4.3.3 Generating orthogonal vectors

Consider two given vectors **u** and **v** in $\mathbb{R}^2$. We can project the vector **v** onto vector **u**.

What does projection mean?

Projection is the procedure, for example, of showing a film on a screen; we say the film has been projected onto a screen. Similarly we project the vector **v** onto **u** as shown in Fig. 4.15.

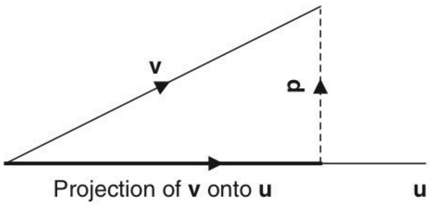

Figure 4.15

*What is the projection of **v** onto **u** equal to?*

Since it is in the direction of vector **u**, it is a scalar multiple of **u**. Let's nominate this scalar by the letter k, so we have

$$\text{Projection of } \mathbf{v} \text{ onto } \mathbf{u} = k\mathbf{u}$$

Let **p** be the perpendicular vector shown in Fig. 4.15. Adding the vectors gives

$$\mathbf{v} = k\mathbf{u} + \mathbf{p}$$

The vector **p** is orthogonal (perpendicular) to the vector **u**, therefore $\langle \mathbf{u}, \mathbf{p} \rangle = 0$. This means that the projection of the perpendicular vector **p** onto vector **u** is zero.

Taking the inner product of this, $\mathbf{v} = k\mathbf{u} + \mathbf{p}$, with the vector **u**, we have

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{u}, (k\mathbf{u} + \mathbf{p}) \rangle \\
&= k\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{p} \rangle \\
&= k\|\mathbf{u}\|^2 + 0 \qquad \left[\begin{array}{l}\text{because } \mathbf{p} \text{ and } \mathbf{u} \text{ are perpendicular,} \\ \text{therefore } \langle \mathbf{u}, \mathbf{p} \rangle = 0 \text{ and } \langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|^2\end{array}\right] \\
&= k\|\mathbf{u}\|^2
\end{aligned}$$

Rearranging this $\langle \mathbf{u}, \mathbf{v} \rangle = k\|\mathbf{u}\|^2$ to make k the subject of the formula gives

$$k = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|^2} \qquad (\dagger)$$

Rearranging the above $\mathbf{v} = k\mathbf{u} + \mathbf{p}$ to make **p** the subject:

$$\begin{aligned}
\text{orthogonal vector } \mathbf{p} &= \mathbf{v} - k\mathbf{u} \qquad && [\mathbf{p} = \mathbf{v} - [\text{projection of } \mathbf{v} \text{ onto } \mathbf{u}]] \\
&= \mathbf{v} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|^2}\mathbf{u} && [\text{by } (\dagger)]
\end{aligned}$$

We have perpendicular vector **p** (Fig. 4.16):

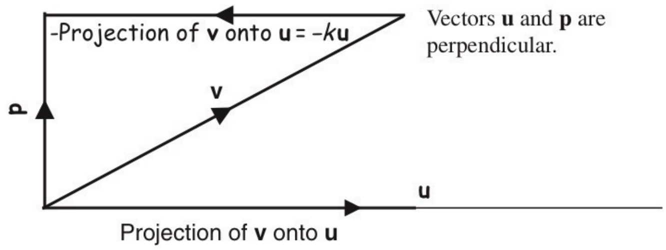

Figure 4.16

Hence we have created orthogonal vectors **p** and **u** out of the given non-orthogonal vectors **u** and **v**.

It is important to note that the projection of orthogonal vectors is zero. This is what we were hinting at in the last section; *orthogonality signifies a certain kind of independence or a complete absence of interference.*

Example 4.15

Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$. Construct a pair of orthogonal (perpendicular) vectors $\{\mathbf{p}_1, \mathbf{p}_2\}$ from this non-orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution

We start with one of the given vectors, $\mathbf{v}_1$ say, and call this vector $\mathbf{p}_1$. Hence $\mathbf{p}_1 = \mathbf{v}_1$. We construct a vector which is orthogonal to $\mathbf{p}_1 = \mathbf{v}_1$.

