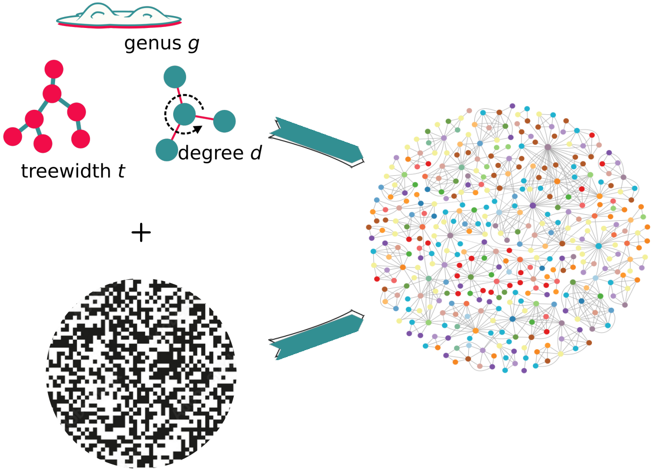
Structural Rounding

Brian Lavallee

18 October 2022

Approximations for Real-World Networks

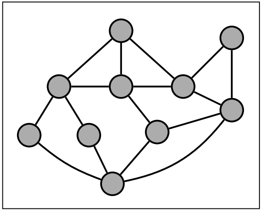


- **Structural rounding** gives better approximations by assuming networks are *close* to a structural class.

Outline

- Structural Rounding Framework
- Extending to Realistic Classes
  - Editing to Bounded Expansion
- Empirical Evaluation
  - Vertex Cover in Near-Bipartite Graphs
  - Dominating Set in Near-Bounded Treewidth Graphs
- Summary and Open Questions

Structural Rounding



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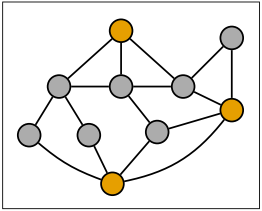

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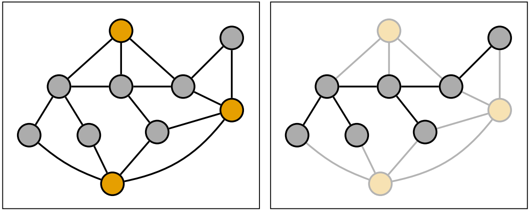

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Solve: Run an exact or approximate algorithm on the edited instance in polynomial time.

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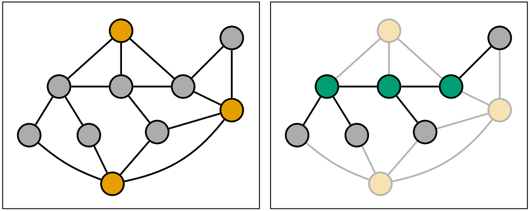

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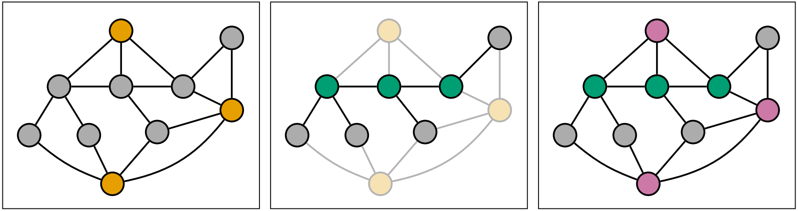

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Lift: Extend the partial solution to G.

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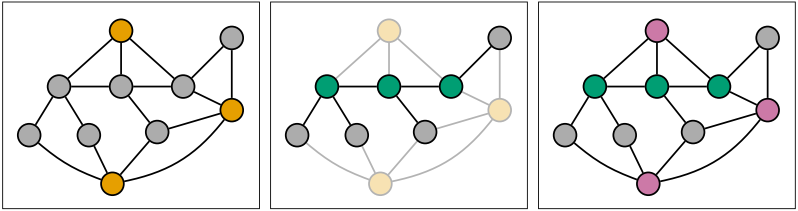

Edit: Use ψ to edit G to be in $\mathcal{C}_\lambda$. A problem is c'-**stable** w.r.t. ψ if $\psi_1, \ldots, \psi_d$ cannot increase $\mathrm{opt}(G)$ by more than $c' \cdot d$.

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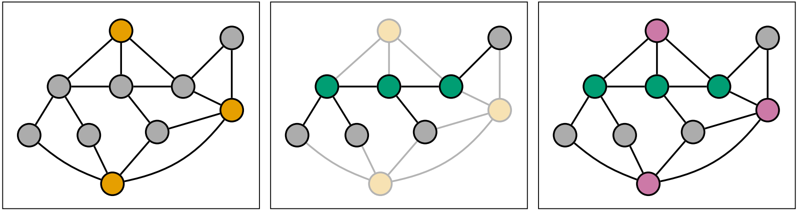

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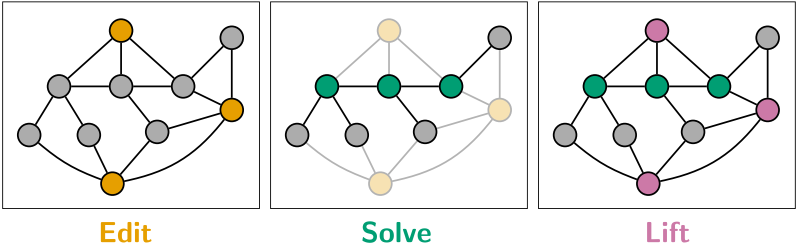
Solve: Run an exact or approximate algorithm on the edited instance in polynomial time.

