Flow

Def: A <u>flow</u> on a graph $G=(V,E)$ is a map $f: V^2 \longrightarrow \mathbb{R}$ such that

- $f(x,y) = -f(y,x)$ $\quad \forall\, x,y \in V$
- $f(x,y) = 0$ $\quad \forall\, xy \notin E$
- $\sum_{y \in N(x)} f(x,y) = 0$ $\quad \forall\, x \in V$ (with some exceptions)

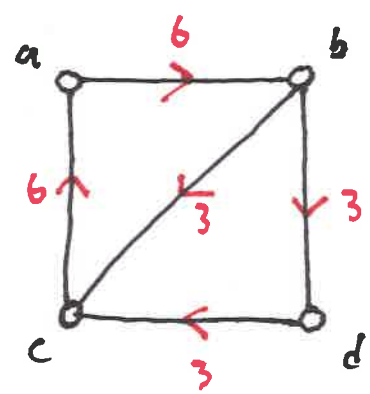
We say f is <u>integral</u> if f is a map onto $\mathbb{Z}$.

Ex: Consider f s.t.

$f(a,b) = 6$ $\qquad f(b,d) = 3$

$f(a,c) = -6$ $\qquad f(c,d) = -3$

$f(b,c) = 3$



Directed Multigraphs

- e:  notated as (e, x, y) ← order of x and y is important

- Let $\vec{E} = \{(e,x,y) \mid e \in E;\ x,y \in V;\ e = xy\}$.

- For $\vec{e} = (e, x, y)$, let $\overleftarrow{e} = (e, y, x)$.

- Similarly for $\vec{F} \subseteq \vec{E}$, let $\overleftarrow{F} = \{\overleftarrow{e} \mid \vec{e} \in \vec{F}\}$. Note $\vec{E} = \overleftarrow{E}$.

- For $X, Y \subseteq V$ and $\vec{F} \subseteq \vec{E}$, let $\vec{F}(X,Y) = \{(e,u,v) \in \vec{F} \mid u \in X,\ v \in Y,\ u \neq v\}$.

- For $S \subseteq V$, the Complement of S $(\bar{S})$ is $\underline{V \setminus S}$.

Flow on Directed Multigraphs

- Given $X, Y \subseteq V$ and a function $f : \vec{E} \to \mathbb{R}$, let

$$f(X,Y) = \sum_{\vec{e} \in \vec{E}(X,Y)} f(\vec{e})$$

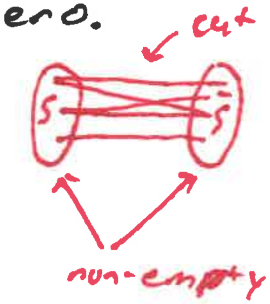
$$f(u, X) = f(\{u\}, X)$$

single vertex

Def: We say f is a <u>circulation</u> of G if

- $f(\vec{e}) = -f(\overleftarrow{e}) \quad \forall \vec{e} \in \vec{E}$ — i.e. a flow with no exceptions
- $f(u, V) = 0 \quad \forall u \in V$

Lemma: In a circulation, the net flow across any cut is zero.



Proof: $f(S, \bar{S}) = f(S, V) - f(S, S)$

net-flow

$$f(S,V) = \sum_{u \in S} f(u, V) = 0 \quad \text{by bullet 2}$$

$$f(S,S) = 0 \quad \text{by bullet 1 (see each edge in both directions)}$$

Corollary: If f is a circulation and $e = xy$ is a bridge (cut edge), then $f(\vec{e}) = f(\overleftarrow{e}) = 0$.

Networks

Def: Let $G=(V,\vec{E})$ be a directed multigraph, let $s,t\in V$ be two fixed vertices called the **source** and **sink**, and let $c:\vec{E}\longrightarrow \mathbb{R}$ be a map called a **capacity function**. $N=(G,s,t,c)$ is a **network**.

Def: A function $f:\vec{E}\longrightarrow\mathbb{R}$ is a **$(s,t\text{-})$ flow** in N if

- $f(\vec{e}) = -f(\overleftarrow{e}) \qquad \forall \vec{e}\in\vec{E}$
- $f(u,V) = 0 \qquad \forall u\in V\setminus\{s,t\}$
- $f(\vec{e}) \le c(\vec{e}) \qquad \forall \vec{e}\in\vec{E}$

Def: If $S\subseteq V$ such that $s\in S$ and $t\in\bar{S}$, we call $(S,\bar{S})$ an **s,t-cut** in N, and the capacity of the cut is defined to be

$$c(S,\bar{S}) = \sum_{\vec{e}\in\vec{E}(S,\bar{S})} c(\vec{e})$$

Max-Flow Min-Cut Theorem

Theorem: (Ford-Fulkerson, 1956) In every network, the maximum total value of a flow is equal to the minimum capacity of an s,t-cut.

little s

Observation: For every cut $(S, \bar{S})$ in N, $f(S, \bar{S}) = f(s, V)$.

Proof:
$$f(S, \bar{S}) = f(S, V) - f(S, S)$$
$$= f(s, V) + \sum_{u \in S \setminus s} f(u, V)$$
$$= f(s, V)$$

$\Rightarrow$ For any cut $(S, \bar{S})$, $f(s, V) \leq c(S, \bar{S})$ since $f(S, \bar{S}) \leq c(S, \bar{S})$.

i.e. max-flow $\leq$ min-cut.

Max-Flow Min-Cut Theorem

Need only show that max-flow is greater than some cut.

Proof: Suppose f is a maximum flow in N. Let S be the set of vertices reachable from s along an s-v walk W in N where $\forall \vec{e} \in W$, $f(\vec{e}) < c(\vec{e})$.

If $t \in S$, then let W_t be the witnessing s-t walk. Let $\varepsilon = \min_{\vec{e} \in W_t} c(\vec{e}) - f(\vec{e})$.

called an (f-) augmenting path.

Define $f'(\vec{e}) = \begin{cases} f(\vec{e}) + \varepsilon & \text{if } \vec{e} \in W_t \\ f(\vec{e}) - \varepsilon & \text{if } \overleftarrow{e} \in W_t \\ f(\vec{e}) & \text{otherwise} \end{cases}$

$f'(s,V) > f(s,V)$ and so f was not a maximum flow. So $t \notin S$.

Thus, $\forall \vec{e} \in \vec{E}(S,\bar{S})$ $f(\vec{e}) = c(\vec{e})$.

$$c(S,\bar{S}) = \sum_{\vec{e} \in \vec{E}(S,\bar{S})} c(\vec{e}) \leq \sum_{\vec{e} \in \vec{E}(S,\bar{S})} f(\vec{e}) = f(S,\bar{S}) = f(s,V)$$