3.10 Exercises

1. Compute the dot product between the following pairs of vectors.

a) $\begin{bmatrix}0\\a\end{bmatrix}, \begin{bmatrix}b\\0\end{bmatrix}$ **b)** $\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}$ **c)** $\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix}$

d) $\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}3\\-3\end{bmatrix}$ **e)** $\begin{bmatrix}1\\4\\0\end{bmatrix}, \begin{bmatrix}3\\-3\\9\end{bmatrix}$ **f)** $\begin{bmatrix}10\\14\\-3\end{bmatrix}, \begin{bmatrix}-1\\1\\-6\end{bmatrix}$

g) $\begin{bmatrix}1\\a\end{bmatrix}, \begin{bmatrix}b\\1\end{bmatrix}$ **h)** $\begin{bmatrix}2i\\4-i\\3\end{bmatrix}, \begin{bmatrix}3\\3\\1\end{bmatrix}$ **i)** $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}$

j) $\begin{bmatrix}1\\-2\\-3\\4\\5\\-6\end{bmatrix}, \begin{bmatrix}-4\\3\\4\\-5\\6\\-2\end{bmatrix}$ **k)** $\begin{bmatrix}2\\-5\\0\\0\\2\\-8\end{bmatrix}, \begin{bmatrix}-1\\5\\0\\-1\\0\\-4\end{bmatrix}$ **l)** $\begin{bmatrix}2.47\\-2.47\end{bmatrix}, \begin{bmatrix}\pi^3\\\pi^3\end{bmatrix}$

m) $\begin{bmatrix}2\\3\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\12\\-8\\4\end{bmatrix}$ **n)** $\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \begin{bmatrix}-2\\1\\-\frac{5}{2}\end{bmatrix}$ **o)** $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}$

1a $\begin{bmatrix}0\\a\end{bmatrix}, \begin{bmatrix}b\\0\end{bmatrix} = 0b + 0a = \boxed{0}$

1b $\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix} = \boxed{ac + bd}$

1c $\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix} = \boxed{ab + cd}$

1d $\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}3\\-3\end{bmatrix} = 1\cdot 3 + 4\cdot -3 = \boxed{-9}$

1e $\begin{bmatrix}1\\4\\0\end{bmatrix} \begin{bmatrix}3\\-3\\9\end{bmatrix} = 3 - 12 + 0 = \boxed{-9}$

1f $\begin{bmatrix}10\\14\\-3\end{bmatrix} \begin{bmatrix}-1\\1\\-6\end{bmatrix} = -10 + 14 + 18 = \boxed{22}$

1g $\begin{bmatrix}1\\a\end{bmatrix} \begin{bmatrix}b\\1\end{bmatrix} = \boxed{b + a}$

1h $\begin{bmatrix}2i\\4-i\\3\end{bmatrix} \begin{bmatrix}3\\3\\1\end{bmatrix} = 6i + 12 - 3i + 3 = \boxed{3i + 15}$

1i $\begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix} = 2 + 2 = 4$

1j $\begin{bmatrix}1\\-2\\-3\\4\\5\\-6\end{bmatrix} \begin{bmatrix}-4\\3\\4\\-5\\6\\-2\end{bmatrix} = -4 - 6 - 12 - 20 + 30 + 12 = \boxed{0}$

k) $\begin{bmatrix}2\\-5\\0\\0\\2\\-8\end{bmatrix}, \begin{bmatrix}-1\\5\\0\\-1\\0\\-4\end{bmatrix}$ $-2 - 25 + 0 + 0 + 0 + 32 = \boxed{5}$

l) $\begin{bmatrix}2.47\\-2.47\end{bmatrix}, \begin{bmatrix}\pi^3\\\pi^3\end{bmatrix}$ $2.47\pi^3 - 2.47\pi^3 = \boxed{0}$

m) $\begin{bmatrix}2\\3\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\12\\-8\\4\end{bmatrix}$ $0 + 36 + 24 + 8 = \boxed{68}$

n) $\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \begin{bmatrix}-2\\1\\-\frac{5}{2}\end{bmatrix}$ $-4 - 1 + \frac{10}{2} = \boxed{0}$

o) $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}$ $\boxed{0}$

2. Assume that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Determine whether each of the following equations is necessarily true, necessarily false, or could be true depending on the elements in $\mathbf{x}$ and $\mathbf{y}$.

