Math 1301 Practice Midterm 2

Vanderbilt University

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Name: Key

Please do not open the exam until instructed to do so.
You are allowed a one-page (double-sided) formula sheet.
Your instructor may ask to see your formula sheet.
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. (*25 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 $\int xe^x \, dx =$ $\boxed{xe^x - e^x}$

$u = x \qquad dv = e^x dx$

$du = dx \qquad V = e^x$

$= xe^x - \int e^x dx = xe^x - e^x$

Q2 $\int \cos(x) \ln(\cos(x)) \, dx =$ $\boxed{}$

$u = \ln(\cos(x)) \qquad dv = \cos(x) dx$

$du = \dfrac{-\sin(x)}{\cos(x)} \qquad V = \sin(x)$

$= \sin(x) \ln(\cos(x)) + \int \dfrac{\sin^2(x)}{\cos(x)} dx$

$\int \dfrac{\sin^2(x)}{\cos(x)} dx = \int \sec(x) - \cos(x) dx$

$= \ln|\sec(x) + \tan(x)| - \sin(x)$

$\int \cos(x) \ln(\cos(x)) dx$

$= \sin(x) \ln(\cos(x)) + \ln|\sec(x) + \tan(x)|$

$- \sin(x).$

Q3 $\int x(x-1)^{2025}\,dx =$ $\boxed{}$

$u = x-1, \quad x = u+1$

$du = dx$

$= \int (u+1)u^{2025}\,dx = \int u^{2026} + u^{2025}\,dx$

$= \frac{1}{2027}(x-1)^{2027} + \frac{1}{2026}(x-1)^{2026}$

Q4 $\int x^2 \tan^{-1}(x)\,dx =$ $\boxed{}$

$u = \tan^{-1}(x) \qquad dv = x^2 dx$

$du = \frac{dx}{1+x^2} \qquad v = \frac{x^3}{3}$

$= \frac{1}{3}x^3 \tan^{-1}(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

$\int \frac{x^3}{1+x^2}\,dx \; ; \quad s = x^2, \; ds = 2x\,dx$

$= \frac{1}{2}\int \frac{s}{1+s}\,ds = \frac{1}{2}\int 1 - \frac{1}{1+s}\,ds$

$= \frac{1}{2}s - \frac{1}{2}\ln|1+s| = \frac{1}{2}x^2 - \frac{1}{2}\ln|1+x^2|$

$\int x^2 \tan^{-1}(x)\,dx = \frac{1}{3}x^3 \tan^{-1}(x) - \frac{1}{6}x^2$

$- \frac{1}{6}\ln|1+x^2|$

Part 2. (*20 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 The implicit solution for the following ODE is given by

$$(1+x^2)\frac{dy}{dx} = (y^2 - 6y + 8)x^2$$

$$\frac{dy}{(y-2)(y-4)} = \frac{x^2}{1+x^2}dx$$

$$\left(\frac{1}{2}\frac{1}{y-4} - \frac{1}{2}\frac{1}{y-2}\right)dy = \left(1 - \frac{1}{1+x^2}\right)dx$$

$$\frac{1}{2}\ln|y-4| - \frac{1}{2}\ln|y-2| = x - \tan^{-1}(x)$$

Q2 The explicit solution for the following IVP is given by

$$\frac{dy}{dx} = (y-1)\ln(x) \qquad y(1) = 2$$

$$\frac{dy}{dx} = (y-1)\ln(x)$$

$$\frac{dy}{y-1} = \ln(x)dx$$

$$\ln|y-1| = x\ln(x) - x + c$$

$$\ln|2-1| = 1\ln|1| - 1 + c$$

$$c = 1$$

$$\ln|y-1| = x\ln(x) - x + 1$$

$$\int \ln(x)dx$$

$$u = \ln(x) \quad dv = dx$$

$$du = \frac{dx}{x} \quad v = x$$

$$= x\ln(x) - x$$

$$y = e^{x\ln(x) - x + 1} + 1.$$

Part 3. (*15 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 The domain for the function $f(p)$ is given by $\boxed{(1,\infty]}$ where

$$f(p) = \int_1^\infty \frac{\ln(x)}{x^p}\,dx$$

$$\int_1^\infty \frac{\ln(x)}{x^p}\,dx$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{p-1}} = \lim_{x\to\infty} \frac{1}{x^{p-1}}$$

$p>1$

$u=\ln(x) \quad dv = x^{-p}dx$

$du = \frac{dx}{x} \quad v = \frac{1}{1-p}x^{1-p}$

$$= \frac{1}{1-p}x^{1-p}\ln(x)\Big|_1^\infty - \underbrace{\int_1^\infty \frac{1}{1-p}x^{-p}dx}_{p>1}$$

