# Matrix row operations

The following table summarizes the three elementary **matrix row operations**.

| Matrix row operation | Example |
|---|---|
| **Switch any two rows** | $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ <br> (Interchange row 1 and row 2.) |
| **Multiply a row by a nonzero constant** | $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \cdot 2 & 3 \cdot 5 & 3 \cdot 3 \\ 3 & 4 & 6 \end{bmatrix}$ <br> (Row 1 becomes 3 times itself.) |
| **Add one row to another** | $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 3+2 & 4+5 & 6+3 \end{bmatrix}$ <br> (Row 2 becomes the sum of rows 2 and 1.) |

Matrix row operations can be used to solve systems of equations, but before we look at why, let's practice these skills.

# Switch any two rows

## Example

Perform the row operation $R_1 \leftrightarrow R_2$ on the following matrix.

$$\begin{bmatrix} 4 & 8 & 3 \\ 2 & 4 & 5 \\ 7 & 1 & 2 \end{bmatrix}$$

## Solution

$R_1 \leftrightarrow R_2$ means to interchange row $1$ and row $2$.

So the matrix $\begin{bmatrix} 4 & 8 & 3 \\ 2 & 4 & 5 \\ 7 & 1 & 2 \end{bmatrix}$ becomes $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix}$.

Sometimes you will see the following notation used to indicate this change.

$$\begin{bmatrix} 4 & 8 & 3 \\ 2 & 4 & 5 \\ 7 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

Notice how row $1$ replaces row $2$ and row $2$ replaces row $1$. The third row is not changed.

PROBLEM 1

**Perform the row operation $R_2 \leftrightarrow R_3$ on the following matrix.**

$$\begin{bmatrix} 7 & 2 & 9 \\ 6 & 4 & 1 \\ 1 & 3 & 12 \end{bmatrix}$$

[ ]

Check

Explain

PROBLEM 2

**Perform the row operation $R_3 \leftrightarrow R_1$ on the following matrix.**

$$\begin{bmatrix} 2 & 11 & -5 \\ 7 & 2 & 18 \\ 0 & -4 & 10 \end{bmatrix}$$

$$\begin{bmatrix} \end{bmatrix}$$

Check

Explain

# Multiply a row by a nonzero constant

## Example

Perform the row operation $3R_2 \to R_2$ on the following matrix.

$$\begin{bmatrix} 6 & 6 & 1 \\ 2 & 3 & 0 \\ 4 & 5 & 9 \end{bmatrix}$$

## Solution

$3R_2 \to R_2$ means to replace the 2nd row with 3 times itself.

$$\begin{bmatrix} 6 & 6 & 1 \\ 2 & 3 & 0 \\ 4 & 5 & 9 \end{bmatrix} \text{ becomes } \begin{bmatrix} 6 & 6 & 1 \\ 3 \cdot 2 & 3 \cdot 3 & 3 \cdot 0 \\ 4 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 1 \\ 6 & 9 & 0 \\ 4 & 5 & 9 \end{bmatrix}$$

To indicate this matrix row operation, we often see the following:

$$\begin{bmatrix} 6 & 6 & 1 \\ 2 & 3 & 0 \\ 4 & 5 & 9 \end{bmatrix} \xrightarrow{3R_2 \rightarrow R_2} \begin{bmatrix} 6 & 6 & 1 \\ 6 & 9 & 0 \\ 4 & 5 & 9 \end{bmatrix}$$

Notice here three times the second row replaces the second row. The other rows remain the same.

PROBLEM 3

**Perform the row operation $2R_1 \rightarrow R_1$ on the following matrix.**

$$\begin{bmatrix} 2 & 6 & 5 & 1 \\ 7 & 4 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix}$$

Check

Explain

PROBLEM 4

**Perform the row operation $-5R_3 \rightarrow R_3$ on the following matrix.**

$$\begin{bmatrix} -2 & 1 \\ 7 & 4 \\ -3 & 6 \end{bmatrix}$$

$\begin{bmatrix} & \end{bmatrix}$

Check

Explain

# Add one row to another

## Example

Perform the row operation $R_1 + R_2 \rightarrow R_2$ on the following matrix.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 8 & 1 \end{bmatrix}$$

## Solution

$R_1 + R_2 \rightarrow R_2$ means to replace the $2\text{nd}$ row with the sum of the $1\text{st}$ and $2\text{nd}$rows.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 8 & 1 \end{bmatrix} \text{ becomes } \begin{bmatrix} 2 & 3 & 4 \\ 2+0 & 3+8 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 11 & 5 \end{bmatrix}$$

To indicate this matrix row operation, we can write the following:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 8 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 2 & 3 & 4 \\ 2 & 11 & 5 \end{bmatrix}$$

Notice how the sum of row $1$ and $2$ replaces row $2$. The other row remains the same.

