Solutions to the Problems for Lecture 16

1. Let v be a vector in the vector space. Then both $0v$ and $v + (-1)v$ must be vectors in the vector space and both of them are the zero vector.

2. In all of the examples, the vector spaces are closed under scalar multiplication and vector addition.

Solutions to the Problems for Lecture 17

1. Only (a) and (b) are linearly independent. (c) is linearly dependent.

Solutions to the Problems for Lecture 18

1. One possible orthonormal basis is

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} \right\}.$$

The dimension of this vector space is two.

Solutions to the Practice quiz: Vector space definitions

1. b. A vector space must be closed under vector addition and scalar multiplication. The set of three-by-one matrices with the sum of all the rows equal to one is not closed under vector addition and scalar multiplication. For example, if you multiply a vector whose sum of all rows is equal to one by the scalar k, then the resulting vector's sum of all rows is equal to k.

2. d. One can find the relations

$$\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad 8\begin{pmatrix}1\\-1\\2\end{pmatrix} + 3\begin{pmatrix}4\\6\\-2\end{pmatrix} = 10\begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix},$$

so that these sets of three matrices are linearly dependent. The remaining set $\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}$ is linearly independent.

3. b. Since a three-by-one matrix has three degrees of freedom, and the constraint that the sum of all rows equals zero eliminates one degree of freedom, the basis should consist of two vectors. We can arbitrarily take the first unnormalized vector to be $\begin{pmatrix}1\\-1\\0\end{pmatrix}$. The vector orthogonal to this first vector with sum of all rows equal to zero is $\begin{pmatrix}1\\1\\-2\end{pmatrix}$. Normalizing both of these vectors, we get $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \ \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}$.

Solutions to the Problems for Lecture 19

1.

$$u_4 = v_4 - \frac{(u_1^T v_4)u_1}{u_1^T u_1} - \frac{(u_2^T v_4)u_2}{u_2^T u_2} - \frac{(u_3^T v_4)u_3}{u_3^T u_3}.$$

Solutions to the Problems for Lecture 20

1. Define

$$\{v_1, v_2\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Let $u_1 = v_1$. Then u_2 is found from

$$\begin{aligned} u_2 &= v_2 - \frac{(u_1^T v_2) u_1}{u_1^T u_1} \\ &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Normalizing, we obtain the orthonormal basis

$$\{\widehat{u}_1, \widehat{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

2. Define

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Let $u_1 = v_1$. Then u_2 is found from

$$\begin{aligned} u_2 &= v_2 - \frac{(u_1^T v_2) u_1}{u_1^T u_1} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \end{aligned}$$

and u_3 is found from

$$\begin{aligned} u_3 &= v_3 - \frac{(u_1^T v_3) u_1}{u_1^T u_1} - \frac{(u_2^T v_3) u_2}{u_2^T u_2} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Normalizing the three vectors, we obtain the orthonormal basis

$$\{\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \widehat{\mathbf{u}}_3\} = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Solutions to the Practice quiz: Gram-Schmidt process

1. a. The vector u_4 is orthogonal to u_1, u_2, and u_3. Since $u_1 = v_1$, then u_4 is orthogonal to v_1.

2. a. Since the vectors are already orthogonal, we need only normalize them to find

$$\{\widehat{u}_1, \widehat{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

3. b. Let $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Then,

$$u_2 = v_2 - \frac{(u_1^T v_2) u_1}{u_1^T u_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

Normalizing the vectors, we have

$$\{\widehat{u}_1, \widehat{u}_2\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Solutions to the Problems for Lecture 21

1. We bring A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The equation $Ax = 0$ with the pivot variables on the left-hand sides is given by

$$x_1 = -2x_4, \quad x_2 = x_4, \quad x_3 = x_4,$$

and a general vector in the nullspace can be written as $\begin{pmatrix} -2x_4 \\ x_4 \\ x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. A basis for the null space is therefore given by the single vector $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Solutions to the Problems for Lecture 22

1. The system in matrix form is given by

$$\begin{pmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix}.$$

We form the augmented matrix and bring the first four columns to reduced row echelon form:

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The null space is found from the first four columns solving $\mathrm{Au} = 0$, and writing the basic variables on the left-hand side, we have the system

$$u_1 = 2u_2 + u_4, \quad u_3 = -2u_4;$$

from which we can write the general form of the null space as

$$\begin{pmatrix} 2u_2 + u_4 \\ u_2 \\ -2u_4 \\ u_4 \end{pmatrix} = u_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

A particular solution is found by solving $\mathrm{Av} = \mathrm{b}$, and we have

$$v_1 - 2v_2 - v_4 = 3, \quad v_3 + 2v_4 = -2.$$

The free variables v_2 and v_4 can be set to zero, and the particular solution is determined to be $v_1 = 3$ and $v_3 = -2$. The general solution to the underdetermined system of equations is therefore given by

$$\mathrm{x} = a \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$

Solutions to the Problems for Lecture 23

1. We find

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

and $\dim(\mathrm{Col}(A)) = 3$, with a basis for the column space given by the first three columns of A.

