CS 4150

Wednesday November 25, 2020

Today: More NP-Hardness

- Subset Sum

- Clique

- Steiner Tree

Subset Sum

An instance of Subset Sum is specified by

- multiset S of integers
- a target integer t

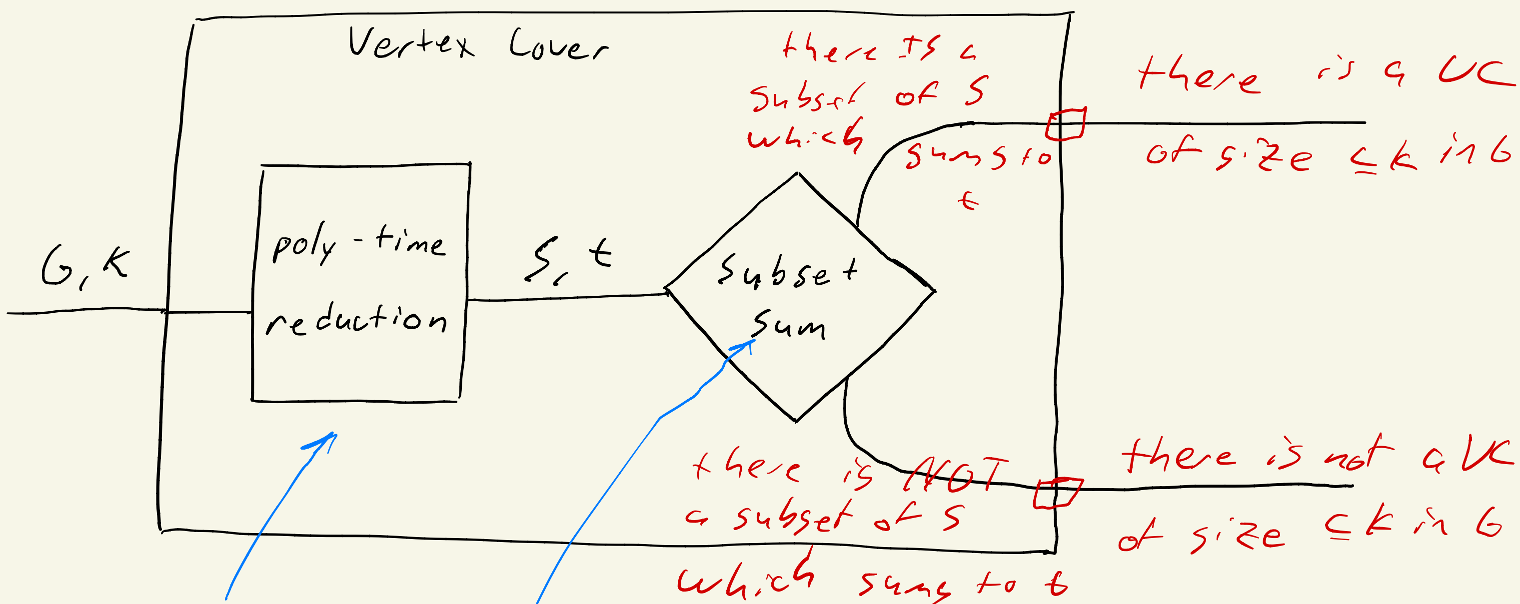
An instance of Subset Sum (S, t) is YES if and only if

- $X \subseteq S$ s.t. $\sum_{y \in X} y = t$

Eg: $S = \{-7, \underline{-3}, \underline{-2}, 9000, \underline{5}, 8\}$ $t = 0$ YES

$\Rightarrow X = \{-3, -2, 5\}$ is a certificate that S, t is a YES-instance

Subset Sum from Vertex Cover



need to do this part

if there's a poly-time algorithm for subset sum

Subset Sum from Vertex Cover

$(G, k) \to (S, t)$

Choices $\to$ Choices

✓ • vertices of G $\to$ integers of S

Constraints $\to$ Constraints

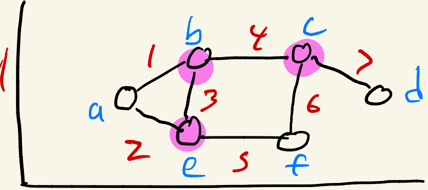
✓ • edges of G $\to$ digits of t

✓ • size $\leq k$ $\to$ digit of t

$\to t = \underline{3}\,2\,2\,2\,2\,2\,2\,2$

$\underline{X_b} + \underline{X_c} + \underline{X_e} + Y_1 + Y_2 + Y_5 + Y_6 + Y_7$

base-4



covers edge 1

$X_a = 11100000$

encodes cost of using a

$- X_b = 11011000$

$- X_c = 10001011$

$X_d = 10000001$

$- X_e = 10110100$

$X_f = 10001100$

$\to Y_1 = 01000000$

no cost

Subset Sum from Vertex Cover

$(G, k) \to (S, t)$

edges numbered from 0

$|E(G)|$

$$x_u = 4^m + \sum_{i \in \Delta(u)} 4^i \qquad y_i = 4^i$$

costs 1

the edge numbers of all incident edges

$$S = \{x_u : u \in V\} \cup \{y_i : i \in E\}$$

$$t = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$$

set most sig digit to k

sets every other digit to 2

Subset Sum from Vertex Cover

Theorem: (S, t) is a YES-instance $\iff$ (G, k) is a YES-instance.

