4.8 Exercises

span = subspace

1. Determine whether each vector is in the span of sets S and T, and if so, what coefficients could produce the given vectors from the sets.

$$S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$$

a) $\begin{bmatrix}2\\2\\6\end{bmatrix}$ **b)** $\begin{bmatrix}0\\4\\12\end{bmatrix}$ **c)** $\begin{bmatrix}0\\0\\0\end{bmatrix}$ **d)** $\begin{bmatrix}-10\\-2\\-3\end{bmatrix}$ **e)** $\begin{bmatrix}-3\\2\\-3\end{bmatrix}$

in span of?

a) $\begin{bmatrix}2\\2\\6\end{bmatrix}$ $S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\}$ start with non zero row ×2

×2 + ×2

$$\begin{bmatrix}0\\2\\6\end{bmatrix} + \begin{bmatrix}2\\0\\6\end{bmatrix} = \begin{bmatrix}2\\2\\12\end{bmatrix}$$

$\boxed{\text{No}}$

in span of?

$\begin{bmatrix}2\\2\\6\end{bmatrix}$ $T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$

×-22 + ×6

$$\begin{bmatrix}-22\\-11\\0\end{bmatrix} + \begin{bmatrix}24\\6\\6\end{bmatrix} = \begin{bmatrix}2\\-5\\6\end{bmatrix}$$

$\boxed{\text{No}}$

in span of?

b) $\begin{bmatrix}0\\4\\12\end{bmatrix}$ ×4 $S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\}$

×4 + ×0

$$\begin{bmatrix}0\\4\\12\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\4\\12\end{bmatrix}$$

$\boxed{\text{Yes, } c_1 = 4,\ c_2 = 0}$

in span of?

$\begin{bmatrix}0\\4\\12\end{bmatrix}$ ×12 $T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$

×-48 ×12

$$\begin{bmatrix}-48\\-24\\0\end{bmatrix} + \begin{bmatrix}48\\12\\12\end{bmatrix} = \begin{bmatrix}0\\-12\\12\end{bmatrix}$$

$\boxed{\text{No}}$

in span of

c) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ $S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\}$ $T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$

$\boxed{\text{Yes}}$ $\boxed{\text{Yes}}$

zero vector is in all spans/subspaces

in span of?

d) $\begin{bmatrix}-10\\-2\\-3\end{bmatrix}$ ×-2 $S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\}$

×-2 ×-10

$$\begin{bmatrix}0\\-2\\-6\end{bmatrix} + \begin{bmatrix}-10\\0\\-30\end{bmatrix} = \begin{bmatrix}-10\\-2\\-36\end{bmatrix}$$

$\boxed{\text{No}}$

in span of?

$\begin{bmatrix}-10\\-2\\-3\end{bmatrix}$ ×-3 $T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$

×2 ×-3

$$\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}-12\\-3\\-3\end{bmatrix} = \begin{bmatrix}-10\\-2\\-3\end{bmatrix}$$

$\boxed{\text{Yes, } c_1 = 2,\ c_2 = -3}$

in span of?

e) $\begin{bmatrix}-3\\2\\-3\end{bmatrix}$ ×-3 ×2 $S = \left\{ \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix} \right\}$

×2 ×-3

$$\begin{bmatrix}0\\2\\6\end{bmatrix} + \begin{bmatrix}-3\\0\\-9\end{bmatrix} = \begin{bmatrix}-3\\2\\-3\end{bmatrix}$$

$\boxed{\text{Yes, } c_1 = 2,\ c_2 = -3}$

in span of?

