Week I

Matrices

In this week's lectures, we learn about matrices. Matrices are rectangular arrays of numbers or other mathematical objects and are fundamental to engineering mathematics. We will define matrices and how to add and multiply them, discuss some special matrices such as the identity and zero matrix, learn about transposes and inverses, and define orthogonal and permutation matrices.

Lecture 1

Definition of a matrix

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An m-by-n matrix is a rectangular array of numbers (or other mathematical objects) with m rows and n columns. For example, a two-by-two matrix A, with two rows and two columns, looks like

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The first row has elements a and b, the second row has elements c and d. The first column has elements a and c; the second column has elements b and d. As further examples, two-by-three and three-by-two matrices look like

$$\mathrm{B} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Of special importance are column matrices and row matrices. These matrices are also called vectors. The column vector is in general n-by-one and the row vector is one-by-n. For example, when $n = 3$, we would write a column vector as

$$\mathrm{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and a row vector as

$$\mathrm{y} = \begin{pmatrix} a & b & c \end{pmatrix}.$$

A useful notation for writing a general m-by-n matrix A is

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Here, the matrix element of A in the ith row and the jth column is denoted as a_{ij}.

Problems for Lecture 1

1. The main diagonal of a matrix A are the entries a_{ij} where $i = j$.

a) Write down the three-by-three matrix with ones on the diagonal and zeros elsewhere.

b) Write down the three-by-four matrix with ones on the diagonal and zeros elsewhere.

c) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere.

Solutions to the Problems

Lecture 2

Addition and multiplication of matrices

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Matrices can be added only if they have the same dimension. Addition proceeds element by element. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Matrices can also be multiplied by a scalar. The rule is to just multiply every element of the matrix. For example,

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

Matrices (other than the scalar) can be multiplied only if the number of columns of the left matrix equals the number of rows of the right matrix. In other words, an m-by-n matrix on the left can only be multiplied by an n-by-k matrix on the right. The resulting matrix will be m-by-k. Evidently, matrix multiplication is generally not commutative. We illustrate multiplication using two 2-by-2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}, \qquad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}.$$

First, the first row of the left matrix is multiplied against and summed with the first column of the right matrix to obtain the element in the first row and first column of the product matrix. Second, the first row is multiplied against and summed with the second column. Third, the second row is multiplied against and summed with the first column. And fourth, the second row is multiplied against and summed with the second column.

In general, an element in the resulting product matrix, say in row i and column j, is obtained by multiplying and summing the elements in row i of the left matrix with the elements in column j of the right matrix. We can formally write matrix multiplication in terms of the matrix elements. Let A be an m-by-n matrix with matrix elements a_{ij} and let B be an n-by-p matrix with matrix elements b_{ij}. Then $\mathrm{C} = \mathrm{AB}$ is an m-by-p matrix, and its ij matrix element can be written as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Notice that the second index of a and the first index of b are summed over.

Problems for Lecture 2

1. Define the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Compute if defined: $B - 2A$, $3C - E$, AC, CD, CB.

2. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Compute AD and DA.

4. Prove the associative law for matrix multiplication. That is, let A be an m-by-n matrix, B an n-by-p matrix, and C a p-by-q matrix. Then prove that $A(BC) = (AB)C$.

Solutions to the Problems

Lecture 3

Special matrices

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The zero matrix, denoted by 0, can be any size and is a matrix consisting of all zero elements. Multiplication by a zero matrix results in a zero matrix. The identity matrix, denoted by I, is a square matrix (number of rows equals number of columns) with ones down the main diagonal. If A and I are the same sized square matrices, then

$$\mathrm{AI} = \mathrm{IA} = \mathrm{A},$$

and multiplication by the identity matrix leaves the matrix unchanged. The zero and identity matrices play the role of the numbers zero and one in matrix multiplication. For example, the two-by-two zero and identity matrices are given by

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A diagonal matrix has its only nonzero elements on the diagonal. For example, a two-by-two diagonal matrix is given by

$$\mathrm{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Usually, diagonal matrices refer to square matrices, but they can also be rectangular.

