LOCAL WELL-POSEDNESS OF THE INCOMPRESSIBLE EULER AND NAVIER-STOKES EQUATIONS

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ABSTRACT. This expository work proves local-in-time well-posedness for the incompressible Euler and Navier-Stokes equations whose spatial domain is $\mathbb{R}^n$. The main method to prove that solutions to these equations exists is using so-called energy methods. We follow closely from the previous works in [6] and [3].

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1. Introduction to the Equations of Motion

This work concerns the analysis of two important equations in fluid mechanics: the *Euler* and *Navier-Stokes* equations. The Euler equations are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p \\ \nabla \cdot u = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.1}$$

and the Navier-Stokes equations are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u \\ \nabla \cdot u = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.2}$$

where $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is the velocity of the fluid, $p : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the pressure of the fluid, and $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the initial velocity of the fluid. The first equation of the Euler and Navier-Stokes equations represents the conservation of momentum of a fluid. The left-hand sides of these equations are the same; mainly $\partial_t u + (u \cdot \nabla)u$. This is called the *material derivative* of a fluid and is commonly denoted as

$$\frac{D}{Dt} = \partial_t + (u \cdot \nabla). \tag{1.3}$$

where the gradient operator is defined as

$$\nabla = (\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n})^{\perp}. \tag{1.4}$$

Physically, this quantity represents the derivative along particle trajectories of the fluid. This term also has the nonlinear expression $(u \cdot \nabla)u$, called the convective term. Both the Euler and Navier-Stokes equations have a negative pressure gradient. However, the Navier-Stokes equations include a Laplacian term, $\nu \Delta u$, where the Laplacian is defined as

$$\Delta = \sum_{j=1}^{n} \partial_{x_j}^2 \tag{1.5}$$

and $\nu > 0$ is a constant called the *kinematic viscosity.* Most engineers are concerned with the dynamic viscosity of a fluid, denoted by μ. This quantity defines how a fluid reacts to forces or changes in the fluid shape. A fluid's kinematic viscosity is defined as the dynamic viscosity and density ratio, denoted by ρ. The dynamic viscosity is useful because it removes the density term from the equations. As the Euler equations do not include a viscosity term, we call them *inviscid*, while the Navier-Stokes equations are called *viscous.* The difference is that the Navier-Stokes equations can see friction forces while the Euler equations cannot. Thus, the Euler equations are primarily used for systems with low viscous, such as vehicles at high altitudes. Conversely, the Navier-Stokes equations are better suited for situations in which viscous forces play a larger role, such as a car driving down the street. A more detailed discussion of the basics of inviscid and viscous fluid mechanics can be found in [2] and [8].

These equations are also equipped with an incompressibility condition. In purely mathematical terms, we call a fluid *incompressible* if it satisfies the compressibility condition: $\nabla \cdot u = 0$. In other words, the velocity field is divergence-free. Physically, it is useful to think of incompressibility as it pertains to the density of a fluid. As the word suggests, incompressibility is equivalent to the density being constant. This equivalence comes from applying the conservation of mass. Thus, the second equation in the Euler and Navier-Stokes equations is simply the conservation of mass. If a fluid is not incompressible, we call it *compressible*. Normally, this adds much complexity to the system as density is now a dependent variable. However, for this work, we only consider incompressible fluids. Of course, the compressibility condition is an approximation, as fluid can't be truly incompressible. Thus, it is natural to ask when this approximation is apt to be used. The incompressible model best approximates a fluid when the density gradient is small (less than five percent). Experimentally, compressible and incompressible fluids behave similarly at a speed at or below Mach 0.3. A more detailed discussion of incompressible fluids can be found in [9].

In this work, we will develop the theory of local-in-time well-posedness for the incompressible Euler and Navier-Stokes equations. A partial differential equation is well-posed if the following propositions are true

(1) **Existence**: Given some initial data u_0, does there exist a solution u to the initial value problem that exists up to some time $T > 0$.
(2) **Uniqueness**: Given two solutions u_1 and u_2 with the same initial condition, u_0, then $u_1 = u_2$.
(3) **Continuous Dependence on Initial Data**: If the initial data u_0 is perturbed, how does the solution react?

We will prove that the Euler and Navier-Stokes equations are well-posed locally in time. For simplicity, we consider the spatial domain to be $\mathbb{R}^n$. Hence, we do not need to consider boundary behavior. This mainly aids us when we use integration by parts and the divergence theorem. This problem has been solved before and we use the same ideas presented in [3] and [6] with some additional details in the proofs. This work is meant for readers just beginning the rigorous mathematical study of fluid mechanics. We will assume knowledge from an introductory real analysis course. However, any important theorems will be included. We now begin by introducing the abstract function spaces in which we wish to work.

2. Functional Spaces for Incompressible Fluids

2.1. **Sobolev Spaces.** Before we can analyze the fluid equations, the appropriate spaces to work in must be understood. It might seem that the Lebesgue spaces are the appropriate spaces; in a sense, this is correct. However, Lebesgue spaces do not give any information about how a function's derivative behaves– or if it has one! Therefore, it is natural to restrict our Lebesgue spaces to those in which the derivatives of its functions also belong to the Lebesgue space. As functions in the Lebesgue spaces are not assumed to be differentiable, we generalize the notion of a derivative to the so-called *weak derivative*.

Definition 2.1. The space $C_c^\infty(U)$ denotes the space of infinitely differentiable functions with compact support in some domain $U \subset \mathbb{R}$. The elements of this space are called *test functions*

Test functions are crucial to the formulation of weak derivatives. We define weak derivatives in the following way.

Definition 2.2. Suppose that $u, v \in L^1_{\mathrm{loc}}(U)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written as

$$D^\alpha u = v$$

provided that

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U v \varphi dx$$

for all test functions $\varphi \in C_c^\infty(U)$.

We are now able to define the main function spaces in which we will work: *Sobolev space*.

Definition 2.3. The Sobolev space $H^m(\mathbb{R}^n)$ for $m \in \mathbb{N}$ is given by

$$H^m(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n) | D^\alpha v \in L^2(\mathbb{R}^N), \quad 0 \leq |\alpha| \leq m\}. \tag{2.1}$$

The norm on the Sobolev space, called the H^m norm is given by

$$\|v\|_{H^m(\mathbb{R}^n)} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha v\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \tag{2.2}$$

This definition of Sobolev spaces provides the most generality. However, as we are working over $\mathbb{R}^n$, we will give an equivalent definition in terms of the Fourier transform. As we have seen, defining Sobolev spaces can be a bit cumbersome. Since we are working over the space $\mathbb{R}^n$, we are going to provide an alternative, but equivalent, way to define Sobolev spaces. Moreover, this definition can be extended to all real numbers by considering an equivalent definition using the Fourier transform. We briefly review a few facts from Fourier analysis that will be of assistance. We define the Fourier transform as the following.

Definition 2.4. For any function $v \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform is defined as

$$\hat{v}(\xi) = \int_{\mathbb{R}} v(x) e^{-2\pi i \xi x} dx. \tag{2.3}$$

Definition 2.5. Consider the functional

$$\|\cdot\| : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}^+$$

given by

$$\|v\|_s = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{v}|^2 \right)^{1/2} \tag{2.4}$$

Theorem 2.6. *Let $v \in L^2(\mathbb{R}^n)$. Then, $v \in H^s(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^s \in L^2(\mathbb{R}^n)$ and the norms*

$$v \mapsto \left(\sum_{0\le|\alpha|\le m} \|D^\alpha v\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \quad \text{and} \quad v \mapsto \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{v}|^2 \right)^{1/2}$$

are equivalent norms.

Proof. For $v \in H^s(\mathbb{R}^n)$, $\widehat{D^\alpha v} = (2\pi i\xi)^\alpha \hat{v}(\xi)$. The Plancherel Theorem provides that

$$\sum_{0\le|\alpha|\le s} \|D^\alpha v\|^2_{L^2(\mathbb{R}^n)} = \sum_{0\le|\alpha|\le s} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |(2\pi\xi)^\alpha|^2 d\xi.$$

Therefore, it suffices to show that the quantities $\sum_{|\alpha|\le k} |\xi^\alpha|^2$ and $(1+|\xi|^2)^k$ are comparable. We have that $|\xi^\alpha| \le 1$ for $|\xi| \le 1$ and $|\xi^\alpha| \le |\xi|^{|\alpha|} \le |\xi|^k$ for $|\xi| \ge 1$ and $|\alpha| \le k$. It follows that

$$\sum_{0\le|\alpha|\le k} |\xi^\alpha|^2 \le C_1 \max(1, |\xi|^{2k}) \le (1+|\xi|^2)^k.$$

Conversely, as $|\xi|^{2k}$ and $\sum_{j=1}^n |\xi_j^k|^2$ are both homogenous of degree k, it follows that $|\xi|^{2k} \le C_2 \sum_{j=1}^n |\xi_j^k|^2$ where C_2 is the reciprocal of the minimum value of $\sum_{j=1}^n |\xi_j^k|^2$ on the unit sphere $|\xi| = 1$. It follows that

$$\begin{aligned}(1+|\xi|^2)^k &\le 2^k \max(1, |\xi)^{2k} \\ &\le 2^k(1+|\xi|^{2k}) \\ &\le 2^k C_2 \left(1 + \sum_{j=1}^n |\xi_j^k|^2\right) \\ &\le 2^k \sum_{|\alpha|\le k} |\xi^\alpha|^2.\end{aligned}$$

Therefore, the two norms are equivalent. □

Theorem 2.7. *The Sobolev space $H^s(\mathbb{R}^n)$ is a Banach space.*

Proof. First, we show that $H^s(\mathbb{R}^n)$ is a normed space.
1. Let $u \in H^s(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}^n$. It follows that

$$\begin{aligned}\|\lambda u\|_{H^s(\mathbb{R}^n)} &= \left(\sum_{|\alpha|\le k} \|D^\alpha(\lambda u)\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \\ &= \left(\lambda^2 \sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \\ &= |\lambda| \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2}\end{aligned}$$

$$= |\lambda| \|u\|_{H^s(\mathbb{R}^n)}.$$

2. Suppose that $u \in H^s(\mathbb{R}^n)$ and $\|u\|_{H^s(\mathbb{R}^n)} = 0$. It follows that

$$\|u\|_{H^s(\mathbb{R}^n)} = 0 \iff \sum_{|\alpha| \leq k} \|D^\alpha(u)\|^2_{L^2(\mathbb{R}^n)} = 0 \iff \text{(as } L^2(\mathbb{R}^n) \text{ is a Banach space)} \ \ u = 0.$$

3. Let $u, v \in H^s(\mathbb{R}^n)$. Then,

$$\begin{aligned}
\|u+v\|_{H^s(\mathbb{R}^n)} &= \left(\sum_{|\alpha| \leq s} \|D^\alpha(u+v)\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \\
&\leq \left(\sum_{|\alpha| \leq s} \left(\|D^\alpha(u)\|_{L^2(\mathbb{R}^n)} + \|D^\alpha(v)\|_{L^2(\mathbb{R}^n)} \right)^2 \right)^{1/2} \\
&\leq \left(\sum_{|\alpha| \leq s} \|D^\alpha(u)\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} + \left(\sum_{|\alpha| \leq s} \|D^\alpha(v)\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2} \\
&= \|u\|_{H^s(\mathbb{R}^n)} + \|v\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $\left(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n)}\right)$ is a normed space.
Next, we show that $H^s(\mathbb{R}^n)$ is a complete normed space under the Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$. Suppose that $\{u_i\}_{\in\mathbb{N}} \subset H^s(\mathbb{R}^n)$ is Cauchy. Then, for each $|\alpha| \leq s$, $\{D^\alpha(u_i)\}_{i\in\mathbb{N}}$ is also a Cauchy sequence in $L^2(\mathbb{R}^n)$. As $L^2(\mathbb{R}^n)$ is a Banach space, there exists u^α such that

$$\|D^\alpha u_i - u^\alpha\|_{L^2(\mathbb{R}^n)} \to 0$$

as $i \to \infty$. In particular, $u_i \to u_{(0,\ldots,0)} = u \in L^2(\mathbb{R}^n)$. Next, fix $\phi \in C^\infty_C(\mathbb{R}^n)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} u \left(D^\alpha \phi\right) dx &= \lim_{i\to\infty} \int_{\mathbb{R}^n} u_i D^\alpha \phi dx \\
&= \lim_{i\to\infty} (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_\alpha \phi dx
\end{aligned}$$

Therefore, $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n)})$ is a Banach space. □

As $H^s(\mathbb{R}^n)$ is a subspace of $L^2(\mathbb{R}^n)$, it is natural to define an inner product on $H^s(\mathbb{R}^n)$. We define the inner product in terms of the Fourier transform for convenience. We then have the following definition.

Definition 2.8. Let $g : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C}$ be defined by

$$g(u,v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \hat{u}(\xi)\hat{v}(\xi)(1+|\xi|^2)^s d\xi. \tag{2.5}$$

Theorem 2.9. *The Sobolev space $H^s(\mathbb{R}^n)$ equipped with the function $g(\cdot,\cdot)_{H^s(\mathbb{R}^n)}$ is an inner-product space.*

Proof. **1. Positive Definiteness**: Let $u \in H^s(\mathbb{R}^n)$. Then, $g(u,u)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \hat{u}^2(\xi)(1+|\xi|^2)^s d\xi \geq 0$. Moreover, suppose that $g(u,u)_{H^s(\mathbb{R}^n)} = 0$. Then,

$$g(u,u)_{H^s(\mathbb{R}^n)} = 0 \iff \int_{\mathbb{R}^n} \hat{u}^2(\xi)(1+|\xi|^2)^s d\xi = 0 \iff \hat{u} = 0.$$

Thus, $g(\cdot,\cdot)_{H^s(\mathbb{R}^n)}$ is positive definite.
2. Linearity in the First Argument: Let $u, v, w \in H^s(\mathbb{R}^n)$. It follows that

$$\begin{aligned} g(u+v,w)_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (\hat{u}(\xi)+\hat{v}(\xi))\hat{w}(\xi)(1+|\xi|^2)^s d\xi \\ &= \int_{\mathbb{R}^n} \hat{u}(\xi)\hat{w}(\xi)(1+|\xi|^2)^s d\xi + \int_{\mathbb{R}^n} \hat{v}(\xi)\hat{w}(\xi)(1+|\xi|^2)^s d\xi \\ &= g(u,w)_{H^s(\mathbb{R}^n)} + g(v,w)_{H^s(\mathbb{R}^n)}. \end{aligned}$$

Thus, $g(\cdot,\cdot)_{H^s(\mathbb{R}^n)}$ is linear in the first argument.
3. Conjugate Symmetry: Let $u, v \in H^s(\mathbb{R}^n)$. It follows that

$$\begin{aligned} \overline{g(u,v)_{H^s(\mathbb{R}^n)}} &= \overline{\int_{\mathbb{R}^n} \hat{u}(\xi)\hat{v}(\xi)(1+|\xi|^2)^s d\xi} \\ &= \int_{\mathbb{R}^n} \hat{u}(\xi)\hat{v}(\xi)(1+|\xi|^2)^s d\xi \\ &= g(v,u)_{H^s(\mathbb{R}^n)}. \end{aligned}$$

Therefore, $(H^s(\mathbb{R}^n), g(\cdot,\cdot)_{H^s(\mathbb{R}^n)})$ is an inner product space. $\square$

Next, we state a very important theorem in the theory of Sobolev spaces: the Sobolev Embedding Theorem. This result will aid us in many of the details in the proof of local well-posedness.

Theorem 2.10. ***(The Sobolev Embedding Theorem)*** *If $s > k + n/2$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$. That is, there exists a constant $C = C(k,s)$ such that*

$$\sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha u| \leq \|u\|_{H^s(\mathbb{R}^n)} \tag{2.6}$$

for $u \in H^s(\mathbb{R}^n)$.

