In the table below, A and B are matrices of equal dimensions, c and d are scalars, and O is a zero matrix.

Property	Example
Associative property of multiplication	$(cd)A = c(dA)$
Distributive properties	$c(A+B) = cA + cB$
	$(c+d)A = cA + dA$
Multiplicative identity property	$1A = A$
Multiplicative properties of zero	$0 \cdot A = O$
	$c \cdot O = O$
Closure property of multiplication	cA is a matrix of the same dimensions as A.

This article explores these properties.

Matrices and scalar multiplication

A **matrix** is a rectangular arrangement of numbers into rows and columns.

When we work with matrices, we refer to real numbers as **scalars**.

The term **scalar multiplication** refers to the product of a real number and a matrix. In scalar multiplication, each entry in the matrix is multiplied by the given scalar.

$$2 \cdot \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 4 \\ 6 & 2 \end{bmatrix}$$

If any of this is new to you, you should check out the following articles before you proceed:

- Intro to matrices
- Adding & subtracting matrices
- Multiplying matrices by scalars
- Intro to zero matrices

Dimensions considerations

Notice that a scalar times a 2×2 matrix is another 2×2 matrix. In general, a scalar multiple of a matrix will be another matrix of the same dimension. This is what is meant by the **closure property** of scalar multiplication!

Matrix scalar multiplication & real number multiplication

Because scalar multiplication relies heavily on real number multiplication, many of the multiplication properties that we know to be true with real numbers are also true in scalar multiplication.

Let's take a look at each property individually.

Associative property of multiplication: $(cd)A = c(dA)$

This property states that if a matrix is multiplied by two scalars, you can multiply the scalars together first, and then multiply by the matrix. Or you can multiply the matrix by one scalar, and then the resulting matrix by the other.

The following example illustrates this property for $c = 2$, $d = 3$, and $A = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}$.

$(c \cdot d)A$	$c(dA)$
$(2 \cdot 3)\begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}$	$2(3\begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix})$
	$\downarrow$
$\downarrow$	$2\begin{bmatrix} (3 \cdot 5) & (3 \cdot 4) \\ (3 \cdot 8) & (3 \cdot 1) \end{bmatrix}$
	$\downarrow$
$\begin{bmatrix} (2 \cdot 3) \cdot 5 & (2 \cdot 3) \cdot 4 \\ (2 \cdot 3) \cdot 8 & (2 \cdot 3) \cdot 1 \end{bmatrix}$	$\begin{bmatrix} 2 \cdot (3 \cdot 5) & 2 \cdot (3 \cdot 4) \\ 2 \cdot (3 \cdot 8) & 2 \cdot (3 \cdot 1) \end{bmatrix}$

$$\begin{bmatrix} (2 \cdot 3) \cdot 5 & (2 \cdot 3) \cdot 4 \\ (2 \cdot 3) \cdot 8 & (2 \cdot 3) \cdot 1 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Real \# multiplication is associative}}}{=} \begin{bmatrix} 2 \cdot (3 \cdot 5) & 2 \cdot (3 \cdot 4) \\ 2 \cdot (3 \cdot 8) & 2 \cdot (3 \cdot 1) \end{bmatrix}$$

In each column we simplified one side of the identity into a single matrix. Notice that these two matrices are equal because of the associative property of multiplication for real numbers. For example, $(2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5)$.

This shows that the original expressions must be equivalent as well!

Distributive properties:

$c(A+B) = cA + cB$

This property states that a scalar can be distributed over matrix addition.

Here's an example where $c = 2$, $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$:

$c(A+B)$	$cA + cB$
$2\left(\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}\right)$	$2\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$
$\downarrow$	$\downarrow$
$2\begin{bmatrix} 5+3 & 2+4 \\ 3+2 & 1+6 \end{bmatrix}$	$\begin{bmatrix} 2\cdot 5 & 2\cdot 2 \\ 2\cdot 3 & 2\cdot 1 \end{bmatrix} + \begin{bmatrix} 2\cdot 3 & 2\cdot 4 \\ 2\cdot 2 & 2\cdot 6 \end{bmatrix}$
$\downarrow$	$\downarrow$
$\begin{bmatrix} 2(5+3) & 2(2+4) \\ 2(3+2) & 2(1+6) \end{bmatrix}$	$= \begin{bmatrix} 2\cdot 5 + 2\cdot 3 & 2\cdot 2 + 2\cdot 4 \\ 2\cdot 3 + 2\cdot 2 & 2\cdot 1 + 2\cdot 6 \end{bmatrix}$

$\uparrow$

The distributive property of real #s

If we compare the last matrix in each column, we see that these are equivalent because of the distributive property for real numbers. For example, $2(5+3) = 2\cdot 5 + 2\cdot 3$.

