5.10 Exercises

Note: ex 5.1 actually uses concepts covered in chapter 6. Look ahead for A^TB or AB^T multiplication size comparisons.

Dot product/inner product → A^TA; outer product → AB^T

$\underset{5\times3}{A}\quad \underset{2\times3}{B} \rightarrow \underset{\substack{5\times3\\3\times5}}{A^T}\underset{5\times3}{A} = 1\times1$ (scalar) $\qquad \underset{5\times3}{A}\underset{\substack{2\times3\\3\times2}}{B^T} = 5\times2$

1. For the following matrix and vectors, solve the given arithmetic problems, or state why they are not solvable.

$$\mathbf{u}=\underset{3\times1}{\begin{bmatrix}2\\4\\1\end{bmatrix}},\quad \mathbf{v}=\underset{4\times1}{\begin{bmatrix}3\\5\\0\\2\end{bmatrix}},\quad \mathbf{w}=\underset{4\times1}{\begin{bmatrix}1\\0\\1\\5\end{bmatrix}},\quad \mathbf{A}=\underset{3\times4}{\begin{bmatrix}-1&-2&-6&-6\\1&-1&0&-2\\-1&0&-1&-4\end{bmatrix}}$$

a) $\mathbf{wu}^T+\mathbf{A}$ b) $\mathbf{wu}^T+\mathbf{A}^T$ c) $\mathbf{uv}^T-\mathbf{A}$

d) $\mathbf{vw}^T-\mathbf{A}$ e) $\mathbf{vw}^T+\mathbf{A}^T$

a) $\mathbf{wu}^T+\mathbf{A}$

$4\times1\cdot3\times1^T+3\times4$

$4\times1\cdot1\times3+3\times4$

$4\times3+3\times4$ $\not\text{O}$

$\boxed{\text{size mismatch}}$ ✓

b) $\mathbf{wu}^T+\mathbf{A}^T$

$4\times1\cdot3\times1^T+3\times4^T$

$4\times1\cdot1\times3+4\times3$

$4\times3+4\times3$ ✓

$$\mathbf{w}=\begin{bmatrix}1\\0\\1\\5\end{bmatrix}\times\mathbf{u}=\begin{bmatrix}2\\4\\1\end{bmatrix}^T+\mathbf{A}=\begin{bmatrix}-1&-2&-6&-6\\1&-1&0&-2\\-1&0&-1&-4\end{bmatrix}^T$$

$$\begin{bmatrix}1\\0\\1\\5\end{bmatrix}\times\begin{bmatrix}2&4&1\end{bmatrix}+\begin{bmatrix}-1&1&-1\\-2&-1&0\\-6&0&-1\\-6&-2&-4\end{bmatrix}$$

$$\begin{bmatrix}2&4&1\\0&0&0\\2&4&1\\10&20&5\end{bmatrix}+\uparrow=\boxed{\begin{bmatrix}1&5&0\\-2&-1&0\\-4&4&0\\4&18&1\end{bmatrix}}$$ ✓

c) $\mathbf{uv}^T-\mathbf{A}$ $3\times4-3\times4$ size OK

$$\mathbf{u}=\begin{bmatrix}2\\4\\1\end{bmatrix}\quad\mathbf{v}=\begin{bmatrix}3\\5\\0\\2\end{bmatrix}^T-\mathbf{A}=\begin{bmatrix}-1&-2&-6&-6\\1&-1&0&-2\\-1&0&-1&-4\end{bmatrix}$$

$$\begin{bmatrix}6&10&0&4\\12&20&0&8\\3&5&0&2\end{bmatrix}-\uparrow=\boxed{\begin{bmatrix}7&12&6&10\\11&21&0&10\\4&5&1&6\end{bmatrix}}$$ ✓

d) $\mathbf{vw}^T-\mathbf{A}$ $4\times4-3\times4$

$\boxed{\text{size mismatch}}$

e) $\mathbf{vw}^T+\mathbf{A}^T$ $4\times4+4\times3$

$\boxed{\text{size mismatch}}$

2. Perform the following matrix operations, when the operation is valid.

$$\mathbf{A}=\begin{bmatrix}2&4&3\\0&1&3\end{bmatrix},\quad\mathbf{B}=\begin{bmatrix}-2&-1&3\\6&-7&7\end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix}0&-6\\-3&-2\\-2&7\end{bmatrix}\quad\mathbf{D}=\begin{bmatrix}1&2\\3&4\\2&4\end{bmatrix}$$

a) $\mathbf{A}+3\mathbf{B}$ b) $\mathbf{A}+\mathbf{C}$ c) $\mathbf{C}-\mathbf{D}$