$\begin{bmatrix}-3\\2\\-3\end{bmatrix}$ ×-3 $T = \left\{ \begin{bmatrix}1\\.5\\0\end{bmatrix}, \begin{bmatrix}4\\1\\1\end{bmatrix} \right\}$

×9 ×-3

$$\begin{bmatrix}9\\4.5\\0\end{bmatrix} + \begin{bmatrix}-12\\-3\\-3\end{bmatrix} = \begin{bmatrix}-3\\1.5\\-3\end{bmatrix}$$

$\boxed{\text{No}}$

2. Determine whether the following vector is in the set spanned by the bracketed vectors, in other words, whether $\mathbf{u} \in S = \{\mathbf{v_1}, ..., \mathbf{v_n}\}$

a) $\begin{bmatrix}5\\1\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$ **b)** $\begin{bmatrix}4\\1\\12\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$

c) $\begin{bmatrix}0\\1\\3\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix} \right\}$

in span of?

a) $\begin{bmatrix}5\\1\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$

$\boxed{\text{No}}$

2D vs 3D

b) $\begin{bmatrix}4\\1\\12\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$

×3 ×1

$$\begin{bmatrix}3\\0\\9\end{bmatrix} + \begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}4\\1\\12\end{bmatrix}$$

$\boxed{\text{Yes}}$

c) $\begin{bmatrix}0\\1\\3\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix} \right\}$

×1 ×1

$$\begin{bmatrix}1\\0\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\5\end{bmatrix}$$

$\boxed{\text{No}}$

3. Label the following sets as independent or dependent. For dependent sets, determine whether it is possible to modify only one element of one vector to change it to an independent set.

a) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}$ b) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\-2\end{bmatrix}\right\}$ d) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-1\\-2\end{bmatrix}\right\}$

e) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix}\right\}$ f) $\left\{\begin{bmatrix}5\\12\end{bmatrix},\begin{bmatrix}-3\\4\end{bmatrix},\begin{bmatrix}10\\23\end{bmatrix}\right\}$

g) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$ h) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}8\\16\\24\end{bmatrix}\right\}$

i) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$

Steps:

1. If vector count $>$ count of elements in each vector, then the set must be dependent
2. A zeros vector makes the set dependent
3. Trial & error educated guesswork. Look for zeros
4. Serious educated guesswork. Try combinations of scaled vectors.

a) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}$

Independent ✓

b) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix}\right\}$

Independent ✓

c) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\-2\end{bmatrix}\right\}$

Dependent ✓

Yes, can change to make independen

d) $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-1\\-2\end{bmatrix}\right\}$

Dependent ✓

Yes, change 1 ✗

Vector count $>$ element count = Dependent

No

e) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix}\right\}$

Dependent ✓

No, can't change 1 ✗

Yes

f) $\left\{\begin{bmatrix}5\\12\end{bmatrix},\begin{bmatrix}-3\\4\end{bmatrix},\begin{bmatrix}10\\23\end{bmatrix}\right\}$

Independent ✗

Dependent

vector count $>$ element count = Dependent

No

g) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$

Independent

h) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\overset{\times 4}{,}\begin{bmatrix}8\\16\\24\end{bmatrix}\right\}$

Dependent

Yes change 1

i) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$

Dependent

Yes change 1

4. Determine the value of λ that would make the following sets of vectors dependent.

a) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4.5\\\lambda\\13.5\end{bmatrix}\right\}$ b) $\left\{\begin{bmatrix}0\\0\\\lambda\\0\\0\end{bmatrix},\begin{bmatrix}a\\b\\5\\c\\d\end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\\lambda\\5\end{bmatrix},\begin{bmatrix}5\\0\\8\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

a) $\left\{\overset{\times 4.5}{\begin{bmatrix}1\\2\\3\end{bmatrix}},\begin{bmatrix}4.5\\\lambda\\13.5\end{bmatrix}\right\}$ $\boxed{\lambda = 9}$

b) $\left\{\begin{bmatrix}0\\0\\\lambda\\0\\0\end{bmatrix},\begin{bmatrix}a\\b\\5\\c\\d\end{bmatrix}\right\}$ $\boxed{\lambda = 0}$

c) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\\lambda\\5\end{bmatrix},\begin{bmatrix}5\\0\\8\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ $\boxed{\lambda = -2}$

5. The following sets of vectors are dependent sets. For each set, determine the number of vectors to remove to create an independent set with the most number of vectors.

