# Appendix A

# Problem and practice quiz solutions

### Solutions to the Problems for Lecture 1

**1.**

$$a)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b)\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad c)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

### Solutions to the Problems for Lecture 2

**1.** $B - 2A = \begin{pmatrix} 0 & -4 & 3 \\ 0 & -2 & -4 \end{pmatrix}$, $\quad 3C - E:$ not defined, $\quad AC:$ not defined,

$$CD = \begin{pmatrix} 11 & 10 \\ 10 & 11 \end{pmatrix}, \quad CB = \begin{pmatrix} 8 & -10 & -3 \\ 10 & -8 & 0 \end{pmatrix}.$$

**2.** $AB = AC = \begin{pmatrix} 4 & 7 \\ 8 & 14 \end{pmatrix}$.

**3.** $AD = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 2 & 9 & 16 \end{pmatrix}$, $\quad DA = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 4 & 12 & 16 \end{pmatrix}$.

**4.** $[A(BC)]_{ij} = \sum_{k=1}^{n} a_{ik}[BC]_{kj} = \sum_{k=1}^{n}\sum_{l=1}^{p} a_{ik}b_{kl}c_{lj} = \sum_{l=1}^{p}\sum_{k=1}^{n} a_{ik}b_{kl}c_{lj} = \sum_{l=1}^{p}[AB]_{il}c_{lj} = [(AB)C]_{ij}$.

## Solutions to the Problems for Lecture 3

**1.** $\begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**2.** Let A be an $m$-by-$p$ diagonal matrix, B a $p$-by-$n$ diagonal matrix, and let $\mathrm{C} = \mathrm{AB}$. The $ij$ element of C is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Since A is a diagonal matrix, the only nonzero term in the sum is $k = i$ and we have $c_{ij} = a_{ii} b_{ij}$. And since B is a diagonal matrix, the only nonzero elements of C are the diagonal elements $c_{ii} = a_{ii} b_{ii}$.

**3.** Let A and B be $n$-by-$n$ upper triangular matrices, and let $\mathrm{C} = \mathrm{AB}$. The $ij$ element of C is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Since A and B are upper triangular, we have $a_{ik} = 0$ when $k < i$ and $b_{kj} = 0$ when $k > j$. Excluding the zero terms from the summation, we have

$$c_{ij} = \sum_{k=i}^{j} a_{ik} b_{kj},$$

which is equal to zero when $i > j$ proving that C is upper triangular. Furthermore, $c_{ii} = a_{ii} b_{ii}$.

## Solutions to the Practice quiz: Matrix definitions

**1.** d. With $a_{ij} = i - j$, we have $a_{11} = a_{22} = 0$, $a_{12} = -1$, and $a_{21} = 1$. Therefore $\mathrm{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

**2.** a. $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

**3.** b. For upper triangular matrices A and B, $a_{ik} = 0$ when $k < i$ and $b_{kj} = 0$ when $k > j$.

## Solutions to the Problems for Lecture 4

**1.** Let A be an $m$-by-$p$ matrix, B a $p$-by-$n$ matrix, and $\mathrm{C} = \mathrm{AB}$ an $m$-by-$n$ matrix. We have

$$c_{ij}^{\mathrm{T}} = c_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} b_{ik}^{\mathrm{T}} a_{kj}^{\mathrm{T}}.$$

With $\mathrm{C}^{\mathrm{T}} = (\mathrm{AB})^{\mathrm{T}}$, we have proved that $(\mathrm{AB})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$.

**2.** The square matrix $\mathrm{A} + \mathrm{A}^{\mathrm{T}}$ is symmetric, and the square matrix $\mathrm{A} - \mathrm{A}^{\mathrm{T}}$ is skew symmetric. Using these two matrices, we can write

$$\mathrm{A} = \frac{1}{2}\left(\mathrm{A} + \mathrm{A}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathrm{A} - \mathrm{A}^{\mathrm{T}}\right).$$

**3.** Let A be a $m$-by-$n$ matrix. Then using $(\mathrm{AB})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$ and $(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}} = \mathrm{A}$, we have

$$(\mathrm{A}^{\mathrm{T}}\mathrm{A})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}}\mathrm{A}.$$

## Solutions to the Problems for Lecture 5

**1.**

$$\mathrm{A}^{\mathrm{T}}\mathrm{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 & ad + be + cf \\ ad + be + cf & d^2 + e^2 + f^2 \end{pmatrix}.$$

**2.** Let A be an $m$-by-$n$ matrix. Then

$$\mathrm{Tr}(\mathrm{A}^{\mathrm{T}}\mathrm{A}) = \sum_{j=1}^{n} (\mathrm{A}^{\mathrm{T}}\mathrm{A})_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ji}^{\mathrm{T}} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2,$$

which is the sum of the squares of all the elements of A.

