

# Math 1301 Practice Final Exam Version 1

Vanderbilt University

8 December 2025

Name: Key

Please do not open the exam until instructed to do so.  
You are allowed a one-page (double-sided) formula sheet.  
Your instructor may ask to see your formula sheet.  
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

## Part 1. (20 points)

Q1  $\frac{d}{dx} [5^{\sqrt{x}} + \log_4(x^x) + \sin^{-1}(4x)] =$   $5^{\sqrt{x}} \frac{\ln(5)}{2\sqrt{x}} + \log_4(x) + \frac{1}{\ln(4)} + \frac{4}{\sqrt{1-16x^2}}$

$$\frac{d}{dx} 5^{\sqrt{x}} = 5^{\sqrt{x}} \ln(5) \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} \log_4(x^x) = \frac{d}{dx} (x \log_4 x) = \log_4 x + \frac{x}{\ln(4)x}$$

$$\frac{d}{dx} \sin^{-1}(4x) = \frac{1}{\sqrt{1-16x^2}} \cdot 4$$

Q2  $\frac{d}{dx} \sec(x)^{\cos^{-1}(x)} =$   $\sec(x)^{\cos^{-1}(x)} \left[ \cos^{-1}(x) \tan(x) - \frac{\ln(\sec(x))}{\sqrt{1-x^2}} \right]$

$$y = \sec(x)^{\cos^{-1}(x)}$$

$$\ln(y) = \cos^{-1}(x) \ln(\sec(x))$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{-\ln(\sec(x))}{\sqrt{1-x^2}} + \frac{\cos^{-1}(x) \sec(x) \tan(x)}{\sec(x)}$$

Q3  $\lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\tan^{-1}(x)} \right) =$  0

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\tan^{-1}(x) - x}{x \tan^{-1}(x)} &\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x^2} - 1}{\tan^{-1}(x) + \frac{x}{1+x^2}} = \lim_{x \rightarrow 0^+} \frac{1 - 1 - x^2}{(1+x^2)\tan^{-1}(x) + x} = \lim_{x \rightarrow 0^+} \frac{-x^2}{(1+x^2)\tan^{-1}(x) + x} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{-2x}{1 + 2x\tan^{-1}(x) + 1} = \frac{0}{2} = 0 \end{aligned}$$

Q4 The point of horizontal tangents for the curve  $y = \ln^2(x+4)$  is

$x = -3$

$$y = \ln^2(x+4)$$

horizontal tangent

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2 \ln(x+4)}{x+4}$$

$$\Leftrightarrow \ln(x+4) = 0$$

$$x = -3$$

**radius of convergence**

**Part 2.** (10 points) Find the ~~interval of convergence~~ for the following power series.

$$p(x) = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x+1)^n$$

$$\text{Let } a_n = \frac{(n!)^2}{(2n)!} (x+1)^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} |x+1|$$

$$= \frac{(n+1)^2}{4n^2 + 6n + 2} |x+1|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4} |x+1| < 1$$

$$\Rightarrow |x+1| < 4$$

$$-5 < x < 3$$

**Part 3** (15 points) Evaluate the following integrals.

$$\text{Q1 } \int \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \sin(e^x) (\ln(e^e))^x dx = \boxed{-e^x \cos(e^x) + \sin(e^x)}.$$

$$\begin{aligned} \int \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \sin(e^x) (\ln(e^e))^x dx &= -t \cos(t) + \sin(t) \\ &= -e^x \cos(e^x) + \sin(e^x). \\ &= \int e^x \sin(e^x) e^x dx \\ t &= e^x \quad dt = e^x dx \\ &= \int t \sin(t) dt = -t \cos(t) + \int \cos(t) dt \\ u &= t \quad dv = \sin(t) dt \\ du &= dt \quad v = -\cos(t) \end{aligned}$$

$$\text{Q2 } \int \ln(x^2 + x + 1) dx = \boxed{\left(x + \frac{1}{2}\right) \ln(x^2 + x + 1) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - 2x}$$