A band (or banded) matrix has nonzero elements only on diagonal bands. For example, a three-by-three band matrix with nonzero diagonals one above and one below a nonzero main diagonal (called a tridiagonal matrix) is given by

$$\mathrm{B} = \begin{pmatrix} d_1 & a_1 & 0 \\ b_1 & d_2 & a_2 \\ 0 & b_2 & d_3 \end{pmatrix}.$$

An upper or lower triangular matrix is a square matrix that has zero elements below or above the diagonal. For example, three-by-three upper and lower triangular matrices are given by

$$\mathrm{U} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \quad \mathrm{L} = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}.$$

Problems for Lecture 3

1. Let $A = \begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}$. Construct a two-by-two matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

2. Verify that $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}$. Prove in general that the product of two diagonal matrices is a diagonal matrix, with elements given by the product of the diagonal elements.

3. Verify that $\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 + a_2 b_3 \\ 0 & a_3 b_3 \end{pmatrix}$. Prove in general that the product of two upper triangular matrices is an upper triangular matrix, with the diagonal elements of the product given by the product of the diagonal elements.

Solutions to the Problems

Practice quiz: Matrix definitions

1. Identify the two-by-two matrix with matrix elements $a_{ij} = i - j$.

a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2. The matrix product $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is equal to

a) $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

c) $\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$

d) $\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}$

3. Let A and B be n-by-n matrices with $(\mathrm{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. If A and B are upper triangular matrices, then $a_{ik} = 0$ or $b_{kj} = 0$ when

A. $k < i$ B. $k > i$ C. $k < j$ D. $k > j$

a) A and C only

b) A and D only

c) B and C only

d) B and D only

Solutions to the Practice quiz

Lecture 4

Transpose matrix

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The transpose of a matrix A, denoted by A^T and spoken as A-transpose, switches the rows and columns of A. That is,

$$\text{if } \mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \text{then } \mathrm{A}^\mathrm{T} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

In other words, we write

$$a_{ij}^\mathrm{T} = a_{ji}.$$

Evidently, if A is m-by-n then A^T is n-by-m. As a simple example, view the following transpose pair:

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}^\mathrm{T} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

The following are useful and easy to prove facts:

$$\left(\mathrm{A}^\mathrm{T}\right)^\mathrm{T} = \mathrm{A}, \text{ and } (\mathrm{A}+\mathrm{B})^\mathrm{T} = \mathrm{A}^\mathrm{T} + \mathrm{B}^\mathrm{T}.$$

A less obvious fact is that the transpose of the product of matrices is equal to the product of the transposes with the order of multiplication reversed, i.e.,

$$(\mathrm{AB})^\mathrm{T} = \mathrm{B}^\mathrm{T}\mathrm{A}^\mathrm{T}.$$

If A is a square matrix, and $\mathrm{A}^\mathrm{T} = \mathrm{A}$, then we say that A is *symmetric*. If $\mathrm{A}^\mathrm{T} = -\mathrm{A}$, then we say that A is *skew symmetric*. For example, 3-by-3 symmetric and skew symmetric matrices look like

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad \begin{pmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{pmatrix}.$$

Notice that the diagonal elements of a skew-symmetric matrix must be zero.

Problems for Lecture 4

1. Prove that $(AB)^T = B^T A^T$.

2. Show using the transpose operator that any square matrix A can be written as the sum of a symmetric and a skew-symmetric matrix.

3. Prove that $A^T A$ is symmetric.

Solutions to the Problems

Lecture 5

Inner and outer products

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The *inner product* (or dot product or scalar product) between two vectors is obtained from the matrix product of a row vector times a column vector. A row vector can be obtained from a column vector by the transpose operator. With the 3-by-1 column vectors u and v, their inner product is given by

$$\mathrm{u}^{\mathrm{T}}\mathrm{v} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1v_1 + u_2v_2 + u_3v_3.$$

If the inner product between two vectors is zero, we say that the vectors are *orthogonal*. The *norm* of a vector is defined by

$$||\mathrm{u}|| = \left(\mathrm{u}^{\mathrm{T}}\mathrm{u}\right)^{1/2} = \left(u_1^2 + u_2^2 + u_3^2\right)^{1/2}.$$

If the norm of a vector is equal to one, we say that the vector is *normalized*. If a set of vectors are mutually orthogonal and normalized, we say that these vectors are *orthonormal*.