Proof. The Fourier Inversion Formula provides that if $\widehat{D^\alpha u}(\xi) \in L^1(\mathbb{R}^n)$, then $D^\alpha u$ is continuous and $\sup_{x\in\mathbb{R}^n} |D^\alpha u| \leq \|\widehat{D^\alpha u}(\xi)\|_{L^1(\mathbb{R}^n)}$. Hence, it suffices to show that $\|\widehat{D^\alpha u}(\xi)\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)}$ for $\|\alpha\| \leq k$. As $\widehat{D^\alpha u}()\xi = (2\pi i\xi)^\alpha \hat{u}(\xi)$, it follows that, for $\alpha \leq k$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |(2\pi i\xi)^\alpha \hat{u}(\xi)| d\xi &\leq (2\pi)^k \int_{\mathbb{R}^n} (1+|\xi|^2)^{k/2} \hat{u}(\xi) d\xi \\ &= (2\pi)^k \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2} |\hat{u}(\xi)| (1+|\xi|^2)^{(k-s)/2} d\xi \\ &\leq (2\pi)^k \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s} \right)^{1/2} \\ &= (2\pi)^k \left(\|u\|^2_{H^s(\mathbb{R}^n)} \int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s} d\xi \right)^{1/2}. \end{aligned}$$

The integral $\int_{\mathbb{R}^n}(1+|\xi|^2)^{k-s}d\xi$ can be written as $\int_{\mathbb{R}^n}(1+|\xi|^2)^{k-s}d\xi = \int_0^\infty (1+r^2)^{k-s}r^{n-1}dr$, which converges precisely when $s > k + n/s$. $\square$

Theorem 2.11. *If $s > n/2$, the Sobolev space $H^s(\mathbb{R}^n)$ is a Banach algebra.*

Proof. We already know that $H^s(\mathbb{R}^n)$ is a Banach space. Hence, it suffices to show that for $u, v \in H^s(\mathbb{R}^n)$, we have $\|uv\|_{H^s(\mathbb{R}^n)} \lesssim \|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$. We begin with the following inequality. For $s > 0$, we have

$$\begin{aligned}
(1+|\xi|^2)^{s/2} &\le (1+|\xi-\eta|^2+|\eta|^2)^{s/2} \\
&\le (1+2|\xi-\eta|^2+2|\eta|^2)^{s/2} \\
&\le 2^s\left((1+|\xi-\eta|^2)^{s/2}+(1+|\eta|^2)^{s/2}\right).
\end{aligned}$$

It follows that for $u, v \in H^s(\mathbb{R}^n)$

$$\begin{aligned}
\|uv\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{uv}(\xi)|^2d\xi \\
&\le (2\pi)^N\int_{\mathbb{R}^n}\left((1+|\xi|^2)^{s/2}(|\hat{u}|*|\hat{v}|)(\xi)\right)^2 d\xi \\
&\le c(N,s)\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left((1+|\xi-\eta|^2)^{s/2}+(1+|\eta|^2)^{s/2}\right)|\hat{u}(\xi-\eta)||\hat{v}(\eta)|d\eta\right)^2 d\xi \\
&\le 2C(n,s)\|u\|^2_{H^s(\mathbb{R}^n)}\|v\|^2_{H^s(\mathbb{R}^n)}
\end{aligned}$$

$\square$

To end this section on the basic theory of Sobolev spaces, we state a few important inequalities for Sobolev spaces known as the *Calculus Inequalities* in the Sobolev spaces.

Proposition 2.12. *For all $s \in \mathbb{N}$, there exists $C > 0$ such that for all $u, v \in L^\infty(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, we have*

$$\|uv\|_{H^s(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)}\|D^s v\|_{L^2(\mathbb{R}^n)} + \|v\|_{L^\infty(\mathbb{R}^n)}\|D^s u\|_{L^2(\mathbb{R}^n)}\right) \tag{2.7}$$

and

$$\sum_{0\le|\alpha|\le s}\|D^\alpha(uv)-uD^\alpha v\|_{L^2(\mathbb{R}^n)} \le C\left(\|\nabla u\|_{L^\infty(\mathbb{R}^n)}\|D^{s-1}v\|_{L^2(\mathbb{R}^n)} + \|\nabla v\|_{L^\infty(\mathbb{R}^n)}\|D^s u\|_{L^2(\mathbb{R}^n)}\right) \tag{2.8}$$

Further reading can be found in [1].

2.2. Mollifiers and the Hodge Decomposition in $H^m(\mathbb{R}^n)$.

Definition 2.13. Given any radial function $\rho(|x|) \in C_0^\infty(\mathbb{R}^N)$ such that $\rho \ge 0$ and $\int_{\mathbb{R}^N}\rho dx = 1$, define the mollification $\mathcal{J}_\epsilon$ for $\epsilon > 0$ by

$$(\mathcal{J}_\epsilon v)(x) = \frac{1}{\epsilon^N}\int_{\mathbb{R}^N}\rho\left(\frac{x-y}{\epsilon}\right)v(y)dy \tag{2.9}$$

for functions $v \in L^p(\mathbb{R}^N)$ where $1 \le p \le \infty$.

Lemma 2.14. *(Properties of Mollifiers) Let $\mathcal{J}_\epsilon$ be the previously defined operator. Then, $\mathcal{J}_\epsilon v \in C^\infty(\mathbb{R}^N)$ and*
(i) for all $v \in C^0(\mathbb{R}^N)$, $\mathcal{J}_\epsilon v \to v$ uniformly on any compact set $\Omega \subset \mathbb{R}^N$ and

$$\|\mathcal{J}_\epsilon v\|_{L^\infty(\mathbb{R}^N)} \leq \|v\|_{L^\infty(\mathbb{R}^N)}. \tag{2.10}$$

(ii) Mollifiers commute with distribution derivatives

$$D^\alpha(\mathcal{J}_\epsilon v) = \mathcal{J}_\epsilon(D^\alpha v) \tag{2.11}$$

for all $v \in H^s(\mathbb{R}^N)$ and $|\alpha| \leq m$.
(iii) For all $u \in L^p(\mathbb{R}^N)$ and $v \in L^q(\mathbb{R}^N)$ such that $1/p + 1/q = 1$,

$$\int_{\mathbb{R}^N} (\mathcal{J}_\epsilon u) v dx = \int_{\mathbb{R}^N} u(\mathcal{J}_\epsilon u) dx. \tag{2.12}$$

(iv) For all $v \in H^s(\mathbb{R}^N)$, $\mathcal{J}_\epsilon v$ converges to v in $H^{\mathbb{R}^N}$ and the rate of convergence in the $H^{s-1}(\mathbb{R}^N)$ norm is linear in ϵ:

$$\lim_{\epsilon \to 0} \|\mathcal{J}_\epsilon v - v\|_{H^s(\mathbb{R}^N)} = 0 \tag{2.13}$$

and

$$\|\mathcal{J}_\epsilon v - v\|_{H^{s-1}(\mathbb{R})} \leq C\epsilon \|v\|_{H^s(\mathbb{R}^N)}. \tag{2.14}$$

(v) For all $v \in H^s(\mathbb{R}^N)$, $k \in \mathbb{N}$, and $\epsilon > 0$,

$$\|\mathcal{J}_\epsilon v\|_{H^{s+k}(\mathbb{R}^N)} \leq \frac{c(m,k)}{\epsilon^k} \|v\|_{H^s(\mathbb{R}^N)}, \tag{2.15}$$

and

$$\|\mathcal{J}_\epsilon D^k v\|_{L^\infty(\mathbb{R}^N)} \leq \frac{c_k}{\epsilon^{N/2+k}} \|v\|_{L^2(\mathbb{R}^N)}. \tag{2.16}$$

Proof. (i). Without loss of generality, assume that $\rho(x/\epsilon) = 0$ for $|x| \geq \epsilon$. For a given compact set $\Omega \in \mathbb{R}^N$, define the set $\Omega_\epsilon := \{x \in \mathbb{R}^N | D(x, \Omega) \leq \epsilon\}$. As $v|_{\Omega_\epsilon}$ is uniformly continuous, given any ϵ', there exists $\Re > 0$ such that for all $x, y \in \Omega_\epsilon$ with $|x - y| < \delta$, then $|v(x) - v(y)| < \epsilon'$. If $x \in \Omega$ and $\epsilon < \delta$, then

$$|\mathcal{J}_\epsilon v - v| = \left| \frac{1}{\epsilon} \int_{\mathbb{R}^N} \frac{\rho(x-y)}{\epsilon} v(y) dy \right| = \left| \frac{1}{\epsilon} \int_{\mathbb{R}^N} \frac{\rho(x-y)}{\epsilon} [v(y) - v(x)] dy \right| \leq \frac{\epsilon'}{\epsilon} \int_{\mathbb{R}^N} \frac{\rho(y)}{\epsilon} dy = \epsilon.'$$

Hence, $\mathcal{J}_\epsilon v \to v$ uniformly on Ω.
(ii). Let $v \in H^s(\mathbb{R}^n)$. It follows from integration by parts that

$$\begin{aligned}
(D^\alpha \mathcal{J}_\epsilon v)(x) &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} D_x^\alpha \rho \left(\frac{x-y}{\epsilon} \right) v(y) dy \\
&= \frac{(-1)^{|\alpha|}}{\epsilon^n} \int_{\mathbb{R}^n} D_y^\alpha \rho \left(\frac{x-y}{\epsilon} \right) v(y) dy \\
&= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \rho \left(\frac{x-y}{\epsilon} \right) D_y^\alpha v(y) dy \\
&= (\mathcal{J}_\epsilon D^\alpha v)(x).
\end{aligned}$$

This concludes the proof of part (ii).
(iii). Let $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$ with $1/p + 1/q = 1$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} (\mathcal{J}_\epsilon u) v dx &= \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} \left[\int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\epsilon}\right) u(y) dy \right] v(x) dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} \rho\left(\frac{x-y}{\epsilon}\right) u(y) v(x) dy dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} \rho\left(\frac{y-x}{\epsilon}\right) u(x) v(y) dy dx \\
&= \int_{\mathbb{R}^n} (\mathcal{J}_\epsilon v) u dx.
\end{aligned}$$

This concludes the proof of part (iii).
(iv). Let $v \in H^s(\mathbb{R}^n)$. We prove the case of $s = 1$. For $\gamma > 0$, we have

$$\mathcal{J}_\epsilon \mathcal{J}_\gamma v(x) - \mathcal{J}_\gamma v(x) = \int$$

□

Proposition 2.15. *(Hodge Decomposition in $\mathbb{R}^n$) Every vector field $v \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ has the following unique orthogonal decomposition*

$$v = w + \nabla q \tag{2.17}$$

such that $div(w) = 0$.

Proposition 2.16. *(Properties of the Hodge Decomposition in $\mathbb{R}^n$) Given a vector field $v \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with the orthogonal decomposition*

$$v = w + \nabla q,$$

the following properties hold:
(i) $w, \nabla q \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$,
(ii) $g(w, \nabla q) = 0$, and
(iii) for any multi-index β of the derivatives D^β, with $|\beta| \geq 0$,

$$\|D^\beta(v)\|^2_{L^2(\mathbb{R}^n)} = \|D^\beta w\|^2_{L^2(\mathbb{R}^n)} + \|\nabla q\|^2_{L^2(\mathbb{R}^n)}.$$

3. Basic Energy Estimates

We begin this section by deriving important *a priori* energy estimates from the Euler and Navier-Stokes equations. The Euler equation is given by

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p & u(x,t) \in \mathbb{R}^n \times [0,\infty) \\ \nabla \cdot u = 0 \\ u(x,0) = u_0 \end{cases} \tag{3.1}$$

The Navier-Stokes equation in $\mathbb{R}^n$ are given by

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu = -\nabla p + \nu \Delta u^\nu & u^\nu(x,t) \in \mathbb{R}^n \times [0,\infty) \\ \nabla \cdot u^\nu = 0 \\ u^\nu(x,0) = u^\nu_0 \end{cases} \tag{3.2}$$

An important system energy is its *kinetic energy*. As it pertains to fluid mechanics, we define the kinetic energy in the following way.

Definition 3.1. The kinetic energy of a fluid is given by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |u(x,t)|^2 dx. \tag{3.3}$$

3.1. **Some Tools for Energy Estimates.** To derive these facts about the energies of these equations, we need a few tools. First, we consider a lemma in dealing with integrals whose integrand involves the product of an expression and a gradient. The second important ingredient is the so-called Reynolds' Transport Theorem. This result extends the Leibniz integral rule to three dimensions and will help us to expand the time derivative of the kinetic energy of a fluid system. Finally, we derive the famous Gronwall inequality. This inequality yields a method to bound an expression according to an inequality of the derivative of the expression. This will provide great utility in time derivatives.

Lemma 3.2. *Let W be a smooth divergence-free vector field in $\mathbb{R}^n$ and let q be a smooth scalar such that*

$$|W(x)||q(x)| = \mathcal{O}(|x|^{1-N})$$

as $|x| \to \infty$. Then, W and ∇q are orthogonal.

Proof. As $\nabla \cdot W = 0$, we have that

$$\int_{|x|\leq R} (W \cdot \nabla q) dx = \int_{|x|=R} q\,(W \cdot \hat{n})\, ds \to 0$$

as $R \to \infty$. □

Proposition 3.3. *(Reynolds Transport Theorem) Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain with a smooth boundary, and let X be a given particle-trajectory mapping of a smooth velocity field v. Then, for any smooth function $f(t,x)$, we have*

$$\frac{d}{dt}\int_{X(\Omega,t)} f(x,t)dx = \int_{X(\Omega,t)} (\partial_t f(t,x) + div_x(f(t,x)v))dx.$$

Lemma 3.4. *(Gronwall's Inequality) If u, q and $c \geq 0$ are continuous on a time interval $[0,T]$, c is differentiable, and*

$$q(t) \leq c(t) + \int_0^t u(s)q(s)ds,$$

then we have

$$q(t) \leq c(0)\exp\left(\int_0^t u(s)ds\right) + \int_0^t c'(s)\exp\left(\int_s^t u(\tau)d\tau\right)ds \tag{3.4}$$

for $t \in [0,T]$.

Proof. Let $Q(t) = c(t) + \int_0^t u(s)q(s)ds$. Then, $Q(t)$ is differentiable and $Q(0) = c(0)$, Hence,

$$Q'(t) = c'(t) + u(t)q(t) \leq c'(t) + u(t)Q(t) \iff Q'(t) - u(t)Q(t) \leq c'(t).$$

Solving the differential inequality yields

$$Q(t) \leq c(0) \exp\left(\int_0^t u(s)ds\right) + \int_o^t c'(t) \exp\left(\int_s^t u(\tau)d\tau\right)dt.$$

□

3.2. **Derivation of the Basic Energy Estimate.** One important feature of the Euler equations is that they conserve kinetic energy.

Proposition 3.5. *Let u be a solution to the Euler equation that is sufficiently smooth in $\mathbb{R}^{\not\models}$ and vanishes sufficiently rapidly as $|x| \to \infty$. Then, the kinetic energy of the fluid is conserved over time.*

Proof. There are two ways to prove this. The first uses the aforementioned Reynolds' Transport Theorem. We recall that this result provides that for some open set Ω with a smooth bounded domain and a particle trajectory X over smooth velocity field V, for any smooth function $f(x,t)$

$$\frac{d}{dt}\int_{X(\Omega,t)} f(x,t)dx = \int_{X(\Omega,t)} (\partial_t f(t,x) + \mathrm{div}_x(f(t,x)v))dx.$$

Let $\Omega = \mathbb{R}^3$ and set $f = \frac{1}{2}u \cdot u$. It follows that

$$\frac{d}{dt}\int_{\mathbb{R}^{\not\models}} \frac{1}{2}|u|^2 dx = -\int_{\mathbb{R}^{\not\models}} (u \cdot \nabla p)dx$$

where the last expression evaluates to zero from Lemma 3.2. Therefore, the derivative of the kinetic energy of the Euler equation evaluates to zero. □

The Navier-Stokes equation does not satisfy such a property. Instead, the kinetic energy of the Navier-Stokes equation is connected with the gradient of the fluid flow.