By the above formula, orthogonal vector $\mathbf{p}_2 = \mathbf{v} - \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|^2}\mathbf{u}$ with $\mathbf{u} = \mathbf{v}_1 = \mathbf{p}_1$ and $\mathbf{v} = \mathbf{v}_2$:

$$\text{orthogonal vector } \mathbf{p}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{p}_1, \mathbf{v}_2 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$$

Substituting $\mathbf{p}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ into this formula gives:

$$\text{orthogonal vector } \mathbf{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad (*)$$

Evaluating each component of (*):

$$\left\langle \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (3 \times 1) + (0 \times 2) = 3, \quad \left\| \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\|^2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3^2 + 0^2 = 9$$

Putting these values into (*) yields

$$\text{orthogonal vector } \mathbf{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{3}{9}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - 1 \\ 2 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Illustrating the orthogonal vector $\mathbf{p}_2$ and the given vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as shown in Fig. 4.17.

(continued...)

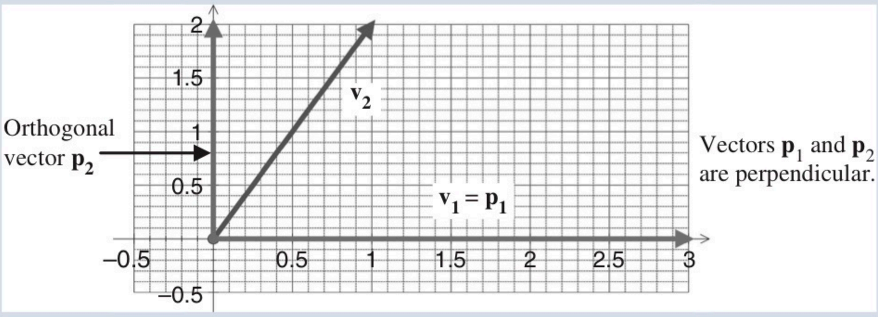

Figure 4.17

Note that we start with one of the given vectors, $\mathbf{v}_1$ say, then we create a vector orthogonal (perpendicular) to it by using the other given vector $\mathbf{v}_2$. We have achieved the following:

$$\left\{\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \Rightarrow \left\{\mathbf{p}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}$$

We can extend this procedure to any finite dimensional vector space and create an **orthogonal basis** for the vector space. Suppose we have a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$, and from this we want create an orthogonal basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_n\}$. The procedure is:

1. Let $\mathbf{p}_1 = \mathbf{v}_1$, that is $\mathbf{p}_1$ equals one of the given vectors.
2. We create vector $\mathbf{p}_2$, which is orthogonal to $\mathbf{p}_1 = \mathbf{v}_1$, by using the second given vector $\mathbf{v}_2$. Hence we apply the above stated formula $\mathbf{p}_2 = \mathbf{v}_2 - \dfrac{\langle \mathbf{v}_2, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1$.
3. We create vector $\mathbf{p}_3$, which is orthogonal to both $\mathbf{p}_1 = \mathbf{v}_1$ and $\mathbf{p}_2$, by using the third given vector $\mathbf{v}_3$. The formula for this is similarly produced:

$$\mathbf{p}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{p}_1 \rangle}{\|\mathbf{p}_1\|^2}\mathbf{p}_1 - \frac{\langle \mathbf{v}_3, \mathbf{p}_2 \rangle}{\|\mathbf{p}_2\|^2}\mathbf{p}_2$$

4. We carry on producing vectors which are orthogonal (perpendicular) to the previous created vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$ by using the next given vector $\mathbf{v}_{k+1}$.

These steps are known as the Gram–Schmidt process.

4.3.4 The Gram–Schmidt process

Given any arbitrary basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ for a finite dimensional inner product space, we can find an orthogonal basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_n\}$ by the Gram–Schmidt process which is described next: