Stats GR5703 HW1 Ex3

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Exercise 3

Problem 1

We know that for distribution that's symmertric about 0, and for which odd-order moment actually exists, the odd-order moments are equal to 0. In our case, since $R_1, ..., R_n \sim N(\mu, \sigma^2)$, $R_1 - \mu \sim N(0, \sigma^2)$, then:

$$\begin{aligned}
\gamma = E[R_1^3] &= E[(R_1 - \mu + \mu)^3] \\
&= E[(R_1 - \mu)^3 + 3(R_1 - \mu)^2\mu + 3(R_1 - \mu)\mu^2 + \mu^3] \\
&= E[3(R_1 - \mu)^2\mu] + E[\mu^3] \text{ since the odd-order moments are 0, and } \mu \text{ is a constant} \\
&= 3\mu E[(R_1 - \mu)^2] + \mu^3 \\
&= 3\mu(var(R_1 - \mu) + E[R_1 - \mu]^2) + \mu^3 \\
&= 3\mu\sigma^2 + \mu^3
\end{aligned}$$

Problem 2

(a) Derive the bias of $\hat{\gamma}$
We have $\bar{R_i} = \frac{1}{n}\sum_{i=1}^{n} R_i \sim N(\mu, \frac{\sigma^2}{n})$ and $\bar{R_i} - \mu \sim N(0, \frac{\sigma^2}{n})$, then

$$\begin{aligned}
E[\hat{\gamma}] = E[\bar{R_i}^3] &= E[(\bar{R_i} - \mu + \mu)^3] \\
&= E[(\bar{R_i} - \mu)^3 + 3(\bar{R_i} - \mu)^2\mu + 3(\bar{R_i} - \mu)\mu^2 + \mu^3] \\
&= 3\mu E[(\bar{R_i} - \mu)^2] + \mu^3 \\
&= 3\mu(var(\bar{R_i} - \mu) + E[\bar{R_i} - \mu]^2) + \mu^3 \\
&= 3\mu\frac{\sigma^2}{n} + \mu^3
\end{aligned}$$

$bias(\hat{\gamma}) = E[\hat{\gamma}] - \gamma = 3\mu\frac{\sigma^2}{n} + \mu^3 - (3\mu\sigma^2 + \mu^3) = 3\mu\frac{1-n}{n}\sigma^2$

(b) Is $\hat{\gamma}$ consistent? Support your answer with a mathematical argument.
For an estimator to be consistent, we want to show it converges in probability to the true parameter as n goes to infinity. In our case, as n goest to infinity, $E[\hat{\gamma}] = 3\mu\frac{\sigma^2}{n} + \mu^3 \to \mu^3$, which does not equal $3\mu\sigma^2 + \mu^3$, so $\hat{\gamma}$ is not consistent. Also, we have

$$\begin{aligned}
P(|\hat{\theta} - \theta| > \epsilon) &= P(|\bar{R_i}^3 - (3\mu\sigma^2 + \mu^3)| > \epsilon) \\
&= 1 - P(|\bar{R_i}^3 - (3\mu\sigma^2 + \mu^3)| \leq \epsilon) = 1 - P(\bar{R_i}^3 - (3\mu\sigma^2 + \mu^3) \leq \epsilon) - (1 - P(\bar{R_i}^3 - (3\mu\sigma^2 + \mu^3) \leq -\epsilon)) \\
&= P(\bar{R_i}^3 \leq 3\mu\sigma^2 + \mu^3 + \epsilon) + P(\bar{R_i}^3 \leq 3\mu\sigma^2 + \mu^3 - \epsilon) \\
&= P(Z \leq \sqrt{n}\frac{(3\mu\sigma^2 + \mu^3 + \epsilon)^{\frac{1}{3}} - \mu}{\sigma}) + P(Z \leq \sqrt{n}\frac{(3\mu\sigma^2 + \mu^3 - \epsilon)^{\frac{1}{3}} - \mu}{\sigma}) \\
&= \phi(\infty) + \phi(-\infty) \\
&= 1 + 0 = 1 \neq 0 \text{ as n goes to infinity}
\end{aligned}$$

Hence, $\hat{\gamma}$ is not consistent

Problem 3

We have $E[\hat{\gamma}] = 3\mu\frac{\sigma^2}{n} + \mu^3$.To get an unbiased estimator of μ^3 from $\hat{\gamma} = (\frac{1}{n}\sum_{i=1}^{n} R_i)^3$, we can simply substrat the bias term from $\hat{\gamma} = (\frac{1}{n}\sum_{i=1}^{n} R_i)^3$. Hence, we can choose $\hat{\gamma} - 3\mu\frac{\sigma^2}{n}$ to be the estimator. Since $3\mu\frac{\sigma^2}{n}$ is a constant,

$$E[\hat{\gamma} - 3\mu\frac{\sigma^2}{n}] = 3\mu\frac{\sigma^2}{n} + \mu^3 - 3\mu\frac{\sigma^2}{n} = \mu^3$$

Problem 4

(a) Derive the bias of $\tilde{\gamma}$

$$\begin{aligned}
bias(\tilde{\gamma}) = E[\tilde{\gamma}] - \gamma &= E[\frac{1}{n}\sum_{i=1}^{n} R_i^3] - \gamma \\
&= \frac{1}{n}\sum_{i=1}^{n} E[R_i^3] - \gamma \\
&= \frac{1}{n} nE[R_1^3] - \gamma \\
&= \gamma - \gamma = 0
\end{aligned}$$

(b) Is $\tilde{\gamma}$ consistent? Support your answer with a mathematical argument.

By the law of large number, as n goes to infinity, $\tilde{\gamma} \xrightarrow{P} E[R_1^3] = \gamma$, so $\tilde{\gamma}$ is consistent.

Problem 5

From problem 4, we know that $U(X) = \tilde{\gamma}$ is an unbiased estimator. Also, we know that the minimal sufficient statistics for normal distributions are $T(R) = (\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i, \bar{R}^2 = \frac{1}{n}\sum_{i=1}^{n} R_i^2)$. Thereby, using Rao-Blackwell Theorem,

$$\begin{aligned}
\gamma_{MVUE} = E[U(R)|T(R)] = E[\tilde{\gamma}|\bar{R}, \bar{R}^2] &= E[\frac{1}{n}\sum_{i=1}^{n} R_i^3|\bar{R}, \bar{R}^2] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}(R_i - \bar{R} + \bar{R})^3|\bar{R}, \bar{R}^2] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}((R_i - \bar{R})^3 + 3(R_i - \bar{R})^2\bar{R} + 3(R_i - \bar{R})\bar{R}^2 + \bar{R}^3)|\bar{R}, \bar{R}^2] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}(3(R_i - \bar{R})^2\bar{R} + \bar{R}^3)|\bar{R}, \bar{R}^2] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}(3R_i^2\bar{R} + 3\bar{R}^3 - 6R_i\bar{R}^2 + \bar{R}^3)|\bar{R}, \bar{R}^2] \\
&= 3\bar{R}E[\frac{1}{n}\sum_{i=1}^{n} R_i^2] + 4E[\bar{R}^3] - 6\bar{R}^2 E[\frac{1}{n}\sum_{i=1}^{n} R_i] \\
&= 3\bar{R}\bar{R}^2 - 2\bar{R}^3
\end{aligned}$$