d) $\mathbf{D}+\mathbf{C}$ e) $\mathbf{A}^T+\mathbf{D}$ f) $(\mathbf{A}+\mathbf{B})^T+2\mathbf{C}$

g) $3\mathbf{A}+(\mathbf{B}^T+\mathbf{C})^T$ h) $-4(\mathbf{A}^T+\mathbf{C})^T+\mathbf{D}$

a) $\mathbf{A}+3\mathbf{B}$ $2\times3+2\times3$ OK

$$\mathbf{A}=\begin{bmatrix}2&4&3\\0&1&3\end{bmatrix}+3\mathbf{B}=\begin{bmatrix}-2&-1&3\\6&-7&7\end{bmatrix}$$

$$\uparrow+\begin{bmatrix}-6&-3&9\\18&-21&21\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}-4&1&12\\18&-20&24\end{bmatrix}}$$ ✓

b) $\mathbf{A}+\mathbf{C}$ $2\times3+3\times2$

$\boxed{\text{size mismatch}}$

c) $\mathbf{C}-\mathbf{D}$ $3\times2-3\times2$ OK

$$\mathbf{C}=\begin{bmatrix}0&-6\\-3&-2\\-2&7\end{bmatrix}-\mathbf{D}=\begin{bmatrix}1&2\\3&4\\2&4\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}-1&-8\\-6&-6\\-4&3\end{bmatrix}}$$ ✓

d) $\mathbf{D}+\mathbf{C}$ size OK

$$\mathbf{D}=\begin{bmatrix}1&2\\3&4\\2&4\end{bmatrix}+\mathbf{C}=\begin{bmatrix}0&-6\\-3&-2\\-2&7\end{bmatrix}$$

$$=\begin{bmatrix}1&-4\\0&2\\0&11\end{bmatrix}$$ ✓

e) $\mathbf{A}^T+\mathbf{D}$ $3\times2+3\times2$ OK

$$\mathbf{A}=\begin{bmatrix}2&4&3\\0&1&3\end{bmatrix}^T+\mathbf{D}=\begin{bmatrix}1&2\\3&4\\2&4\end{bmatrix}$$

$$\begin{bmatrix}2&0\\4&1\\3&3\end{bmatrix}+\uparrow$$

$$=\begin{bmatrix}3&2\\7&5\\5&7\end{bmatrix}$$ ✓

f) $(\mathbf{A}+\mathbf{B})^T+2\mathbf{C}$ $(2\times3+2\times3)^T+3\times2$ OK

$$\left(\mathbf{A}=\begin{bmatrix}2&4&3\\0&1&3\end{bmatrix},+\mathbf{B}=\begin{bmatrix}-2&-1&3\\6&-7&7\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}0&3&6\\6&-6&10\end{bmatrix}^T+2\mathbf{C}=\begin{bmatrix}0&-6\\-3&-2\\-2&7\end{bmatrix}$$

$$=\begin{bmatrix}0&6\\3&-6\\6&10\end{bmatrix}+\begin{bmatrix}0&-12\\-6&-4\\-4&14\end{bmatrix}$$

$$=\begin{bmatrix}0&-6\\-3&-10\\2&24\end{bmatrix}$$ ✓

Skipping last 2. too easy

3. An N×N matrix has N^2 elements. For a symmetric matrix, however, not all elements are unique. Create 2×2 and 3×3 symmetric matrices and count the number of total elements and the number of possible unique elements. Then work out a formula for the number of possible unique elements in such a matrix.

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \quad \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \quad \boxed{n(n+1)/2} \checkmark$$

$N=2$
4 elements
3 unique

$N=3$
9 elements
6 unique

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4. Identify the following types of matrices from the list provided in the section "A zoo of matrices." Note that some matrices can be given multiple labels.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

d) $\begin{bmatrix} a & b & c \\ -b & d & e \\ -c & -e & f \end{bmatrix}$ **e)** $\begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -e & 0 \end{bmatrix}$ **f)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 32 & 0 & 0 \\ 0 & 0 & 42 & 0 \end{bmatrix}$

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
Square
Diagonal
Symmetric

b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$
Square
triangle
(upper)

c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$
Square
Symmetric

d) $\begin{bmatrix} a & b & c \\ -b & d & e \\ -c & -e & f \end{bmatrix}$
Square
skew symmetric

e) $\begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -e & 0 \end{bmatrix}$ Square

f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 32 & 0 & 0 \\ 0 & 0 & 42 & 0 \end{bmatrix}$ rectangular
diagonal