An *outer product* is also defined, and is used in some applications. The outer product between u and v is given by

$$\mathrm{u}\mathrm{v}^{\mathrm{T}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}.$$

Notice that every column is a multiple of the single vector u, and every row is a multiple of the single vector v^{T}.

Problems for Lecture 5

1. Let A be a rectangular matrix given by $\mathrm{A} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. Compute $\mathrm{A}^{\mathrm{T}}\mathrm{A}$ and show that it is a symmetric square matrix and that the sum of its diagonal elements is the sum of the squares of all the elements of A.

2. The trace of a square matrix B, denoted as $\mathrm{Tr}\,\mathrm{B}$, is the sum of the diagonal elements of B. Prove that $\mathrm{Tr}(\mathrm{A}^{\mathrm{T}}\mathrm{A})$ is the sum of the squares of all the elements of A.

Solutions to the Problems

Lecture 6

Inverse matrix

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Square matrices may have inverses. When a matrix A has an inverse, we say it is invertible and denote its inverse by A^{-1}. The inverse matrix satisfies

$$\mathrm{AA}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}.$$

If A and B are invertible matrices, then $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$. Furthermore, if A is invertible then so is A^{T}, and $(\mathrm{A}^{\mathrm{T}})^{-1} = (\mathrm{A}^{-1})^{\mathrm{T}}$.

It is illuminating to derive the inverse of a general 2-by-2 matrix. Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and try to solve for x_1, y_1, x_2 and y_2 in terms of a, b, c, and d. There are two inhomogeneous and two homogeneous linear equations:

$$\begin{aligned} ax_1 + by_1 &= 1, & cx_1 + dy_1 &= 0, \\ cx_2 + dy_2 &= 1, & ax_2 + by_2 &= 0. \end{aligned}$$

To solve, we can eliminate y_1 and y_2 using the two homogeneous equations, and find x_1 and x_2 using the two inhomogeneous equations. The solution for the inverse matrix is found to be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The term $ad - bc$ is just the definition of the determinant of the two-by-two matrix:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The determinant of a two-by-two matrix is the product of the diagonals minus the product of the off-diagonals. Evidently, a two-by-two matrix A is invertible only if $\det \mathrm{A} \neq 0$. Notice that the inverse of a two-by-two matrix, in words, is found by switching the diagonal elements of the matrix, negating the off-diagonal elements, and dividing by the determinant.

Later, we will show that an n-by-n matrix is invertible if and only if its determinant is nonzero. This will require a more general definition of the determinant.

Problems for Lecture 6

1. Find the inverses of the matrices $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 4 \\ 3 & 3 \end{pmatrix}$.

2. Prove that if A and B are same-sized invertible matrices , then $(AB)^{-1} = B^{-1}A^{-1}$.

3. Prove that if A is invertible then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

4. Prove that if a matrix is invertible, then its inverse is unique.

Solutions to the Problems

Practice quiz: Transpose and inverses

1. $(ABC)^T$ is equal to

a) $A^TB^TC^T$

b) $A^TC^TB^T$

c) $C^TA^TB^T$

d) $C^TB^TA^T$

2. Which matrix is not symmetric?

a) $A + A^T$

b) AA^T

c) $A - A^T$

d) A^TA

3. Which matrix is the inverse of $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$?

a) $\frac{1}{2}\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

b) $\frac{1}{2}\begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix}$

c) $\frac{1}{2}\begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix}$

d) $\frac{1}{2}\begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}$

Solutions to the Practice quiz

Lecture 7

Orthogonal matrices

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A square matrix Q with real entries that satisfies

$$Q^{-1} = Q^{T}$$

is called an *orthogonal matrix*.

Since the columns of Q^{T} are just the rows of Q, and $QQ^{T} = I$, the row vectors that form Q must be orthonormal. Similarly, since the rows of Q^{T} are just the columns of Q, and $Q^{T}Q = I$, the column vectors that form Q must also be orthonormal.

Orthogonal matrices preserve norms. Let Q be an n-by-n orthogonal matrix, and let x be an n-by-one column vector. Then the norm squared of Qx is given by

$$||Qx||^{2} = (Qx)^{T}(Qx) = x^{T}Q^{T}Qx = x^{T}x = ||x||^{2}.$$

The norm of a vector is also called its length, so we can also say that orthogonal matrices preserve lengths.

