STAT5703 HW1 Exercise 1

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Exercise 1.

Question 1.

To compute p^{th} population quantile, we need to find point $Q_D(p)$ such that $P(D \leq Q_D(p)) = p$. So for the exponential distribution:

$$\int_0^{Q_D(p)} \lambda e^{-\lambda D} dD = p = 1 - e^{-\lambda Q_D(p)}$$

$$1 - p = e^{-\lambda Q_D(p)}$$

$$\ln(1-p) = -\lambda Q_D(p)$$

$$Q_D(p) = -\frac{1}{\lambda} \ln(1-p)$$

Question 2.

To find a method of moments-based estimator of $Q_D(p)$, we need to finf MM of λ first, because $Q_D(p) = -\frac{1}{\lambda} \ln(1-p)$. The first moment of D is:

$$\mu_1 = \mathbf{E}[D] = \frac{1}{\lambda} = \psi(\lambda)$$

The first empirical moment is:

$$\hat{\mu}_1 = \bar{D_n}$$

The MM estimatoe of *lambda* is:

$$\hat{\lambda}^{MM} = \psi^{-1}(\bar{D_n}) = \frac{1}{\bar{D_n}}$$

Therefore the MM-based estimator of $Q_D(p)$:

$$\hat{Q}_D(p)^{MM} = -\frac{1}{\hat{\lambda}^{MM}} \ln(1-p) = -\bar{D_n} \ln(1-p)$$

.

Question 3.

We arealdy know $D_1, ..., D_n \overset{i.i.d.}{\sim} Exp(\lambda)$, hence by CLN,

$$\sqrt{n}(\bar{D_n} - \mu) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

$$\sqrt{n}(\bar{D_n} - \frac{1}{\lambda}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{\lambda^2})$$

Then by Delta Method,

$$\sqrt{n}(\ln(1-p)\bar{D_n} + Q_D(p)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(\ln(1-p))^2}{\lambda^2})$$

From question 2, we already obtaint the method of moments estimator of $Q_D(p)$ is $\hat{Q}_D(p)^{MM} = -\bar{D_n}\ln(1-p)$, hence

$$\frac{\sqrt{n}\lambda(Q_D(p) - \hat{Q}_D(p)^{MM})}{\ln(1-p)} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1)$$

For *approximate* $(1-\alpha)$-confidence interval, we can get:

$$L(D) = \hat{Q}_D(p)^{MM} - \frac{z_{1-\alpha/2}\ln(1-p)}{\lambda\sqrt{n}}$$

$$U(D) = \hat{Q}_D(p)^{MM} + \frac{z_{1-\alpha/2}\ln(1-p)}{\lambda\sqrt{n}}$$

the *approximate* $(1-\alpha)$-confidence interval for $Q_D(p)$ is $[\hat{Q}_D(p)^{MM} - \frac{z_{1-\alpha/2}\ln(1-p)}{\lambda\sqrt{n}}, \hat{Q}_D(p)^{MM} + \frac{z_{1-\alpha/2}\ln(1-p)}{\lambda\sqrt{n}}]$

Question 4.

Since $D_1, ..., D_n \overset{i.i.d.}{\sim} Exp(\lambda)$, from addition rule for the exponential distribution, $\sum_{i=1}^{n} D_i \sim \Gamma(n, \lambda)$, then

$$\lambda\bar{D_n} = \frac{\lambda}{n}\sum_{i=1}^{n} D_i \sim \Gamma(n, n)$$

Because distribution of $\lambda\bar{D_n}$ is independent of the parameter λ, $\lambda\bar{D_n}$ is an exact pivot.

To construct an exact confidence interval of median, we can get the confidence interval of λ first. We know that

$$2n\lambda\bar{D_n} \sim \Gamma(n, \frac{1}{2}) \sim \chi^2_{2n}$$

Therefore, $P(\chi^2_{1-\alpha/2,2n} < 2n\lambda\bar{D_n} < \chi^2_{\alpha/2,2n}) = 1-\alpha$ for any $\alpha \in (0,1)$

$$P(\frac{\chi^2_{1-\alpha/2,2n}}{2n\bar{D_n}} < \lambda < \frac{\chi^2_{\alpha/2,2n}}{2n\bar{D_n}}) = 1-\alpha$$

Because $Q_D(0.5) = -\frac{1}{\lambda}\ln(0.5)$,

$$P(\frac{-2n\bar{D_n}\ln(0.5)}{\chi^2_{1-\alpha/2,2n}} < Q_D(0.5) < \frac{-2n\bar{D_n}\ln(0.5)}{\chi^2_{\alpha/2,2n}}) = 1-\alpha$$

Hence, the $(1-\alpha)$ exact confidence interval of the median $Q_D(0.5)$is,

$$[\frac{-2n\bar{D_n}\ln(0.5)}{\chi^2_{1-\alpha/2,2n}}, \frac{-2n\bar{D_n}\ln(0.5)}{\chi^2_{\alpha/2,2n}}]$$