Proposition 3.6. *Let u be a smooth solution to the Navier-Stokes equation that vanishes sufficiently rapidly as $|x| \to \infty$. Then, the time derivative of the kinetic energy satisfies*

$$\frac{d}{dt}E(t) = -\nu \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Proof. We again employ the Reynolds' Transport Theorem to $\Omega = \mathbb{R}^n$ and $f(x,t) = \frac{1}{2}u \cdot u$. Note that

$$\partial_t f(x,t) = \partial_t \left(\frac{1}{2}u(x,t) \cdot u(x,t)\right) = (\partial_t u)u \quad \text{and} \quad \mathrm{div}_x(f(x,t)u) = u \cdot ((u \cdot \nabla)u).$$

Then, the Reynolds' Transport Theorem yields

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}|u|^2 dx &= \int_{\mathbb{R}^n} u \cdot (\partial_t u + (u \cdot \nabla)u)\, dx \\ &= \int_{\mathbb{R}^n} u\,(-\nabla p + \nu \Delta u)\, dx \qquad \text{(from the NS equation definition)} \\ &= -\int_{\mathbb{R}^n} u \cdot (\nabla p) dx + \nu \int_{\mathbb{R}^n} u \cdot (\Delta u) dx \end{aligned}$$

$$= 0 - \nu \int_{\mathbb{R}^n} |\nabla u|^2 dx \qquad \text{(from Lemma 3.2 and Green's formula).}$$

□

3.2.1. *Basic Energy Estimate for the Euler Equations.* We now provide the framework for the *a priori* energy estimates for the Euler and Navier-Stokes equations. We begin with the Euler equations and then easily adapt this procedure to produce a similar result for the Navier-Stokes equation. We first suppose that we have two smooth solutions u_1 and u_2 and corresponding pressure solutions p_1 and p_2 to the Euler equation. Further, assume that both solutions exist on a common time interval $[0, T]$ and that each solution defines the difference of the fluid velocity and fluid pressure as $\tilde{u} = u_1 - u_2$ and $\tilde{p} = p_1 - p_2$. Taking the difference between the two corresponding equations yields

$$\partial_t \tilde{u} + (u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2 = -\nabla \tilde{p}$$

which can be written as

$$\partial_t \tilde{u} + (u_1 \cdot \nabla)\tilde{u} - (\tilde{u} \cdot \nabla)u_2 = -\nabla \tilde{p}. \tag{3.5}$$

A standard technique to derive energy estimates is to multiply the difference $\tilde{u}$ and equation 3.5 and take the spacial L^2 norm. It follows that

$$\int_{\mathbb{R}^n} (\partial_t \tilde{u}) \cdot \tilde{u} dx + \int_{\mathbb{R}^n} (u_1 \cdot \nabla \tilde{u}) \cdot \tilde{u} dx + \int_{\mathbb{R}^n} (\tilde{u} \cdot \nabla u_2) \cdot \tilde{u} dx = -\int_{\mathbb{R}^n} (\nabla p) \cdot \tilde{u} dx$$

and the following simplifications using integration by parts and the incompressibility condition yield

$$\int_{\mathbb{R}^n} (u_1 \cdot \nabla \tilde{u}) \cdot \tilde{u} dx = \int_{\mathbb{R}^n} u_1 \cdot \left(\frac{1}{2} \nabla (\tilde{u}^2) \right) dx = \int_{\mathbb{R}^n} \nabla \cdot \left(\frac{1}{2} u_1 \tilde{u}^2 \right) dx = 0$$

and

$$\int_{\mathbb{R}^n} (\nabla p) \cdot \tilde{u} dx = -\int_{\mathbb{R}^n} p (\nabla \cdot \tilde{u}) dx = 0.$$

We arrive at the so-called *Basic Energy Identity for the Euler Equation*:

$$g(\partial_t \tilde{u}, \tilde{u})_{L^2(\mathbb{R}^n)_x} + g(\tilde{u} \cdot \nabla u_2, \tilde{u})_{L^2(\mathbb{R}^n)_x} = 0 \tag{3.6}$$

Using the Cauchy-Schwartz inequality provides,

$$\frac{d}{dt} \|\tilde{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u_2(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \|\tilde{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \tag{3.7}$$

Theorem 3.7. *(Basic Energy Estimate) Let u_1 and u_2 be smooth solutions to the Euler equation. Suppose that each solution exists on a common time interval $[0, T]$, and, for a fixed time t, decays sufficiently at infinity to belong to $L^2(\mathbb{R}^n)$. Then,*

$$\sup_{0 \leq t \leq T} \|u_1 - u_2\|_{L^2(\mathbb{R}^n)} \leq \|(u_1 - u_2)_{t=0}|\|_{L^2(\mathbb{R}^n)} \exp \left(\int_0^t |\nabla u_2|_{L^\infty(\mathbb{R}^n)} dt \right) \tag{3.8}$$

Proof. Set $\tilde{u} = u_1 - u_2$. Then,

$$\frac{d}{dt}\|\tilde{u}\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u_2\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}\|_{L^2(\mathbb{R}^n)}$$

$$\iff \frac{d}{dt}\|\tilde{u}\|_{L^2(\mathbb{R}^3)} - \|\nabla u_2\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}\|_{L^2(\mathbb{R}^n)} = \frac{d}{dt}\left(\|\tilde{u}\|_{L^2(\mathbb{R}^n)} \exp\left(-\int_0^T \|\nabla u_2\|_{L^\infty(\mathbb{R}^n)}dt\right)\right) \leq 0$$

$$\iff \|\tilde{u}\|_{L^2(\mathbb{R}^n)} \leq \|\tilde{u}(0,\cdot)\|_{L^2(\mathbb{R}^n)} \exp\left(\int_0^T \|\nabla u_2\|_{L^\infty(\mathbb{R}^n)}dt\right).$$

□

An important consequence of the Basic Energy Estimate is that solutions to the Euler equations are unique.

Corollary 3.8. *(Uniqueness of Solutions) Let u_1 and u_2 be smooth L^2 solutions to the Euler equation on a common time interval $[0,T]$ with the same initial data. Then, $u_1 = u_2$.*

Proof. Set $\tilde{u} = u_1 - u_2$. Then, $\tilde{u}(0,\cdot) = u_1(0,\cdot) - u_2(0,\cdot) = 0$. Applying the Basic Energy Estimate yields that $\|\tilde{u}\|_{L^2(\mathbb{R}^n)} \leq 0$ for all $t \in [0,T]$. Therefore, $u_1 = u_2$. □

3.2.2. *Basic Energy Estimate for the Navier-Stokes Equations.* A similar procedure for the Navier-Stokes equations can be performed. To avoid repetition, we see that for smooth solutions u_1^ν and u_2^ν and corresponding pressure solutions p_1 and p_2 to the Navier-Stokes equations, using the same difference equations, and taking the spacial L^2 norm yields

$$g(\partial_t\tilde{u},\tilde{u})_{L^2(\mathbb{R}^n)_x} + g(\tilde{u}\cdot\nabla u_2,\tilde{u})_{L^2(\mathbb{R}^n)_x} = \nu g(\Delta\tilde{u},\tilde{u})_{L^2(\mathbb{R}^n)_x}.$$

We can simplify the right-hand side of the above equation using integration by parts and then the divergence theorem in the following way

$$\nu g(\Delta\tilde{u},\tilde{u})_{L^2(\mathbb{R}^n)_x} = \nu\int_{\mathbb{R}^n}(\Delta\tilde{u}_i)\tilde{u}_i dx = \nu\int_{\mathbb{R}^n}(\nabla\cdot\nabla\tilde{u}_i)\tilde{u}_i dx = -\nu\int_{\mathbb{R}^n}|\nabla\tilde{u}_i|^2 dx.$$

This calculation provides the *Basic Energy Identity for the Navier Stokes Equations*

$$g(\partial_t\tilde{u},\tilde{u})_{L^2(\mathbb{R}^n)_x} + g(\tilde{u}\cdot\nabla u_2,\tilde{u})_{L^2(\mathbb{R}^n)_x} + \nu g(\nabla\tilde{u},\nabla\tilde{u})_{L^2(\mathbb{R}^n)_x} = 0. \tag{3.9}$$

Using the Cauchy-Schwartz inequality provides,

$$\frac{1}{2}\|\tilde{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \nu\|\nabla\tilde{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u_2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \tag{3.10}$$

Remark 3.9. We can easily obtain the basic energy identity for the Euler equation from the basic energy identity for the Navier-Stokes equations from the assumption that $\nu > 0$. It follows that the Basic Energy Estimate for the Navier-Stokes equations is identical to Theorem 3.7 and the uniqueness of solutions presented in Theorem

3.3. **Energy in Two-Dimensions.** The above analysis requires that the velocity fields decay fast enough at infinity. This assumption works well in three dimensions. However, this assumption is usually not fulfilled in two dimensions. To adapt, we review how the vorticity and the velocity connect in two dimensions through the vorticity stream formulation.

3.3.1. *The Vorticity-Stream Formulation for Two-Dimensional Flows.* The Navier-Stokes equations can also be expressed in terms of the vorticity, ω, as the so-called *vorticity equation.* This can be derived by taking the curl of the Navier-Stokes equations to yield

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \Delta \omega. \tag{3.11}$$

However, equation 3.11 simplifies in two dimensions. Indeed, consider the convective term and note that $\omega = (0, 0, u_{x_1} - u_{x_2})^{\perp}$

$$\omega \cdot \nabla u = 0$$

which reduces the vorticity equation in two-dimensions to

$$\frac{D\omega}{Dt} = \nu \Delta \omega. \tag{3.12}$$

This simplified version of the vorticity equation can be solved through the following proposition.

Proposition 3.10. ***(The Vorticity-Stream Formulation in $\mathbb{R}^2$)****. For a two-dimensional flow vanishing sufficiently rapidly as $|x| \to \infty$, the Navier-Stokes equations are equivalent to the vorticity-stream formulation*

$$\begin{cases} \frac{D\omega}{Dt} = \nu\omega & (x,t) \in \mathbb{R}^2 \times [0, \infty) \\ \omega | t = 0 = \omega_0, \end{cases} \tag{3.13}$$

where the velocity is determined explicitly from the vorticity way of the Biot-Savart law

$$u(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy, \qquad x \in \mathbb{R}^2 \tag{3.14}$$

where the kernel is given by $K_2(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^{\perp}$.

Although solutions do not have finite kinetic energy in two dimensions, they have finite *local kinetic energy* in two dimensions. We have the following definition.

Definition 3.11. For a solution u to the two-dimension Euler or Navier-Stokes equations, u is said to have finite *local kinetic energy* if for any $R > 0$, we have

$$\int_{|x|<R} |u|^2 dx < \infty \tag{3.15}$$

It turns out that in two dimensions, the vorticity of the fluid plays a crucial component to its kinetic energy. Consider a smooth vorticity ω with compact support. This means that the the vorticity is nonzero in a compact subset of $\mathbb{R}^2$. Recall that in a two-dimensional fluid, the velocity field is given by

$$u(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy$$

Proposition 3.12. *A two dimensional incompressible velocity field with vorticity of compact support in $\mathbb{R}^2$ has finite kinetic energy if and only if*

$$\int_{\mathbb{R}^2} \omega(y,t)dy = 0. \tag{3.16}$$

Proof. We consider the kernel $K_2(x)$ and note that $|x| \neq 0$,

$$|x-y|^{-2} = |x|^{-2}\left(1 - \frac{2x\cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^{-1}.$$

Also, if $|y| \leq R$ and $|x| \geq 2R$, then for large x,

$$|x-y|^{-2} = |x|^{-2} + O(|x|^{-3}).$$

From the compact support of $\omega(y)$ inside $|y| < R$, we see that for large $|x|$,

$$u(x,t) = K_2(x)\int_{\mathbb{R}^2}\omega(y,t)dy + O(|x|^{-2}).$$

Combining this fact with

$$\int_{\mathbb{R}^n}(1+|x|^2)^{-l/2}dx \leq \infty \iff l > n$$

yields the result. □

There are many two dimensional solutions to the Euler and Navier-Stokes equations in which the vorticity does not satisfy equation 3.16. As previously discussed, we can not use the previously derived energy estimate as the kinetic energy is not finite over $\mathbb{R}^2$. However, we can derive a similar estimate by splitting the velocity in two dimensions into a radial and energy part. We have the following definition.

Definition 3.13. Let u be a smooth incompressible velocity field in $\mathbb{R}^2$. Then, we define the *radial-energy decomposition* if there exists a smooth radially symmetric vorticity $\overline{\omega}(|x|)$ such that

$$u(x) = v(x) + w(x) \tag{3.17}$$

such that

$$\int_{\mathbb{R}^2}|v(x)|^2dx \leq \infty \quad \text{and} \quad \nabla\cdot v = 0$$

where $\overline{u}$ is defined as

$$w(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}|x|^{-2}\int_0^{|x|} s\overline{\omega}(s)ds. \tag{3.18}$$

Note that this radial energy decomposition is not unique. Indeed, it depends on the choice of $\overline{\omega}$. Moreover, the decomposition is *not* an orthogonal decomposition. We have the following lemma about the existence of the radial-energy decomposition.

Lemma 3.14. *Any smooth incompressible velocity field with vorticity*

$$\omega = \nabla\times u \in L^1(\mathbb{R}^2)$$

has a radial-energy decomposition.

3.3.2. *Two Dimensional Basic Energy Estimate.* We now discuss special radial-energy decompositions for two dimensional time-dependent solutions to the Euler and Navier-Stokes solutions. In the Euler case,

For the Navier-Stokes equations, the viscous eddies are exact radial solutions with initial vorticity $\omega_0(r)$. The vorticity $\omega(x,t)$ satisfies the *heat equation*:

$$\begin{cases} \partial_t \omega = \nu \Delta \omega \\ \omega|_{t=0} = \omega_0 \end{cases} \tag{3.19}$$

The solution to the heat equation is well known [4] and given by

$$\omega(x,t) = \frac{1}{4\pi\nu t}\int_{\mathbb{R}^2} e^{\frac{|x-y|}{4\nu t}}\omega_0(|y|)dy. \tag{3.20}$$

The velocity is also recovered by the relation

$$u(x,t) = \left(\frac{-x_2}{r^2}, \frac{x_1}{r^2}\right)^{\perp}\int_0^r s\omega_0(s)ds. \tag{3.21}$$

This allows us to construct explicit time-dependent decompositions for solutions to both the Euler and Navier-Stokes equations. Given an initial smooth vorticity $\omega_0(x) \in L^1(\mathbb{R}^2)$ and respective initial velocity given by

$$u_0(x) = \int_{\mathbb{R}^2} K_2(x-y)\omega_0(y)dy,$$

let $\tilde{\omega}_0(r)$ be the radial part of an energy decomposition of $\omega_0(x)$. Also, let $w(r,t)$ and $\tilde{\omega}(r,t)$ be the exact radial solutions to the Euler and Navier-Stokes equations, respectively, whose initial data is $\omega_0(r)$. We also have that the quantity $v(x,t) = u(x,t) - w(x,t)$ satisfies

$$\partial_t v + v\cdot\nabla v + w\cdot\nabla v + v\cdot\nabla w = -\nabla p + \nu\Delta v + F.$$

As w is smooth and known velocity field obtained from the initial data, the classical energy estimate on v yields estimates for the full solution u. To derive the two dimensional energy estimate, we consider the radial-energy decomposition of two solutions: u_1 and u_2. Let $u_1 = v_1 + w_1$ and $u_2 = v_2 + w_2$ be two finite energy decompositions for two solutions to the Navier-Stokes equations in two dimensions. As before, set $\tilde{u} = u_1 - u_2$, $\tilde{v} = v_1 - v_2$, $\tilde{w} = w_1 - w_2$, $\tilde{F} = F_1 - F_2$, and $\tilde{u} = u_1 - u_2$. We then have that $\tilde{v}$ satisfies

$$\partial_t\tilde{v} + v_1\cdot\nabla\tilde{v} + \tilde{v}\cdot\nabla v_2 + w_1\cdot\nabla\tilde{v} + \tilde{w}\cdot\nabla v_2 + \tilde{v}\cdot\nabla w_1 + v_2\cdot\nabla\tilde{w} = -\nabla\tilde{p} + \nu\Delta\tilde{v} + \tilde{F}. \tag{3.22}$$

As in the previous basic energy estimate, multiplying the above equation by $\tilde{u}$ and using integration by parts

Proposition 3.15. ***(Basic Energy Estimate and Gradient Control in Two Dimensions)*** *Let u_1 and u_2 be two smooth divergence-free solutions to the Navier-Stokes equations with radial-energy decomposition $u_i(x,t) = v_i(x,t) + w_i(x,t)$ and with forces F_1, F_2 and pressures p_1, p_2. Then, we have the following basic energy estimate in two dimensions,*

$$\sup_{0\le t\le T} \|\|\|_{L^2} \tag{3.23}$$

4. The Pressure Gradient

One awkward feature of the Euler and the Navier-Stokes equation is the inclusion of a pressure gradient. Instead, the pressure term should be considered as a Lagrange multiplier. Further, we can derive an expression of the pressure from the velocity field.

