

3.10 Exercises

1. Compute the dot product between the following pairs of vectors.

a) $\begin{bmatrix} 0 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ c) $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ e) $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix}$ f) $\begin{bmatrix} 10 \\ 14 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix}$

g) $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 1 \end{bmatrix}$ h) $\begin{bmatrix} 2i \\ 4-i \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ i) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

j) $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 4 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 4 \\ -5 \\ 6 \\ -2 \end{bmatrix}$ k) $\begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 0 \\ -1 \\ 0 \\ -4 \end{bmatrix}$ l) $\begin{bmatrix} 2.47 \\ -2.47 \end{bmatrix}, \begin{bmatrix} \pi^3 \\ \pi^3 \end{bmatrix}$

m) $\begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ -8 \\ 4 \end{bmatrix}$ n) $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -\frac{5}{2} \end{bmatrix}$ o) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}$

a) $\begin{bmatrix} 0 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} = 0b + 0a = \boxed{0}$

b) $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} = \boxed{ac + bd}$

c) $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} = \boxed{ab + cd}$

d) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 1 \cdot 3 + 4 \cdot (-3) = \boxed{-9}$

e) $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix} = 3 - 12 + 0 = \boxed{-9}$ f) $\begin{bmatrix} 10 \\ 14 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} = -10 + 14 + 18 = \boxed{22}$

g) $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} b \\ 1 \end{bmatrix} = \boxed{b+a}$

h) $\begin{bmatrix} 2i \\ 4-i \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = 6i + 12 - 3i + 3 = \boxed{3i + 15}$

i) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 + 2 = 4$

j) $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 4 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 4 \\ -5 \\ 6 \\ -2 \end{bmatrix} = -4 - 6 - 12 - 20 + 30 + 12 = \boxed{0}$

k) $\begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 0 \\ -1 \\ 0 \\ -4 \end{bmatrix} = -2 - 25 + 0 + 0 + 0 + 32 = \boxed{5}$

l) $\begin{bmatrix} 2.47 \\ -2.47 \end{bmatrix}, \begin{bmatrix} \pi^3 \\ \pi^3 \end{bmatrix} = 2.47\pi^3 - 2.47\pi^3 = \boxed{0}$

m) $\begin{bmatrix} 2 \\ 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ -8 \\ 4 \end{bmatrix} = 0 + 36 + 24 + 8 = \boxed{68}$

n) $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -\frac{5}{2} \end{bmatrix} = -4 - 1 + \frac{10}{2} = \boxed{0}$

o) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \boxed{0}$

2. Assume that $\|x\| = \|y\| = 1$. Determine whether each of the following equations is necessarily true, necessarily false, or could be true depending on the elements in x and y .

a) $(x - y)^T(x - y) = 2(1 - x^T y)$ b) $\|(x - \frac{1}{2}y)\|^2 = \frac{5}{4} - x^T y$

c) $\frac{x^T y}{x^T x} - \frac{y^T y}{y^T y} = 0$ d) $\|y + \frac{2}{3}x\|^2 = \frac{4}{9}(1 + 4x^T y)$

e) $x^T x + x^T x - 2y^T y = x^T y$ f) $\frac{\frac{1}{2}x^T x \frac{1}{2}y^T y}{y^T(\frac{1}{4}y)} + \frac{\|x\|^2}{\|x\|^2 \|y\|^3} - \frac{y^T y y^T y}{x^T x} = 1$

a) $(x - y)^T(x - y) = 2(1 - x^T y)$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $([1, -1])^T([1, -1]) = 2(1 - 0)$
 $2 = 2$
True

b) $\|(x - \frac{1}{2}y)\|^2 = \frac{5}{4} - x^T y$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $\|([1, 0] - [0, \frac{1}{2}])\|^2 = \frac{5}{4} - 0$
 $\|[1, -\frac{1}{2}]\|^2 = \frac{5}{4}$
 $1 + \frac{1}{4} = \frac{5}{4}$
True

