Properties of matrix multiplication

In this table, A, B, and C are $n \times n$ matrices, I is the $n \times n$ identity matrix, and O is the $n \times n$ zero matrix

Property	Example
The commutative property of multiplication does not hold!	$AB \neq BA$
Associative property of multiplication	$(AB)C = A(BC)$
Distributive properties	$A(B + C) = AB + AC$ $(B + C)A = BA + CA$
Multiplicative identity property	$IA = A$ and $AI = A$
Multiplicative property of zero	$OA = O$ and $AO = O$
Dimension property	The product of an $m \times n$ matrix and an $n \times k$ matrix is an $m \times k$ matrix.

Let's take a look at matrix multiplication and explore these properties.

What you should be familiar with before taking this lesson

In **matrix multiplication**, each entry in the product matrix is the dot product of a row in the first matrix and a column in the second matrix.

$$\begin{array}{c} b_1 \quad b_2 \\ \downarrow \quad \downarrow \end{array}$$

$$\begin{array}{c} a_1 \rightarrow \\ a_2 \rightarrow \end{array} \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} \vec{a_1} \cdot \vec{b_1} & \vec{a_1} \cdot \vec{b_2} \\ \vec{a_2} \cdot \vec{b_1} & \vec{a_2} \cdot \vec{b_2} \end{bmatrix}$$

$$\quad\quad\quad A \quad\quad\quad B \quad\quad\quad\quad\quad C$$

If this is new to you, we recommend that you check out our matrix multiplication article.

Here are other relevant articles:

- Matrix multiplication dimensions
- Intro to identity matrices
- Properties of matrix addition

Matrix multiplication is not commutative

One of the biggest differences between real number multiplication and matrix multiplication is that matrix multiplication is **not commutative**.

In other words, in matrix multiplication, the order in which two matrices are multiplied matters!

See for yourselves!

Let's take a look at a concrete example with the following matrices.

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \quad\quad B = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}$$

1) Find AB **and** BA.

$$AB = \begin{bmatrix} & \end{bmatrix} \qquad BA = \begin{bmatrix} & \end{bmatrix}$$

Check

[I need help!]

Notice that the products are not the same! Since $AB \neq BA$, matrix multiplication is not commutative!

Other than this major difference, however, the properties of matrix multiplication are mostly similar to the properties of real number multiplication.

Associative property of multiplication: $(AB)C = A(BC)$

This property states that you can change the grouping surrounding matrix multiplication.

For example, you can multiply matrix A by matrix B, and then multiply the result by matrix C, **or** you can multiply matrix B by matrix C, and then multiply the result by matrix A.

When using this property, be sure to pay attention to the order in which the matrices are multiplied, since we know that the commutative property does not hold for matrix multiplication!

[I'd like to see an example, please!]

Distributive properties

We can distribute matrices in much the same way we distribute real numbers.

- $A(B + C) = AB + AC$

- $(B + C)A = BA + CA$

If a matrix A is distributed from the left side, be sure that each product in the resulting sum has A on the left! Similarly, if a matrix A is distributed from the right side, be sure that each product in the resulting sum has A on the right!

[I'd like to see an example, please!]

Multiplicative identity property

The $n \times n$ **identity matrix**, denoted I_n, is a matrix with n rows and n columns. The entries on the diagonal from the upper left to the bottom right are all 1's, and all other entries are 0.

For example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The multiplicative identity property states that the product of any $n \times n$ matrix A and I_n is always A, regardless of the order in which the multiplication was performed. In other words, $A \cdot I = I \cdot A = A$.

[I'd like to see an example, please!]

The role that the $n \times n$ identity matrix plays in matrix multiplication is similar to the role that the number 1 plays in the real number system. If a is a real number, then we know that $a \cdot 1 = a$ and $1 \cdot a = a$.

Multiplicative property of zero

A **zero matrix** is a matrix in which all of the entries are 0. For example, the 3×3 zero matrix is $O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

A zero matrix is indicated by O, and a subscript can be added to indicate the dimensions of the matrix if necessary.
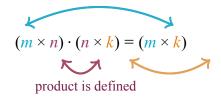
The multiplicative property of zero states that the product of any $n \times n$ matrix and the $n \times n$ zero matrix is the $n \times n$ zero matrix. In other words, $A \cdot O = O \cdot A = O$.