## Solutions to the Problems for Lecture 6

**1.** $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -6 \\ -4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 4 \\ 3 & 3 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -4 \\ -3 & 6 \end{pmatrix}$.

**2.** From the definition of an inverse,

$$(AB)^{-1}(AB) = I.$$

Multiply on the right by $B^{-1}$, and then by $A^{-1}$, to obtain

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**3.** We assume that A is invertible so that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

Taking the transpose of both sides of these two equations, using both $I^T = I$ and $(AB)^T = B^T A^T$, we obtain

$$(A^{-1})^T A^T = I \quad \text{and} \quad A^T (A^{-1})^T = I.$$

We can therefore conclude that $A^T$ is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

**4.** Let A be an invertible matrix, and suppose B and C are its inverse. To prove that $B = C$, we write

$$B = BI = B(AC) = (BA)C = C.$$

### Solutions to the Practice quiz: Transpose and inverses

**1.** d. $(ABC)^T = ((AB)C)^T = C^T(AB)^T = C^TB^TA^T$.

**2.** c. A symmetric matrix C satisfies $C^T = C$. We can test all four matrices.
$(A + A^T)^T = A^T + A = A + A^T$;
$(AA^T)^T = AA^T$;
$(A - A^T)^T = A^T - A = -(A - A^T)$;
$(A^TA)^T = A^TA$.
Only the third matrix is not symmetric. It is a skew-symmetric matrix, where $C^T = -C$.

**3.** a. Exchange the diagonal elements, negate the off-diagonal elements, and divide by the determinant.
We have $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.

## Solutions to the Problems for Lecture 7

**1.** Let $Q_1$ and $Q_2$ be orthogonal matrices. Then

$$(Q_1Q_2)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^TQ_1^T = (Q_1Q_2)^T.$$

**2.** Since $I\,I = I$, we have $I^{-1} = I$. And since $I^T = I$, we have $I^{-1} = I^T$ and $I$ is an orthogonal matrix.

## Solutions to the Problems for Lecture 8

**1.** $\mathrm{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathrm{R}(\theta)^{-1}$.

**2.** The $z$-coordinate stays fixed, and the vector rotates an angle $\theta$ in the $x$-$y$ plane. Therefore,

$$\mathrm{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

## Solutions to the Problems for Lecture 9

**1.**

$$\mathrm{P}_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{P}_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{P}_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{P}_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{P}_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{P}_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

**2.**

$$\mathrm{P}_{123}^{-1} = \mathrm{P}_{123}, \quad \mathrm{P}_{132}^{-1} = \mathrm{P}_{132}, \quad \mathrm{P}_{213}^{-1} = \mathrm{P}_{213}, \quad \mathrm{P}_{321}^{-1} = \mathrm{P}_{321},$$
$$\mathrm{P}_{231}^{-1} = \mathrm{P}_{312}, \quad \mathrm{P}_{312}^{-1} = \mathrm{P}_{231}.$$

The matrices that are their own inverses correspond to either no permutation or a single permutation of rows (or columns), e.g., $\{1,3,2\}$, which permutes row (column) two and three. The matrices that are not their own inverses correspond to two permutations, e.g., $\{2,3,1\}$, which permutes row (column) one and two, and then two and three. For example, commuting rows by left multiplication, we have

$$P_{231} = P_{132}P_{213},$$

so that the inverse matrix is given by

$$P_{231}^{-1} = P_{213}^{-1}P_{132}^{-1} = P_{213}P_{132}.$$

Because matrices in general do not commute, $P_{231}^{-1} \neq P_{231}$. Note also that the permutation matrices are orthogonal, so that the inverse matrices are equal to the transpose matrices. Therefore, only the symmetric permutation matrices can be their own inverses.

## Solutions to the Practice quiz: Orthogonal matrices

**1.** d. An orthogonal matrix has orthonormal rows and columns. The rows and columns of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ are not orthonormal and therefore this matrix is not an orthogonal matrix.

**2.** a. The rotation matrix representing a counterclockwise rotation around the $x$-axis in the $y$-$z$ plane can be obtained from the rotation matrix representing a counterclockwise rotation around the $z$-axis in the $x$-$y$ plane by shifting the elements to the right one column and down one row, assuming a periodic extension of the matrix. The result is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

**3.** b. Interchange the rows of the identity matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.