a) $(\mathbf{x}-\mathbf{y})^{\mathrm{T}}(\mathbf{x}-\mathbf{y}) = 2(1-\mathbf{x}^{\mathrm{T}}\mathbf{y})$ **b)** $\|(\mathbf{x}-\frac{1}{2}\mathbf{y})\|^2 = \frac{5}{4}-\mathbf{x}^{\mathrm{T}}\mathbf{y}$

c) $\frac{\mathbf{x}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\mathbf{y}} = 0$ **d)** $\|\mathbf{y}+\frac{2}{3}\mathbf{x}\|^2 = \frac{4}{9}(1+4\mathbf{x}^{\mathrm{T}}\mathbf{y})$

e) $\mathbf{x}^{\mathrm{T}}\mathbf{x}+\mathbf{x}^{\mathrm{T}}\mathbf{x}-2\mathbf{y}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ **f)** $\frac{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x}\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}(\frac{1}{4}\mathbf{y})} + \frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2\|\mathbf{y}\|^3} - \frac{\mathbf{y}^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = 1$

a) $(\mathbf{x}-\mathbf{y})^{\mathrm{T}}(\mathbf{x}-\mathbf{y}) = 2(1-\mathbf{x}^{\mathrm{T}}\mathbf{y})$

try $x = [1\ 0]$ $y = [0,1]$

$([1,-1])^T([1,-1]) = 2(1-0)$

$2 = 2$

$\boxed{\text{True}}$

b) $\|(\mathbf{x}-\frac{1}{2}\mathbf{y})\|^2 = \frac{5}{4}-\mathbf{x}^{\mathrm{T}}\mathbf{y}$

try $x = [1,0]$ $y = [0,1]$

$\|([1,0]-[0,\frac{1}{2}])\|^2 = \frac{5}{4} - 0$

$\|[1,-\frac{1}{2}]\|^2 = \frac{5}{4}$

$1\frac{1}{4} = \frac{5}{4}$

$\boxed{\text{True}}$

c) $\frac{\mathbf{x}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} - \frac{\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\mathbf{y}} = 0$

try $x = [1,0]$ $y = [0,1]$

$\frac{0}{1} - \frac{1}{0} = \text{undefined}$

try $x = [1,0]$ $y = [-1,0]$

$\frac{-1}{1} - \frac{1}{-1} = \text{true}$

$\boxed{\text{depends}}$

d) $\|\mathbf{y}+\frac{2}{3}\mathbf{x}\|^2 = \frac{4}{9}(1+4\mathbf{x}^{\mathrm{T}}\mathbf{y})$

try $x = [1,0]$ $y = [0,1]$

$\|[0,1]+[\frac{2}{3},0]\|^2 = \frac{4}{9}(1+0)$

$\|[\frac{2}{3},1]\|^2 = \frac{4}{9}$

$1\frac{4}{9} = \frac{4}{9}$

$\boxed{\text{False}}$

e) $\mathbf{x}^{\mathrm{T}}\mathbf{x}+\mathbf{x}^{\mathrm{T}}\mathbf{x}-2\mathbf{y}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$

try $x = [1,0]$ $y = [0,1]$

$1+1-2 = 0$

$0 = 0$ true

try $x = [1,0]$ $y = [-1,0]$

$1+1-2 = -1$

$0 = -1$ false

$\boxed{\text{depends}}$

f) $\frac{\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{x}\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}(\frac{1}{4}\mathbf{y})} + \cancel{\frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2\|\mathbf{y}\|^3}} - \cancel{\frac{\mathbf{y}^{\mathrm{T}}\mathbf{y}\mathbf{y}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}} = 1$

try $x = [1,0]$ $y = [0,1]$

$\frac{\frac{1}{2}\cdot\frac{1}{2}}{[0,1]^T[0,\frac{1}{4}]} + \cancel{\frac{1}{1} - \frac{1}{1}} = 1$

