Probabilistic Method

Idea: Show existance of an object by defining a probability space and an event A (corresponding to object) s.t. $p(A) > 0$.

Theorem: (Erdös 1947) For $k \geq 3$, $R(k) > 2^{k/2}$.

Proof: Suffices to show $\forall n \leq 2^{k/2}$, there is a graph G on n vertices with $\omega < k$ and $\alpha < k$. Use probabilistic method: show $p(\omega \geq k) + p(\alpha \geq k) < 1$. Consider the random graph model where each edge is included independently at random with probability $1/2$.

$$p(\omega \geq k) = \binom{n}{k} \cdot 2^{-\binom{k}{2}} = p(\alpha \geq k)$$

Need only show $\binom{n}{k} \cdot 2^{-\binom{k}{2}} < 1/2$ for $n \leq 2^{k/2}$ and $k \geq 3$.

$k=3$: $R(3) = 6 > 2\sqrt{2}$

$k \geq 4$: $k! > 2^k$

$$\binom{n}{k} \cdot 2^{-\binom{k}{2}} < \frac{n^k}{2^k} \cdot 2^{-\frac{k}{2}(k-1)} \leq \frac{2^{k^2/2}}{2^k} \cdot 2^{-\frac{k}{2}(k-1)} = 2^{-k/2} \leq 1/2$$

$\binom{k}{2} \downarrow$ $\quad$ $n \leq 2^{k/2} \downarrow$

$\binom{n}{k}$, drop $(n-k)!$, use $k! > 2^k$

Linearity of Expectation

Pigeonhole principle: There is an element $x_1 \le E(X)$ and an element $x_2 \ge E(X)$.

existance → probabilistic method

Theorem: (Szele 1943) There $\boxed{\text{is}}$ an n-vertex tournament with at least $n!/2^{n-1}$ Hamiltonian paths.

Proof: Probability space: for each of the $\binom{n}{2}$ pairs $\{i,j\} \in [n]^2$, choose $i \to j$ or $j \to i$ independently at random with probability $1/2$.

Let X be the number of Hamiltonian paths in a random tournament G.

Let L be a permutation of $V(G)$.

$$X = \sum_L X_L \longleftarrow \text{indicator, 1 if } L \text{ is a Hamiltonian path and 0 otherwise.}$$

sum over all permutations

linearity of expectation

$$E(X) = \sum_L E(X_L) = n! \cdot 2^{-(n-1)} = n!/2^{n-1}$$

$n!$ permutations

$E(X_L) = 2^{-(n-1)}$

By pigeonhole principle, there exists a G with $\ge n!/2^{n-1}$ Hamiltonian paths.

"The" Random Graph

Model: (Erdös-Renyi) Graphs on $\{v_1 \dots v_n\}$ where each edge is present independently at random with probability p. Denoted $\mathcal{G}(n,p)$.

- event in $\mathcal{G}(n,p)$ is a set of graphs on n vertices.
- for a fixed G_0, $\{G_0\}$ is an event in $\mathcal{G}(n,p)$ with probability

  n vertices
  m edges

  $$p^m (1-p)^{\binom{n}{2}-m}$$

- Note G_0 is a labelled graph. Probabilities of isomorphism classes may be different.

Alteratives:
- $\mathcal{G}(n,m)$: all graphs on n vertices and m edges with uniform probability.
- Chung-Lu: inhomogeneous edge probabilies.
- configuration model: guaranteed degree distribution, requires multiedges.
- preferential attachment: constructed iteratively, more likely to attach to high degree
- small world: grid + "long range" edges.

Almost All Graphs

Def: A <u>graph property</u> is a class of graphs closed under isomorphism.

graph property

If $P(G \in Q)$ for $G \in \mathcal{G}(n,p) \to 1$ as $n \to \infty$, then <u>almost all graphs</u> (in $\mathcal{G}(n,p)$) have the property Q.

If $P(G \in Q)$ for $G \in \mathcal{G}(n,p) \to 0$ as $n \to \infty$, then <u>almost no graphs</u> (in $\mathcal{G}(n,p)$) have the property Q.

Note: If almost all graphs have Q, then almost no graphs have $\bar{Q}$.

doesn't depend on n

Lemma: For constant $p \in (0,1)$ and any graph H, almost all graphs in $\mathcal{G}(n,p)$ contain an induced copy of H.

↳ contains an induced subgraph isomorphic to H.

Proof: Let $k = |H|$ and $U \subseteq \{v_1 \dots v_n\}$ of size k. $G[U]$ is isomorphic to H with probability $r > 0$. G contains $\lfloor n/k \rfloor$ disjoint such sets U. Thus, probability that none are isomorphic to H is $\leq (1-r)^{\lfloor n/k \rfloor}$ by disjointness of Us.
Thus, $P(H \not\subseteq G) \leq (1-r)^{\lfloor n/k \rfloor} \longrightarrow 0$ as $n \longrightarrow \infty$.

Almost All Graphs

Theorem: For every constant $p \in (0,1)$ and $k \in \mathbb{N}$, almost all graphs in $\mathcal{G}(n,p)$ are k-connected.

Strategy: Let Q_{ij} be the property that for any disjoint vertex sets X, Y s.t. $|X| \le i$ and $|Y| \le j$, $\exists\, v \notin X, Y$ adjacent to all vertices in X and none in Y.

Note: Q_{ij} only contains graphs with at least $i+j+1$ vertices.

Claim: If $G \in Q_{2,k-1}$, then G is k-connected.

Proof: Assume not. Then there exists a separator of size $\le k-1$. Let u, v be two vertices in different connected components of $G \setminus S$.
Let $X = \{u, v\}$ and let $Y = S$. $G \in Q_{2,k-1}$ implies u and v have a common neighbor which is not in S. This contradicts our choice of u and v.

$\Rightarrow$ If almost all graphs in $\mathcal{G}(n,p)$ have Q_{ij}, then almost all graphs in $\mathcal{G}(n,p)$ are k-connected.

Almost All Graphs

Lemma: For every constant $p \in (0,1)$ and $i, j \in \mathbb{N}$, almost all graphs in $\mathcal{G}(n,p)$ have the property Q_{ij}.

Proof: Fix X, Y, and v. The probability that v is adjacent to every vertex in X and none in Y is

$$r = p^{|X|}(1-p)^{|Y|} \geq p^i(1-p)^j$$

Then, the probability that there is no v for X and Y is

$$(1-r)^{n-|X|-|Y|} \leq (1-r)^{n-i-j}$$

There are at most n^{i+j} sets X and Y to consider. $\leftarrow$ from $\binom{n}{i}\binom{n}{j}$ not tight

The probability that there are sets X and Y with no suitable v is

$$n^{i+j}(1-r)^{n-i-j} \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ since } 1-r < 1.$$