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5. To "decompose" a matrix means to represent a matrix using the sum or product of other matrices. Let's consider an additive decomposition, thus starting from some matrix $\mathbf{A}$ and setting $\mathbf{A} = \mathbf{B} + \mathbf{C}$. You can also use more matrices: $\mathbf{A} = \mathbf{B} + ... + \mathbf{N}$. Decompose the following matrices $\mathbf{A}$. Are your decompositions unique? (A decomposition is "unique" if it has exactly one solution.)

error in text
shouldn't be here

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \quad \cancel{\begin{bmatrix} 1 & 3 & 3 \\ 2 & 5 & 7 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$$

additive is
not unique

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6. Create a Hankel matrix from the vector

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 & 1 & 2 \end{bmatrix}$$

Hankel
Matrix:

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b & c & d \\ b & c & d & 0 \\ c & d & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix}$$

A Hankel matrix can also "wrap around" to produce a full matrix, like this:

$$\begin{bmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{bmatrix}$$

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7. Determine whether the following matrices are orthogonal.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **b)** $\frac{1}{\sqrt{9}} \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ -2 & 1 \end{bmatrix}$ **c)** $\mathbf{I}_{17}$

Orthogonal A matrix is called orthogonal if it satisfies the following two criteria.

1. All of its columns are pairwise orthogonal. That means that the dot product between any two columns is exactly 0.
2. Each column i has $\|\mathbf{Q}_i\| = 1$, in other words, each column is unit magnitude. Remember that the magnitude of a vector (in this case, the column of a matrix) is computed as the dot product of that vector with itself.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
No
3rd col
not unit
magnitude

b) $\frac{1}{\sqrt{9}} \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ -2 & 1 \end{bmatrix}$
No
columns not
pairwise
orthogonal
(but are
unit length)

c) $\mathbf{I}_{17}$
Yes
Identity
matrix is
orthogonal

8. Consider two $M \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$. Is the sum of their traces equal to the trace of their sum? That is, $tr(\mathbf{A}) + tr(\mathbf{B}) = tr(\mathbf{A}+\mathbf{B})$? Try for a few examples, then see if you can work out a proof for this property. Trace = sum of diagonal

a) $\begin{bmatrix} 3 & -4 \\ 3 & -9 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 6 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 2 & 5 \\ -5 & 0 & 5 \\ -9 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 5 & 5 \\ 4 & 3 & 4 \\ 1 & 1 & 9 \end{bmatrix}$

c) $\begin{bmatrix} a & d & e \\ f & b & g \\ h & i & c \end{bmatrix}, \begin{bmatrix} j & m & n \\ o & k & p \\ q & r & l \end{bmatrix}$

a) $\overset{A}{\begin{bmatrix} 3 & -4 \\ 3 & -9 \end{bmatrix}}, \overset{B}{\begin{bmatrix} 4 & 4 \\ 6 & 0 \end{bmatrix}}$

$tr(A) = -6 \quad tr(B) = 4$

$-6 + 4 = \boxed{-2}$ ✓

$tr(A+B) = 7 - 9 = \boxed{-2}$ ✓

b) $\begin{bmatrix} 1 & 2 & 5 \\ -5 & 0 & 5 \\ -9 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 5 & 5 \\ 4 & 3 & 4 \\ 1 & 1 & 9 \end{bmatrix}$

$tr(A) = 4 \quad tr(B) = 17$

$4 + 17 = \boxed{21}$ ✓

$tr(A+B) = 6 + 3 + 12 = \boxed{21}$ ✓

c) $\overset{A}{\begin{bmatrix} a & d & e \\ f & b & g \\ h & i & c \end{bmatrix}}, \overset{B}{\begin{bmatrix} j & m & n \\ o & k & p \\ q & r & l \end{bmatrix}}$

$tr(A) = a+b+c$

$tr(B) = j+k+l$

$tr(A+B) = (a+j)+(b+k)+(c+l)$ ✓

$\boxed{\text{yes, the sum of traces always equal the trace of the sum}}$ ✓

9. Here's another neat property of the trace: The trace of the outer product is equal to the dot product: $tr(\mathbf{v}\mathbf{w}^{\mathrm{T}}) = \mathbf{v}^{\mathrm{T}}\mathbf{w}$. Demonstrate this property using the following sets of vectors.

