

## Math 1301 Practice Final Exam Version 2

Vanderbilt University

8 December 2025

Name: Key

Please do not open the exam until instructed to do so.

You are allowed a one-page (double-sided) formula sheet.

Your instructor may ask to see your formula sheet.

No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

## Part 1. (20 points)

Q1  $\frac{d}{dx} [\log_9(5^{\sin(e^x)}) + \cos^{-1}(\ln(\sqrt[50]{x}))] =$

$$\log_9(5) \cos(e^x) e^x - \frac{1}{x \sqrt{1 - 1/50^2 \ln^2(x)}} \cdot \frac{1}{50}$$

$$\log_9(5^{\sin(e^x)}) + \cos^{-1}(\ln(\sqrt[50]{x}))$$

$$= \sin(e^x) \log_9(5) + \cos^{-1}\left(\frac{1}{50} \ln(x)\right)$$

Q2  $\frac{d}{dx} \ln^{5^x}(x) =$

$$\ln^{5^x}(x) \left[ 5^x \ln(5) \ln(\ln(x)) + \frac{5^x}{x \ln(x)} \right]$$

$$y = \ln^{5^x}(x)$$

$$\ln(y) = 5^x \ln(\ln(x))$$

$$\frac{1}{y} \frac{dy}{dx} = 5^x \ln(5) \ln(\ln(x)) + \frac{5^x}{x \ln(x)}$$

$$\frac{dy}{dx} = \ln^{5^x}(x) \left[ 5^x \ln(5) \ln(\ln(x)) + \frac{5^x}{x \ln(x)} \right]$$

Q3  $\lim_{x \rightarrow 0^+} \left( \frac{\sin(x)}{x} \right)^{1/x^2} =$

$$e^{-1/6}$$

$$\left( \frac{\sin(x)}{x} \right)^{1/x^2} = e^{\frac{1}{x^2} \ln(\sin(x)) - \frac{1}{x^2} \ln(x)}$$

$$\lim_{x \rightarrow 0^+} \frac{\ln(\sin(x)) - \ln(x)}{x^2} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\cot(x) - 1/x}{2x} = \lim_{x \rightarrow 0^+} \frac{x \cot(x) - 1}{2x^2}$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\cot(x) - x \csc^2(x)}{4x} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{1}{4} (-\csc^2(x) - \csc^2(x) + 2x \cot(x) \csc^2(x))$$

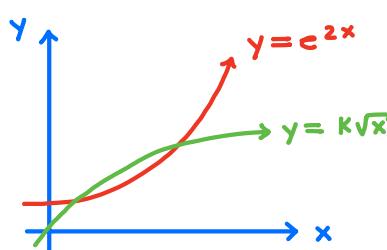
$$= \lim_{x \rightarrow 0^+} \frac{1}{4} (2x \frac{\cos(x)}{\sin(x)} - 2) \frac{1}{\sin^2(x)} = \lim_{x \rightarrow 0^+} \frac{2x \cos(x) - 2 \sin(x)}{4 \sin^3(x)} \stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{-2x \sin(x)}{12 \sin^2(x) \cos(x)}$$

$$= -\frac{1}{6} \lim_{x \rightarrow 0^+} \frac{x}{\sin(x) \cos(x)} \stackrel{L'H}{=} -\frac{1}{6} \lim_{x \rightarrow 0^+} \frac{1}{\cos^2(x) - \sin^2(x)} = -1/6. \text{ As } e^x \text{ is continuous,}$$

$$\lim_{x \rightarrow 0^+} \left( \frac{\sin(x)}{x} \right)^{1/x^2} = \frac{1}{e^{1/6}}$$

Q4 The value of  $k$  such that the equation  $e^{2x} = k\sqrt{x}$  has one solution is

$$k = 2\sqrt{e}$$



Key idea: One solution means derivatives are also equal!

$$e^{2x} = k\sqrt{x}$$

$$2e^{2x} = \frac{k}{2\sqrt{x}} \Rightarrow e^{2x} = 4x e^{2x} \Rightarrow x = 1/4$$

$$e^{1/2} = k \frac{1}{2} \Rightarrow k = 2\sqrt{e}$$

**Part 2.** (10 points) Consider the two power series given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \quad \text{and} \quad J_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$$

These are called Bessel Functions of the first kind.

