Solutions to the Practice quiz: Orthogonal matrices

1. d. An orthogonal matrix has orthonormal rows and columns. The rows and columns of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ are not orthonormal and therefore this matrix is not an orthogonal matrix.

2. a. The rotation matrix representing a counterclockwise rotation around the x-axis in the y-z plane can be obtained from the rotation matrix representing a counterclockwise rotation around the z-axis in the x-y plane by shifting the elements to the right one column and down one row, assuming a periodic extension of the matrix. The result is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

3. b. Interchange the rows of the identity matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Solutions to the Problems for Lecture 10

1.

(a) Row reduction of the augmented matrix proceeds as follows:

$$\begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{pmatrix}.$$

Solution by back substitution is given by

$$\begin{aligned} x_3 &= -6, \\ x_2 &= -\frac{1}{2}(x_3 - 2) = 4, \\ x_1 &= \frac{1}{3}(7x_2 + 2x_3 - 7) = 3. \end{aligned}$$

The solution is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}.$$

(b) Row reduction of the augmented matrix proceeds as follows:

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ -1 & 3 & -1 & -1 \\ 2 & -5 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Solution by back substitution is given by

$$\begin{aligned} x_3 &= -1, \\ x_2 &= -2x_3 = 2, \\ x_1 &= 2x_2 - 3x_3 + 1 = 8. \end{aligned}$$

The solution is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 11

1.

(a) Row reduction proceeds as follows:

$$A = \begin{pmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 3/2 & 0 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{pmatrix} \to$$
$$\begin{pmatrix} 3 & 0 & 0 & 9 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & -1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{pmatrix}.$$

Here, columns one, two, and three are pivot columns.

(b) Row reduction proceeds as follows:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, columns one and three are pivot columns.

Solutions to the Problems for Lecture 12

1.

$$
\begin{pmatrix}
3 & -7 & -2 & 1 & 0 & 0\\
-3 & 5 & 1 & 0 & 1 & 0\\
6 & -4 & 0 & 0 & 0 & 1
\end{pmatrix}
\to
\begin{pmatrix}
3 & -7 & -2 & 1 & 0 & 0\\
0 & -2 & -1 & 1 & 1 & 0\\
0 & 10 & 4 & -2 & 0 & 1
\end{pmatrix}
\to
$$
$$
\begin{pmatrix}
3 & 0 & 3/2 & -5/2 & -7/2 & 0\\
0 & -2 & -1 & 1 & 1 & 0\\
0 & 0 & -1 & 3 & 5 & 1
\end{pmatrix}
\to
\begin{pmatrix}
3 & 0 & 3/2 & -5/2 & -7/2 & 0\\
0 & -2 & -1 & 1 & 1 & 0\\
0 & 0 & -1 & 3 & 5 & 1
\end{pmatrix}
\to
$$
$$
\begin{pmatrix}
1 & 0 & 0 & 2/3 & 4/3 & 1/2\\
0 & 1 & 0 & 1 & 2 & 1/2\\
0 & 0 & 1 & -3 & -5 & -1
\end{pmatrix}.
$$

Therefore,

$$
\begin{pmatrix}
3 & -7 & -2\\
-3 & 5 & 1\\
6 & -4 & 0
\end{pmatrix}^{-1}
=
\begin{pmatrix}
2/3 & 4/3 & 1/2\\
1 & 2 & 1/2\\
-3 & -5 & -1
\end{pmatrix}.
$$

Solutions to the Practice quiz: Gaussian elimination

1. a. $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix}$.

2. c. A matrix in reduced row echelon form has all its pivots equal to one, and all the entries above and below the pivots eliminated. The only matrix that is not in reduced row echelon form is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. The pivot in the third row, third column has a one above it in the first row, third column.

3. d. There are many ways to do this computation by hand, and here is one way:
$\begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 6 & -4 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 10 & 4 & -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 & 5 & 1 \end{pmatrix} \to$
$\begin{pmatrix} 3 & -7 & 0 & -5 & -10 & -2 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & 0 & -5 & -10 & -2 \\ 0 & -2 & 0 & -2 & -4 & -1 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 0 & 2 & 4 & 3/2 \\ 0 & 1 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix} \to$
$\begin{pmatrix} 1 & 0 & 0 & 2/3 & 4/3 & 1/2 \\ 0 & 1 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -3 & -5 & -1 \end{pmatrix}$. Therefore, $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2/3 & 4/3 & 1/2 \\ 1 & 2 & 1/2 \\ -3 & -5 & -1 \end{pmatrix}$.

Solutions to the Problems for Lecture 13

1.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 14

1.

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 15

1. We know

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU.$$

To solve $LUx = b$, we let $y = Ux$, solve $Ly = b$ for y, and then solve $Ux = y$ for x.

(a)

$$b = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$$

The equations $Ly = b$ are given by

$$\begin{aligned} y_1 &= -3, \\ -y_1 + y_2 &= 3, \\ 2y_1 - 5y_2 + y_3 &= 2, \end{aligned}$$

with solution $y_1 = -3$, $y_2 = 0$, and $y_3 = 8$. The equations $Ux = y$ are given by

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= -3 \\ -2x_2 - x_3 &= 0 \\ -x_3 &= 8, \end{aligned}$$

with solution $x_3 = -8$, $x_2 = 4$, and $x_1 = 3$.

(b)

$$b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

The equations $Ly = b$ are given by

$$\begin{aligned} y_1 &= 1, \\ -y_1 + y_2 &= -1, \\ 2y_1 - 5y_2 + y_3 &= 1, \end{aligned}$$

with solution $y_1 = 1$, $y_2 = 0$, and $y_3 = -1$. The equations $Ux = y$ are given by

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= 1 \\ -2x_2 - x_3 &= 0 \\ -x_3 &= -1, \end{aligned}$$

with solution $x_3 = 1$, $x_2 = -1/2$, and $x_1 = -1/6$.

Solutions to the Practice quiz: LU decomposition

1. c. Start with the identity matrix. In the third row (changed row) and second column (the row which is multiplied by 2) , place a 2. The elementary matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

2. b.

$$A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} = M_1A, \quad \text{where } M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} = M_2M_1A, \quad \text{where } M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = M_3M_2M_1A, \quad \text{where } M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}.$$

Therefore,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

3. b. To solve $LUx = b$, let $y = Ux$. Then solve $Ly = b$ for y and $Ux = y$ for x. The equations given by $Ly = b$ are

$$\begin{aligned} y_1 &= 1, \\ -y_1 + y_2 &= -1, \\ 2y_1 - 5y_2 + y_3 &= 1. \end{aligned}$$

Solution by forward substitution gives $y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The equations given by $Ux = y$ are

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= 1, \\ -2x_2 - x_3 &= 0, \\ -x_3 &= -1. \end{aligned}$$

Solution by backward substitution gives $x = \begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$.