Q2 (True/False): The integral $\int_1^\infty \frac{\sin(x)}{x}\,dx$ diverges. Justify your answer.

$$\int_1^\infty \frac{\sin(x)}{x}\,dx$$

$u=\frac{1}{x} \quad dv=\sin(x)dx$

$du = \frac{-dx}{x^2} \quad v=-\cos(x)$

$$= -\frac{\cos(x)}{x}\Big|_1^\infty - \int_1^\infty \frac{\cos(x)}{x^2}\,dx$$

$\downarrow$

Squeeze Thm $\frac{-1}{x^2} \le \frac{\cos(x)}{x^2} \le \frac{1}{x^2}$; $\pm\int_1^\infty \frac{dx}{x^2}$ conv.

$\Rightarrow \int_1^\infty \frac{\cos(x)}{x^2}\,dx$ converges, too.

Part 4. (*25 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 Construct a sequence $\{a_n\}$ such that the following properties hold:

- $\{a_n\}$ is monotonically increasing for $n = 1, 2, 3$
- $\{a_n\}$ converges to e
- $\{a_n\}$ is greater than e for some $n > 3$. (for example, $a_6 = 7$)
- $\{a_n\}$ is less than e for some $n > 3$. (for example, $a_{12} = 1$)

$$a_n = \begin{cases} n^2 & ; n = 1, 2, 3 \\ e + \dfrac{(-1)^n}{n} & ; n > 3 \end{cases}$$

Q2 Find the limits of the following sequences or state the limit does not exist:

(a) $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = \boxed{0}$

$$a_n = \frac{(-1)^n}{n^2} \Rightarrow \frac{-1}{n^2} \leq a_n \leq \frac{1}{n^2}$$

as $\frac{-1}{n^2}, \frac{1}{n^2} \to 0$ as $n \to \infty$, we

have $a_n \to 0$ as $n \to \infty$.

(b) $\lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}} = \boxed{0}$

consider $\lim_{x\to\infty} \frac{\ln(x)}{x^\epsilon} \xrightarrow{L'H} \lim_{x\to\infty} \frac{1}{\epsilon x^\epsilon} \to 0 \ \forall \epsilon > 0$

$\Rightarrow \ln(x) \lesssim x^\epsilon \ \forall \epsilon > 0$. Set $a_n = \frac{\ln(n)}{n^{1/2}}$.

From $\ln(x) \lesssim x^\epsilon$, $a_n \lesssim \frac{n^\epsilon}{n^{1/2}}$. Set $\epsilon = 1/3$

$\Rightarrow a_n \lesssim \frac{n^{1/3}}{n^{1/2}} \to 0$.

(c) $\lim_{n\to\infty} \sqrt[n]{2} = \boxed{1}$

Let $a_n = \sqrt[n]{2}$. Then, $\ln(a_n) = \frac{1}{n}\ln(2)$. So,

$\lim_{n\to\infty} \ln(a_n) = \ln(2) \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore,

$\lim_{n\to\infty} a_n = e^{\ln(\lim_{n\to\infty} a_n)} = e^0 = 1$

(d) $\lim_{n\to\infty} \frac{(2n)!}{(2n+1)!} = \boxed{0}$

$a_n = \frac{(2n)!}{(2n+1)!} = \frac{(2n)!}{(2n+1)\cdot(2n)!} = \frac{1}{2n+1}$

$\Rightarrow \lim_{n\to\infty} a_n = 0$.

Part 5. (*15 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 The partial fraction decompsition form for $\dfrac{x^2-304x-5052}{(x+\pi)^3(x^2-3)(x+2)}$ is

Q2 The implicit solution for the following ODE is given by

$$2^{-x}\frac{dy}{dx} = yx + y$$

$2^{-x}\dfrac{dy}{dx} = yx+y$

$2^{-x}\dfrac{dy}{dx} = y(x+1)$

$\dfrac{dy}{y} = (2^x x + 2^x)\,dx$

$c + \ln|y| = \dfrac{x2^x}{\ln(2)} - \dfrac{1}{\ln^2(2)}2^x + \dfrac{2^x}{\ln(2)}$

$\int x2^x\,dx$

$u = x \qquad dv = 2^x dx$

$du = dx \qquad v = \dfrac{2^x}{\ln(2)}$

$= \dfrac{x2^x}{\ln(2)} - \dfrac{1}{\ln(2)}\int 2^x dx$

$= \dfrac{x2^x}{\ln(2)} - \dfrac{1}{\ln^2(2)}2^x$

End of Exam. Check your work!