Problems for Lecture 7

1. Show that the product of two orthogonal matrices is orthogonal.

2. Show that the n-by-n identity matrix is orthogonal.

Solutions to the Problems

Lecture 8

Rotation matrices

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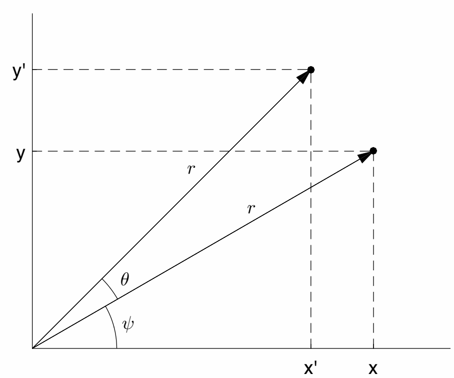
A matrix that rotates a vector in space doesn't change the vector's length and so should be an orthog-



Rotating a vector in the x-y plane.

onal matrix. Consider the two-by-two rotation matrix that rotates a vector through an angle θ in the x-y plane, shown above. Trigonometry and the addition formula for cosine and sine results in

$$\begin{aligned} x' &= r\cos(\theta+\psi) \\ &= r(\cos\theta\cos\psi - \sin\theta\sin\psi) \\ &= x\cos\theta - y\sin\theta \end{aligned} \qquad \begin{aligned} y' &= r\sin(\theta+\psi) \\ &= r(\sin\theta\cos\psi + \cos\theta\sin\psi) \\ &= x\sin\theta + y\cos\theta. \end{aligned}$$

Writing the equations for x' and y' in matrix form, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The above two-by-two matrix is a rotation matrix and we will denote it by R_θ. Observe that the rows and columns of R_θ are orthonormal and that the inverse of R_θ is just its transpose. The inverse of R_θ rotates a vector by $-\theta$.

Problems for Lecture 8

1. Let $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Show that $R(-\theta) = R(\theta)^{-1}$.

2. Find the three-by-three matrix that rotates a three-dimensional vector an angle θ counterclockwise around the z-axis.

Solutions to the Problems

Lecture 9

Permutation matrices

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Another type of orthogonal matrix is a permutation matrix. An n-by-n permutation matrix, when multiplying on the left permutes the rows of a matrix, and when multiplying on the right permutes the columns. Clearly, permuting the rows of a column vector will not change its norm.

For example, let the string $\{1, 2\}$ represent the order of the rows or columns of a two-by-two matrix. Then the permutations of the rows or columns are given by $\{1, 2\}$ and $\{2, 1\}$. The first permutation is no permutation at all, and the corresponding permutation matrix is simply the identity matrix. The second permutation of the rows or columns is achieved by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

The rows or columns of a three-by-three matrix have $3! = 6$ possible permutations, namely $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, $\{2, 3, 1\}$, $\{3, 1, 2\}$, $\{3, 2, 1\}$. For example, the row or column permutation $\{3, 1, 2\}$ is obtained by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}, \qquad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c & a & b \\ f & d & e \\ i & g & h \end{pmatrix}.$$

Notice that the permutation matrix is obtained by permuting the corresponding rows (or columns) of the identity matrix. This is made evident by observing that

$$\mathrm{PA} = (\mathrm{PI})\mathrm{A}, \qquad \mathrm{AP} = \mathrm{A}(\mathrm{PI}),$$

where P is a permutation matrix and PI is the identity matrix with permuted rows. The identity matrix is orthogonal, and so is the matrix obtained by permuting its rows.

Problems for Lecture 9

1. Write down the six three-by-three permutation matrices corresponding to the permutations $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, $\{2, 3, 1\}$, $\{3, 1, 2\}$, $\{3, 2, 1\}$.

2. Find the inverses of all the three-by-three permutation matrices. Explain why some matrices are their own inverses, and others are not.

Solutions to the Problems

Practice quiz: Orthogonal matrices

1. Which matrix is not orthogonal?

a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

2. Which matrix rotates a three-by-one column vector an angle θ counterclockwise around the x-axis?

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

b) $\begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix}$

c) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. Which matrix, when multiplying another matrix on the left, moves row one to row two, row two to row three, and row three to row one?

a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Solutions to the Practice quiz