$$\begin{aligned} \int \ln(x^2 + x + 1) dx &= \int \frac{2x^2 + x}{x^2 + x + 1} dx = \int \frac{2x^2 + x + x + 2 - x - 2}{x^2 + x + 1} dx \\ &= \int \frac{2(x^2 + x + 1)}{x^2 + x + 1} dx - \frac{1}{2} \int \frac{2x + 1}{x^2 + x + 1} dx - \frac{3}{2} \int \frac{dx}{x^2 + x + 1} \\ &= 2x - \frac{1}{2} \ln(x^2 + x + 1) - \frac{3}{2} \int \frac{dx}{x^2 + x + 1} \\ x^2 + x + 1 &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \rightarrow \frac{1}{x^2 + x + 1} = \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ \frac{3}{2} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} &= \frac{3}{2} \int \frac{du}{u^2 + \frac{3}{4}} = 2 \int \frac{du}{\left(\frac{2}{\sqrt{3}}u\right)^2 + 1} \\ &= \frac{2}{\sqrt{3}} \int \frac{dt}{t^2 + 1} = \sqrt{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \end{aligned}$$

$$\text{Q3 } \int \tan(x) \ln(\cos(x)) dx = \boxed{-\frac{1}{2} \ln(\cos(x))}$$

$$\begin{aligned} \int \tan(x) \ln(\cos(x)) dx &= - \int s ds \\ &= - \frac{s^2}{2} \\ &= - \frac{1}{2} \ln(\cos(x)) \\ u &= \cos(x), \quad du = -\sin(x) dx \\ &= - \int \frac{\ln(u)}{u} du \\ s &= \ln(u) \\ ds &= \frac{du}{u} \end{aligned}$$

## Part 4. (15 points)

Q1 The value of  $\sum_{n=1}^{\infty} \frac{\sin^n(x) \overbrace{\sin(\pi n)}^{\cos(\pi n)}}{3^n}$  is

$$\frac{-\sin(x)/3}{1 + \sin(x)/3}$$

$$\sum_{n=1}^{\infty} \frac{\sin^2(x) \cos(\pi n)}{3^n} = \sum_{n=1}^{\infty} \left( \frac{-\sin(x)}{3} \right)^n = \frac{-\sin(x)/3}{1 + \sin(x)/3}$$

Q2 The values of  $p$  such that  $\sum_{n=3}^{\infty} \frac{1}{n \ln(n) [\ln(\ln(n))]^p}$  converges are

$$p > 1$$

$$\text{Let } f(x) = \frac{1}{x \ln(x) (\ln(\ln(x)))^p}$$

1)  $f(x)$  continuous

2)  $f(x)$  positive

3)  $f'(x) < 0$

$$\int_3^{\infty} f(x) dx = \int_3^{\infty} \frac{dx}{x \ln(x) (\ln(\ln(x)))^p}$$

$$u = \ln(x) \quad s = \ln(u) \\ = \int_{\ln(3)}^{\infty} \frac{du}{u \ln^p(u)} = \int_{\ln(\ln(3))}^{\infty} \frac{ds}{s^p}$$

converges if  $p > 1$

Q3 The limit of the sequence defined by  $a_n = \begin{cases} 2 & n = 1 \\ \frac{1}{3-a_n} & n > 1 \end{cases}$  is

**Part 5.** (15 points)

- Q1** Find a parametric equation for the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and find the slope of the tangent line at an arbitrary point using the parametric equations.

$$x = a \cos(t) \quad 0 \leq t \leq 2\pi$$

$$y = b \sin(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b \cos(t)}{-a \sin(t)} = -\frac{b}{a} \cot(t)$$

- Q2** Find the length of the polar curve of  $r = e^{\theta/2}$  from  $0 \leq \theta \leq \pi/2$

$$r = e^{\theta/2} \quad r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\frac{dr}{d\theta} = \frac{e^{\theta/2}}{2} = e^{\theta} + \frac{1}{4} e^{\theta} = \frac{5}{4} e^{\theta}$$

$$L = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_0^{\pi/2} \frac{\sqrt{5}}{2} e^{\theta/2} d\theta = \sqrt{5} e^{\theta/2} \Big|_0^{\pi/2} = \sqrt{5} (e^{\pi/4} - 1)$$

