Lecture 28

Two-by-two and three-by-three determinants

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We already showed that a two-by-two matrix A is invertible when its determinant is nonzero, where

$$\det \mathrm{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If A is invertible, then the equation $\mathrm{A}x = b$ has the unique solution $x = \mathrm{A}^{-1}b$. But if A is not invertible, then $\mathrm{A}x = b$ may have no solution or an infinite number of solutions. When $\det \mathrm{A} = 0$, we say that the matrix A is singular.

It is also straightforward to define the determinant for a three-by-three matrix. We consider the system of equations $\mathrm{Ax} = 0$ and determine the condition for which $\mathrm{x} = 0$ is the only solution. With

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

one can do the messy algebra of elimination to solve for x_1, x_2, and x_3. One finds that $x_1 = x_2 = x_3 = 0$ is the only solution when $\det \mathrm{A} \neq 0$, where the definition, apart from a constant, is given by

$$\det \mathrm{A} = aei + bfg + cdh - ceg - bdi - afh.$$

An easy way to remember this result is to mentally draw the following picture:

$$\begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix} - \begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix}.$$

The matrix A is periodically extended two columns to the right, drawn explicitly here but usually only imagined. Then the six terms comprising the determinant are made evident, with the lines slanting down towards the right getting the plus signs and the lines slanting down towards the left getting the minus signs. Unfortunately, this mnemonic only works for three-by-three matrices.