Solutions to the Problems for Lecture 28

1.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 = 1.$$

2.

$$\begin{aligned} \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} &= dbi + ecg + fah - fbg - eai - dch \\ &= -(aei + bfg + cdh - ceg - bdi - afh) = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}. \end{aligned}$$

3. Let

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\mathrm{AB} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

and

$$\begin{aligned} \det \mathrm{AB} &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= (acef + adeh + bcfg + bdgh) - (acef + adfg + bceh + bdgh) \\ &= (adeh + bcfg) - (adfg + bceh) \\ &= ad(eh - fg) - bc(eh - fg) \\ &= (ad - bc)(eh - fg) \\ &= \det \mathrm{A} \det \mathrm{B}. \end{aligned}$$

Solutions to the Problems for Lecture 29

1. We first expand in minors across the fourth row:

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & 4 & 1 & 0 \\ 8 & -5 & 6 & 7 & -2 \\ -2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 3 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ -5 & 6 & 7 & -2 \\ 0 & 3 & 2 & 0 \end{vmatrix}.$$

We then expand in minors down the fourth column:

$$2 \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ -5 & 6 & 7 & -2 \\ 0 & 3 & 2 & 0 \end{vmatrix} = 4 \begin{vmatrix} 3 & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{vmatrix}.$$

Finally, we expand in minors down the first column:

$$4 \begin{vmatrix} 3 & 2 & 4 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{vmatrix} = 12 \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 60.$$

Solutions to the Problems for Lecture 30

1. For each element chosen from the first row, there is only a single way to choose nonzero elements from all subsequent rows. Considering whether the columns chosen are even or odd permutations of the ordered set $\{1, 2, 3, 4\}$, we obtain

$$\begin{vmatrix} a & b & c & d \\ e & f & 0 & 0 \\ 0 & g & h & 0 \\ 0 & 0 & i & j \end{vmatrix} = afhj - behj + cegj - degi.$$

Solutions to the Problems for Lecture 31

1. Suppose the square matrix A has two zero rows. If we interchange these two rows, the determinant of A changes sign according to Property 2, even though A doesn't change. Therefore, $\det A = -\det A$, or $\det A = 0$.

2. To prove that the determinant is a linear function of row i, interchange rows 1 and row i using Property 2. Use Property 3, then interchange rows 1 and row i again.

3. Consider a general n-by-n matrix. Using the linear property of the jth row, and that a matrix with two equal rows has zero determinant, we have

$$\begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1}+ka_{i1} & \dots & a_{jn}+ka_{in} \\ \vdots & \ddots & \vdots \end{vmatrix} = \begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \end{vmatrix} + k \begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \end{vmatrix} = \begin{vmatrix} \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \end{vmatrix}.$$

Therefore, the determinant doesn't change by adding k times row-i to row-j.

4.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & 5/2 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & 5/2 \\ 0 & 0 & 7/2 \end{vmatrix} = 2 \times 1 \times 7/2 = 7.$$

Solutions to the Practice quiz: Determinants

1. a. To find the determinant of a matrix with many zero elements, perform a Laplace expansion across the row or down the column with the most zeros. Choose the correct sign. We have for one expansion choice,

$$\begin{vmatrix} -3 & 0 & -2 & 0 & 0 \\ 2 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 3 & 0 & -3 & 2 & -3 \\ -3 & 3 & 3 & 0 & -2 \end{vmatrix} = 2 \begin{vmatrix} -3 & 0 & -2 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ -3 & 3 & 3 & -2 \end{vmatrix} = -4 \begin{vmatrix} -3 & 0 & -2 \\ 2 & -2 & -2 \\ 0 & 0 & -2 \end{vmatrix} = 8 \begin{vmatrix} -3 & 0 \\ 2 & -2 \end{vmatrix} = 48.$$

2. b. We can apply the Leibniz formula by going down the first column. For each element in the first column there is only one possible choice of elements from the other three columns. We have

$$\begin{pmatrix} a & e & 0 & 0 \\ b & f & g & 0 \\ c & 0 & h & i \\ d & 0 & 0 & j \end{pmatrix} = afhj - behj + cegj - degi.$$

The signs are obtained by considering whether the following permutations of the rows $\{1, 2, 3, 4\}$ are even or odd: $afhj = \{1, 2, 3, 4\}$ (even); $behj = \{2, 1, 3, 4\}$ (odd); $cegj = \{3, 1, 2, 4\}$ (even); $degi = \{4, 1, 2, 3\}$ (odd).

