms of the Taylor series of $g(x)$ centered at $x = 3$.

$$\begin{aligned} g(x) &= a^x, \quad g(3) = a^3 \\ g'(x) &= \ln(a) a^x, \quad g'(3) = \ln(a) a^3 \\ g''(x) &= \ln^2(a) a^x, \quad g''(3) = \ln^2(a) a^3 \\ g'''(x) &= \ln^3(a) a^x, \quad g'''(3) = \ln^3(a) a^3 \end{aligned}$$

$$g(x) \approx a^3 + \ln(a) a^3 (x-3) + \ln^2(a) a^3 (x-3)^2 + \ln^3(a) a^3 (x-3)^3$$

Q3 Using **Q1**, write down a differential equation that $g(x)$ satisfies. Is the differential equation separable?

$$\frac{dy}{dx} = \ln(a) y$$

Yes, this ODE is separable.

Q4 The values of a such that $\int_0^\infty g(x) dx$ converges are

$$0 < a < 1$$

$$\begin{aligned} \int_0^\infty g(x) dx &= \lim_{c \rightarrow \infty} \int_0^c a^x dx \\ &= \lim_{c \rightarrow \infty} \frac{1}{\ln(a)} a^x \Big|_0^c = \underbrace{\frac{1}{\ln(a)} \lim_{c \rightarrow \infty} a^c - 1}_{\text{converges if and only if } 0 < a < 1} \end{aligned}$$

Part 10 (25 points) Consider the power series $h(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$.

Q1 The radius of convergence of $h(x)$ is

$$\text{ROC} = 1$$

$$\text{Set } a_n = (-1)^{n+1} \frac{x^{2n}}{n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} |x|^2 \rightarrow |x|^2 \text{ as } n \rightarrow \infty$$

Q2 The power series $h(x)$ is given by the function

$$\ln(1+x^2)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

Q3 The 41 partial sum of $h(1/2)$ is an **OVERESTIMATE** or **UNDERESTIMATE** of $h(1/2)$.

$$S_{41} = \sum_{n=1}^{41} (-1)^{n+1} \frac{(1/2)^{2n}}{n}$$

The 41st term is $\frac{(1/2)^{82}}{41} > 0 \rightarrow \text{overestimate}$

Q4 The number of terms required for to estimate $g(0.1)$ with an error less than $\frac{1}{100}$ is

$$2$$

$$|a_1| = \frac{(0.1)^2}{1} = \frac{1}{100}$$

$$|a_2| = \frac{(0.1)^4}{2} = \frac{1}{2 \cdot 10^4} < \frac{1}{100}$$

Q5 Using Q1, $\int_0^1 h(x) dx =$

$$\ln(2) - 2 + \pi/2$$

$$\int_0^1 \ln(1+x^2) dx$$

$$u = \ln(1+x^2) \quad dv = dx$$

$$du = \frac{2x}{1+x^2} dx \quad v = x$$

$$= \underbrace{\int_0^1 x \ln(1+x^2) dx}_{= \ln(2)} - 2 \int_0^1 \frac{x^2}{1+x^2} dx$$

$$\int_0^1 \frac{x^2}{1+x^2} dx = \int_0^1 1 - \frac{1}{1+x^2} dx$$

$$= x - \tan^{-1}(x) \Big|_0^1 = 1 - \tan^{-1}(1)$$

$$= 1 - \pi/4$$

Part 11 *Extra space for work*

End of Exam. Check your work!