4.1. Removal of the Pressure Gradient.

We will first work with the momentum equation from the Euler equations. To compute the pressure term, take the divergence of the momentum equation to find

$$\nabla \cdot \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = \nabla \cdot (-\nabla p)$$

Switching the order of differentiation and applying the incompressibility condition provides

$$\nabla \cdot \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (\nabla \cdot u) = 0.$$

Also, we have

$$\begin{aligned} \nabla \cdot [(u \cdot \nabla) u] &= \partial_i (u_j \partial_j u_i) \\ &= \partial_i u_j \partial_j u_i + u_j \partial_i \partial_j u_i \\ &= \partial_i u_j \partial_j u_i + u_j \partial_j \partial_i u_i \\ &= \partial_i u_j \partial_j u_i \end{aligned}$$

Thus, the pressure is given by the following Poisson equation

$$-\Delta p = \partial_i u_j \partial_j u_i \tag{4.1}$$

We can invert the Laplacian to solve for p. Instead, we will apply the Fourier transform to solve for p. Using definition 2.4 and applying it to the above equation 4.1 formally yields

$$|\xi|^s \hat{p}(\xi) = -\xi_i \xi_j \widehat{u_j u_i}(\xi) = -\xi_i \xi_j \int_{\mathbb{R}^d} u_j(\eta) u_i(\xi - \eta) d\eta$$

and thus the pressure has the following Fourier transform

$$\hat{p}(\xi) = -\frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_j u_i}(\xi)$$

4.2. The Leray Projection Operator.

The inclusion of the pressure gradient makes the Euler equations *nonlocal*. This means that at a given point and time $(x, t) \in \mathbb{R}^n \times [0, \infty)$, the solution $u(x, t)$ depends on the entire solution u through the pressure gradient. To correct this, the idea is to project the equations onto a space in which the pressure gradient can be negated. The obvious choice is to consider the subspace of $H^s(\mathbb{R}^n)$ in which every vector is divergence-free. In this space, the compressibility condition will be We define the subspace of $H^s(\mathbb{R}^n)$ that will provide the appropriate space to work in to prove local-in-time well-posedness.

Definition 4.1. Let $s > 0$ and define the *Divergence Free Sobolev Space* as

$$V^s(\mathbb{R}^n) = \{ v \in H^s(\mathbb{R}^n) \mid \nabla \cdot v = 0 \}. \tag{4.2}$$

The *Leray Projection Operator* is the path from the Sobolev space $H^s(\mathbb{R}^n)$ to the divergence-free Sobolev space we would like to work in. The key to defining the Leray projection relies on the Hodge decomposition. Define the projection by

$$P : H^s(\mathbb{R}^n) \to V^s(\mathbb{R}^n)$$

given by $P(v) = w$ where $H^s(\mathbb{R}^n) \ni v = w + \nabla q$. In other words, the Leray projection disregards the gradient of the decomposition of the vector v. It is clear how this is useful in removing the pressure gradient from the Euler and Navier-Stokes equations. We now present some important properties about this projection operator.

Proposition 4.2. ***(Properties of the Leray Projection Operator)*** *Let $v \in H^s(\mathbb{R}^n)$ have the Hodge decomposition of*

$$v = w + \nabla\varphi$$

where $\nabla \cdot w = 0$. Then, we have the following properties.

(1) $P(v), \nabla\varphi \in H^s(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} P(v) \cdot \nabla w dx = 0$, $\nabla \cdot (P(v)) = 0$, *and*

$$\|P(v)\|^2_{H^s(\mathbb{R}^n)} + \|\nabla\varphi\|^2_{H^s(\mathbb{R}^n)} = \|v\|^2_{H^s(\mathbb{R}^n)}, \tag{4.3}$$

(2) *P commutes with weak derivatives*

$$P(D^\alpha(v)) = D^\alpha(P(v)) \qquad v \in H^s(\mathbb{R}^n) \quad |\alpha| \leq s, \tag{4.4}$$

(3) *P commutes with mollifiers $\mathcal{J}_\epsilon$,*

$$P(\mathcal{J}_\epsilon(v)) = \mathcal{J}_\epsilon(P(v)) \qquad v \in H^s(\mathbb{R}^n) \quad \epsilon > 0, \tag{4.5}$$

(4) *P is symmetric,*

$$g(P(u), v)_{H^s(\mathbb{R}^n)} = g(u, P(v))_{H^s(\mathbb{R}^n)}. \tag{4.6}$$

5. Roadmap to Local Well-Posedness

Before we begin the theory of local-in-time well-posedness of the Euler and Navier-Stokes equations, we provide a roadmap to outline the forthcoming sections.

(1) **(Constructing an Approximate Equation)** For a family of mollifiers $\{\mathcal{J}_\epsilon\}_{\epsilon>0}$, consider the mollified Navier-Stokes equations given as

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + P\mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla\left(\mathcal{J}_\epsilon u^\epsilon\right)\right) = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon \\ u^\epsilon|_{t=0} = u_0 \end{cases} \tag{5.1}$$

The mollified Navier-Stokes equations satisfy the hypotheses of the Picard Theorem (locally Lipschitz and mapping into a Banach space), and hence for every $\epsilon > 0$, there exists a $T_\epsilon > 0$ and a unique solution $u^\epsilon \in C^1\left([0, T_\epsilon]; V^s(\mathbb{R}^n)\right)$ that satisfies the energy estimate on any time interval $[0, T]$ with $T \leq T_\epsilon$

$$\sup_{0 \leq t \leq T} \|u^\epsilon\|_{V^0(\mathbb{R}^3)} \leq \|u_0\|_{V^0(\mathbb{R}^3)}.$$

(2) **(Extend the Approximate Solutions to $T_\epsilon = \infty$)** For each $\epsilon > 0$, as the solution u^ϵ remains in the $V^s(\mathbb{R}^n)$, u^ϵ exists globally in time.

(3) **(The $H^s(\mathbb{R}^n)$ Energy Estimate)** We have the a priori estimate

$$\frac{d}{dt}\|u^\epsilon(\cdot,t)\|_{H^s(\mathbb{R}^n)} \leq C(\|u_0\|_{L^2(\mathbb{R}^n)},\epsilon,n)\|u^\epsilon\|_{H^s(\mathbb{R}^n)}.$$

However, this estimate depends badly on ϵ as it decreases. We derive a better *higher-order energy estimate* in $H^s(\mathbb{R}^n)$ as

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)} + \nu\|\mathcal{J}_\epsilon\nabla u^\epsilon\|^2_{H^s(\mathbb{R}^n)} \leq C(s)\|\nabla\mathcal{J}_\epsilon u^\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)}. \tag{5.2}$$

(4) **($\{u^\epsilon\}_{\epsilon>0}$ is Uniformly Bounded in $C([0,T];H^s(\mathbb{R}^n))$)** Using the previous energy estimate in $H^s(\mathbb{R}^n)$ and the Sobolev Embedding Theorem provide for all $\epsilon>0$,

$$\sup_{0\leq t\leq T}\|u^\epsilon\|_{H^s(\mathbb{R}^n)} \leq \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1-C(s)\|u_0\|_{H^s(\mathbb{R}^3)}} \tag{5.3}$$

for $s>n/2$ and $T\leq(C(M)\|u_0\|_{H^s(\mathbb{R}^n)})$. Furthermore, the family $\{du^\epsilon/dt\}_{\epsilon>0}$ is uniformly bounded in $H^{s-2}(\mathbb{R}^n)$

(5) **($\{u^\epsilon\}_{\epsilon>0}$ forms a Contraction in $C([0,T];L^2(\mathbb{R}^n))$)** There exists a constant C that depends only on $\|u_0\|_{H^s(\mathbb{R}^n)}$ and time T such that for all ϵ_1 and ϵ_2.

$$\sup_{0<t<T}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \leq C\left(\|u_0\|_{H^s(\mathbb{R}^n)}\right)\max(\epsilon_1,\epsilon_2).$$

Form the completeness of $C([0,T];L^2(\mathbb{R}^n))$, the limit $u\in C([0,T];L^2(\mathbb{R}^n))$.

(6) **(Intermediate Norms)** To show strong convergence in all the intermediate norms, consider the following interpolation theorem: given $s>0$, there exists a constant $C(s)$ such that for all $v\in H^s(\mathbb{R}^n)$ and $0<s'<s$,

$$\|v\|_{H^{s'}(\mathbb{R}^n)} \leq C(s)\|v\|^{1-s'/s}_{L^2(\mathbb{R}^n)}\|v\|^{s'/s}_{H^s(\mathbb{R}^n)} \tag{5.4}$$

Applying the difference $u^\epsilon - u$ to the interpolation identity yields strong convergence in $C\left([0,T];H^{s'}(\mathbb{R}^n)\right)$ for all $s'<s$.

(7) **(The Vector u Satisfies the Euler and Navier-Stokes Equations)** The Cauchy limit of the family $\{u^\epsilon\}_{\epsilon>0}$ given as u satisfies

$$\left\|u(t)-u_0-\int_0^t P(u(\tau)\cdot\nabla u(\tau))d\tau + \nu\int_0^t \Delta u(\tau)d\tau\right\|_{L^2(\mathbb{R}^3)}$$
$$\lesssim \|u-u^\epsilon\|_{L^\infty([0,T;L^2(\mathbb{R}^3)])} + \|\nabla u - \nabla u^\epsilon\|_{L^\infty([0,T;L^\infty(\mathbb{R}^3)])}$$
$$+\epsilon\|u_0\|_{H^s(\mathbb{R}^n)} + \nu\|u-u^\epsilon\|_{L^\infty([0,T;H^2(\mathbb{R}^3)])} + \nu\epsilon^\alpha$$

where $\alpha\in(0,s-2]$ and $t\in[0,T]$. As $\epsilon\to 0$, we see that the Cauchy limit u is indeed a solution to the Euler or Navier Stokes equations.

6. Global Existence of the Regularization of the Euler and Navier-Stokes Equations

Theorem 6.1. ***(Picard Theorem on a Banach Space)****. Let $O\subset B$ be an open subset of a Banach space B and let $F:O\to B$ be a mapping that satisfies the following properties:*
(i) $F(X)$ maps O to B, and

(ii) F is locally Lipschitz continuous; that is, for any $x \in O$, there exists $L > 0$ and a neighborhood $U_x \subset O$ such that

$$\|F(x) - F(y)\|_B \leq L\|x - y\|_B \qquad \text{for all } x, y \in U_x.$$

Then, for any $x_0 \in O$, there exists $T > 0$ such that the Cauchy problem

$$\begin{cases} \frac{dX}{dt} = F(X) \\ X|_{t=0} = x_0 \end{cases}$$

has a unique local solution $X(t) \in C^1((-T, T); O)$.

Proposition 6.2. ***(Local Existence of Solutions to the Regularized Equations)*** *Consider an initial condition $u_0 \in V^n$, for $n \in \mathbb{N}$. Then,*
(i) for any $\epsilon > 0$, there is a unique solution $u^\epsilon \in C^1([0, T_\epsilon], V^n)$ to the ODE

$$\begin{cases} \frac{du^\epsilon}{dt} = F(X) \\ X|_{t=0} = x_0 \end{cases}$$

where $F_\epsilon(u^\epsilon) = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon - P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla(\mathcal{J}_\epsilon u^\epsilon))$, and
(ii) on any time interval $[0, T]$ on which the solution u^ϵ belongs to $C^1([0, T], V^0)$,

$$\sup_{0 \leq t \leq T} \|u^\epsilon\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Proof. We begin with part (i). Let the initial data u_0 belong to V^s for $n \in \mathbb{N}$ and let $F_\epsilon^1(u^\epsilon) = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon$, $F_\epsilon^2(u^\epsilon) = P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla(\mathcal{J}_\epsilon u^\epsilon))$ and $F_\epsilon = F_\epsilon^1 + F_\epsilon^2$. For $u^\epsilon \in V^s$, the Leray projection operator P maps into divergence free vector fields and $\nabla \cdot u^\epsilon = 0$. Then, $F : V^s \to V^s$. To show local in-time existence, we would like to employ the Picard Theorem. Hence, we will show that F_ϵ is locally Lipschitz. First, consider F_ϵ^1 and two solutions u_1 and u_2 to see that

$$\begin{aligned} \|F_\epsilon^1(u_1) - F_\epsilon^2(u_2)\|_{H^s(\mathbb{R}^n)} &= \nu\|\mathcal{J}_\epsilon^2 \Delta u_1 - \mathcal{J}_\epsilon^2 \Delta u_2\|_{H^s(\mathbb{R}^n)} \\ &\leq \nu\|\mathcal{J}_\epsilon^2 (u_1 - u_2)\|_{H^2(\mathbb{R}^3)} \\ &\lesssim \frac{\nu}{\epsilon}\|u_1 - u_2\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

In a similar way, consider F_ϵ^1 and two solutions u_1 and u_2 to see that

$$\begin{aligned} \|F_\epsilon^2(u_1) - F_\epsilon^2(u_2)\|_{H^s(\mathbb{R}^n)} &\leq \|P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u_1) \cdot \nabla(cJe(u_1 - u_2)))\|_{H^s(\mathbb{R}^n)} \\ &+ \|P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u_1) \cdot \nabla(\mathcal{J}_\epsilon(u_1 - u_2)))\|_{H^s(\mathbb{R}^n)} \\ &\lesssim \|\mathcal{J}_\epsilon u_1\|_{L^\infty(\mathbb{R}^n)}\|D^s \mathcal{J}_\epsilon \nabla(u_1 - u_2)\|_{L^2(\mathbb{R}^n)} \\ &+ \|\mathcal{J}_\epsilon(u_1 - u_2)\|_{L^\infty(\mathbb{R}^n)}\|D^s \mathcal{J}_\epsilon \nabla u_2\|_{L^2(\mathbb{R}^n)} \\ &+ \|\mathcal{J}_\epsilon \nabla u_2\|_{L^\infty(\mathbb{R}^n)}\|D^s \mathcal{J}_\epsilon(u_1 - u_2)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \frac{1}{\epsilon^{N/2+m+1}}\left(\|u_1\|_{L^2(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}\right)\|u_1 - u_2\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

Combining each result yields that

$$\|F_\epsilon(u_1) - F_\epsilon(u_2)\|_{H^s(\mathbb{R}^n)} \leq c(\|u_1\|_{L^2(\mathbb{R}^n)}, \|u_2\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u_1 - u_2\|_{H^s(\mathbb{R}^n)}$$

and therefore, F_ϵ is locally Lipschitz on any open set given by

$$\mathcal{O}^M = \{u \in V^s(\mathbb{R}^n) | \|u\|_{H^s(\mathbb{R}^n)} \leq M\}.$$

Therefore, the Picard Theorem provides that, given any initial condition $u_0 \in H^s(\mathbb{R}^n)$, there exists a unique solution $u^\epsilon \in C^1\left([0, T_\epsilon]; V^s \cap \mathcal{O}^M\right)$, for some $T_\epsilon > 0$.
We turn to part (ii). First, we take the $L^2(\mathbb{R}^n)$ spacial inner product with each term of the regularized equation and the solution u^ϵ which yields

$$\int_{\mathbb{R}^n} (\partial_t u_\epsilon)\, u^\epsilon dx = \nu \int_{\mathbb{R}^n} \left(\mathcal{J}_\epsilon^2 \Delta u^\epsilon\right) u^\epsilon dx - \int_{\mathbb{R}^n} \left(P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla(\mathcal{J}_\epsilon u^\epsilon))\right) u^\epsilon dx.$$