3. c. The only identity which is false is $\det(\mathrm{A} + \mathrm{B}) = \det \mathrm{A} + \det \mathrm{B}$.

Solutions to the Problems for Lecture 32

1.

$$
\begin{aligned}
0 = \det(\mathrm{A} - \lambda\mathrm{I}) &= \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} \\
&= (a-\lambda)(e-\lambda)(i-\lambda) + bfg + cdh - c(e-\lambda)g - bd(i-\lambda) - (a-\lambda)fh \\
&= -\lambda^3 + (a+e+i)\lambda^2 - (ae+ai+ei-bd-cg-fh)\lambda + aei + bfg + cdh - ceg - bdi - afh.
\end{aligned}
$$

Solutions to the Problems for Lecture 33

1. Let $A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$. The eigenvalues of A are found from

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 49.$$

Therefore, $2 - \lambda = \pm 7$, and the eigenvalues are $\lambda_1 = -5$, $\lambda_2 = 9$. The eigenvector for $\lambda_1 = -5$ is found from

$$\begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

or $x_1 + x_2 = 0$. The eigenvector for $\lambda_2 = 9$ is found from

$$\begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

or $x_1 - x_2 = 0$. The eigenvalues and corresponding eigenvectors are therefore given by

$$\lambda_1 = -5, \quad \mathrm{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \lambda_2 = 9, \quad \mathrm{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

2. The eigenvalues are found from

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2 - 2\right).$$

Therefore, $\lambda_1 = 2$, $\lambda_2 = 2 - \sqrt{2}$, and $\lambda_3 = 2 + \sqrt{2}$. The eigenvector for $\lambda_1 = 2$ are found from

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

or $x_2 = 0$ and $x_1 + x_3 = 0$, or $\mathrm{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The eigenvector for $\lambda_2 = 2 - \sqrt{2}$ is found from

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Gaussian elimination gives us

$$\operatorname{rref} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $x_1 = x_3$ and $x_2 = -\sqrt{2}x_3$ and an eigenvector is $\mathrm{x}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$. Similarly, the third eigenvector is $\mathrm{x}_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

Solutions to the Problems for Lecture 34

34. Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The eigenvalues of A are found from

$$0 = \det\left(A - \lambda I\right) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1.$$

Therefore, $1 - \lambda = \pm i$, and the eigenvalues are $\lambda_1 = 1 - i$, $\lambda_2 = 1 + i$.

Solutions to the Practice quiz: The eigenvalue problem

1. b. The characteristic equation $\det(A - \lambda I) = 0$ for a two-by-two matrix A results in the quadratic equation $\lambda^2 - \mathrm{Tr}A\,\lambda + \det A = 0$, which for the given matrix yields $\lambda^2 - 3\lambda + 1 = 0$. Application of the quadratic formula results in $\lambda_\pm = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$.

2. d. The characteristic equation is $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 1 = 0$. With $i = \sqrt{-1}$, the solution is $\lambda_\pm = 3 \pm i$.