$\Rightarrow$ Let $X \subseteq S$ such that $\sum_{x \in X} x = t$ be a certificate that (S, t) is a YES-instance. Let $C = \{u : x_u \in X\}$. $|C| \le k$ because $x_u \ge 4^m$ and $t < (k+1)4^m$. Suppose $\exists\, uv$ st $u \notin C$ and $v \notin C$. Let i be the index of uv. $\sum_{x \in X} x \neq t$ because the ith digit of the sum cannot be 2. Only x_u, x_v, and y_{uv} have a 1 in the ith digit, and x_u and $\cancel{x_v}$ are not in X.

Subset Sum from Vertex Cover

Important Fact: If G has a vertex cover of size i and $i < n$, then G has a vertex cover of size $i+1$.

$\Leftarrow$ Let $C \subseteq V$ be a vertex cover of G of size at most k.

Let C' be a vertex cover of G of size $= k$ by IF,

XOR

Let $Q = \{uv : u \in C' \oplus v \in C'\}$.

Let $X = \{x_u : u \in C'\} \cup \{y_{uv} : uv \in Q\}$.

First m digits are all 2, Last digit $= k$ because $|\{x_u : u \in C'\}| = k$

But Subset Sum has an $O(nt)$ DP-algorithm similar to

$|S|$

Knapsack, $\xrightarrow{?} \leftarrow$

Weakly NP-hard

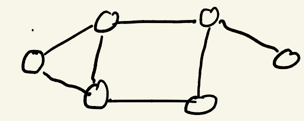
To be "truly" polynomial-time, the running time must be polynomial in the input size.

$O(nt)$

input size: $\underbrace{n \cdot \log X}_{\text{bits needed to encode } S} + \underbrace{\log t}_{\text{bits to encode } t}$

This is called pseudo-polynomial time and NP-hard problems with such an algorithm is weakly NP-hard.

Clique from Independent Set



Problem: Given G, K, does G have a complete subgraph of size at least K?

Steiner Tree

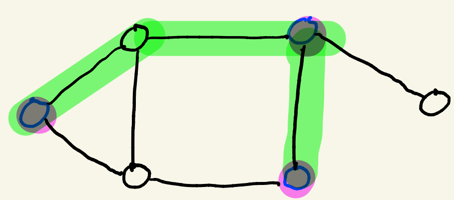
An instance of Steiner Tree is specified by

- a graph H (undirected)
- a set $S \subseteq V(H)$ of terminals
- an integer w

An instance of Steiner Tree is a YES-instance if and only if

- there is a subtree of H, T, such that $S \subseteq T$ and $|T| \leq w$

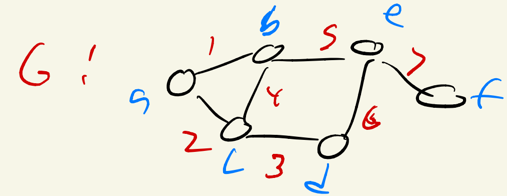
Eg:



$w = 4$

Steiner Tree from Vertex Cover

$(G, k) \to (H, S, w)$



Constraints → Constraints

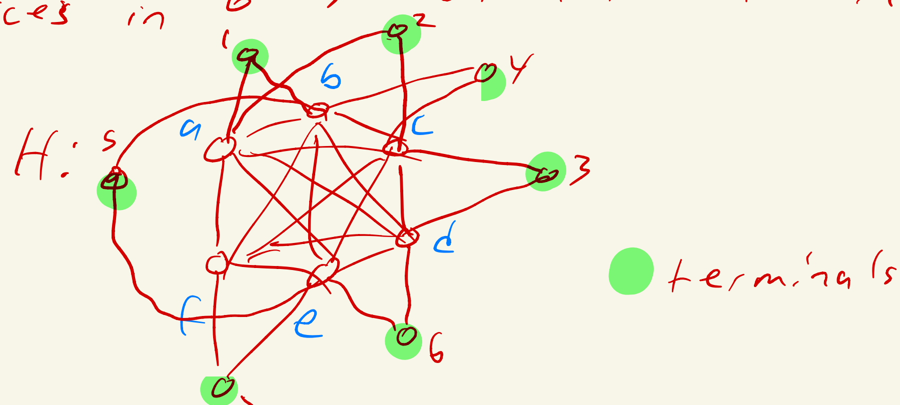
- edges in G → vertices in S (ie terminals)
- size $\leq k$ → size $\leq w$

Choices → Choices

- vertices in G → non-terminal vertices in H

1. add u to H for each u in G
2. make it a clique
3. add S_{uv} to H for $uv \in G$ and connect to u and v
4. $S = \{S_{uv}\}$

→ 5. $w = m + k$



Steiner Tree from Vertex Cover

Theorem: (H, S, w) is a YES-instance $\Longleftrightarrow$ (G, k) is a YES-instance.

$\Longrightarrow$ Let T be a subtree of H such that $S \subseteq T$ and $|T| \leq w$.

Let $C = \{u : u \in T\}$. $|C| \leq k$ because $|T| \leq m+k$

in G in H

$|S| \geq m$.

Suppose $\exists\, uv \in G$ st $u \notin C$ and $v \notin C$. S_{uv} must be in T since its a terminal and therefore one of its neighbors (u or v) must be in T.

Steiner Tree from Vertex Cover

$\Leftarrow$ Let $C \subseteq V(G)$ be a vertex cover of G of size at most k.

Let $c_1 \ldots c_k$ be the vertices of C.

For each edge $c_i - u$, add $c_i - s_{c_i u}$ to T.

For each edge $c_i - c_j$ $(i < j)$, add $c_i - s_{c_i c_j}$ to T.

add $\underline{c_1 - c_i}$ to T for $1 < i \leq k$.

$|T| \leq m + k$ because all edges added only have endpoints in $c_1 \ldots c_k$ and s_{uv}.

T is connected by construction.

T contains all the terminals.