Week II

Systems of linear equations

In this week's lectures, we learn about solving a system of linear equations. A system of linear equations can be written in matrix form, and we can solve using Gaussian elimination. We will learn how to bring a matrix to reduced row echelon form, and how this can be used to compute a matrix inverse. We will also learn how to find the LU decomposition of a matrix, and how to use this decomposition to efficiently solve a system of linear equations.

Lecture 10

Gaussian elimination

View this lecture on YouTube

Consider the linear system of equations given by

$$\begin{aligned}
-3x_1 + 2x_2 - x_3 &= -1, \\
6x_1 - 6x_2 + 7x_3 &= -7, \\
3x_1 - 4x_2 + 4x_3 &= -6,
\end{aligned}$$

which can be written in matrix form as

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix},$$

or symbolically as $\mathrm{Ax} = \mathrm{b}$.

The standard numerical algorithm used to solve a system of linear equations is called *Gaussian elimination*. We first form what is called an *augmented matrix* by combining the matrix A with the column vector b:

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{pmatrix}.$$

Row reduction is then performed on this augmented matrix. Allowed operations are (1) interchange the order of any rows, (2) multiply any row by a constant, (3) add a multiple of one row to another row. These three operations do not change the solution of the original equations. The goal here is to convert the matrix A into upper-triangular form, and then use this form to quickly solve for the unknowns x.

We start with the first row of the matrix and work our way down as follows. First we multiply the first row by 2 and add it to the second row. Then we add the first row to the third row, to obtain

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix}.$$

We then go to the second row. We multiply this row by -1 and add it to the third row to obtain

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

The original matrix A has been converted to an upper triangular matrix, and the transformed equations can be determined from the augmented matrix as

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1, \\ -2x_2 + 5x_3 &= -9, \\ -2x_3 &= 2. \end{aligned}$$

These equations can be solved by back substitution, starting from the last equation and working backwards. We have

$$\begin{aligned} x_3 &= -1, \\ x_2 &= -\frac{1}{2}(-9 - 5x_3) = 2, \\ x_1 &= -\frac{1}{3}(-1 + x_3 - 2x_2) = 2. \end{aligned}$$

We have thus found the solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

When performing Gaussian elimination, the diagonal element that is used during the elimination procedure is called the *pivot*. To obtain the correct multiple, one uses the pivot as the divisor to the matrix elements below the pivot. Gaussian elimination in the way done here will fail if the pivot is zero. If the pivot is zero, a row interchange must first be performed.

Even if no pivots are identically zero, small values can still result in an unstable numerical computation. For very large matrices solved by a computer, the solution vector will be inaccurate unless row interchanges are made. The resulting numerical technique is called Gaussian elimination with partial pivoting, and is usually taught in a standard numerical analysis course.

Problems for Lecture 10

1. Using Gaussian elimination with back substitution, solve the following two systems of equations:

(a)

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= -7, \\ -3x_1 + 5x_2 + x_3 &= 5, \\ 6x_1 - 4x_2 &= 2. \end{aligned}$$

(b)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1, \\ -x_1 + 3x_2 - x_3 &= -1, \\ 2x_1 - 5x_2 + 5x_3 &= 1. \end{aligned}$$

Solutions to the Problems

Lecture 11

Reduced row echelon form

View this lecture on YouTube

If we continue the row elimination procedure so that all the pivots are one, and all the entries above and below the pivots are eliminated, then we say that the resulting matrix is in reduced row echelon form. We notate the reduced row echelon form of a matrix A as rref(A). For example, consider the three-by-four matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}.$$

Row elimination can proceed as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

and we therefore have

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We say that the matrix A has two pivot columns, that is two columns that contain a pivot position with a one in the reduced row echelon form. Note that rows may need to be exchanged when computing the reduced row echelon form.

Problems for Lecture 11

1. Put the following matrices into reduced row echelon form and state which columns are pivot columns:

(a)

$$A = \begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{pmatrix}$$

Solutions to the Problems

Lecture 12

Computing inverses

View this lecture on YouTube

By bringing an invertible matrix to reduced row echelon form, that is, to the identity matrix, we can compute the matrix inverse. Given a matrix A, consider the equation

$$\mathrm{A}\mathrm{A}^{-1} = \mathrm{I},$$

for the unknown inverse A^{-1}. Let the columns of A^{-1} be given by the vectors a_1^{-1}, a_2^{-1}, and so on. The matrix A multiplying the first column of A^{-1} is the equation

$$\mathrm{A}a_1^{-1} = \mathrm{e}_1, \quad \text{with } \mathrm{e}_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^{\mathrm{T}},$$

and where e_1 is the first column of the identity matrix. In general,

$$\mathrm{A}a_i^{-1} = \mathrm{e}_i,$$

for $i = 1, 2, \dots, n$. The method then is to do row reduction on an augmented matrix which attaches the identity matrix to A. To find A^{-1}, elimination is continued until one obtains $\mathrm{rref}(\mathrm{A}) = \mathrm{I}$.