c) $\frac{x^T y}{x^T x} - \frac{y^T y}{y^T y} = 0$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $\frac{0}{1} - \frac{1}{0} = \text{undefined}$
 $+ny \ x = [1, 0] \ y = [-1, 0]$
 $\frac{-1}{1} - \frac{1}{-1} = \text{true}$
depends

d) $\|y + \frac{2}{3}x\|^2 = \frac{4}{9}(1 + 4x^T y)$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $\|[0, 1] + [\frac{2}{3}, 0]\|^2 = \frac{4}{9}(1 + 0)$
 $\|[\frac{2}{3}, 1]\|^2 = \frac{4}{9}$
 $1 + \frac{4}{9} = \frac{13}{9} \neq \frac{4}{9}$
False

e) $x^T x + x^T x - 2y^T y = x^T y$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $1 + 1 - 2 = 0$
 $0 = 0 \text{ true}$
 $+ny \ x = [1, 0] \ y = [-1, 0]$
 $1 + 1 - 2 = 0$
 $0 = 0 \text{ true}$
depends

f) $\frac{\frac{1}{2}x^T x \frac{1}{2}y^T y}{y^T(\frac{1}{4}y)} + \frac{\|x\|^2}{\|x\|^2 \|y\|^3} - \frac{y^T y y^T y}{x^T x} = 1$
 $+ny \ x = [1, 0] \ y = [0, 1]$
 $\frac{\frac{1}{2} \cdot \frac{1}{2}}{[0, 1]^T [0, \frac{1}{4}]} + \frac{1}{1} - \frac{1}{1} = 1$
 $\frac{\frac{1}{4}}{\frac{1}{4}} = 1$
True

3. Compute the angle θ between the following pairs of vectors. See if you can do without a calculator!

a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} 10 \\ 12 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.5 \\ 3 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$

Dot product: $\|a\| \|b\| \cos(\theta_{ab})$
 Law of cosines: $c^2 = a^2 + b^2 - 2ab \cdot \cos(\theta_{ab})$
 $\therefore \|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\| \|b\| \cos(\theta_{ab})$

a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ Dot product
 $-2 + 2 + 0 = 0 = 90^\circ$

b) $\begin{bmatrix} 10 \\ 12 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.5 \\ 3 \\ 1 \end{bmatrix}$
 co-linear so $\theta = 0^\circ$

c) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$
 co-linear in opp direction so $\theta = 180^\circ$

4. Compute the outer product between the following pairs of vectors. Implement problems a-c element-wise (the i, j^{th} element in the product matrix is the product of the i^{th} element in the left vector and the j^{th} element in the right vector); problems d-f row-wise (each row in the product matrix is the right-vector scaled by each element in the left vector); and problems g-i column-wise (each column in the product matrix is the left-vector scaled by each element in the right-vector).

a-c = element-wise

$$\text{a) } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 2 \\ 2 \cdot 1 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\text{b) } \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -1 \cdot 0 & -1 \cdot 1 & -1 \cdot 3 \\ 3 \cdot 0 & 3 \cdot 1 & 3 \cdot 3 \\ 0 \cdot 0 & 0 \cdot 1 & 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{c) } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 1 \cdot 0 & 1 \cdot 1 & 1 \cdot 0 \\ 0 \cdot 0 & 0 \cdot 1 & 0 \cdot 0 \\ 0 \cdot 0 & 0 \cdot 1 & 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d-f = row-wise

$$\text{d) } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}^T = \begin{bmatrix} 1 \cdot [5 \ 6 \ 7 \ 8] \\ 2 \cdot [5 \ 6 \ 7 \ 8] \\ 3 \cdot [5 \ 6 \ 7 \ 8] \\ 4 \cdot [5 \ 6 \ 7 \ 8] \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \\ 15 & 18 & 21 & 24 \\ 20 & 24 & 28 & 32 \end{bmatrix}$$

$$\text{e) } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}^T = \begin{bmatrix} 1 \cdot [10 \ 20 \ 30 \ 40] \\ 2 \cdot [10 \ 20 \ 30 \ 40] \\ 3 \cdot [10 \ 20 \ 30 \ 40] \\ 4 \cdot [10 \ 20 \ 30 \ 40] \end{bmatrix} = \begin{bmatrix} 10 & 20 & 30 & 40 \\ 20 & 40 & 60 & 80 \\ 30 & 60 & 90 & 120 \\ 40 & 80 & 120 & 160 \end{bmatrix}$$

