Orthogonal matrix

For matrices with orthogonality over the complex number field, see unitary matrix.

An **orthogonal matrix** is a square matrix whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors), i.e.

$$Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I,$$

where I is the identity matrix.

This leads to the equivalent characterization: a matrix Q is orthogonal if its transpose is equal to its inverse:

$$Q^{\mathrm{T}} = Q^{-1}.$$

An orthogonal matrix Q is necessarily invertible (with inverse $Q^{-1} = Q^{\mathrm{T}}$), unitary ($Q^{-1} = Q^*$) and therefore normal ($Q^*Q = QQ^*$) in the reals. The determinant of any orthogonal matrix is either +1 or −1. As a linear transformation, an orthogonal matrix preserves the dot product of vectors, and therefore acts as an isometry of Euclidean space, such as a rotation, reflection or rotoreflection. In other words, it is a unitary transformation.

The set of $n \times n$ orthogonal matrices forms a group $\mathrm{O}(n)$, known as the orthogonal group. The subgroup $\mathrm{SO}(n)$ consisting of orthogonal matrices with determinant +1 is called the special orthogonal group, and each of its elements is a **special orthogonal matrix**. As a linear transformation, every special orthogonal matrix acts as a rotation.

Matrix properties

A real square matrix is orthogonal if and only if its columns form an orthonormal basis of the Euclidean space $\mathbb{R}^n$ with the ordinary Euclidean dot product, which is the case if and only if its rows form an orthonormal basis of $\mathbb{R}^n$. It might be tempting to suppose a matrix with orthogonal (not orthonormal) columns would be called an orthogonal matrix, but such matrices have no special interest and no special name; they only satisfy $M^{\mathrm{T}}M = D$, with D a diagonal matrix.

The determinant of any orthogonal matrix is +1 or −1. This follows from basic facts about determinants, as follows:

$$1 = \det(I) = \det\left(Q^{\mathrm{T}}Q\right) = \det\left(Q^{\mathrm{T}}\right)\det(Q) = \left(\det(Q)\right)^2.$$

The converse is not true; having a determinant of ±1 is no guarantee of orthogonality, even with orthogonal columns, as shown by the following counterexample.

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

With permutation matrices the determinant matches the signature, being +1 or −1 as the parity of the permutation is even or odd, for the determinant is an alternating function of the rows.

Stronger than the determinant restriction is the fact that an orthogonal matrix can always be diagonalized over the complex numbers to exhibit a full set of eigenvalues, all of which must have (complex) modulus 1.

Examples

Below are a few examples of small orthogonal matrices and possible interpretations.

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(identity transformation)}$

An instance of a 3×3 rotation matrix:

- $R(16.26^\circ) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix} \qquad \text{(rotation by } 16.26^\circ\text{)}$
- $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{(reflection across } x\text{-axis)}$
- $\begin{bmatrix} 0 & -0.80 & -0.60 \\ 0.80 & -0.36 & 0.48 \\ 0.60 & 0.48 & -0.64 \end{bmatrix} \qquad \left(\begin{array}{l} \text{rotoinversion:} \\ \text{axis } \left(0, -\frac{3}{5}, \frac{4}{5}\right), \text{ angle } 90^\circ \end{array}\right)$
- $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \text{(permutation of coordinate axes)}$