We illustrate below:

$$\begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 6 & -6 & 7 & 0 & 1 & 0 \\ 3 & -4 & 4 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} -3 & 0 & 4 & 3 & 1 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} -3 & 0 & 0 & 1 & -1 & 2 \\ 0 & -2 & 0 & -1/2 & -3/2 & 5/2 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 1/4 & 3/4 & -5/4 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{pmatrix};$$

and one can check that

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} -1/3 & 1/3 & -2/3 \\ 1/4 & 3/4 & -5/4 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problems for Lecture 12

1. Compute the inverse of
$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}.$$

Solutions to the Problems

Practice quiz: Gaussian elimination

1. Perform Gaussian elimination without row interchange on the following augmented matrix:
$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -2 \end{pmatrix}$. Which matrix can be the result?

a) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$

2. Which matrix is not in reduced row echelon form?

a) $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. The inverse of $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$ is

a) $\begin{pmatrix} 4/3 & 2/3 & 1/2 \\ 2 & 1 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 2/3 & 1/2 & 4/3 \\ 1 & 1/2 & 2 \\ -3 & -5 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -5 & -3 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$

Solutions to the Practice quiz

Lecture 13

Elementary matrices

View this lecture on YouTube

The row reduction algorithm of Gaussian elimination can be implemented by multiplying elementary matrices. Here, we show how to construct these elementary matrices, which differ from the identity matrix by a single elementary row operation. Consider the first row reduction step for the following matrix A:

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1A, \quad \text{where } M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To construct the elementary matrix M_1, the number two is placed in column-one, row-two. This matrix multiplies the first row by two and adds the result to the second row.

The next step in row elimination is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2M_1A, \quad \text{where } M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Here, to construct M_2 the number one is placed in column-one, row-three, and the matrix multiplies the first row by one and adds the result to the third row.

The last step in row elimination is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_3M_2M_1A, \quad \text{where } M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Here, to construct M_3 the number negative-one is placed in column-two, row-three, and this matrix multiplies the second row by negative-one and adds the result to the third row.

We have thus found that

$$M_3M_2M_1A = U,$$

where U is an upper triangular matrix. This discussion will be continued in the next lecture.

Problems for Lecture 13

1. Construct the elementary matrix that multiplies the second row of a four-by-four matrix by two and adds the result to the fourth row.

Solutions to the Problems

Lecture 14

LU decomposition

View this lecture on YouTube

In the last lecture, we have found that row reduction of a matrix A can be written as

$$M_3M_2M_1A = U,$$

where U is upper triangular. Upon inverting the elementary matrices, we have

$$A = M_1^{-1}M_2^{-1}M_3^{-1}U.$$

Now, the matrix M_1 multiples the first row by two and adds it to the second row. To invert this operation, we simply need to multiply the first row by negative-two and add it to the second row, so that

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly,

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}; \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$L = M_1^{-1}M_2^{-1}M_3^{-1}$$

is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix},$$

which is lower triangular. Also, the non-diagonal elements of the elementary inverse matrices are simply combined to form L. Our LU decomposition of A is therefore

$$\begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}.$$

Problems for Lecture 14

1. Find the LU decomposition of

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}.$$

Solutions to the Problems

Lecture 15

Solving (LU)x = b

View this lecture on YouTube

The LU decomposition is useful when one needs to solve $Ax = b$ for many right-hand-sides. With the LU decomposition in hand, one writes

$$(LU)x = L(Ux) = b,$$

and lets $y = Ux$. Then we solve $Ly = b$ for y by forward substitution, and $Ux = y$ for x by backward substitution. It is possible to show that for large matrices, solving $(LU)x = b$ is substantially faster than solving $Ax = b$ directly.

We now illustrate the solution of $LUx = b$, with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

With $y = Ux$, we first solve $Ly = b$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}.$$

Using forward substitution

$$\begin{aligned} y_1 &= -1, \\ y_2 &= -7 + 2y_1 = -9, \\ y_3 &= -6 + y_1 - y_2 = 2. \end{aligned}$$

We then solve $Ux = y$, that is

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix}.$$

Using back substitution,

$$\begin{aligned}
x_3 &= -1, \\
x_2 &= -\frac{1}{2}(-9 - 5x_3) = 2, \\
x_1 &= -\frac{1}{3}(-1 - 2x_2 + x_3) = 2,
\end{aligned}$$

and we have found

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Problems for Lecture 15

1. Using

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU,$$

compute the solution to $Ax = b$ with

(a) $b = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$, (b) $b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solutions to the Problems

Practice quiz: LU decomposition

1. Which of the following is the elementary matrix that multiplies the second row of a four-by-four matrix by 2 and adds the result to the third row?

a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

2. Which of the following is the LU decomposition of $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$?

a) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1/2 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 2 & -10 & 6 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ -6 & 14 & 3 \end{pmatrix}$

3. Suppose $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Solve $LUx = b$ by letting $y = Ux$. The solutions for y and x are

a) $y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} 1/6 \\ 1/2 \\ -1 \end{pmatrix}$

b) $y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$

c) $y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $x = \begin{pmatrix} 1/6 \\ -1/2 \\ 1 \end{pmatrix}$

d) $y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} -1/6 \\ 1/2 \\ 1 \end{pmatrix}$

Solutions to the Practice quiz