- Q3** Find the values of theta that the polar curve  $r = 1 + \cos(\theta)$  has vertical and horizontal tangent lines.

$$r = 1 + \cos(\theta)$$

$$\frac{dr}{d\theta} = -\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}$$

$$\frac{dy}{d\theta} = -\sin^2(\theta) + \cos^2(\theta) + \cos(\theta)$$

$$= -(1 - \cos^2(\theta)) + \cos(\theta)$$

$$= 2\cos^2(\theta) + \cos(\theta) - 1$$

$$= (2\cos(\theta) - 1)(\cos(\theta) + 1)$$

$$\frac{dx}{d\theta} = -\sin(\theta) \cos(\theta) - \sin(\theta) - \sin(\theta) \cos(\theta)$$

$$= -2\sin(\theta) \cos(\theta) - \sin(\theta)$$

$$= -\sin(\theta) (2\cos(\theta) + 1)$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \begin{aligned} 2\cos(\theta) - 1 &= 0 & \cos(\theta) + 1 &= 0 \\ \cos(\theta) &= 1/2 & \cos(\theta) &= -1 \\ \theta &= \pi/3, 5\pi/3 & \theta &= \pi \end{aligned}$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \begin{aligned} \sin(\theta) &= 0 & 2\cos(\theta) + 1 &= 0 \\ \theta &= 0, \pi & \cos(\theta) &= -1/2 \\ & & \theta &= 2\pi/3, 4\pi/3 \end{aligned}$$

$$\pi \text{ is solution to } \frac{dx}{d\theta} = 0 \text{ and}$$

$$\frac{dy}{d\theta} = 0, \text{ so have to check}$$

$$\lim_{\theta \rightarrow \pi} \frac{dy}{dx}.$$

## Part 6. (10 points)

Q1 The general solution to  $\cos(y) \frac{dy}{dx} = x e^{x^2 + \ln(1 + \sin^2(y))}$  is

$$\tan^{-1}(\sin(y)) = \frac{1}{2} e^{x^2} + C$$

$$\cos(y) \frac{dy}{dx} = x e^{x^2 + \ln(1 + \sin^2(y))}$$

$$\cos(y) \frac{dy}{dx} = x e^{x^2} e^{\ln(1 + \sin^2(y))}$$

$$\frac{\cos(y)}{1 + \sin^2(y)} dy = x e^{x^2} dx$$

$$\int \frac{\cos(y)}{1 + \sin^2(y)} dy = \int x e^{x^2} dx$$

$$u_1 = \sin(y) \quad u_2 = x^2$$

$$du_1 = \cos(y) dy \quad du_2 = 2x dx$$

$$\int \frac{du_1}{1 + u_1^2} = \frac{1}{2} \int e^{u_2} du_2$$

$$\tan^{-1}(u_1) = \frac{1}{2} e^{u_2} + C$$

$$\tan^{-1}(\sin(y)) = \frac{1}{2} e^{x^2} + C.$$

Q2 The integral  $\int_1^{\infty} \frac{\cos(x)}{x} dx$  **CONVERGES** or **DIVERGES**. (explain your response)

$$\int_1^{\infty} \frac{\cos(x)}{x} dx$$

$$u = \frac{1}{x} \quad dv = \cos(x) dx$$

$$du = -\frac{dx}{x^2} \quad v = \sin(x)$$

$$= \lim_{c \rightarrow \infty} \underbrace{\frac{\sin(x)}{x} \Big|_1^c}_{\frac{\sin(c)}{c} - \sin(1)} + \underbrace{\int_1^{\infty} \frac{\sin(x)}{x^2} dx}_{\int_1^{\infty} \frac{dx}{x^2}}$$

$$= \lim_{c \rightarrow \infty} \frac{\sin(c)}{c} - \sin(1) \leq \int_1^{\infty} \frac{dx}{x^2} \rightarrow \text{converges via p-integral test.}$$

$$= -\sin(1)$$

**Part 7 (20 points)** Determine if the following series converge or diverge. Show your work and state any necessary hypotheses.