**Q1** Determine the radii of convergence of  $J_0(x)$  and  $J_1(x)$ .

i)  $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n}$

Then,  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n!)^2 4^n}{(n+1)^2 (n!) 4^{n+1}} x^2 \right| = \lim_{n \rightarrow \infty} \frac{1}{4(n+1)^2} x^2 = 0 \quad \forall x \in \mathbb{R}.$   
 $ROC = \infty$ .

ii) Same idea for  $J_1(x)$  to see  $ROC = \infty$

**Q2** Show that  $y(x) = J_0(x)$  solves the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0.$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n} = y(x) \quad \text{Set } a_n = \frac{(-1)^n}{(n!)^2 4^n}$$

ii)  $x^2 y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n!)^2 4^n}$

iii)  $xy'(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(n!)^2 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n}}{(n!)^2 4^n}$

iv)  $x^2 y''(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(n!)^2 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n}}{(n!)^2 4^n}$

Summing (ii) and (iv) =  $\sum_{n=0}^{\infty} 4n^2 a_n x^{2n}$

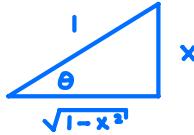
Summing the above result with (i) yields

$$\underbrace{\sum_{n=0}^{\infty} a_n x^{2n+2}}_{= \sum_{n=1}^{\infty} a_{n-1} x^{2n}} + \sum_{n=0}^{\infty} 4n^2 a_n x^{2n}$$

hence consider  $4n^2 a_n + a_{n-1}$   
 $= \frac{4n^2 (-1)^n}{(n!)^2 4^n} + \frac{(-1)^{n-1}}{((n-1)!)^2 4^{n-1}} = 0.$

**Part 3 (15 points)** Evaluate the following integrals.

Q1  $\int_0^1 \tan(\sin^{-1}(x)) dx =$



$$\tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$$

$$\int \tan(\sin^{-1}(x)) dx = \int \frac{x dx}{\sqrt{1-x^2}}$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$= -\frac{1}{2} \int u^{-1/2} du = -u^{1/2} = -\sqrt{1-x^2}$$

$$\int_0^1 \tan(\sin^{-1}(x)) dx = -\sqrt{1-x^2} \Big|_0^1 = -(0-1) = 1$$

Q2  $\int \sqrt{1-x^2} dx =$

$$\frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin(x)$$

$$\int \sqrt{1-x^2} dx$$

$$u = \sqrt{1-x^2} \quad dv = dx$$

$$du = \frac{-x}{\sqrt{1-x^2}} dx \quad v = x$$

$$= x \sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{x^2 - 1 + 1}{\sqrt{1-x^2}} dx$$

$$= - \int \frac{1-x^2}{\sqrt{1-x^2}} dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$= - \int \sqrt{1-x^2} dx + \arcsin(x)$$

$$\Rightarrow \underbrace{\int \sqrt{1-x^2} dx}_{\text{combine to } 2 \int \sqrt{1-x^2} dx} = - \int \sqrt{1-x^2} dx + x \sqrt{1-x^2} + \arcsin(x) \Rightarrow \int \sqrt{1-x^2} dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin(x)$$

Q3  $\int \frac{\cot(x)}{\ln(\sin(x))} dx =$

$$\ln(\ln(\sin(x)))$$

$$\int \frac{\cot(x)}{\ln(\sin(x))} dx = \int \frac{\cos(x)}{\sin(x) \ln(\sin(x))} dx$$

$$u = \sin(x)$$

$$du = \cos(x) dx$$

$$= \int \frac{du}{u \ln(u)}$$

$$s = \ln(u)$$

$$ds = \frac{du}{u}$$

$$= \int \frac{ds}{s} = \ln(s) = \ln(\ln(\sin(x)))$$

## Part 4. (15 points)

Q1 If  $f(x) = \sum_{n=1}^{\infty} a_n x^n$ , then  $\sum_{n=1}^{\infty} n^2 a_n x^n =$