$$\text{f) } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T = \begin{bmatrix} 1 \cdot [a \ b \ c \ d] \\ 2 \cdot [a \ b \ c \ d] \\ 2 \cdot [a \ b \ c \ d] \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ 2a & 2b & 2c & 2d \\ 2a & 2b & 2c & 2d \end{bmatrix}$$

$$\begin{aligned} \text{a) } & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \\ \text{b) } & \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}^T \\ \text{c) } & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \\ \text{d) } & \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}^T \\ \text{e) } & \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}^T \\ \text{f) } & \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T \\ \text{g) } & \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix}^T \\ \text{h) } & \begin{bmatrix} 3 \\ 40 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 30 \\ 4 \end{bmatrix}^T \\ \text{i) } & \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

g-i = column-wise

$$\text{g) } \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix}^T = \begin{bmatrix} 4 \cdot [6 \ 7 \ 5] \\ 2 \cdot [6 \ 7 \ 5] \\ 3 \cdot [6 \ 7 \ 5] \\ 1 \cdot [6 \ 7 \ 5] \end{bmatrix} = \begin{bmatrix} 24 & 28 & 20 \\ 12 & 14 & 10 \\ 18 & 21 & 15 \\ 6 & 7 & 5 \end{bmatrix}$$

$$\text{h) } \begin{bmatrix} 3 \\ 40 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 30 \\ 4 \end{bmatrix}^T = 10 \begin{bmatrix} 3 \\ 40 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 40 \end{bmatrix} + 30 \begin{bmatrix} 3 \\ 40 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 40 \end{bmatrix} = \begin{bmatrix} 30 & 6 & 90 & 12 \\ 400 & 80 & 1200 & 160 \end{bmatrix}$$

$$\text{i) } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \boxed{a + b + 2c + 2d}$$

5. Determine whether the following vectors are unit vectors.

$$\begin{aligned} \text{a) } & \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \text{b) } & \begin{bmatrix} .2 \\ .64 \\ .1 \\ .4 \end{bmatrix} & \text{c) } & \begin{bmatrix} 3/5 \\ 0 \\ 0 \\ .8 \end{bmatrix} & \text{d) } & \frac{1}{\sqrt{90}} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{bmatrix} \end{aligned}$$

Unit vectors: Dot product = 1 (with itself) ($A^T A = 1$)

$$\text{a) } \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{A^T A} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Yes

$$\text{b) } \begin{bmatrix} .2 \\ .64 \\ .1 \\ .4 \end{bmatrix} \xrightarrow{A^T A} .04 + .4096 + .01 + .16 = .6196$$

No

$$\text{c) } \begin{bmatrix} 3/5 \\ 0 \\ 0 \\ .8 \end{bmatrix} \xrightarrow{A^T A} \frac{9}{25} + 0 + 0 + .64 = 1$$

Yes

$$\text{d) } \frac{1}{\sqrt{90}} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{bmatrix} \xrightarrow{A^T A} \frac{1}{90} (1 + 9 + 25 + 4 + 16 + 36) = \frac{91}{90} = 1 \frac{1}{90}$$

No

6. What is the magnitude of vector $\mu \mathbf{v}$ for the following μ ?

$$\text{a) } \mu = 0 \quad \text{b) } \mu = \|\mathbf{v}\| \quad \text{c) } \mu = 1/\|\mathbf{v}\| \quad \text{d) } \mu = 1/\|\mathbf{v}\|^2$$

$$\text{a) } \mu = 0$$

0, obviously

$$\text{b) } \mu = \|\mathbf{v}\|$$

$\|\mu \mathbf{v}\| = \|\mathbf{v}\|^2$

$$\text{c) } \mu = 1/\|\mathbf{v}\|$$

mag of vector divided by it's own mag = **1**

$$\text{d) } \mu = 1/\|\mathbf{v}\|^2$$

$\|\mu \mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}$