PROBLEM 5

**Perform the row operation $R_1 + R_3 \rightarrow R_3$ on the following matrix.**

$$\begin{bmatrix} -1 & 6 & -2 \\ -3 & 5 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \end{bmatrix}$$

Check

Explain

PROBLEM 6

**Perform the row operation $R_2 + R_3 \rightarrow R_2$ on the following matrix.**

$$\begin{bmatrix} -4 & 12 & 9 \\ 7 & 4 & 2 \\ 1 & 5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} \end{bmatrix}$$

Check

Explain

CHALLENGE PROBLEM

**Perform the row operation $R_1 + 2R_3 \to R_1$ on the following matrix.**

$$\begin{bmatrix} -5 & 7 & 3 \\ -2 & -1 & 4 \\ 8 & 8 & -6 \end{bmatrix}$$

$$\begin{bmatrix} \end{bmatrix}$$

Check

Explain

## Systems of equations and matrix row operations

Recall that in an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.

For example, the system on the left corresponds to the augmented matrix on the right.

| System | Matrix |
|---|---|
| $1x + 3y = 5$<br>$2x + 5y = 6$ | $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$ |

When working with augmented matrices, we can perform any of the **matrix row operations** to create a new augmented matrix that produces an equivalent system of equations. Let's take a look at why.

## Switching any two rows

| Equivalent Systems | Augmented matrix |
|---|---|
| $1x+3y=5$<br>$2x+5y=6$ | $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$ |
| | $\downarrow$ |
| $2x+5y=6$<br>$1x+3y=5$ | $\begin{bmatrix} 2 & 5 & 6 \\ 1 & 3 & 5 \end{bmatrix}$ |

The two systems in the above table are equivalent, because the order of the equations doesn't matter. This means that when using an augmented matrix to solve a system, we can **interchange any two rows**.

## Multiply a row by a nonzero constant

We can multiply both sides of an equation by the same nonzero constant to obtain an equivalent equation.

In solving systems of equations, we often do this to eliminate a variable. Because the two equations are equivalent, we see that the two systems are also equivalent.

| Equivalent Systems | Augmented matrix |
|---|---|
| $1x+3y=5$<br>$2x+5y=6$ | $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$ |
| | $\downarrow$ |
| $-2x+(-6)y=-10$<br>$2x+5y=6$ | $\begin{bmatrix} -2 & -6 & -10 \\ 2 & 5 & 6 \end{bmatrix}$ |

This means that when using an augmented matrix to solve a system, we can **multiply any row by a nonzero constant**.

## Add one row to another

We know that we can add two equal quantities to both sides of an equation to obtain an equivalent equation.

So if $A = B$ and $C = D$, then $A + C = B + D$.

We do this often when solving systems of equations. For example, in this system $\begin{aligned} -2x - 6y &= -10 \\ 2x + 5y &= 6 \end{aligned}$, we can add the equations to obtain $-y = -4$.

Pairing this new equation with either original equation creates an equivalent system of equations. *[Why do we keep one of the original equations?]*

| Equivalent Systems | Augmented matrix |
|---|---|
| $\begin{aligned} -2x - 6y &= -10 \\ 2x + 5y &= 6 \end{aligned}$ | $\left[\begin{array}{ccc} -2 & -6 & -10 \\ 2 & 5 & 6 \end{array}\right]$ |
| | $\downarrow$ |
| $\begin{aligned} -2x + (-6)y &= -10 \\ 0x + (-1)y &= -4 \end{aligned}$ | $\left[\begin{array}{ccc} -2 & -6 & -10 \\ 0 & -1 & -4 \end{array}\right]$ |

So when using an augmented matrix to solve a system, we can **add one row to another**.

CONCLUDING CHALLENGE PROBLEM

A sequence of row operations is performed on the matrix $\begin{bmatrix} 2 & 2 & 10 \\ -2 & -3 & 3 \end{bmatrix}$.

The table below describes the result of each step in the sequence.

**Arrange the row operations according to each step.**

Original matrix: $\begin{bmatrix} 2 & 2 & 10 \\ -2 & -3 & 3 \end{bmatrix}$

| Step | Row operation |
|---|---|
| Step 1: $\begin{bmatrix} 2 & 2 & 10 \\ 0 & -1 & 13 \end{bmatrix}$ | $R_1 - R_2 \longrightarrow R_1$ |
| Step 2: $\begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 13 \end{bmatrix}$ | $\frac{1}{2}R_1 \longrightarrow R_1$ |
| Step 3: $\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -13 \end{bmatrix}$ | $R_1 + R_2 \longrightarrow R_2$ |
| Step 4: $\begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & -13 \end{bmatrix}$ | $-1R_2 \longrightarrow R_2$ |

Check

Explain

Notice that the original matrix corresponds to $\begin{aligned} 2x + 2y &= 10 \\ -2x - 3y &= 3 \end{aligned}$, while the final matrix corresponds to $\begin{aligned} x &= 18 \\ y &= -13 \end{aligned}$ which simply gives the solution.

The system was solved entirely by using augmented matrices and row operations!