a) $\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 \\ -3 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \\ 4 \end{bmatrix}$ **c)** $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$

a) $\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}^T = \begin{bmatrix} -6 & 3 & 15 \\ 10 & -5 & -25 \\ -2 & 1 & 5 \end{bmatrix}$

trace $= \boxed{-6}$ ✓

$\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}^T \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = -6 - 5 + 5 = \boxed{-6}$ ✓

b) $\begin{bmatrix} 1 \\ -3 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 6 & 1 & 4 \\ -15 & -18 & -3 & -12 \\ 25 & 30 & 5 & 20 \\ 10 & 12 & 2 & 8 \end{bmatrix}$

trace $= \boxed{0}$ ✓

$\begin{bmatrix} 1 \\ -3 \\ 5 \\ 2 \end{bmatrix}^T \begin{bmatrix} 5 \\ 6 \\ 1 \\ 4 \end{bmatrix} = 5 - 18 + 5 + 8 = \boxed{0}$ ✓

c) $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}^T = \begin{bmatrix} ae & af & ag & ah \\ be & bf & bg & bh \\ ce & cf & cg & ch \\ de & df & dg & dh \end{bmatrix}$ $\quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \boxed{ae + bf + cg + dh}$ ✓

trace $= \boxed{ae + bf + cg + dh}$ ✓

10. Perform the indicated matrix operations using the following matrices and scalars. Determine the underlying principle regarding trace, matrix addition, and scalar multiplication.

$\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -4 & -1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad \lambda = 5, \quad \alpha = -3$

a) $tr(\mathbf{A})$	**b)** $tr(\mathbf{B})$	**c)** $tr(\mathbf{A}+\mathbf{B})$
d) $tr(\lambda\mathbf{C})$	**e)** $\lambda\, tr(\mathbf{C})$	**f)** $\lambda\, tr(\alpha\mathbf{C})$
g) $\alpha\, tr(\lambda\mathbf{C})$	**h)** $tr(\alpha\mathbf{A}+\lambda\mathbf{B})$	**i)** $(\lambda\alpha)\, tr(\mathbf{A}+\mathbf{B})$
j) $tr(\lambda\mathbf{A}+\lambda\mathbf{B})$	**k)** $\lambda\, tr(\mathbf{A}+\mathbf{B})$	**l)** $tr(\mathbf{A}+\mathbf{B}^{\mathrm{T}})$

a) $tr(\mathbf{A})$ $\quad \mathbf{A} = \begin{bmatrix} 5 & -3 \\ 2 & -3 \end{bmatrix}$

$\boxed{2}$ ✓

b) $tr(\mathbf{B})$ $\quad \mathbf{B} = \begin{bmatrix} -4 & -1 \\ 1 & 3 \end{bmatrix}$

$\boxed{-1}$ ✓

c) $tr(\mathbf{A}+\mathbf{B})$

$= 5 - 4 - 3 + 3$

$= \boxed{1}$ ✓

d) $tr(\lambda\mathbf{C})$

$\boxed{5a + 5d}$ ✓

e) $\lambda\, tr(\mathbf{C})$

$\boxed{5(a+d)}$ ✓

f) $\lambda\, tr(\alpha\mathbf{C})$

$\boxed{5(-3a-3d)}$ ✓

g) $\alpha\, tr(\lambda\mathbf{C})$

$\boxed{-3(5a+5d)}$ ✓

h) $tr(\alpha\mathbf{A}+\lambda\mathbf{B})$

$tr\left(-3\begin{bmatrix} 5 & -3 \\ 2 & -3 \end{bmatrix} + 5\begin{bmatrix} -4 & -1 \\ 1 & 3 \end{bmatrix}\right)$

$= -15 + 9 - 20 + 15$

$= \boxed{-11}$ ✓

i) $(\lambda\alpha)\, tr(\mathbf{A}+\mathbf{B})$

$-15\, tr(A+B)$

$-15 \cdot 1$

$= \boxed{-15}$ ✓

j) $tr(\lambda\mathbf{A}+\lambda\mathbf{B})$

$tr\left(5\begin{bmatrix} 5 & -3 \\ 2 & -3 \end{bmatrix} + 5\begin{bmatrix} -4 & -1 \\ 1 & 3 \end{bmatrix}\right)$

$= 25 - 15 - 20 + 15$

$= \boxed{5}$ ✓

k) $\lambda\, tr(\mathbf{A}+\mathbf{B})$

$5 \cdot 1$

$= \boxed{5}$ ✓

l) $tr(\mathbf{A}+\mathbf{B}^{\mathrm{T}})$ – no change diagonal is same

$tr\left(\begin{bmatrix} 5 & -3 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 1 & 3 \end{bmatrix}^T\right)$

$= 5 - 4 - 3 + 3$

$= \boxed{1}$ ✓