Note that

$$\int_{\mathbb{R}^n} (\partial_t u_\epsilon)\, u^\epsilon dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u^\epsilon|^2 dx$$

and hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u^\epsilon|^2 dx &= \nu \int_{\mathbb{R}^n} (\mathcal{J}_\epsilon u^\epsilon)\, \Delta\, (\mathcal{J}_\epsilon u^\epsilon)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} (\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla\, (\mathcal{J}_\epsilon u^\epsilon)^2\, dx \\ &= -\nu \int_{\mathbb{R}^n} (\mathcal{J}_\epsilon \nabla u^\epsilon)^2\, dx - \frac{1}{2}\int_{\mathbb{R}^n} (\nabla \cdot (\mathcal{J}_\epsilon u^\epsilon))^2\, (\mathcal{J}_\epsilon u^\epsilon)^2\, dx.\end{aligned}$$

As $u^\epsilon \in V^s(\mathbb{R}^n)$, it follows that $\nabla \cdot u^\epsilon = 0$. Thus,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u_\epsilon|^2 dx + \nu \int_{\mathbb{R}^n} (\nabla \mathcal{J}_\epsilon u^\epsilon)^2\, dx = 0$$

which provides

$$\frac{d}{dt}\|u^\epsilon\|^2_{L^2(\mathbb{R}^n)} + 2\nu\|\nabla \mathcal{J}_\epsilon u^\epsilon\|^2_{L^2(\mathbb{R}^n)} = 0.$$

Integrating from 0 to $\tilde{t}$ where $0 \leq \tilde{t} \leq T_\epsilon$

$$\int_0^{\tilde{t}} \frac{d}{dt}\|u^\epsilon\|^2_{L^2(\mathbb{R}^n)} + \int_0^{\tilde{t}} 2\nu\|\nabla \mathcal{J}_\epsilon u^\epsilon\|^2_{L^2(\mathbb{R}^n)} = 0.$$

Finally, using the Fundamental Theorem of Calculus and $\nu > 0$ gives

$$\|u^\epsilon(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq \|u_0\|^2_{L^2(\mathbb{R}^n)}$$

and we arrive at the following energy estimate

$$\sup_{0 \leq t \leq T} \|u^\epsilon\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}.$$

$\square$

Theorem 6.3. ***(Continuation of an Autonomous ODE on a Banach Space)*** *Let $\mathcal{O} \subset B$ be an open subset of a Banach space B and let $F : \mathcal{O} \to B$ be a locally Lipschitz operator. Then, the unique solution $x \in C^1([0, T]; \mathcal{O})$ to the autonomous ODE,*

$$\begin{cases} \frac{dX}{dt} = F(X) \\ X|_{t=0} = x_0 \in \mathcal{O}, \end{cases}$$

either exists globally in time or $T < \infty$ and one of the following conditions holds
(i) $\lim_{t\to T} \|F(X)\|_B = \infty$
(ii) $\lim_{t\to T} \|F(X)\|_B \notin \mathcal{O}$.

Proof. Let $T < \infty$ and suppose that condition (i) does not hold. Set $M = 1+\lim_{t\to T} \|F(X)\|_B$, which is finite. Then, there exists $\delta > 0$ such that

$$\|x(t) - x(s)\|_B \leq \int_s^t \|F(X(\tau))\|_B d\tau \leq M(t-s)$$

whenever $T - \delta < s \leq t < T$. Hence, the limit converges to $X(T) = \lim_{t\to T} X(t) \in \mathcal{O}$. We can employ the Picard Theorem to find a local in-time solution at $X(T)$. However, such a solution will not belong to $\mathcal{O}$. □

Theorem 6.4. ***(Global Existence of Solutions to the Regularized Equations)*** *Given an initial condition $u_0 \in V^s(\mathbb{R}^n)$ for $s \in \mathbb{N}$, for every $\epsilon > 0$, there exists for all time a unique solution $u^\epsilon \in C^1([0,\infty); V^s(\mathbb{R}^n))$ to the equation*

$$\begin{cases} \frac{du^\epsilon}{dt} = F(u^\epsilon) \\ u^\epsilon|_{t=0} = u_0 \end{cases}$$

where $F_\epsilon(u^\epsilon) = \nu \mathcal{J}_\epsilon \Delta u^\epsilon - P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla(\mathcal{J}_\epsilon u^\epsilon))$.

Proof. We have previously shown via the Picard Theorem that local in-time solutions exist to the regularized equations. That result also provided the following Lipschitz condition

$$\|F_\epsilon(u_1) - F_\epsilon(u_2)\|_{H^s(\mathbb{R}^n)} \leq c(\|u_1\|_{L^2(\mathbb{R}^n)}, \|u_2\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u_1 - u_2\|_{H^s(\mathbb{R}^n)}.$$

Set $u_1 = u^\epsilon(t,x)$ and $u_2 = 0$. This yields the Lipschitz conditions of

$$\|F_\epsilon(u^\epsilon)\|_{H^s(\mathbb{R}^n)} \leq c(\|u^\epsilon\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u^\epsilon\|_{H^s(\mathbb{R}^n)}$$

and hence

$$\frac{d}{dt}\|u^\epsilon(\cdot,t)\|_{H^s(\mathbb{R}^n)} \leq c(\|u^\epsilon\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u^\epsilon\|_{H^s(\mathbb{R}^n)}.$$

Applying the basic energy estimate gives

$$\frac{d}{dt}\|u^\epsilon(\cdot,t)\|_{H^s(\mathbb{R}^n)} \leq c(\|u_0\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u^\epsilon\|_{H^s(\mathbb{R}^n)}$$

and applying Gronwall's inequality provides

$$\|u^\epsilon(\cdot,x)\|_{H^s(\mathbb{R}^n)} \leq e^{cT}$$

where $c = c(\|u_0\|_{L^2(\mathbb{R}^n)}, \epsilon, N)$. We now apply the previous theorem within the setting of the Banach space $V^s(\mathbb{R}^n)$. Neither condition in the theorem statement will hold. Therefore, the solution u^ϵ must exist globally in time. □

7. Local-in-Time Existence of Solutions to the Euler and Navier-Stokes Equations

7.1. **Local-in-Time Existence.** We now want to show that these approximate solutions approach the solution to the Euler and Navier-Stokes equations. That is, we want to let $\epsilon \to 0$. The estimate derived above is given as

$$\|F_\epsilon(u^\epsilon)\|_{H^s(\mathbb{R}^n)} \leq c(\|u^\epsilon\|_{L^2(\mathbb{R}^n)}, \epsilon, N)\|u^\epsilon\|_{H^s(\mathbb{R}^n)}.$$

This is a problem as the bound depends poorly upon ϵ. Thus, to perform the correct analysis, we need to derive a different estimate that does not depend upon ϵ. This estimate is given as the following higher-order energy estimate.

Proposition 7.1. *(**The** $H^s(\mathbb{R}^n)$ **Energy Estimate**) Let $u_0 \in V^s(\mathbb{R}^n)$. Then, the unique regularized solution $u^\epsilon \in C^1([0,\infty]; V^s(\mathbb{R}^n))$ satisfies*

$$\frac{d}{dt}\frac{1}{2}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)} + \nu\|\mathcal{J}_\epsilon \nabla u^\epsilon\|^2_{H^s(\mathbb{R}^n)} \leq C(s)\|\nabla \mathcal{J}_\epsilon u^\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)} \tag{7.1}$$

Proof. Let u^ϵ be a solution to the Cauchy problem

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon - P\mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)\right) \\ u^\epsilon(0,x) = u_0 \end{cases}$$

First, take the distributive derivative D^α, for $|\alpha| \leq s$ of the mollified equation above. Next, take the L^2 inner product with the term $D^\alpha u^\epsilon$. It follows that

$$\begin{aligned} g\left(D^\alpha u^\epsilon_t, D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)} &= g(\nu D^\alpha \mathcal{J}_\epsilon^2 \Delta u^\epsilon, D^\alpha u^\epsilon)_{L^2(\mathbb{R}^n)} - \left(D^\alpha P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)), D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)} \\ &= -\nu\|\mathcal{J}_\epsilon D^\alpha \nabla u^\epsilon\|_{L^2(\mathbb{R}^n)} - \left(P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(D^\alpha \mathcal{J}_\epsilon u^\epsilon)), D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)} \\ &- \left(D^\alpha P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)) - P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(D^\alpha\mathcal{J}_\epsilon u^\epsilon)), D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)}. \end{aligned}$$

We calculate the second term to be

$$\begin{aligned} g\left(P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(D^\alpha\mathcal{J}_\epsilon u^\epsilon)), D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)} &= \frac{1}{2}g(\mathcal{J}_\epsilon u^\epsilon, \nabla(\mathcal{J}_\epsilon D^\alpha u^\epsilon)^2)_{L^2(\mathbb{R}^n)} \\ &= -\frac{1}{2}g(\nabla\mathcal{J}_\epsilon u^\epsilon, |\mathcal{J}_\epsilon D^\alpha u^\epsilon|^2)_{L^2(\mathbb{R}^n)} \\ &= 0. \end{aligned}$$

Similarly, if we sum over $|\alpha| \leq s$, we have that

$$\sum_{|\alpha|\leq s} g\left(D^\alpha u^\epsilon_t, D^\alpha u^\epsilon\right)_{L^2(\mathbb{R}^n)} = \frac{1}{2}\frac{d}{dt}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)}$$

and by integration by parts

$$\sum_{|\alpha|\leq s} g(\nu D^\alpha \mathcal{J}_\epsilon^2\Delta u^\epsilon, D^\alpha u^\epsilon)_{L^2(\mathbb{R}^n)} = -\nu\|\mathcal{J}_\epsilon\nabla u^\epsilon\|^2_{H^s(\mathbb{R}^n)}$$

Combining these results yields

$$\frac{1}{2}\frac{d}{dt}\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)} + \nu\|\mathcal{J}_\epsilon\nabla u^\epsilon\|^2_{H^s(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\leq \|u^\epsilon\|_{H^s(\mathbb{R}^n)} \sum_{|\alpha|\leq s} \|D^\alpha((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)) - ((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(D^\alpha \mathcal{J}_\epsilon u^\epsilon))\|_{L^2(\mathbb{R}^3)} \\
&\leq C(s)\|u^\epsilon\|_{H^s(\mathbb{R}^n)} \left(\|\nabla \mathcal{J}_\epsilon u^\epsilon\|_{L^\infty(\mathbb{R}^n)}\|D^{s-1}\nabla \mathcal{J}_\epsilon u^\epsilon\|_{L^2(\mathbb{R}^n)} + \|D^s \mathcal{J}_\epsilon u^\epsilon\|_{L^2(\mathbb{R}^n)}\|\nabla \mathcal{J}_\epsilon u^\epsilon\|\right) \\
&\leq C(s)\left(\|\mathcal{J}_\epsilon \nabla u^\epsilon\|_{L^\infty(\mathbb{R}^n)}\|\mathcal{J}_\epsilon u^\epsilon\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}
$$

□

This energy estimate yields the following result.

Corollary 7.2. *For $u_0^\nu \in V^s(\mathbb{R}^n)$, we have*

$$
\nu \int_0^{T_0} \|\nabla \mathcal{J}_\epsilon u^\epsilon(t)\|^2_{H^s(\mathbb{R}^n)} dt \leq C\|u_0\|^3_{H^s(\mathbb{R}^n)} \tag{7.2}
$$

Proof. The energy estimate 7.1 provides

$$
2\nu\|\nabla \mathcal{J}_\epsilon u^\epsilon\|^2_{H^s(\mathbb{R}^n)} \leq C\|u^\epsilon\|^3_{H^s(\mathbb{R}^n)}
$$

□

To show that the sequence of the approximate solutions converge, we show that $\{u^\epsilon\}_{\epsilon>0}$ is Cauchy in the space $C([0,T];L^2(\mathbb{R}^n))$.

Proposition 7.3. *The family $\{u^\epsilon\}$ of solutions to the regularized Navier-Stokes equations forms a Cauchy sequence in $C([0,T];L^2(\mathbb{R}^3))$. In particular, there exists a constant C that depends only on $\sup_{0\leq t\leq T}\|u_0\|_{H^s(\mathbb{R}^n)}$ and a time T such that, for all ϵ_1 and ϵ_2,*

$$
\sup_{0\leq t\leq T} \|u^{\epsilon_1} - u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)} \leq C max\{\epsilon_1, \epsilon_2\}.
$$

Proof. We have that

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u^{\epsilon_1} - u^{\epsilon_2}\|^2_{L^2(\mathbb{R}^n)} &= \nu g\left(\mathcal{J}^2_{\epsilon_1}\Delta u^{\epsilon_1} - \mathcal{J}^2_{\epsilon_1}\Delta u^{\epsilon_2}, u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^n)} \\
&\quad - g\left(P\mathcal{J}_{\epsilon_1}\left((\mathcal{J}_{\epsilon_1}u^{\epsilon_1})\cdot(\mathcal{J}_{\epsilon_1}u^{\epsilon_1})\right) - P\mathcal{J}_{\epsilon_2}\left((\mathcal{J}_{\epsilon_2}u^{\epsilon_2})\cdot(\mathcal{J}_{\epsilon_2}u^{\epsilon_2})\right), u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^n)} \\
&= T_1 + T_2.
\end{aligned}
$$

We first estimate T_1 and see that

$$
\begin{aligned}
T_1 &= g\left(\mathcal{J}^2_{\epsilon_1}\Delta u^{\epsilon_1} - \mathcal{J}^2_{\epsilon_1}\Delta u^{\epsilon_2}, u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)} \\
&= g\left(\left(\mathcal{J}^2_{\epsilon_1} - \mathcal{J}^2_{\epsilon_2}\right)\Delta u^{\epsilon_1}, u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)} + g\left(\mathcal{J}^2_{\epsilon_2}\left(\Delta u^{\epsilon_1} - u^{\epsilon_2}\right), u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)} \\
&= g\left(\left(\mathcal{J}^2_{\epsilon_1} - \mathcal{J}^2_{\epsilon_2}\right)\Delta u^{\epsilon_1}, u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)} - \|\mathcal{J}^2_{\epsilon_2}\nabla(u^{\epsilon_1} - u^{\epsilon_2})\|^2_{L^2(\mathbb{R}^3)} \quad \text{(by integration by parts)} \\
&\leq g\left(\left(\mathcal{J}^2_{\epsilon_1} - \mathcal{J}^2_{\epsilon_2}\right)\Delta u^{\epsilon_1}, u^{\epsilon_1} - u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}
\end{aligned}
$$

Next, we estimate $T2$ by writing it to include five terms.

$$\begin{aligned}
T_2 &= g\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\\
&+g\left(\mathcal{J}_{\epsilon_2}\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\\
&+g\left(\mathcal{J}_{\epsilon_2}\left(\mathcal{J}_{\epsilon_1}\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\\
&+\left(\mathcal{J}_{\epsilon_2}\left(\mathcal{J}_{\epsilon_2}\left(u^{\epsilon_2}\cdot\nabla\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)u^{\epsilon_1}\right)\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\\
&+g\left(\mathcal{J}_{\epsilon_2}\left(\mathcal{J}_{\epsilon_2}\left(u^{\epsilon_2}\cdot\nabla\left(\mathcal{J}_{\epsilon_2}\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\right)\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\\
&=R_1+R_2+R_3+R_4+R_5.
\end{aligned}$$

We estimate each R_i separately.