$$\text{Q1 } \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$$

$$\text{Set } a_n = (-1)^n \frac{\pi^{2n}}{(2n)!}$$

$\sum a_n$  converges via the ratio test

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\pi^{2n+2}}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{\pi^{2n}} \right|$$

$$= \frac{\pi^2}{(2n+2)(2n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\text{Q2 } \sum_{n=1}^{\infty} \frac{1}{2 + \sin(n)}$$

$$\text{Set } b_n = \frac{1}{2 + \sin(n)}. \text{ As } \frac{1}{3} \leq b_n \leq 1$$

$\lim_{n \rightarrow \infty} b_n \neq 0$ . Hence,  $\sum b_n$  diverges via the divergence test.

$$\text{Q3 } \sum_{n=1}^{\infty} \left(1 + \frac{1}{\pi n}\right)^{n^2}$$

Set  $c_n = \left(1 + \frac{1}{\pi n}\right)^{n^2}$ . Then,  $\sqrt[n]{|c_n|} = \left(1 + \frac{1}{\pi n}\right)^n$  and  $\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = e^{1/\pi}$ . As  $e^{1/\pi} > 1$ , the root test yields divergence.

$$\text{Q4 } \sum_{n=1}^{\infty} \frac{n}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$$

$$\text{We have } n^3 < n^3 + \cos^2(n)n^2 + 3^n n + 4$$

$$\Rightarrow \frac{1}{n^3} > \frac{1}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$$

$$\Rightarrow \frac{1}{n^2} > \frac{n}{n^3 + \cos^2(n)n^2 + 3^n n + 4}$$

As  $\sum \frac{1}{n^2}$  converges via p-series test ( $p=2$ ), the direct comparison test yields convergence.



**Part 8** (20 points) Define the function  $f(x) = \arctan(\ln(x+1)) - \arccos(\ln(\sqrt[4]{x^2+2x+1}))$

Q1 The domain of  $f(x)$  is

$$e^{-2}-1 \leq x \leq e^2-1$$

$$f(x) = \underbrace{\tan^{-1}(\ln(x+1))}_{\substack{x+1 > 0 \\ x > -1}} - \underbrace{\cos^{-1}\left(\frac{1}{2} \ln(x+1)\right)}_{\substack{-1 \leq \frac{1}{2} \ln(x+1) \leq 1 \\ -2 \leq \ln(x+1) \leq 2 \\ e^{-2}-1 \leq x \leq e^2-1}}$$

Q2  $f'(x) =$

$$\frac{1}{1+(\ln(x+1))^2} \cdot \frac{1}{1+x} + \frac{1}{\sqrt{1-\frac{1}{4}\ln^2(x+1)}} \cdot \frac{1}{2} \cdot \frac{1}{1+x}$$

Q3  $f(x)$  has an inverse on

$$e^{-2}-1 \leq x \leq e^2-1$$

(justify your answer)!

$$f'(x) = \underbrace{\frac{1}{1+x}}_{<0} \left[ \underbrace{\frac{1}{1+\ln^2(x+1)}}_{<0} + \underbrace{\frac{1/2}{\sqrt{1-\frac{1}{4}\ln^2(x+1)}}}_{<0} \right]$$

As  $f'(x) > 0$  for all  $x$  in the domain of  $f(x)$ ,  
 $f$  has an inverse on its entire domain.

Q4  $f(e^{\sqrt{3}}-1) =$

$$\pi/6$$

$$\begin{aligned} f(e^{\sqrt{3}}-1) &= \tan^{-1}(\ln(e^{\sqrt{3}}-1+1)) - \cos^{-1}\left(\frac{1}{2} \ln(e^{\sqrt{3}}-1+1)\right) \\ &= \tan^{-1}(\sqrt{3}) - \cos^{-1}(\sqrt{3}/2) \\ &= \frac{\pi}{3} - \frac{\pi}{6} = \pi/6 \end{aligned}$$