$x f'(x) + x^2 f''(x)$

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n x^n, \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow x f'(x) = \sum_{n=1}^{\infty} n a_n x^n \\ \Rightarrow \frac{d}{dx}(x f'(x)) &= f'(x) + x f''(x) = \sum_{n=1}^{\infty} n^2 a_n x^{n-1} \\ \Rightarrow x f'(x) + x^2 f''(x) &= \sum_{n=1}^{\infty} n^2 a_n x^n \end{aligned}$$

Q2 The values of  $p$  and  $q$  such that  $\sum_{n=3}^{\infty} \frac{1}{n^p \ln^q(n)}$  converges are

$$\begin{cases} p=1 \\ \int_3^{\infty} \frac{dx}{x \ln^q(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u^q} \end{cases}$$

converges if and  
and only if  $q > 1$ .

$$p=1, q>1$$

$$\begin{cases} p>1 \\ n^p \ln^q(n) > n^p \\ \Rightarrow \frac{1}{n^p \ln^q(n)} < \frac{1}{n^p} \\ \downarrow \\ \text{converges via } p\text{-series} \\ \text{Direct comparison test} \\ \text{yields convergence for} \end{cases}$$

$$p>1, q \in \mathbb{R}$$

$$p=1, q>1$$

$$p>1, q \in \mathbb{R}$$

$$\begin{cases} p<1 \\ \int_3^{\infty} \frac{dx}{x^p \ln^2(x)} \\ u = \ln(x) \Rightarrow x = e^u \\ du = \frac{dx}{x} \\ = \int_{\ln(3)}^{\infty} \frac{e^{(1-p)u}}{u^p} du \rightarrow +\infty \quad \forall p \in (0,1), q \in \mathbb{R} \end{cases}$$

Q3 Evaluate the following integral:  $\int \frac{\ln(\ln(x))}{x} dx =$

$\ln(x) \ln(\ln(x)) - \ln(x)$

$$\int \frac{\ln(\ln(x))}{x} dx$$

$$t = \ln(x)$$

$$dt = \frac{dx}{x}$$

$$= \int \ln(t) dt$$

$$u = \ln(t) \quad dv = dt$$

$$du = \frac{dt}{t} \quad v = t$$

$$= t \ln(t) - \int dt$$

$$= t \ln(t) - t$$

$$= \ln(x) \ln(\ln(x)) - \ln(x)$$

**Part 5. (15 points)**

**Q1** Consider the parametric equations

$$x(t) = e^t \cos(t) \quad y(t) = e^t \sin(t)$$

Eliminate the parameter and find  $\frac{dy}{dx}$  in terms of  $t$ .

$$\text{i)} \quad x^2 + y^2 = e^{2t} \cos^2(t) + e^{2t} \sin^2(t)$$

$$= e^{2t}$$

$$\Rightarrow t = \frac{1}{2} \ln(x^2 + y^2)$$

$$\text{ii)} \quad \frac{x}{y} = \frac{e^t \cos(t)}{e^t \sin(t)} = \cot(t)$$

$$\Rightarrow t = \cot^{-1}(x/y)$$

$$\cot^{-1}(x/y) = \frac{1}{2} \ln(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t \sin(t) + e^t \cos(t)}{e^t \cos(t) - e^t \sin(t)}$$

$$= \frac{y+x}{x-y}$$

**Q2** Find the length of the polar curve of  $r = \frac{1}{\theta}$  from  $\cancel{\theta} \leq \theta \leq 2\pi$

$$r(\theta) = \frac{1}{\theta} \quad \Rightarrow \quad r^2 + (r'(\theta))^2 = \frac{1}{\theta^2} + \frac{1}{\theta^4}$$

$$r'(\theta) = \frac{-1}{\theta^2} \quad \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}} = \frac{\sqrt{\theta^2 + 1}}{\theta^2}$$

$$\int \sqrt{r^2 + (r'(\theta))^2} d\theta$$

$$= \int \frac{\sqrt{\theta^2 + 1}}{\theta^2} d\theta = -\frac{\sqrt{1+\theta^2}}{\theta} + \int \frac{d\theta}{\sqrt{1+\theta^2}}$$

$$u = \sqrt{1+\theta^2} \quad dv = \frac{d\theta}{\theta^2}$$

$$du = \frac{\theta d\theta}{\sqrt{1+\theta^2}} \quad v = -\theta^{-1}$$

$$= \ln(\theta + \sqrt{1+\theta^2}) - \frac{\sqrt{1+\theta^2}}{\theta}$$

$$\int \frac{d\theta}{\sqrt{1+\theta^2}} = \int \frac{d\theta}{\theta \sqrt{1/\theta^2 + 1}}$$

$$t^2 = 1 + \frac{1}{\theta^2} \Rightarrow \theta^2 = \frac{1}{t^2-1}$$

$$2t dt = -2 \frac{d\theta}{\theta^3} \Rightarrow \frac{d\theta}{\theta} = -t \theta^2 dt$$