$$\begin{aligned}
|R_1| &= \left|g\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\right|\\
&\le \|\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\quad\text{(Cauchy-Shwartz Inequality)}\\
&= \|\left(\mathcal{J}_{\epsilon_1}-1\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)-\left(\mathcal{J}_{\epsilon_2}-1\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\|\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{H^3(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{H^3(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le C\max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|^2_{H^3(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

$$\begin{aligned}
|R_2| &= \left|g\left(\mathcal{J}_{\epsilon_2}\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right),u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\right|\\
&\le \|\left(\mathcal{J}_{\epsilon_2}\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\quad\text{(Cauchy-Shwartz Inequality)}\\
&\le \|\left(\left(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2}\right)u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&= \|\left(\mathcal{J}_{\epsilon_1}-1\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)-\left(\mathcal{J}_{\epsilon_2}-1\right)\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\|\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\cdot\nabla\left(\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\mathcal{J}_{\epsilon_1}u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|\nabla u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{H^3(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{L^\infty(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le \max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|_{H^3(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}+\|\nabla u^{\epsilon_1}\|_{H^2(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^1(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le C\max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|^2_{H^3(\mathbb{R}^3)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}\\
&\le C\max(\epsilon_1,\epsilon_2)\left(\|u^{\epsilon_1}\|^2_{H^s(\mathbb{R}^n)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

$$\begin{aligned}
|R_3| &= \left| g\left(\mathcal{J}_{\epsilon_2}\left(\mathcal{J}_{\epsilon_1}\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right), u^{\epsilon_1}-u^{\epsilon_2}\right)_{L^2(\mathbb{R}^3)}\right| \\
&\leq \|\mathcal{J}_{\epsilon_2}\left(\mathcal{J}_{\epsilon_1}\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\|\mathcal{J}_{\epsilon_1}\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\|\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\cdot\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\|_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^3)} \\
&\leq C\|\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\|^2_{L^2(\mathbb{R}^3)}\|\nabla\left(\mathcal{J}_{\epsilon_2}u^{\epsilon_1}\right)\|_{L^2(\mathbb{R}^3)} \\
&\leq C\|\left(u^{\epsilon_1}-u^{\epsilon_2}\right)\|^2_{L^2(\mathbb{R}^3)}\|u^{\epsilon_1}\|_{H^s(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
|R_4| &= g(\mathcal{J}_{\epsilon_2}(\mathcal{J}_{\epsilon_2}(u^{\epsilon_2}\cdot\nabla((\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1}))), u^{\epsilon_1}-u^{\epsilon_2})_{L^2(\mathbb{R}^n)} \\
&\leq \|\mathcal{J}_{\epsilon_2}(\mathcal{J}_{\epsilon_2}(u^{\epsilon_2}\cdot\nabla((\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1})))\|_{L^2(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&\leq C\|\mathcal{J}_{\epsilon_2}(u^{\epsilon_2}\cdot\nabla((\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1})))\|_{L^2(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&\leq C\|u^{\epsilon_2}\cdot\nabla((\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1}))\|_{L^2(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&\leq C\|u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\|\nabla((\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1}))\|_{L^2(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&\leq C\|u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\|(\mathcal{J}_{\epsilon_1}-\mathcal{J}_{\epsilon_2})u^{\epsilon_1})\|_{H^1(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&= C\|u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\|(\mathcal{J}_{\epsilon_1}-1)u^{\epsilon_1}-(\mathcal{J}_{\epsilon_2}-1)u^{\epsilon}\|_{H^1(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \\
&\leq C\max(\epsilon_1,\epsilon_2)\|u^{\epsilon_1}\|_{H^s(\mathbb{R}^n)}\|u^{\epsilon_2}\|_{H^s(\mathbb{R}^n)}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
|R_5| &= g(\mathcal{J}_{\epsilon_2}(\mathcal{J}_{\epsilon_2}(u^{\epsilon_2}\cdot\nabla(\mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2})))), u^{\epsilon_1}-u^{\epsilon_2})_{L^2(\mathbb{R}^n)} \\
&= g(\mathcal{J}_{\epsilon_2}(u^{\epsilon_2}\cdot\nabla(\mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2}))), \mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2}))_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n}\mathcal{J}_{\epsilon_2}u^{\epsilon_2}\cdot\nabla(\mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2}))\mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2})dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n}\mathcal{J}_{\epsilon_2}u^{\epsilon_2}\nabla(|\mathcal{J}_{\epsilon_2}(u^{\epsilon_1}-u^{\epsilon_2})|^2)dx \\
&= 0.
\end{aligned}$$

Combining each of the estimates yields that

$$\frac{1}{2}\frac{d}{dt}\|u^{\epsilon_1}-u^{\epsilon_2}\|^2_{L^2(\mathbb{R}^n)} \leq C(M)\left(\max(\epsilon_1\epsilon_2)+\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\right)\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}$$

and hence

$$\frac{d}{dt}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} \leq C(M)\left(\max(\epsilon_1\epsilon_2)+\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\right)$$

where M is a uniform bound for $\|u^\epsilon\|_{H^s(\mathbb{R}^n)}$ to be derived later. Using the integrating factor of $\mu = \exp\left(\int_0^T C(M)dt\right) = \exp(C(M)T)$, we solve the above differential inequality as

$$\begin{aligned}
\sup_{0\leq t\leq T}\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)} &\leq \exp\left(C(M)T\right)\left(\max(\epsilon_1,\epsilon_2)+\|u^{\epsilon_1}-u^{\epsilon_2}\|_{L^2(\mathbb{R}^n)}\right) \\
&\leq C(M,T)\max(\epsilon_1,\epsilon_2).
\end{aligned}$$

Therefore, $\{u^\epsilon\}_{\epsilon>0}$ is a Cauchy sequence in the Sobolev space $C([0,T];L^2(\mathbb{R}^n))$, so that it converges strongly to some vector $u^\nu \in C([0,T];L^2(\mathbb{R}^n))$. □

Lemma 7.4. ***(Sobolev Interpolation Theorem)*** *Given $s > 0$, there exists a constant $C(s)$ such that for all $v \in H^s(\mathbb{R}^n)$ and $s_1 < s < s_2$ with $s = \theta s_1 + (1-\theta)s_2$ for $0 \leq \theta \leq 1$,*

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C(s)\|v\|^{\theta}_{H^{s_1}(\mathbb{R}^n)}\|v\|^{1-\theta}_{H^{s_2}(\mathbb{R}^n)} \tag{7.3}$$

Proof. Let $v \in H^s(\mathbb{R}^n)$ and $s_1 < s < s_2$ with $s = \theta s_1 + (1-\theta)s_2$ for $0 \leq \theta \leq 1$. It follows that

$$\begin{aligned}
\|v\|^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{v}(\xi)|^{2\theta} |\hat{v}(\xi)|^{2(1-\theta)} d\xi \\
&\leq \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s_1} |\hat{v}(\xi)|^2 d\xi\right)^{\theta} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s_2} |\hat{v}(\xi)|^2 d\xi\right)^{1-\theta} \\
&= \|v\|^{2\theta}_{H^s(\mathbb{R}^n)} \|v\|^{2(1-\theta)}_{H^s(\mathbb{R}^n)}
\end{aligned}$$

where the inequality follows from Holder's inequality with $p = 1/\theta$ and $q = 1/(1-\theta)$. □

Theorem 7.5. ***(Local-in-Time Existence of Solutions to the Euler and Navier-Stokes Equations)*** *Given an initial condition $u_0 \in V^s(\mathbb{R}^n)$ such that $s \geq n/2 + 2$, then we have the following*

1. *There exists a time $T > 0$ with a rough upper bound*

$$T \leq \frac{1}{C(s)\|u_0\|_{H^s(\mathbb{R}^n)}}, \tag{7.4}$$

such that for any viscosity $0 \leq \nu < \infty$, there exists a unique solution $u^\nu \in C([0,T]; C^2(\mathbb{R}^n)) \cap C^1([0,T]; C^2(\mathbb{R}^n))$ to the equation

$$\begin{cases} \frac{\partial u^\nu}{\partial t} + (u^\nu \cdot \nabla) u^\nabla = -\nabla p + \nu \Delta u^\nu \\ \nabla \cdot u^\nu = 0 \\ u^\nu|_{t=0} = u_0. \end{cases}$$

The solution u^ν is the limit of a subsequence of the approximate solutions $\{u^\epsilon\}_{\epsilon>0}$ to the equation

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + P\mathcal{J}_\epsilon\left((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla\left(\mathcal{J}_\epsilon u^\epsilon\right)\right) = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon \\ u^\epsilon|_{t=0} = u_0. \end{cases}$$

2. *The approximate solutions $\{u^\epsilon\}_{\epsilon>0}$ and the limit u^ν satisfy the following higher-order energy estimates*

$$\sup_{0 \leq t \leq T} \|u^\nu\|_{H^s(\mathbb{R}^n)} \leq \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1 - C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}}. \tag{7.5}$$

and

$$\sup_{0 \leq t \leq T} \|u^\epsilon\|_{H^s(\mathbb{R}^n)} \leq \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1 - C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}} \tag{7.6}$$

Proof. We begin with demonstrating part (2), that the family of approximate solutions $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in the $H^s(\mathbb{R}^n)$. From energy estimate 7.1 and $s > n/2+1$, we have

$$\begin{aligned}\frac{d}{dt}\|u^\epsilon\|_{H^s(\mathbb{R}^n)} &\leq C(s)\|\mathcal{J}_\epsilon \nabla u^\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u^\epsilon\|_{H^s(\mathbb{R}^n)} \\ &\leq C(s)\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)}\end{aligned}$$

Solving the differential inequality provides that for some time T, we have

$$\sup_{0\leq t\leq T}\|u^\epsilon\|_{H^s(\mathbb{R}^n)} \leq \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1-C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}}.$$

Thus, the family of approximate solutions $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in the space $C([0,T];H^s(\mathbb{R}^n))$ for $s > n/2$. This also gives a rough upper bound on the time T given as

$$T < \frac{1}{C(s)\|u_0\|_{H^s(\mathbb{R}^n)}}.$$

Next, we show that the limit of the approximate solutions $\{u^\epsilon\}_{\epsilon>0}$ is indeed a solution to the Euler and Navier-Stokes equations. We first consider the case of the Euler equations and then extend the estimates to easily show the case of the Navier-Stokes equations. In the space $V^s(\mathbb{R}^n)$, the Euler equation and mollified Euler equation are given by

$$\partial_t u + P(u\cdot\nabla)u = 0.$$

and

$$\partial_t u^\epsilon + P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)) = 0$$

Integrating to some time t yields

$$\begin{aligned}&u(t) - u_0 + \int_0^t P\left[(u\cdot\nabla)u\right]d\tau \\ &\qquad = (u(t)-u^\epsilon(t)) - (1-\mathcal{J}_\epsilon)u_0 + \int_0^t P((u(\tau)-u^\epsilon(\tau))\cdot\nabla u(\tau))d\tau \\ &\qquad + \int_0^t P(u^\epsilon(\tau)\cdot\nabla(u(\tau)-u^\epsilon\tau))d\tau + \int_0^t P(1-\mathcal{J}_\epsilon)(u^\epsilon(\tau)\cdot u^\epsilon(\tau))d\tau \\ &\qquad + \int_0^t P((1-\mathcal{J}_\epsilon)u^\epsilon(\tau)\cdot\nabla u^\epsilon)d\tau + \int_0^t P(\mathcal{J}_\epsilon u^\epsilon(\tau)\cdot\nabla(1-\mathcal{J}_\epsilon)u^\epsilon(\tau))d\tau\end{aligned}$$

for all $t\in[T,T]$. Taking the $L^2(\mathbb{R}^n)$ norm yields

$$\begin{aligned}&\left\|u(t)-u_0+\int_0^t P\left[(u\cdot\nabla)u\right]d\tau\right\|_{L^2(\mathbb{R}^n)} \\ &\qquad \leq \|u(t)-u^\epsilon(t)\|_{L^2(\mathbb{R}^n)} + C\epsilon\|u_0\|_{H^1(\mathbb{R}^n)} \\ &\qquad + \int_0^t \|u(\tau)-u^\epsilon(\tau)\|_{L^2(\mathbb{R}^n)}\|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^n)}d\tau \\ &\qquad + \int_0^t \|u_0\|_{L^2(\mathbb{R}^n)}\|\nabla u(\tau)-\nabla u^\epsilon(\tau)\|_{L^\infty(\mathbb{R}^n)}d\tau\end{aligned}$$

$$
\begin{aligned}
&+ C\epsilon \int_0^t \|u^\epsilon(\tau) \otimes u^\epsilon(\tau)\|_{H^2(\mathbb{R}^n)} d\tau \\
&+ C\epsilon \int_0^t \|u^\epsilon(\tau)\|_{H^1(\mathbb{R}^n)} \|\nabla u^\epsilon(\tau)\|_{L^\infty(\mathbb{R}^n)} d\tau \\
&+ C\epsilon \int_0^t \|u^\epsilon(\tau)\|_{L^\infty(\mathbb{R}^n)} \|u^\epsilon(\tau)\|_{H^2(\mathbb{R}^n)} d\tau \\
&\leq \|u - u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} (1 + CT\|u_0\|_{H^s(\mathbb{R}^n)}) \\
&+ T\|u_0\|_{L^2(\mathbb{R}^n)} \|\nabla u - \nabla u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} \\
&+ C\epsilon(\|u_0\|_{H^s(\mathbb{R}^n)} + T\|u_0\|^2_{H^s(\mathbb{R}^n)}) \\
&\leq C\|u - u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} + C\|\nabla u - \nabla u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} \\
&+ C\epsilon\|u_0\|_{H^s(\mathbb{R}^n)}
\end{aligned}
$$

for all $t \in [-T, T]$. Letting $\epsilon \to 0$, we have that

$$u(t) + \int_0^t P\left[(u(\tau) \cdot \nabla) u(\tau)\right] d\tau = u_0.$$

This shows the case for the Euler equations. We can easily extend this case to the Navier-Stokes equations. The diffusive term has the $L^2(\mathbb{R}^n)$ norm of

$$
\begin{aligned}
&\left\| \int_0^t \left(\Delta u(\tau) - \mathcal{J}_\epsilon^2 \Delta u^\epsilon(\tau) \right) d\tau \right\|_{L^2(\mathbb{R}^n)} \\
&\qquad \leq C \int_0^t \|\Delta(u(t) - u^\epsilon(\tau))\|_{L^2(\mathbb{R}^n)} d\tau + C \int_0^t \|(1 - \mathcal{J}_\epsilon) \Delta u^\epsilon(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\qquad \leq CT\|u - u^\epsilon\|_{L^\infty([0,T];H^2)(\mathbb{R}^n)} + C\epsilon^\alpha T\|u^\epsilon\|_{L^\infty([0,T];H^s(\mathbb{R}^n)}
\end{aligned}
$$

which holds for $\alpha \in (0, s-2]$ and all $t \in (0, T]$. Combining with the Euler case yields that

$$
\begin{aligned}
&\left\| u(t) - u_0 + \int_0^t P\left[(u \cdot \nabla) u\right] d\tau + \nu \int_0^t \Delta u(\tau) d\tau \right\|_{L^2(\mathbb{R}^n)} \\
&\qquad \leq C\|u - u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \\
&\qquad + C\|\nabla u - \nabla u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} \\
&\qquad + C\epsilon\|u_0\|_{H^s(\mathbb{R}^n)} + C\nu\epsilon^\alpha \\
&\qquad + C\nu\|u - u^\epsilon\|_{L^\infty([0,T];H^2(\mathbb{R}^n))}
\end{aligned}
$$

where $C = C(T) > 0$. Therefore, for $s > 2$, we have that

$$u(t) + \int_0^t P\left[(u \cdot \nabla) u\right] d\tau + \nu \int_0^t \Delta u(\tau) d\tau = u_0$$

for all $t \in [0, T]$. $\square$

7.2. **Uniform Bounds.**

Definition 7.6. (Weak Convergence in a Hilbert Space) In a Hilbert space $\mathcal{H}$ with corresponding inner-product $g(\cdot,\cdot)_{\mathcal{H}}$, we say that a sequence $\{u_i\} \subset \mathcal{H}$ *converges weakly* to an element $u \in \mathcal{H}$ (written as $\{u_i\} \rightharpoonup u$) if

$$g(u_i, v)_{\mathcal{H}} \to g(u, v)_{\mathcal{H}}$$

for all $v \in \mathcal{H}$.