[I'd like to see an example, please!]

The role that the $n \times n$ zero matrix plays in matrix multiplication is similar to the role that the number 0 plays in the real number system. If a is a real number, then we know that $a \cdot 0 = 0$ and $0 \cdot a = 0$.

The dimension property

One property that is unique to matrices is the dimension property. This property has two parts:

$$(m \times n) \cdot (n \times k) = (m \times k)$$

product is defined

1. The product of two matrices will be defined if the number of columns in the first matrix is equal to the number of rows in the second matrix.

2. If the product is defined, the resulting matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix.

For example, if A is a 3×2 matrix and if B is a 2×4 matrix, the dimension property tells us:
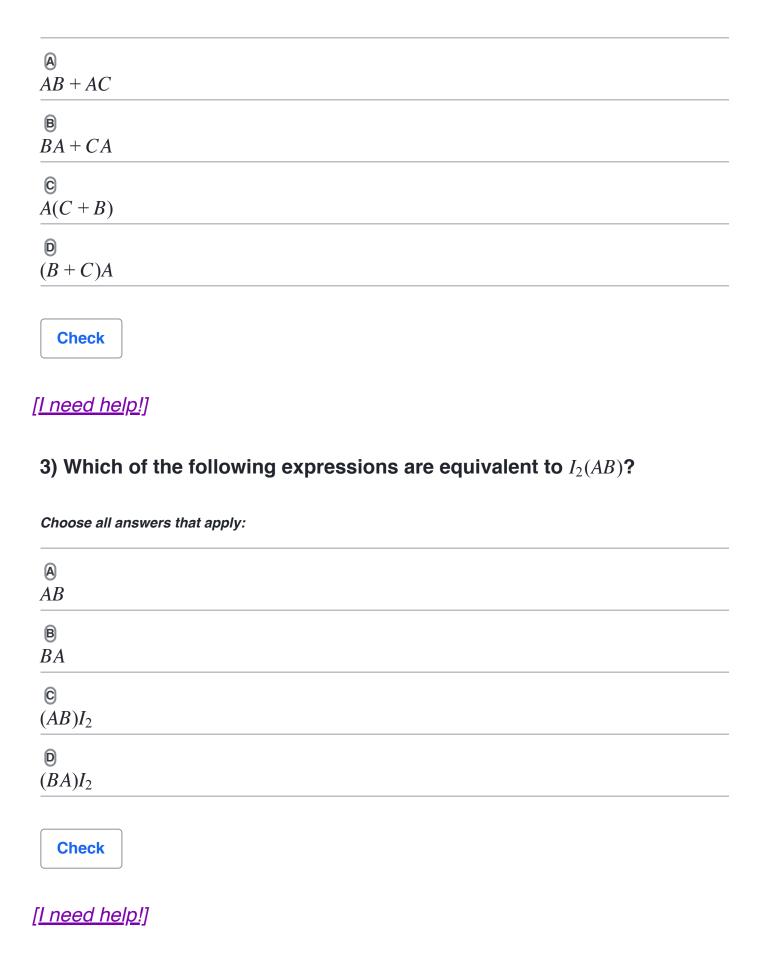
- The product AB is defined.
- AB will be a 3×4 matrix.

Check your understanding

Now that you are familiar with matrix multiplication and its properties, let's see if you can use them to determine equivalent matrix expressions.

For the problems below, let A, B, and C be 2×2 matrices and let O be the 2×2 zero matrix.

2) Which of the following expressions are equivalent to $A(B+C)$?

Choose all answers that apply:

3) Which of the following expressions are equivalent to $I_2(AB)$?

Choose all answers that apply:

(A)
AB

(B)
BA

(C)
$(AB)I_2$

(D)
$(BA)I_2$

Check

[I need help!]

4) Which of the following expressions are equivalent to $O(A + B)$?

Choose all answers that apply:

A

O

B

$A + B$

C

$B + A$

D

$(A + B)O$