**Part 9** (20 points) Define the function  $g(x) = a^x$ .

**Q1** Use logarithmic differentiation to prove  $g'(x)$ .

$$\begin{aligned}
 y &= a^x & \frac{dy}{dx} &= y \ln(a) \\
 \ln(y) &= x \ln(a) & \frac{dy}{dx} &= \ln(a) a^x \\
 \frac{d}{dx} \ln(y) &= \frac{d}{dx} x \ln(a) \\
 \frac{1}{y} \frac{dy}{dx} &= \ln(a)
 \end{aligned}$$

**Q2** Find the first four terms of the Taylor series of  $g(x)$  centered at  $x = 3$ .

$$\begin{aligned}
 g(x) &= a^x, \quad g(3) = a^3 \\
 g'(x) &= \ln(a) a^x, \quad g'(3) = \ln(a) a^3 \\
 g''(x) &= \ln^2(a) a^x, \quad g''(3) = \ln^2(a) a^3 \\
 g'''(x) &= \ln^3(a) a^x, \quad g'''(3) = \ln^3(a) a^3
 \end{aligned}$$

$$g(x) \approx a^3 + \ln(a) a^3 (x-3) + \frac{\ln^2(a) a^3}{2} (x-3)^2 + \frac{\ln^3(a) a^3}{6} (x-3)^3$$

**Q3** Using **Q1**, write down a differential equation that  $g(x)$  satisfies. Is the differential equation separable?

$$\frac{dy}{dx} = \ln(a) y$$

Yes, this ODE is separable.

**Q4** The values of  $a$  such that  $\int_0^\infty g(x) dx$  converges are

$$0 < a < 1$$

$$\begin{aligned}
 \int_0^\infty g(x) dx &= \lim_{c \rightarrow \infty} \int_0^c a^x dx \\
 &= \lim_{c \rightarrow \infty} \frac{1}{\ln(a)} a^x \Big|_0^c = \frac{1}{\ln(a)} \underbrace{\lim_{c \rightarrow \infty} a^c - 1}_{\text{converges if and only if } 0 < a < 1}
 \end{aligned}$$

**Part 10** (25 points) Consider the power series  $h(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$ .

**Q1** The radius of convergence of  $h(x)$  is

$$\text{ROC} = 1$$

$$\text{Set } a_n = (-1)^{n+1} \frac{x^{2n}}{n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} |x|^2 \rightarrow |x|^2 \text{ as } n \rightarrow \infty$$

**Q2** The power series  $h(x)$  is given by the function

$$\ln(1+x^2)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

**Q3** The 41 partial sum of  $h(1/2)$  is an **OVERESTIMATE** or **UNDERESTIMATE** of  $h(1/2)$ .

$$S_{41} = \sum_{n=1}^{41} (-1)^{n+1} \frac{(1/2)^{2n}}{n}$$

$$\text{The 41st term is } \frac{(1/2)^{82}}{41} > 0 \rightarrow \text{overestimate}$$

**Q4** The number of terms required for to estimate  $g(0.1)$  with an error less than  $\frac{1}{100}$  is

$$2$$

$$|a_1| = \frac{(0.1)^2}{1} = \frac{1}{100}$$

$$|a_2| = \frac{(0.1)^4}{2} = \frac{1}{2 \cdot 10^4} < \frac{1}{100}$$

**Q5** Using **Q1**,  $\int_0^1 h(x) dx =$

$$\ln(2) - 2 + \pi/2$$

$$\int_0^1 \ln(1+x^2) dx$$

$$u = \ln(1+x^2) \quad dv = dx$$

$$du = \frac{2x}{1+x^2} dx \quad v = x$$

$$= \underbrace{x \ln(1+x^2) \Big|_0^1}_{= \ln(2)} - 2 \int_0^1 \frac{x^2}{1+x^2} dx$$

$$\int_0^1 \frac{x^2}{1+x^2} dx = \int_0^1 1 - \frac{1}{1+x^2} dx$$

$$= x - \tan^{-1}(x) \Big|_0^1 = 1 - \tan^{-1}(1)$$

$$= 1 - \pi/4$$

**Part 11** *Extra space for work*

*End of Exam. Check your work!*