Thus the original two expressions must be equivalent as well!

$(c+d)A = cA + dA$

This property states that a matrix can be distributed over scalar addition.

Here's an example where $c = 2$, $d = 3$, and $A = \begin{bmatrix} 6 & 9 \\ 7 & 4 \end{bmatrix}$:

$(c+d)A$	$cA+dA$
$(2+3)\begin{bmatrix} 6 & 9 \\ 7 & 4 \end{bmatrix}$	$2\begin{bmatrix} 6 & 9 \\ 7 & 4 \end{bmatrix} + 3\begin{bmatrix} 6 & 9 \\ 7 & 4 \end{bmatrix}$
$\downarrow$	$\downarrow$
	$\begin{bmatrix} 2 \cdot 6 & 2 \cdot 9 \\ 2 \cdot 7 & 2 \cdot 4 \end{bmatrix} + \begin{bmatrix} 3 \cdot 6 & 3 \cdot 9 \\ 3 \cdot 7 & 3 \cdot 4 \end{bmatrix}$
	$\downarrow$
$\begin{bmatrix} (2+3) \cdot 6 & (2+3) \cdot 9 \\ (2+3) \cdot 7 & (2+3) \cdot 4 \end{bmatrix}$ =	$\begin{bmatrix} 2 \cdot 6 + 3 \cdot 6 & 2 \cdot 9 + 3 \cdot 9 \\ 2 \cdot 7 + 3 \cdot 7 & 2 \cdot 4 + 3 \cdot 4 \end{bmatrix}$

$\uparrow$

The distributive property of real #s

Once again, we see that the last matrix in each column are equivalent because of the distributive property for real numbers, making the original expressions equivalent as desired!

Multiplicative identity property: $1A = A$

This property says that when you multiply any matrix A by the scalar 1, the result is simply the original matrix A.

So, for example, if $A = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$, then we have:

$$1\begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 & 1 \cdot 5 \\ 1 \cdot 1 & 1 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}$$

Notice that because $1 \cdot a = a$ for *any* real number a, the scalar 1 will *always* be the multiplicative identity in scalar multiplication!

Multiplicative properties of zero:

$0 \cdot A = O$

This property states that in scalar multiplication, 0 times any $m \times n$ matrix A is the $m \times n$ **zero matrix**.

This is true because of the multiplicative properties of zero in the real number system. If a is a real number, we know $0 \cdot a = 0$. The following example illustrates this.

$$\color{green}{0}\begin{bmatrix} 3 & 8 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} \color{green}{0} \cdot 3 & \color{green}{0} \cdot 8 \\ \color{green}{0} \cdot 6 & \color{green}{0} \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c \cdot O = O$

This property states that any scalar times a zero matrix is the same zero matrix.

Again, this property is true because of the multiplicative properties of zero in the real number system. Here's an example where $c = 3$ and O is the 2×2zero matrix.

$$\color{green}{3}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \color{green}{3} \cdot 0 & \color{green}{3} \cdot 0 \\ \color{green}{3} \cdot 0 & \color{green}{3} \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Check your understanding

Now that you are familiar with all of the scalar multiplication properties, let's see if you can use them to determine equivalent matrix expressions.

For the problems below, let A and B be 2×2 matrices and let c and d be scalars.

1) Which of the following are equivalent to $c(1A + B)$?

Choose all answers that apply:

Ⓐ
$1Ac + B$

Ⓑ
$cA + cB$

Ⓒ
$cB + cA$

Check

[I need help!]

2) Which of the following are equivalent to $(cd)A + 0A$?

Choose all answers that apply:

Ⓐ
A

Ⓑ
$c(dA)$

Ⓒ

$(cd + 0)A$