Solutions to the Problems for Lecture 24

1. We find the reduced row echelon from of A and A^T:

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 & 2 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 1 & -2 & 3 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A^T = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 3 & -1 & 2 & -2 \\ -1 & 0 & -1 & 3 \\ 1 & -1 & 1 & -1 \\ 2 & 1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Columns one, two, and four are pivot columns of A and columns one, two, and three are pivot columns of A^T. Therefore, the column space of A is given by

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\};$$

and the row space of A (the column space of A^T) is given by

$$\mathrm{Row}(A) = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The null space of A are found from the equations

$$x_1 = -x_3 + x_5, \quad x_2 = x_3 - 2x_5, \quad x_4 = 2x_5,$$

and a vector in the null space has the general form

$$\begin{pmatrix} -x_3 + x_5 \\ x_3 - 2x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, the null space of A is given by

$$\mathrm{Null}(A) = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

The null space of A^T are found from the equations

$$x_1 = -2x_4, \quad x_2 = 2x_4, \quad x_3 = 5x_4,$$

and a vector in the null space has the general form

$$\begin{pmatrix} -2x_4 \\ 2x_4 \\ 5x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -2 \\ 2 \\ 5 \\ 1 \end{pmatrix}.$$

Therefore, the left null space of A is given by

$$\mathrm{LeftNull}(A) = \mathrm{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

It can be checked that the null space is the orthogonal complement of the row space and the left null space is the orthogonal complement of the column space. The $\mathrm{rank}(A) = 3$, and A is not of full rank.

Solutions to the Practice quiz: Fundamental subspaces

1. d. To find the null space of a matrix, bring it to reduced row echelon form. We have

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With $x_1 = -2x_2$, $x_3 = 0$, and $x_4 = 0$, a basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

2. b. This system of linear equations is underdetermined, and the solution will be a general vector in the null space of the multiplying matrix plus a particular vector that satisfies the underdetermined system of equations. The linear system in matrix form is given by

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We bring the augmented matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 & 1 \\ 3 & 6 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

A basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, and a particular solution can be found by setting the free variable $x_2 = 0$. Therefore, $x_1 = x_3 = 0$ and $x_4 = 1$, and the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where a is a free constant.

3. c. The matrix in reduced row echelon form is $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The number of pivot columns is three, and this is the rank.

Solutions to the Problems for Lecture 25

1. Using the Gram-Schmidt process, an orthonormal basis for W is found to be

$$\mathrm{s}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \mathrm{s}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

The projection of v onto W is then given by

$$\mathrm{v}_{\mathrm{proj}_W} = (\mathrm{v}^{\mathrm{T}}\mathrm{s}_1)\mathrm{s}_1 + (\mathrm{v}^{\mathrm{T}}\mathrm{s}_2)\mathrm{s}_2 = \frac{1}{3}(a+b+c)\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{6}(-2a+b+c)\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

When $a = 1$, $b = c = 0$, we have

$$\mathrm{v}_{\mathrm{proj}_W} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix};$$

and when $b = 1$, $a = c = 0$, we have

$$\mathrm{v}_{\mathrm{proj}_W} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-2\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Solutions to the Problems for Lecture 26

1.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 4 \end{pmatrix}$$

Solutions to the Problems for Lecture 27

1. The normal equations are given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 4 \end{pmatrix},$$

or

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 11 \\ 21 \end{pmatrix}.$$

The solution is $\beta_0 = 7/5$ and $\beta_1 = 9/10$, and the least-squares line is given by $y = 7/5 + 9x/10$.

Solutions to the Practice quiz: Orthogonal projections

1. b. We first normalize the vectors in W to obtain the orthonormal basis

$$\mathrm{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathrm{w}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Then the orthogonal projection of $\mathrm{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ onto W is given by

$$\mathrm{v}_{\mathrm{proj}_W} = (\mathrm{v}^{\mathrm{T}}\mathrm{w}_1)\mathrm{w}_1 + (\mathrm{v}^{\mathrm{T}}\mathrm{w}_2)\mathrm{w}_2 = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

2. d. The overdetermined system of equations is given by

$$\begin{aligned} \beta_0 + \beta_1 x_1 &= y_1, \\ \beta_0 + \beta_1 x_2 &= y_2, \\ \beta_0 + \beta_1 x_3 &= y_3. \end{aligned}$$

Substituting in the values for x and y, and writing in matrix form, we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

3. b. The normal equations are given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Multiplying out, we have

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}.$$

Inverting the two-by-two matrix, we have

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -2 \\ 6 \end{pmatrix}.$$

The best fit line is therefore $y = -\frac{1}{3} + x$.