Substituting yields

$$= \int \frac{dt}{1-t^2} = \int \frac{A}{1-t} dt + \int \frac{B}{1+t} dt$$

$$= \frac{-1}{2} \int \frac{dt}{1-t} + \frac{1}{2} \int \frac{dt}{1+t} = \frac{1}{2} \ln | \frac{1+t}{1-t} |$$

$$\frac{1+t}{1-t} \cdot \frac{(1+t)}{(1+t)} = \frac{(1+t)^2}{1-t^2} = -\theta^2(1+\theta^2)$$

$$= \ln(\theta(1+\theta)) = \ln(\theta + \theta\sqrt{1+\theta^2})$$

$$= \ln(\theta + \sqrt{1+\theta^2}).$$

**Q3** Eliminate the parameter of the parametric equations  $x = 2 \cos(\theta)$ ,  $y = 1 + \sin(\theta)$ . Find the points of the horizontal and vertical tangent lines.

$$x = 2 \cos(\theta) \Rightarrow \cos(\theta) = \frac{x}{2}$$

$$y = 1 + \sin(\theta) \Rightarrow \sin(\theta) = y - 1$$

$$\text{Hence, } \frac{x^2}{4} + y^2 - 2y + 1 = 1.$$

$$\frac{dx}{d\theta} = -2 \sin(\theta)$$

$$-2 \sin(\theta) = 0$$

$$\theta = 0, \pi$$

$$\frac{dy}{d\theta} = \cos(\theta)$$

$$\cos(\theta) = 0$$

$$\theta = \pi/2, 3\pi/2$$

**Part 6. (10 points)**

**Q1** The general solution to  $\frac{dy}{dx} = \frac{3^x y (\ln^2(y) + 3 \ln(y) + 2)}{1 + 9^x}$  is

$$\frac{\ln(y) + 2}{\ln(y) + 1} = e^{\tan^{-1}(3^x)} + C$$

$$\frac{dy}{dx} = \frac{3^x y (\ln^2(y) + 3 \ln(y) + 2)}{1 + 9^x}$$

$$\frac{1}{(u_1+2)(u_1+1)} = \frac{A}{u_1+2} + \frac{B}{u_1+1}$$

$$A = -1, B = 1$$

$$\int \frac{dy}{y(\ln^2(y) + 3 \ln(y) + 2)} = \int \frac{3^x}{1 + (3^x)^2} dx$$

$$u_1 = \ln(y) \quad u_2 = 3^x$$

$$du_1 = \frac{1}{y} dy \quad du = \ln(3) 3^x dx$$

$$\int \frac{du_1}{(u_1+1)(u_1+2)} = \frac{1}{\ln(3)} \int \frac{du_2}{1+(u_2)^2}$$

$$\int \frac{1}{u_1+2} - \frac{1}{u_1+1} du = \frac{1}{\ln(3)} \int \frac{du_2}{1+(u_2)^2}$$

$$\ln(u_1+2) - \ln(u_1+1) = \frac{1}{\ln(3)} \tan^{-1}(u_2) + C$$

$$\ln(\ln(y)+2) - \ln(\ln(y)+1) = \frac{1}{\ln(3)} \tan^{-1}(3^x) + C$$

**Q2** Use power series to evaluate:  $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{1 + x - e^x} =$

$$-1$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{1 + x - e^x} &= \lim_{x \rightarrow 0} \frac{1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{1 + x + \sum_{n=0}^{\infty} \frac{x^n}{n!}} = \lim_{x \rightarrow 0} \frac{1 - (1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots)}{1 + x - (1 + x + x^2/2 + \dots)} \\ &= \lim_{x \rightarrow 0} \frac{x^2/2 - x^4/4! + \dots}{-x^2/2 - x^3/3! - x^4/4! - \dots} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{x - x^3/3! + x^5/5! - \dots}{-x - x^2/2! - x^3/3! - \dots} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{1 - x^2/2! + x^4/4! - \dots}{-1 - x - x^2/2! - \dots} \\ &= -1 \end{aligned}$$

