7.1.2 Characteristic equation

From the above formula (7.1) $\mathbf{Au} = \lambda\mathbf{u}$ we have

$$\mathbf{Au} = \lambda\mathbf{Iu}$$

[$\lambda\mathbf{Iu} = \lambda\mathbf{u}$ – multiplying by the identity keeps it the same]
where $\mathbf{I}$ is the identity matrix. We can rewrite this as

$$\mathbf{Au} - \lambda\mathbf{Iu} = \mathbf{O}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$$

Under what condition is the non-zero vector $\boldsymbol{u}$ a solution of this equation?

By question 26 of Exercises 6.3:

$$\mathbf{Ax} = \mathbf{O} \text{ has an infinite number of solutions} \Leftrightarrow \det(\mathbf{A}) = 0.$$

Applying this result to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$ means that we must have a non-zero vector $\mathbf{u}$ (because there are an infinite number of solutions which satisfy this equation) $\Leftrightarrow$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is an important equation because we use this to find the eigenvalues and it is called the **characteristic equation**:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{7.2}$$

The procedure for determining eigenvalues and eigenvectors is:

1. Solve the characteristic equation (7.2) for the scalar λ.
2. For the eigenvalue λ determine the corresponding eigenvector $\mathbf{u}$ by solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$.