a) $\left\{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\3\\4\\5\\0\end{bmatrix},\begin{bmatrix}2\\5\\4\\5\\0\end{bmatrix},\begin{bmatrix}0\\6\\8\\10\\0\end{bmatrix},\begin{bmatrix}1\\4\\4\\5\\1\end{bmatrix}\right\}$ b) $\left\{\begin{bmatrix}4\\3\\2\end{bmatrix},\begin{bmatrix}4\\4\\3\end{bmatrix},\begin{bmatrix}7\\1\\2\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix},\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$

a) $\left\{\cancel{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}},\begin{bmatrix}0\\3\\4\\5\\0\end{bmatrix},\begin{bmatrix}2\\5\\4\\5\\0\end{bmatrix},\cancel{\begin{bmatrix}0\\6\\8\\10\\0\end{bmatrix}},\begin{bmatrix}1\\4\\4\\5\\1\end{bmatrix}\right\}$

$\boxed{\text{remove 2}}$

b) $\left\{\begin{bmatrix}4\\3\\2\end{bmatrix},\begin{bmatrix}4\\4\\3\end{bmatrix},\begin{bmatrix}7\\1\\2\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix},\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$

vectors = 5

elements = 3

$\boxed{\text{remove 2}}$

6. Determine whether the following are descriptions of subspaces and subsets, or only subsets.

a) The set of points y such that $y = 2x$.

b) The set of points y such that $y = 2x + .01$.

c) The point at the origin in $\mathbb{R}^5$.

d) The set of all points in $\mathbb{R}^3$ with positive magnitude.

Subspaces are infinite, subsets have boundaries

a) The set of points y such that $y = 2x$.
subspace & subset

b) The set of points y such that $y = 2x + .01$.
subspace & subset

c) The point at the origin in $\mathbb{R}^5$.
subset

d) The set of all points in $\mathbb{R}^3$ with positive magnitude.
subset

7. What is the dimensionality of the subspace spanned by the following vector subspaces?

a) $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\0\\0\\0\end{bmatrix}\right\}$ **b)** $\left\{\begin{bmatrix}1\\2\\6\\0\end{bmatrix}, \begin{bmatrix}2\\4\\12\\6\end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ **d)** $\left\{\begin{bmatrix}6\\9\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}4\\6\\2\end{bmatrix}\right\}$

a) $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\0\\0\\0\end{bmatrix}\right\}$ $\boxed{1D}$

b) $\left\{\begin{bmatrix}1\\2\\6\\0\end{bmatrix}, \begin{bmatrix}2\\4\\12\\6\end{bmatrix}\right\}$ ×2 ×3 $\boxed{2D}$

c) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ $\boxed{3D}$

d) $\left\{\begin{bmatrix}6\\9\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}4\\6\\2\end{bmatrix}\right\}$ ×3 ×3 $r_1 + r_3 = r_2$ $\boxed{1D}$

8. Remove one vector in the following sets to create a basis set for a 2D subspace.

a) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ **b)** $\left\{\begin{bmatrix}1\\4\\3\end{bmatrix}, \begin{bmatrix}0\\-4\\5\end{bmatrix}, \begin{bmatrix}3\\12\\9\end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix}-3\\2\\13\end{bmatrix}, \begin{bmatrix}4.5\\-3\\-19.5\end{bmatrix}, \begin{bmatrix}-1.5\\1\\6\end{bmatrix}\right\}$

a) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \cancel{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$

b) $\left\{\begin{bmatrix}1\\4\\3\end{bmatrix}, \begin{bmatrix}0\\-4\\5\end{bmatrix}, \cancel{\begin{bmatrix}3\\12\\9\end{bmatrix}}\right\}$ ×3

c) $\left\{\begin{bmatrix}-3\\2\\13\end{bmatrix}, \cancel{\begin{bmatrix}4.5\\-3\\-19.5\end{bmatrix}}, \begin{bmatrix}-1.5\\1\\6\end{bmatrix}\right\}$ x-1.5