**Part 7 (20 points)** Determine if the following series converge or diverge. Show your work and state any necessary hypotheses.

$$Q1 \sum_{n=0}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

Method 1 (Direct Comparison Test)

$$\begin{aligned} n^2+1 &> n^2 & \text{As } \frac{1}{n^2} \text{ converges via the} \\ \Rightarrow \sqrt{n^2+1} &> n & p\text{-series test, the direct} \\ \Rightarrow n\sqrt{n^2+1} &> n^2 & \text{comparison test yields} \\ \Rightarrow \frac{1}{n\sqrt{n^2+1}} &< \frac{1}{n^2} & \frac{1}{n\sqrt{n^2+1}} \text{ converges, too.} \end{aligned}$$

Method 2 (Limit Comparison Test)

$$\begin{aligned} \text{Set } a_n &= \frac{1}{n\sqrt{n^2+1}} \text{ and } b_n = \frac{1}{n^2}. \text{ As} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= 1 > 0, \text{ the limit comparison test} \\ &\text{yields that } \sum a_n \text{ converges as } \sum b_n \text{ converges.} \end{aligned}$$

$$Q2 \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

$$\begin{aligned} \text{set } a_n &= \frac{(n!)^n}{n^{4n}}. \text{ Then, } \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^n (n!)^n}{(n+1)^{4n+4}} \cdot \frac{n^{4n}}{(n!)^n} \right| \\ &= \left| \frac{n^{4n}}{(n+1)^{3n+4}} \right| = \left| \frac{n^n \cdot n^{3n}}{(n+1)^4 (n+1)^{3n}} \right| = \underbrace{\frac{n^n}{(n+1)^4}}_{\rightarrow \infty} \underbrace{\left( \frac{n}{n+1} \right)^{3n}}_{\rightarrow 1/e^3} \end{aligned}$$

By the ratio test yields divergence

$$Q3 \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n} \text{ (Use the alternating series test)}$$

$$\begin{aligned} \text{Set } a_n &= \frac{n^2}{2^n} \text{ and consider } f(x) = \frac{x^2}{2^x}. \text{ Then, } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{2^x} \\ &= \lim_{x \rightarrow \infty} \frac{2x}{2\ln(2)2^x} = \lim_{x \rightarrow \infty} \frac{2}{2\ln^2(2)2^x} = 0. \text{ Hence, } \lim_{n \rightarrow \infty} a_n = 0. \text{ Thus, the Alternating} \\ &\text{Series Test yields convergence.} \end{aligned}$$

$$Q4 \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6}\right)$$

$\lim_{n \rightarrow \infty} \sin\left(\frac{n\pi}{6}\right) = \text{DNE}$ . By the  
divergence test  $\sum \sin\left(\frac{n\pi}{6}\right)$  diverges.

**Part 8 (20 points)** Define the function  $f(x) = \int_0^x \frac{1}{(t+1)\sqrt{1-\ln^2(t+1)}} dt$

Q1 The domain of  $f(x)$  is

$$\{ x \in \mathbb{R} \mid e^{-1}-1 \leq x \leq e-1 \}$$

$$f(x) = \int_0^x \frac{dt}{(t+1)\sqrt{1-\ln^2(t+1)}} = \int_0^x \frac{du}{u\sqrt{1-\ln^2(u)}} \quad \begin{matrix} u=t+1 \\ s=\ln(u) \end{matrix} = \int_0^x \frac{ds}{\sqrt{1-s^2}} = \sin^{-1}(\ln(x+1))$$

$$f(x) = \arcsin(\ln(x+1))$$

$$-1 \leq \ln(x+1) \leq 1$$

$$e^{-1}-1 \leq x \leq e-1$$

Q2  $f(x)$  has an inverse on

$$\{ x \in \mathbb{R} \mid e^{-1}-1 \leq x \leq e-1 \}$$

$$f'(x) = \frac{1}{(x+1)\sqrt{1-\ln^2(x+1)}} = \underbrace{\frac{1}{x+1}}_{<0} \cdot \underbrace{\frac{1}{\sqrt{1-\ln^2(x+1)}}}_{<0}$$

Q3  $f\left(\sum_{n=1}^{\infty} \frac{(3)^{n/2}}{n! 2^n}\right) = \boxed{\frac{\pi}{3}}$

$$\sum_{n=0}^{\infty} \frac{3^{n/2}}{n! 2^n} = \sum_{n=0}^{\infty} \frac{(\sqrt{3}/2)^n}{n!} = e^{\sqrt{3}/2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3^{n/2}}{n! 2^n} = e^{\sqrt{3}/2} - 1$$

$$f(e^{\sqrt{3}/2} - 1) = \arcsin(\ln(e^{\sqrt{3}/2} - 1 + 1)) \\ = \sin^{-1}(\sqrt{3}/2) = \pi/3$$

Q4  $(f^{-1})'(1/2) = \boxed{e^{\sin(\pi/2)} \cos(\pi/2)}$

$$(f^{-1})'(1/2) = \frac{1}{f'(f^{-1}(1/2))} = e^{\sin(\pi/2)} \sqrt{1-\sin^2(\pi/2)} = e^{\sin(\pi/2)} \cos(\pi/2)$$

$$\arcsin(\ln(x+1)) = \pi/2$$

$$\ln(x+1) = \sin(\pi/2)$$

$$x = e^{\sin(\pi/2)} - 1$$

**Part 9 (20 points)** Suppose that  $y = f(x)$  is a solution to the differential equation

$$\frac{dy}{dx} = yx \ln(x) + k \sin(x)$$

where  $k > 0$  and  $f(1) = 4$ .

**Q1** The second order Taylor polynomial of  $f(x)$  at  $x = 1$  is given by

$$f(x) \approx 4 + k \sin(1)(x-1) + (4 + k \cos(1))(x-1)^2$$

$$f(1) = 4$$

$$f'(x) = \frac{dy}{dx} = yx \ln(x) + k \sin(x)$$

$$f'(1) = (4)(1) \ln(1) + k \sin(1) = k \sin(1)$$

$$f''(x) = \frac{d^2y}{dx^2} = x \ln(x) + y(\ln(x) + 1) + k \cos(x)$$

$$f''(1) = (k \sin(1))(1) \ln(1) + 4(\ln(1) + 1) + k \cos(1) = 4 + k \cos(1)$$

$$f(x) \approx 4 + k \sin(1)(x-1) + (4 + k \cos(1))(x-1)^2$$

**Q2** If  $k = 0$ , then the solution to the initial value problem is given by

$$\text{If } k=0, \quad \frac{dy}{dx} = yx \ln(x) \quad c = \ln(4) + \frac{1}{4}$$

$$\frac{dy}{y} = x \ln(x) dx$$

$$\ln|y| = \frac{x^2}{2} (\ln(x) - \frac{1}{2}) + \ln(4) + \frac{1}{4}$$

$$\int \frac{dy}{y} = \int x \ln(x) dx$$

$$u = \ln(x) \quad dv = x dx$$

$$du = \frac{dx}{x} \quad v = \frac{x^2}{2}$$

$$\ln|y| = \frac{x^2}{2} \ln(x) - \frac{1}{2} \int x dx$$

$$y = \exp\left(\frac{x^2}{2}(\ln(x) - \frac{1}{2}) + \ln(4) + \frac{1}{4}\right)$$

$$\ln|y| = \frac{x^2}{2} (\ln(x) - \frac{1}{2}) + C$$

$$f(1) = 4$$

$$\ln(4) = \frac{1}{4}(-1) + C$$

**Q3** The domain of  $f(x)$  is

$$x > 0$$

**Part 10 (25 points)** Define the function  $g(x) = xe^x$ .

Q1 Calculate  $\int_0^1 g(x) dx$  in two ways: an integration technique and a power series technique.

$$\text{i)} \int_0^1 xe^x dx$$

$$u=x \quad dv=e^x dx$$

$$du=dx \quad v=e^x$$

$$= xe^x \Big|_0^1 - \int_0^1 e^x dx$$

$$= e - (e - 1) = 1$$

$$\text{ii)} xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int_0^1 xe^x dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{n+1}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+2)n!}$$

Q2  $y = g(x)$  satisfies the following differential equation

$$\frac{dy}{dx} = xe^x + e^x$$

$$\frac{dy}{dx} = y + \frac{y}{x} = \left(\frac{x+1}{x}\right)y$$

$$\frac{dy}{dx} = \left(\frac{x+1}{x}\right)y$$

Q3 The above differential equation is **SEPARABLE** or **NON-SEPARABLE**.