3. b. One can either compute the eigenvalues and eigenvectors of the matrix, or test the given possible answers. If we test the answers, then only one is an eigenvector, and we have

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ 2+2\sqrt{2} \\ \sqrt{2}+2 \end{pmatrix} = (2+\sqrt{2}) \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Solutions to the Problems for Lecture 35

1. Let λ_1 and λ_2 be distinct eigenvalues of A, with corresponding eigenvectors x_1 and x_2. Write

$$c_1\mathrm{x}_1 + c_2\mathrm{x}_2 = 0.$$

To prove that x_1 and x_2 are linearly independent, we need to show that $c_1 = c_2 = 0$. Multiply the above equation on the left by A and use $\mathrm{Ax}_1 = \lambda_1\mathrm{x}_1$ and $\mathrm{Ax}_2 = \lambda_2\mathrm{x}_2$ to obtain

$$c_1\lambda_1\mathrm{x}_1 + c_2\lambda_2\mathrm{x}_2 = 0.$$

By eliminating x_1 or by eliminating x_2, we obtain

$$(\lambda_1 - \lambda_2)c_1\mathrm{x}_1 = 0, \quad (\lambda_2 - \lambda_1)c_2\mathrm{x}_2 = 0,$$

from which we conclude that if $\lambda_1 \neq \lambda_2$, then $c_1 = c_2 = 0$ and x_1 and x_2 are linearly independent.

2. Let A be an n-by-n matrix. We have

$$\dim(\mathrm{Col}(\mathrm{A})) + \dim(\mathrm{Null}(\mathrm{A})) = n.$$

Since the columns of A are linearly independent, we have $\dim(\mathrm{Col}(\mathrm{A})) = n$ and $\dim(\mathrm{Null}(\mathrm{A})) = 0$. If the only solution to $\mathrm{Ax} = 0$ is the zero vector, then $\det \mathrm{A} \neq 0$ and A is invertible.

Solutions to the Problems for Lecture 36

1. The eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ are

$$\lambda_1 = 2,\; x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \lambda_2 = 2 - \sqrt{2},\; x_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}; \quad \lambda_2 = 2 + \sqrt{2},\; x_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Notice that the three eigenvectors are mutually orthogonal. This will happen when the matrix is symmetric. If we normalize the eigenvectors, the matrix with eigenvectors as columns will be an orthogonal matrix. Normalizing the orthogonal eigenvectors (so that $S^{-1} = S^T$) , we have

$$S = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}.$$

We therefore find

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}$$

Solutions to the Problems for Lecture 37

1.

$$
\begin{aligned}
e^{\mathrm{A}} &= e^{\mathrm{S}\Lambda\mathrm{S}^{-1}} \\
&= \mathrm{I} + \mathrm{S}\Lambda\mathrm{S}^{-1} + \frac{\mathrm{S}\Lambda^2\mathrm{S}^{-1}}{2!} + \frac{\mathrm{S}\Lambda^3\mathrm{S}^{-1}}{3!} + \dots \\
&= \mathrm{S}\left(\mathrm{I} + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \dots\right)\mathrm{S}^{-1} \\
&= \mathrm{S}e^{\Lambda}\mathrm{S}^{-1}.
\end{aligned}
$$

Because Λ is a diagonal matrix, the powers of Λ are also diagonal matrices with the diagonal elements raised to the specified power. Each diagonal element of e^{Λ} contains a power series of the form

$$1 + \lambda_i + \frac{\lambda_i^2}{2!} + \frac{\lambda_i^3}{3!} + \dots,$$

which is the power series for e^{λ_i}.

Solutions to the Problems for Lecture 38

1. We use the result

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}$$

to find

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}.$$

Solutions to the Practice quiz: Matrix diagonalization

1. c. Eigenvectors with distinct eigenvalues must be linearly independent. All the listed pairs of eigenvectors are linearly independent except $x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where $x_2 = -x_1$.

2. d. A simple calculation shows that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Therefore $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{100} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \right)^{50} = I^{50} = I$.

A more complicated calculation diagonalizes this symmetric matrix. The eigenvalues and orthonormal eigenvectors are found to be $\lambda_1 = 1$, $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The diagonalization then takes the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then,
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{100} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{100} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = I$.

3. a. We have

$$e^I = I + I + \frac{I^2}{2!} + \frac{I^3}{3!} + \cdots = I \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \right) = Ie^1 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$