Definition 7.7. The space $C_W[0,T]; H^s(\mathbb{R}^n)$ denotes continuity on the interval $[0,T]$ with values in the weak topology of $H^s(\mathbb{R}^n)$, that is, for any fixed $\varphi \in H^s(\mathbb{R}^n)$, $g(\varphi, u(t))_{H^s(\mathbb{R}^n)}$ is a continuous scalar function on $[0,T]$, where the inner product is given by

$$g(u,v)_{H^s(\mathbb{R}^n)} = \sum_{\alpha<s} \int_{\mathbb{R}^n} D^\alpha u \cdot D^\alpha v dx. \tag{7.7}$$

Theorem 7.8. ***(Uniform Bounds)*** *Given an initial condition $u_0 \in V^s(\mathbb{R}^n)$ such that $s \geq n/2+2$, then the approximate solutions and the limit u^ν are uniformly bounded in the spaces*

(1) $L^\infty([0,\infty], H^s(\mathbb{R}^n))$
(2) $Lip([0,T]; H^{s-2}(\mathbb{R}^n))$
(3) $C_W[0,T]; H^s(\mathbb{R}^n)$.

Proof of (1). We begin with a corollary of the Banach Alaoglu theorem: if a sequence $\{u_\epsilon\} \in H^s(\mathbb{R}^n)$ is bounded, then there exists a subsequence that converges weakly to some limit in $H^s(\mathbb{R}^n)$. We have that

$$\sup_{0\leq t\leq T} \|u^\epsilon\|_{H^s(\mathbb{R}^n)} \leq \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1 - C(s)T\|u_0\|_s} = M_1(s, T, \|u_0\|_{H^s(\mathbb{R}^n)}).$$

Thus, u^ϵ is uniformly bounded in the Sobolev space $L^2([0,T]; H^s(\mathbb{R}^n))$. Hence, there exists a subsequence that weakly converges to some limit

$$u^\epsilon \in L^2([0,T]; H^s(\mathbb{R}^n)).$$

Moreover, fix $t \in [0,T]$. Then, the sequence $\{u^\epsilon(\cdot,t)\}$ is uniformly bounded in $H^s(\mathbb{R}^n)$. Again, it has a subsequence that converges weakly to some $u(t) \in H^s(\mathbb{R}^n)$. Thus, for each t, $\|u\|_{H^s(\mathbb{R}^n)}$ is bounded. It follows that $u \in L^\infty([0,T]; H^s(\mathbb{R}^n))$.

Proof of (2). **Case 1: $\boldsymbol{\nu = 0}$**: We begin with the Euler case. Showing that a function is Lipschitz can be achieved by appealing to the fundamental theorem of calculus. Hence, we need a bound on the time derivative of the solutions. It follows that

$$\begin{aligned}
\left\|\frac{\partial u^\epsilon}{\partial t}\right\|_{H^{s-1}(\mathbb{R}^n)} &= \|P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)]\|_{H^{s-1}(\mathbb{R}^n)} && \text{(mollified Euler equation definition)}\\
&\leq C(s)\,\|(\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)\|_{H^{s-1}(\mathbb{R}^n)} && \text{(mollifier property 2.14)}\\
&\leq C(s)\|\mathcal{J}_\epsilon u^\epsilon\|_{H^{s-1}(\mathbb{R}^n)}\|\nabla\mathcal{J}_\epsilon u^\epsilon\|_{H^{s-1}(\mathbb{R}^n)} && \text{(Banach algebra)}\\
&\leq C(s)\|u^\epsilon\|_{H^{s-1}(\mathbb{R}^n)}\|u^\epsilon\|_{H^s(\mathbb{R}^n)} && \text{(mollifier property 2.14)}\\
&\leq C(s)\|u^\epsilon\|^2_{H^s(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)}) && \text{(uniform } H^s(\mathbb{R}^n) \text{ bound)}
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial u^0}{\partial t}\right\|_{H^{s-1}(\mathbb{R}^n)} &= \left\|P[(u^0\cdot\nabla)u^0]\right\|_{H^{s-1}(\mathbb{R}^n)} && \text{(Euler equation definition)}\\
&\leq C(s)\|u^0\|_{H^{s-1}(\mathbb{R}^n)}\|\nabla u^0\|_{H^{s-1}(\mathbb{R}^n)} && \text{(Banach algebra)}\\
&\leq C(s)\|u^0\|_{H^{s-1}(\mathbb{R}^n)}\|u^0\|_{H^{s}(\mathbb{R}^n)} && \text{(mollifier property 2.14)}\\
&\leq C(s)\|u^0\|^2_{H^{s}(\mathbb{R}^n)}\\
&\leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)}) && \text{(uniform } H^s(\mathbb{R}^n) \text{ bound).}
\end{aligned}
$$

Hence, the mollified Euler and Euler solutions' time derivatives are bounded in $H^{s-1}(\mathbb{R}^n)$. Integrating from $0 \leq t < s \leq T$, we have that

$$
\|u^\epsilon(t)-u^\epsilon(s)\|_{H^{s-1}(\mathbb{R}^n)} \leq \int_t^s \left\|\frac{\partial u^\epsilon}{\partial t}\right\|_{H^{s-1}(\mathbb{R}^n)} d\tau \leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)})(s-t)
$$

and

$$
\|u^0(t)-u^0(s)\|_{H^{s-1}(\mathbb{R}^n)} \leq \int_t^s \left\|\frac{\partial u^0}{\partial t}\right\|_{H^{s-1}(\mathbb{R}^n)} d\tau \leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)})(s-t).
$$

Thus, the mollified Euler and Euler solutions belong to $\mathrm{Lip}([0,T],H^{s-2}(\mathbb{R}^n))$.
Case 2: $\nu > 0$: To show that the approximate and actual solutions to the Navier-Stokes euqations are Lipschitz, we can piggyback off the first case. As these equations contain a Laplacian, we begin in $H^{s-2}(\mathbb{R}^n)$. It follows that

$$
\begin{aligned}
\left\|\frac{\partial u^\epsilon}{\partial t}\right\|_{H^{s-2}(\mathbb{R}^n)} &= \left\|P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)] + \nu\mathcal{J}_\epsilon^2\Delta u^\epsilon\right\|_{H^{s-2}(\mathbb{R}^n)} && \text{(mollified NS equation definition)}\\
&\leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)}) + \|\nu\mathcal{J}_\epsilon^2\Delta u^\epsilon\|_{H^{s-2}(\mathbb{R}^n)} && \text{(previous Euler case)}\\
&\leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)}) + \nu\|\Delta u^\epsilon\|_{H^{s-2}(\mathbb{R}^n)} && \text{(mollifier property 2.14)}\\
&\leq C(s,T,\|u_0\|_{H^s(\mathbb{R}^n)}) + \nu\|u^\epsilon\|_{H^{s}(\mathbb{R}^n)}\\
&\leq \tilde{C}(s,T,\|u_0\|_{H^s(\mathbb{R}^n)},\nu) && \text{(uniform } H^s(\mathbb{R}^n) \text{ bound)}
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial u^\nu}{\partial t}\right\|_{H^{s-2}(\mathbb{R}^n)} &= \|P[(u^\nu\cdot\nabla)u^\nu] + \nu\Delta u^\nu\|_{H^{s-2}(\mathbb{R}^n)} && \text{(NS equation definition)}\\
&\leq C(s,T,\|u_0^\nu\|_{H^s(\mathbb{R}^n)}) + \nu\|\Delta u^\epsilon\|_{H^{s-2}(\mathbb{R}^n)} && \text{(previous Euler case)}\\
&\leq C(s,T,\|u_0^\nu\|_{H^s(\mathbb{R}^n)}) + \nu\|u^\nu\|_{H^{s}(\mathbb{R}^n)}\\
&\leq \tilde{C}(s,T,\|u_0^\nu\|_{H^s(\mathbb{R}^n)},\nu)
\end{aligned}
$$

Employing the same method used in the Euler case, it is immediate that the mollified Navier-Stokes and the Navier-Stokes solutions belong to $\mathrm{Lip}([0,T];H^{s-2}(\mathbb{R}^n))$.

Proof of (3). **Case 1:** $\nu = \mathbf{0}$: As before, we begin with the Euler case. To show that $u^0 \in C_W([0,T]; H^s(\mathbb{R}^n))$, we must show that $g(u^0(t), \varphi)_{L^2(\mathbb{R}^n)}$ is a continuous function for a fixed test function $\varphi \in H^{-s}(\mathbb{R}^n)$. Indeed, fix a test function $\varphi \in H^{-s}(\mathbb{R}^n)$ such that $\|\varphi\|_{H^{-s}(\mathbb{R}^n)} = 1$ and let $\delta \in (0, 1/4]$ be arbitrary. As $H^{-s+1}(\mathbb{R}^n)$ is dense in $H^{-s}(\mathbb{R}^n)$ and is the dual of $H^{s-1}(\mathbb{R}^n)$, we can then find an element $\psi \in H^{-s+1}(\mathbb{R}^n)$ such that

$$\|\varphi - \psi\|_{H^{-s}(\mathbb{R}^n)} = \left\|(1 + |\xi|^2)^{-s/2}(\hat{\varphi}(\xi) - \hat{\psi}(\xi))\right\|_{L^2(\mathbb{R}^n)} \leq \delta$$

Note that for $f \in H^s(\mathbb{R}^n)$ and $g \in H^{-s}(\mathbb{R}^n)$, identify the duality inner product written as $g(f,g)_{H^s(\mathbb{R}^n)\times H^{-s}(\mathbb{R}^n)}$ with the $L^2(\mathbb{R}^n)$ inner product $g\left((1 + |\xi|^2)^{s/2}\hat{f}(\xi), (1 + |\xi|^2)^{-s/2}\hat{g}(\xi)\right)_{L^2(\mathbb{R}^n)}$

As $u^{\epsilon_n} \to u^0$ in $C([0,T]; H^s(\mathbb{R}^n))$ as $n \to \infty$, we have that

$$g(u^\epsilon(t), \psi)_{H^{s-1}(\mathbb{R}^n)\times H^{1-s}(\mathbb{R}^n)} \to g(u^0(t), \psi)_{H^{s-1}(\mathbb{R}^n)\times H^{1-s}(\mathbb{R}^n)}$$

as $\epsilon \to 0$, uniformly in $t \in [-T, T]$. Thus, there exists $\epsilon > 0$ sufficiently small such that

$$\sup_{t\in[-T,T]} |g(u(t) - u^\epsilon, \psi)| \leq \delta.$$

Additionally, from the construction of u^ϵ, we have that $u^\epsilon \in \mathrm{Lip}([-T,T]; H^{s-1}(\mathbb{R}^n))$, and hence there exists $\tau > 0$ such that

$$\sup_{t,s\in[-T,T];|t-s|\leq\tau} \|u^\epsilon(t) - u^\epsilon(s)\|_{H^{s-1}(\mathbb{R}^n)} \leq \delta(1 + \|\psi\|_{H^{1-s}(\mathbb{R}^n)})^{-1}.$$

We therefore arrive at for $|t - s| \leq \tau$, we have

$$\begin{aligned}
|g(u(t) - u(s), \psi)_{L^2(\mathbb{R}^n)}| &\leq |g(u^\epsilon(t) - g^\epsilon(s), \psi)_{L^2(\mathbb{R}^n)}| + |g(u(t) - u(s), \varphi - \psi)_{L^2(\mathbb{R}^n)}| \\
&\quad + |g(u^0(t) - u^\epsilon(t), \psi)_{L^2(\mathbb{R}^n)}| + |g(u^0(s) - u^\epsilon(s))_{L^2(\mathbb{R}^n)}| \\
&\leq \delta + 2\delta\|u^0\|_{L^\infty([-T,T;H^s(\mathbb{R}^n)])} + 2\delta
\end{aligned}$$

As $\delta \in (0, 1/4]$ was arbitrary, u^0 is weakly uniformly continuous with values of $H^s(\mathbb{R}^n)$.
Case 2: $\nu > \mathbf{0}$: For the Navier-Stokes case, we need to take the test function $\psi \in H^{-s+2}(\mathbb{R}^n)$ which approximates $\varphi \in H^{-s}(\mathbb{R}^n)$ up to an error of size ϵ. As $s > 2$, we can now employ the same argument as above. Therefore, $u^\nu \in C_W([0,T]; H^s(\mathbb{R}^n))$.
Additionally, from corollary 7.2, we have that

$$u^\nu \in L^2([0,T]; H^{s+1}(\mathbb{R}^n)). \tag{7.8}$$

This will become important in the next section to show that the solution is continuous in the high norm. We begin this discussion now.

7.3. **Continuity in the High Norm.** The final part of the well-posedness argument is to show that the solutions are continuous in the high norm: $H^s(\mathbb{R}^n)$. This is answered in theorem 7.9.

Theorem 7.9. ***(Continuity in the High Norm)*** *Let u^ν be the solution described in Theorem 7.5. Then,*

$$u^\nu \in C([0,T]; V^s(\mathbb{R}^n)) \cap C^1([0,T]; V^{s-2}(\mathbb{R}^n)).$$

Proof. As before, we consider the Euler and Navier-Stokes solutions separately. We also employ two different methods to demonstrate continuity in the high norm for the two equations. We begin with the Euler solution. It suffices to show that $u^\nu \in C([0,T];H^s(\mathbb{R}^n))$. As $u^\nu \in C_W([0,T];H^s(\mathbb{R}^n))$, it suffices to show that the norm $\|u^\nu\|_{H^s(\mathbb{R}^n)}$ is continuous in time. We begin with the proof for the Euler equation.
Case 1: $\nu = \mathbf{0}$. We have shown that the that the norm $\|u\|_{H^s(\mathbb{R}^n)}$ is uniformly bounded. That is

$$\sup_{0\le t\le T} \|u^0\|_{H^s(\mathbb{R}^n)} \le \frac{\|u_0\|_{H^s(\mathbb{R}^n)}}{1 - C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}} = \|u_0\|_{H^s(\mathbb{R}^n)} + \frac{\|u_0\|^2_{H^s(\mathbb{R}^n)}C(s)T}{1 - C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}}.$$

Also, it is easy to see that $\limsup_{\epsilon\to 0} \|u^\epsilon\|_{H^s(\mathbb{R}^n)} \ge \|u^\nu\|_{H^s(\mathbb{R}^n)}$. Passing the limit into the equation yields that

$$\sup_{0\le t\le T} \|u^0\|_{H^s(\mathbb{R}^n)} - \|u_0\|_{H^s(\mathbb{R}^n)} \le \frac{\|u_0\|^2_{H^s(\mathbb{R}^n)}C(s)T}{1 - C(s)T\|u_0\|_{H^s(\mathbb{R}^n)}}.$$

Next, as $u^0 \in C_W(0,T;H^s(\mathbb{R}^n))$, we have that $\liminf_{t\to 0^+} \|u^0(\cdot,t)\| \le \|u_0\|_{H^s(\mathbb{R}^n)}$. In particular, $\liminf_{t\to 0^+} \|u^0(\cdot,t)\| = \|u_0\|_{H^s(\mathbb{R}^n)}$. We arrive at strong continuity of u^0 at $t = 0$. Now, we must prove continuity of the $\|\cdot\|_{H^s(\mathbb{R}^n)}$ at all other times. Consider a time $T_0 \in [0,T]$ and the solution $u^0(\cdot,T_0)$. At this fixed time, $u^0(\cdot,T_0) = u_0^{T_0} \in H^s(\mathbb{R}^n)$. From uniform boundedness in $H^s(\mathbb{R}^n)$, we have that

$$\|u_0^{T_0}\|_{H^s(\mathbb{R}^n)} \le \|u_0\|_{H^s(\mathbb{R}^n)} + \frac{\|u_0\|^2_{H^s(\mathbb{R}^n)}C(s)T_0}{1 - C(s)T_0\|u_0\|_{H^s(\mathbb{R}^n)}}.$$

Thus, we can take $u_0^{T_0}$ as initial data and construct a forward and backward in time solution as above by solving the regularized Euler equation. In doing so, we obtain a family of approximate solutions, $\{u^\epsilon_{T_0}\}_{\epsilon>0}$ that satisfy the $H^s(\mathbb{R}^n)$ energy estimate, that is

$$\frac{d}{dt}\|u^\epsilon_{T_0}\|_{H^s(\mathbb{R}^n)} \le C(s)\|\mathcal{J}_\epsilon \nabla u^\epsilon_{T_0}\|_{L^\infty(\mathbb{R}^n)}\|u^\epsilon_{T_0}\|_{H^s(\mathbb{R}^n)} \le C(s)\|u^\epsilon_{T_0}\|^2_{H^s(\mathbb{R}^n)}.$$

If we pass to a limit in $\{u^\epsilon_{T_0}\}$ as before to find a solution $\tilde{u}$ to the Euler equation on some time interval $[T_0 - T', T_0 + T']$ with initial data $u_0^{T_0}$. The result then follows from uniqueness.
Case 2: $\nu > 0$. We now show that the solution u^ν to the Navier-Stokes equations belong to $C([0,T];V^s(\mathbb{R}^n)) \cap C^1([0,T];V^{s-2}(\mathbb{R}^n))$. Using the argument from case 1, it follows that u^ν has a strong right continuity at $t = 0$. However, for $\nu > 0$, the equation is time irreversible and hence we cannot prove left continuity using the previous section. The solution comes from the addition of the diffusion term in the Navier-Stokes equations. In short, the initial data in $V^s(\mathbb{R}^n)$ will yield a solition that is more regular than $V^s(\mathbb{R}^n)$. This can be seen in equation 7.8. Thus, for almost every $T \in [0,T]$, we have that $u()$ □

8. Local-in-Time Existence of 2-Dimensional Solutions with Locally Finite Energy Decomposition

We have previously mentioned the issues that arrive when considering the energy of a fluid in two dimensions. Mainly, the velocity fields do not decay fast enough at infinity. Thus, this section requires that the velocity fields have globally finite kinetic energy in $L^2(\mathbb{R}^n)$.

