2.3 LINEAR INDEPENDENCE, BASIS, AND DIMENSION

By themselves, the numbers m and n give an incomplete picture of the true size of a linear system. The matrix in our example had three rows and four columns, but the third row was only a combination of the first two. After elimination it became a zero row. It had no effect on the homogeneous problem $Ax = 0$. The four columns also failed to be independent, and the column space degenerated into a two-dimensional plane.

The important number that is beginning to emerge (the true size) is the ***rank r***. The rank was introduced as the *number of pivots* in the elimination process. Equivalently, the final matrix U has r nonzero rows. This definition could be given to a computer. But it would be wrong to leave it there because the rank has a simple and intuitive meaning: *The rank counts the number of genuinely independent rows in the matrix A*. We want definitions that are mathematical rather than computational.

The goal of this section is to explain and use four ideas:

1. Linear independence or dependence.
2. Spanning a subspace.
3. Basis for a subspace (a set of vectors).
4. Dimension of a subspace (a number).

The first step is to define ***linear independence***. Given a set of vectors $v_1, \dots, v_k$, we look at their combinations $c_1v_1 + c_2v_2 + \cdots + c_kv_k$. The trivial combination, with all weights $c_i = 0$, obviously produces the zero vector: $0v_1 + \cdots + 0v_k = 0$. The question is whether this is the *only way* to produce zero. If so, the vectors are independent.

If any other combination of the vectors gives zero, they are *dependent*.

2E Suppose $c_1v_1 + \cdots + c_kv_k = 0$ only happens when $c_1 = \cdots = c_k = 0$. Then the vectors $v_1, \dots, v_k$ are ***linearly independent***. If any c's are nonzero, the v's are ***linearly dependent***. One vector is a combination of the others.

Linear dependence is easy to visualize in three-dimensional space, when all vectors go out from the origin. Two vectors are dependent if they lie on the same line. *Three vectors are dependent if they lie in the same plane*. A random choice of three vectors, without any special accident, should produce linear independence (not in a plane). Four vectors are always linearly dependent in $\mathbf{R}^3$.

Example 1 If v_1 = zero vector, then the set is linearly dependent. We may choose $c_1 = 3$ and all other $c_i = 0$; this is a nontrivial combination that produces zero.

Example 2 The columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent, since the second column is three times the first. The combination of columns with weights $-3, 1, 0, 0$ gives a column of zeros.

The rows are also linearly dependent; row 3 is two times row 2 minus five times row 1. (This is the same as the combination of b_1, b_2, b_3, that had to vanish on the