Q4 The domain of  $h(p) = \int_1^\infty \frac{g(x)}{e^{px}} dx$  is p > 1.

$$h(p) = \int_1^\infty \frac{g(x)}{e^{px}} dx = \int_1^\infty xe^{(1-p)x} dx = \lim_{c \rightarrow \infty} \underbrace{\frac{1}{p-1} xe^{(1-p)x} \Big|_1^c}_{\text{converges if and}} - \frac{1}{p-1} \int_1^\infty e^{(1-p)x} dx$$

$$u=x \quad dv=e^{(1-p)x} dx$$

$$du=dx \quad v=\frac{1}{1-p} e^{(1-p)x}$$

only if  $p > 1$ .

Q5  $g(x)$  has an inverse on the domain

$$(-\infty, -1) \cup (-1, \infty)$$

$$g'(x) = (x+1)e^x$$

$$g'(x) > 0 \quad \forall x > -1$$

As  $e^x > 0 \quad \forall x \in \mathbb{R}$ , the sign of  $x+1$  determines if  $g(x)$  has an inverse.

$$g'(x) < 0 \quad \forall x < -1$$

Q6  $\int_0^e g^{-1}(x) dx = \boxed{e-1}$ . (Hint: what is an inverse?)

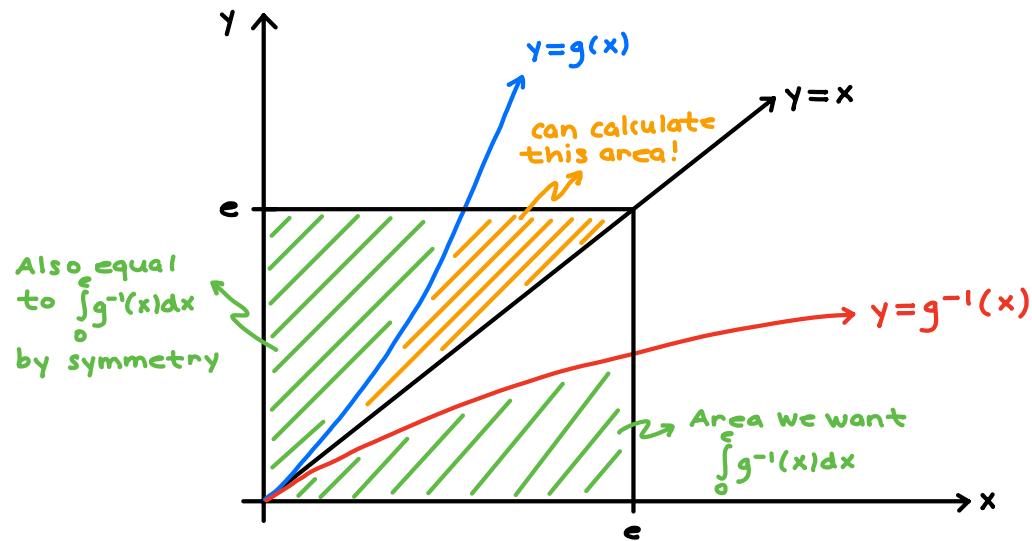
$$\text{triangle area} = \frac{1}{2} e^2$$

$$\text{area of orange region} = \int_0^1 g(x) - x dx + \int_1^e e - x dx$$

$$= 1 - \frac{1}{2} + e(e-1) - \frac{1}{2}(e^2-1)$$

$$= 1 + \frac{1}{2}e^2 - e$$

$$\Rightarrow \int_0^e g^{-1}(x) dx = \frac{1}{2}e^2 - (1 + \frac{1}{2}e^2 - e) = e - 1$$

**Part 11** Extra space for work

End of Exam. Check your work!