$\frac{\frac{1}{4}}{\frac{1}{4}} = 1$ $\boxed{\text{true}}$

3. Compute the angle θ between the following pairs of vectors. See if you can do with without a calculator!

a) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}$ **b)** $\begin{bmatrix}10\\12\\4\end{bmatrix}, \begin{bmatrix}2.5\\3\\1\end{bmatrix}$ **c)** $\begin{bmatrix}-1\\-2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\-9\end{bmatrix}$

Dot product: $\|a\|\|b\|\cos(\theta_{ab})$

Law of cosines: $c^2 = a^2+b^2-2ab\cdot\cos(\theta_{ab})$

$: \|a-b\|^2 = \|a\|^2+\|b\|^2-2\|a\|\|b\|\cos(\theta_{ab})$

a) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}$ Dot product

$-2+2+0 = 0 = \boxed{90^\circ}$

b) $\begin{bmatrix}10\\12\\4\end{bmatrix} \div 4, \begin{bmatrix}2.5\\3\\1\end{bmatrix}$

Co-linear so $\boxed{\theta = 0^\circ}$

c) $\begin{bmatrix}-1\\-2\\3\end{bmatrix} \times -3, \begin{bmatrix}3\\6\\-9\end{bmatrix}$

co-linear in opp direction so $\boxed{\theta = 180^\circ}$

4. Compute the outer product between the following pairs of vectors. Implement problems a-c element-wise (the i, j^{th} element in the product matrix is the product of the i^{th} element in the left vector and the j^{th} element in the right vector); problems d-f row-wise (each row in the product matrix is the right-vector scaled by each element in the left vector); and problems g-i column-wise (each column in the product matrix is the left-vector scaled by each element in the right-vector).

a) $\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}^T$ **b)** $\begin{bmatrix}-1\\3\\0\end{bmatrix}\begin{bmatrix}0\\1\\3\end{bmatrix}^T$ **c)** $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}^T$

d) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}5\\6\\7\\8\end{bmatrix}^T$ **e)** $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}10\\20\\30\\40\end{bmatrix}^T$ **f)** $\begin{bmatrix}1\\1\\2\\2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}^T$

g) $\begin{bmatrix}4\\2\\3\\1\end{bmatrix}\begin{bmatrix}6\\7\\5\end{bmatrix}^T$ **h)** $\begin{bmatrix}3\\40\end{bmatrix}\begin{bmatrix}10\\2\\30\\4\end{bmatrix}^T$ **i)** $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}^T\begin{bmatrix}1\\1\\2\\2\end{bmatrix}$

a-c = element-wise

a) $\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}^T = \begin{bmatrix}1\cdot1 & 1\cdot2\\ 2\cdot1 & 2\cdot2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

b) $\begin{bmatrix}-1\\3\\0\end{bmatrix}\begin{bmatrix}0\\1\\3\end{bmatrix}^T = \begin{bmatrix}-1\cdot0 & -1\cdot1 & -1\cdot3\\ 3\cdot0 & 3\cdot1 & 3\cdot3\\ 0\cdot0 & 0\cdot1 & 0\cdot3\end{bmatrix} = \begin{bmatrix}0 & -1 & -3\\ 0 & 3 & 9\\ 0 & 0 & 0\end{bmatrix}$

c) $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}^T = \begin{bmatrix}1\cdot0 & 1\cdot1 & 1\cdot0\\ 0\cdot0 & 0\cdot1 & 0\cdot0\\ 0\cdot0 & 0\cdot1 & 0\cdot0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

d-f = row-wise

d) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}5\\6\\7\\8\end{bmatrix}^T = \begin{matrix}1\cdot[5 & 6 & 7 & 8]\\ 2\cdot[5 & 6 & 7 & 8]\\ 3\cdot[5 & 6 & 7 & 8]\\ 4\cdot[5 & 6 & 7 & 8]\end{matrix} = \begin{bmatrix}5 & 6 & 7 & 8\\ 10 & 12 & 14 & 16\\ 15 & 18 & 21 & 24\\ 20 & 24 & 28 & 32\end{bmatrix}$

e) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}10\\20\\30\\40\end{bmatrix}^T = \begin{matrix}1\cdot[10 & 20 & 30 & 40]\\ 2\cdot[10 & 20 & 30 & 40]\\ 3\cdot[10 & 20 & 30 & 40]\\ 4\cdot[10 & 20 & 30 & 40]\end{matrix} = \begin{bmatrix}10 & 20 & 30 & 40\\ 20 & 40 & 60 & 80\\ 30 & 60 & 90 & 120\\ 40 & 80 & 120 & 160\end{bmatrix}$

f) $\begin{bmatrix}1\\1\\2\\2\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}^T = \begin{matrix}1\cdot[a & b & c & d]\\ 1\cdot[a & b & c & d]\\ 2\cdot[a & b & c & d]\\ 2\cdot[a & b & c & d]\end{matrix} = \begin{bmatrix}a & b & c & d\\ a & b & c & d\\ 2a & 2b & 2c & 2d\\ 2a & 2b & 2c & 2d\end{bmatrix}$

g-i = column-wise

g) $\begin{bmatrix}4\\2\\3\\1\end{bmatrix}\begin{bmatrix}6\\7\\5\end{bmatrix}^T = 6\begin{bmatrix}4\\2\\3\\1\end{bmatrix} 7\begin{bmatrix}4\\2\\3\\1\end{bmatrix} 5\begin{bmatrix}4\\2\\3\\1\end{bmatrix} = \begin{bmatrix}24 & 28 & 20\\ 12 & 14 & 10\\ 18 & 21 & 15\\ 6 & 7 & 5\end{bmatrix}$

h) $\begin{bmatrix}3\\40\end{bmatrix}\begin{bmatrix}10\\2\\30\\4\end{bmatrix}^T = 10\begin{bmatrix}3\\40\end{bmatrix} 2\begin{bmatrix}3\\40\end{bmatrix} 30\begin{bmatrix}3\\40\end{bmatrix} 4\begin{bmatrix}3\\40\end{bmatrix} = \begin{bmatrix}30 & 6 & 90 & 12\\ 400 & 80 & 1200 & 160\end{bmatrix}$

i) $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}^{T}\begin{bmatrix}1\\1\\2\\2\end{bmatrix} = \boxed{a+b+2c+2d}$ Dot product

5. Determine whether the following vectors are unit vectors.

a) $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ **b)** $\begin{bmatrix}.2\\.64\\.1\\.4\end{bmatrix}$ **c)** $\begin{bmatrix}3/5\\0\\0\\.8\end{bmatrix}$ **d)** $\frac{1}{\sqrt{90}}\begin{bmatrix}1\\3\\5\\2\\4\\6\end{bmatrix}$

Unit vectors: Dot product with itself = 1 $(A^TA = 1)$

a) $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \xrightarrow{A^TA} \begin{bmatrix}\frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3}\end{bmatrix}$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ $\boxed{\text{Yes}}$

b) $\begin{bmatrix}.2\\.64\\.1\\.4\end{bmatrix} \xrightarrow{A^TA} .04 + .4096 + .01 + .16$

$= .6196$ $\boxed{\text{No}}$

c) $\begin{bmatrix}3/5\\0\\0\\.8\end{bmatrix}$ $A^TA = 1?$ $\rightarrow \frac{9}{25} + 0 + 0 + .64$

$= 1$ $\boxed{\text{Yes}}$

d) $\frac{1}{\sqrt{90}}\begin{bmatrix}1\\3\\5\\2\\4\\6\end{bmatrix}$ $A^TA = 1?$ $\rightarrow \frac{1}{90}(1+9+25+4+16+36)$

$= \frac{1}{90}(91) = \frac{91}{90} = 1\frac{1}{90}$ $\boxed{\text{No}}$

6. What is the magnitude of vector $\mu\mathbf{v}$ for the following μ?

a) $\mu = 0$ **b)** $\mu = \|\mathbf{v}\|$ **c)** $\mu = 1/\|\mathbf{v}\|$ **d)** $\mu = 1/\|\mathbf{v}\|^2$

a) $\mu = 0$

0 obviously

b) $\mu = \|\mathbf{v}\|$

$\|\mu v\| = \|v\|^2$

c) $\mu = 1/\|\mathbf{v}\|$

mag of vector divided by it's own mag $= \boxed{1}$

d) $\mu = 1/\|\mathbf{v}\|^2$

$\|\mu v\| = \boxed{1/\|v\|}$