Recall from definition 3.13 that a fluid in two dimensions has a radial-energy decomposition of the form

$$u(x) = v(x) + w(x)$$

where

$$\int_{\mathbb{R}^2} |v(x)|^2 dx < \infty \qquad \nabla \cdot v = 0$$

and

$$w(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} |x|^{-2} \int_0^{|x|} s\overline{\omega}(s)ds.$$

where $\overline{\omega}$ is a smooth radially symmetric vorticity. For the Euler equations, $\overline{\omega} = \overline{\omega}_0(r)$. For the Navier-Stokes equations, $\overline{\omega}$ is an exact solution of the Navier-Stokes equations. We now summarize the theory of local-in-time well-posedness for two-dimensional solutions. As most of the proofs are the same, we will omit them and provide only the results. First, we must write the equations in light of the radial-energy decomposition of the velocity. The momentum equation becomes

$$\partial_t v + v \cdot \nabla v + w \cdot \nabla v + v \cdot \nabla w = -\nabla p + \nu \Delta v. \tag{8.1}$$

Therefore, the Navier-Stokes equations become

$$\begin{cases} \partial_t v + v \cdot \nabla v + w \cdot \nabla v + v \cdot \nabla w = -\nabla p + \nu \Delta v \\ \nabla \cdot v = 0 \\ v(x, 0) = v_0 \end{cases} \tag{8.2}$$

As before, we have to consider a mollification of the momentum equation. We write this as

$$\partial_t v^\epsilon + P[\mathcal{J}_\epsilon(\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla \mathcal{J}_\epsilon u^\epsilon) + \mathcal{J}_\epsilon(w \cdot \nabla \mathcal{J}_\epsilon u^\epsilon) + \mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla w)] = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon. \tag{8.3}$$

This equation will be crucial in obtaining the two forthcoming results.

Theorem 8.1. *Given an initial two dimensional velocity u_0 and a corresponding local finite-energy decomposition $v_0 + w_0$, $v_0 \in V^s(\mathbb{R}^2)$ where $s \in \mathbb{N}$ and w_0 is a smooth function of the form defined above, for any $\epsilon > 0$, there exists for all time a unique solution $v^\epsilon \in C^1([0, \infty]; V^s(\mathbb{R}^2))$ to the regularized equation above.*

Proposition 8.2. *Consider an initial two dimensional velocity that satisfies the conditions of theorem 8.1. Then, the following hold.*

(1) *For any $\epsilon > 0$, there exists a unique solution $u^\epsilon \in C^1([0, T_\epsilon]; V^s(\mathbb{R}^2))$ to the equation*

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} = F(u^\epsilon) \\ u^\epsilon|_{t=0} = u_0 \end{cases}$$

where $F(u^\epsilon) = \nu \mathcal{J}_\epsilon^2 \Delta u^\epsilon - P[\mathcal{J}_\epsilon(\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla \mathcal{J}_\epsilon u^\epsilon) + \mathcal{J}_\epsilon(w \cdot \nabla \mathcal{J}_\epsilon u^\epsilon) + \mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon) \cdot \nabla w)]$ and $T_\epsilon = T(\|v_0\|_{H^s(\mathbb{R}^n)}, \epsilon)$.

(2) *On any time interval $[0, T]$ on which the solution belongs to $C^1([0, T]; V^0(\mathbb{R}^2))$, we have the following inequality*

$$\sup_{0 \le t \le T} \|v^\epsilon(\cdot, t)\|_{L^2(\mathbb{R}^2)} \le \|v^\epsilon(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \exp\left(\int_0^T \|\nabla w\|_{L^\infty(\mathbb{R}^2)} dt \right). \tag{8.4}$$

The proofs to these the above theorem and proposition are so similar to the previous sections that we omit them here. We now provide the $H^s(\mathbb{R}^n)$ energy estimate for solutions to the regularized equation.

Proposition 8.3. ***(The** $H^s(\mathbb{R}^2)$**) Energy Estimate in Two Dimensions)*** *Let $v_0 \in V^s(\mathbb{R}^2)$ where $s \in \mathbb{N}$. The regularized solution $v^\epsilon \in C^1([0,T_\epsilon], V^s(\mathbb{R}^2))$ to the above equation satisfied*

$$\frac{d}{dt}\|v^\epsilon\|_{H^s(\mathbb{R}^2)} \leq C_s \left[\|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{L^\infty(\mathbb{R}^2)} + \|\nabla w\|_{L^\infty(\mathbb{R}^2)}\right] \|v^\epsilon\|_{H^s(\mathbb{R}^2)} \tag{8.5}$$

9. Global Solutions to the Euler and Navier-Stokes Equations

We have used energy methods in the previous section to demonstrate local-in-time well-posedness for the Euler and Navier-Stokes equations. It is natural to consider when a solution exists globally. It is not known if smooth solutions exist globally in general in time. However, there is a connection between the accumulation of vorticity and the global existence of solutions. Thus, to control the smoothness of the Euler or Navier-Stokes equations, it is enough to control the size of the vorticity. Specifically, we will show that independent of the size of vorticity, if for any $T > 0$ there exists $M > 0$ such that

$$\int_0^t \|\omega(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} ds \leq M,$$

for all $0 \leq t \leq T$, then the classical solution u exists globally in time. Before we show this, we prove a lemma from potential theory.

Lemma 9.1. *(**Potential Theory Estimate**) Let $u \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a smooth, divergence-free vector field and set $\omega = \nabla \times u$. Then,*

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq C\left(1 + \log\left(\|u\|_{H^3(\mathbb{R}^n)}\right) + \log\left(\|\omega\|_{L^2(\mathbb{R}^n)}\right)\right)\left(1 + \|\omega\|_{L^\infty(\mathbb{R}^n)}\right) \tag{9.1}$$

Proof. We begin with the relation between the velocity and the vorticity given by

$$u(x,t) = \int_{\mathbb{R}^n} K_N(x-y)\omega(y,t)dy \qquad x \in \mathbb{R}^n$$

where the Kernel is homogeneous of degree $1-n$ and given by

$$K_n(\lambda x) = \lambda^{1-n} K_n(x) \qquad \lambda > 0 \;\; 0 \neq x \in \mathbb{R}^n.$$

Also, the gradient of the velocity can be retrieved from the vorticity as

$$\nabla u(x) = C\omega(x) + P_3\omega(x)$$

where $P_3\omega(x)$ is defined by

$$P_3\omega(x) = p.v. \int_{\mathbb{R}^3} \nabla K_3(x-y)\omega(y)dy.$$

Consider the bump function $\rho(r)$ such that for $R_0 > 0$, we have $\rho(r) = 0$ for $r > 2R_0$, $\rho(r) = 1$ for $r < R_0$, and $\rho \geq 0$. Decompose $P_3\omega(x)$ into two parts:

$$P_3\omega(x) = p.v. \int_{\mathbb{R}^3} \nabla K_3(x-y)\rho(|x-y|)\omega(y)dy$$

$$
\begin{aligned}
&+ p.v. \int_{\mathbb{R}^3} \nabla K_3(x-y)[1-\rho(|x-y|)]\omega(y)dy \\
&= (\nabla u)_1(x) + (\nabla u)_2(x).
\end{aligned}
$$

Now, we use another potential theory estimate given by

$$
\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \le C\left(\|\omega\|_{C^\gamma} e^\gamma + \max\left(1, \log\left(\frac{R_0}{\epsilon}\right)\right)\|\omega\|_{L^\infty(\mathbb{R}^n)}\right).
$$

Applying this estimate to $(\nabla u)_1$ implies that, for any $\epsilon > 0$,

$$
\|(\nabla u)_1\|_{L^\infty(\mathbb{R}^n)} \le C\left(\|\omega\|_{C^\gamma} e^\gamma + \max\left(1, \log\left(\frac{R_0}{\epsilon}\right)\right)\|\omega\|_{L^\infty(\mathbb{R}^n)}\right)
$$

where $0 < \gamma < 1$ and $\|\cdot\|_{C^\gamma}$ denotes the Holder norm. The Sobolev Embedding Theorem yields $\|\omega\|_{C^\gamma} \le C\|\omega\|_{H^2(\mathbb{R}^n)}$ for all $\omega \in H^2(\mathbb{R}^3)$. It follows that

$$
\begin{aligned}
\|(\nabla u)_1\|_{L^\infty(\mathbb{R}^n)} &\le C\left(\|\omega\|_{H^2(\mathbb{R}^n)} e^\gamma + \max\left(1, \log\left(\frac{R_0}{\epsilon}\right)\right)\|\omega\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\le C\left(\|u\|_{H^3(\mathbb{R}^n)} e^\gamma + \max\left(1, \log\left(\frac{R_0}{\epsilon}\right)\right)\|\omega\|_{L^\infty(\mathbb{R}^n)}\right)
\end{aligned}
$$

The Cauchy-Schwartz inequality also provides

$$
\|(\nabla u)_2\|_{L^\infty(\mathbb{R}^n)} \le C R_0^{-n/2}\|\omega\|_{L^2(\mathbb{R}^n)}.
$$

If we take $0 < \epsilon < R_0$ as $\epsilon = 1$ if $\|u\|_{H^3(\mathbb{R}^n)} \le 1$ and $\|u\|_{H^3(\mathbb{R}^n)}^{-\gamma}$ otherwise. Also, set $R_0^{n/2} = \|\omega\|_{L^2(\mathbb{R}^n)}$. We then arrive at

$$
\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \le C\left(1 + \log\left(\|u\|_{H^3(\mathbb{R}^n)}\right) + \log\left(\|\omega\|_{L^2(\mathbb{R}^n)}\right)\right)\left(1 + \|\omega\|_{L^\infty(\mathbb{R}^n)}\right).
$$

□

Theorem 9.2. ***(The Beale-Kato-Majda Theorem)*** *Let the initial velocity $u_0 \in V^s(\mathbb{R}^n)$ where $s > n/2 + 2$, so that there exists a classical solution $u^\nu \in C^1([0,T]; C^2 \cap V^s)$ to the three dimensional Euler or Navier-Stokes equations. Then, (i) If for any $T > 0$, there exists $M > 0$ such that the vorticity satisfies*

$$
\int_0^T \|\omega(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} d\tau \le M, \tag{9.2}
$$

then the solution u exists globally in time.
(ii) If the maximal time T of existence of solutions $u \in C^1([0,T]; C^2 \cap V^s)$ is finite, then necessarily the vorticity accumulates so rapidly that

$$
\lim_{t\to T} \int_0^T \|\omega(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} d\tau = \infty. \tag{9.3}
$$

Proof. The $H^s(\mathbb{R}^n)$ energy estimate provides that

$$
\frac{d}{dt}\|u\|_{H^s(\mathbb{R}^n)} \le C(s)\|\nabla u\|_{L^\infty(\mathbb{R}^3)}\|u\|_{H^s(\mathbb{R}^n)}.
$$

Using Gronwall's lemma, we have that any solution $u \in C^1([0,T]; C^2 \cap V^s)$ satisfies

$$\|u(\cdot,T)\|_{H^s(\mathbb{R}^n)} \leq \|u_0\|_{H^s(\mathbb{R}^n)} \exp\left(\int_0^T C(s)\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} d\tau\right).$$

Hence, the solution exists in $V^s(\mathbb{R}^3)$ provides that we have an a priori bound on $\int_0^T C(s)\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} d\tau$. A similar bound holds for the $L^2(\mathbb{R}^n)$ norm of the vorticity:

$$\|\omega(\cdot,t)\|_{H^s(\mathbb{R}^n)} \leq \|\omega_0\|_{H^s(\mathbb{R}^n)} \exp\left(\int_0^T C(s)\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} d\tau\right).$$

Applying the previous potential theory lemma yields the result. □

Appendix A. The Lebesgue Spaces

We briefly review the *Lesbegue spaces*, also called the L^p spaces. We define these as the following.

Definition A.1. Suppose that (X,μ) is a measure space and $0 < p < \infty$. Define the space $L^p(X,\mu)$ as the collection of all measurable functions $v \in \mathcal{M}(X,\mu)$ such that $|v|^p \in L^1(X,\mu)$. The space $L^p(X,\mu)$ is a normed space with the norm

$$\|v\|_{L^p(X,\mu)} = \left(\int_X |v|^p d\mu\right)^{1/p}. \tag{A.1}$$

Next, we will provide the following important results that will be useful later on. To avoid clogging this section with proofs from real analysis, we point anyone to [5].

Proposition A.2. ***(Young's Inequality)*** *Suppose that $0 \leq a,b < \infty$ and $1 < p,q < \infty$ such that $1/p + 1/q = 1$. Then,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A.2}$$

Proposition A.3. ***(Holder's Inequality)*** *Let (X,μ) be a measure space and suppose that $1 < p,q < \infty$ such that $1/p + 1/q = 1$. Then, for $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$, we have $fg \in L^1(X,\mu)$ and*

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)}\|g\|_{L^q(X)}. \tag{A.3}$$

Proposition A.4. ***(Minkowski's Inequality)****: Let (X,μ) be a measure space and suppose that $1 \leq p \leq \infty$. Then, $L^p(X,\mu)$ is a vector space and $\|\cdot\|_{L^p(X)}$ gives a norm. That is, for $f,g \in L^p(X)$, we have that $f+g \in L^p(X)$ and*

$$\|f+g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}. \tag{A.4}$$

Theorem A.5. *Let (X,μ) be a measure space and suppose that $1 \leq p \leq \infty$. Then, $L^p(X,\mu)$ is a vector space and the norm $\|\cdot\|_{L^p(X)}$ is complete.*

Appendix B. Functional Analysis

Some of the forthcoming arguments require a bit of knowledge from functional analysis. We give a few definitions and important results now. First, we consider linear functionals and the idea of a weak topology. In this section, for a vector space V over some base space K, we assume that $K = \mathbb{R}$ or $K = \mathbb{C}$. A further review of this can be found in [5] and [7].

Definition B.1. Let V be a vector space over a base space K. A linear map from X to K is called a *linear functional* on V. If V is a normed vector space, then the space of bounded linear functionals on V is called the *dual space* on V. It is denoted as V^*.

Definition B.2. Let V be a normed vector space. A net $\{x_n\} \subset V$ is said to *converge weakly* to a vector x if and only if $\varphi(x_n) \to \varphi(x)$ for every $\varphi \in V^*$.

Proposition B.3. *Let V be a Banach space and $\{x_n\} \subset V$. If $\|x_n\|_V$ is bounded, then $\{x_n\}$ has a subsequence that weakly converges in the space V.*

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