Lift: Extend the partial solution to G. A problem can be c-**lifted** w.r.t. ψ if any partial solution S can be extended to G by undoing $\psi_1, \ldots, \psi_d$ without increasing $\text{cost}(S)$ by more than $c \cdot d$.

Structural Rounding



Theorem (Demaine, **L** et al, 2019)

Let Π *be a problem*

- *that is* c'*-stable under* ψ *and*
- *that can be* c*-lifted w.r.t.* ψ.

If Π *has a poly-time* $\rho(\lambda)$*-approximation in* $\mathcal{C}_\lambda$ *and* $(\mathcal{C}_\lambda, \psi)$-Editing *has a poly-time* (α, β)*-approximation, then there is a poly-time* $((1 + c'\alpha\delta) \cdot \rho(\beta\lambda) + c\alpha\delta)$*-approximation for* Π *on graphs that are* $(\delta \cdot \mathrm{opt}_\Pi(G))$*-close to* $\mathcal{C}_\lambda$.

Extending to Realistic Classes

- Editing to Bounded Expansion
- Generalized Coloring Numbers

Why Editing?

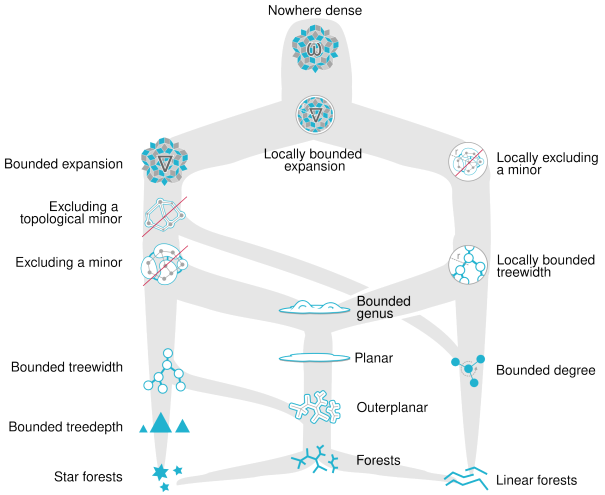
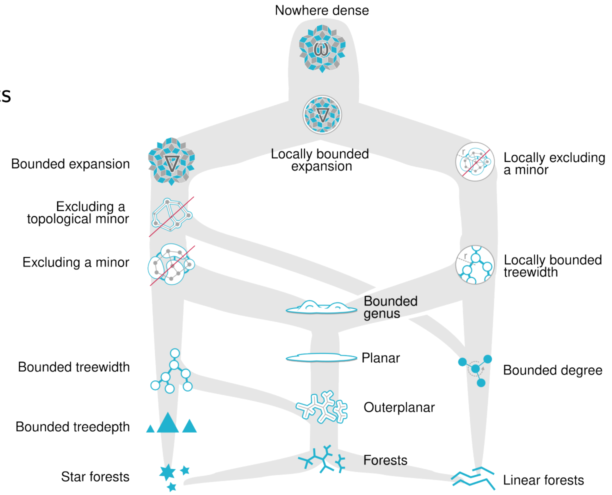


image credit: Felix Reidl

Why Editing?

Smaller Edit Sets



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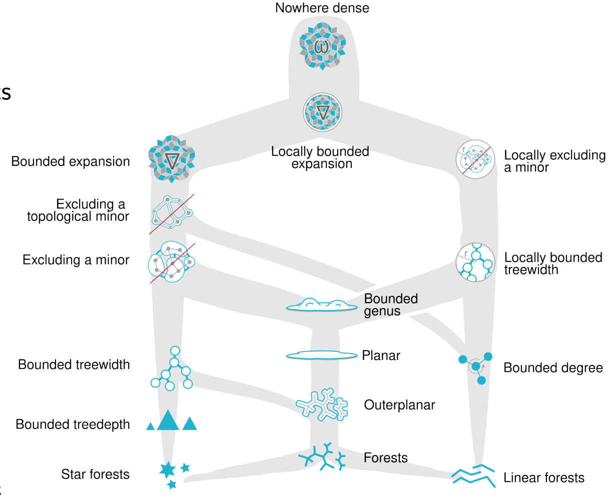

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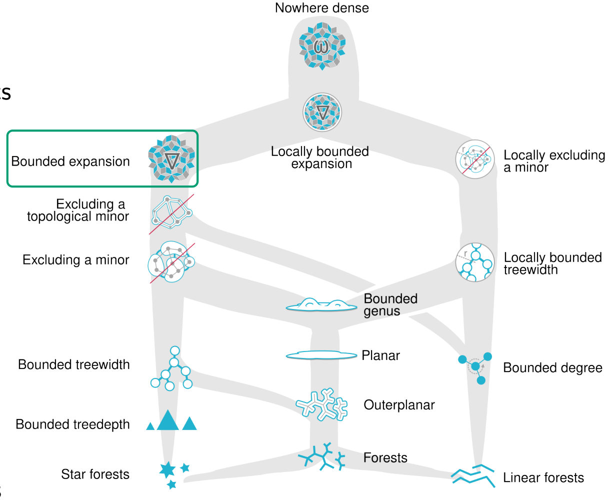

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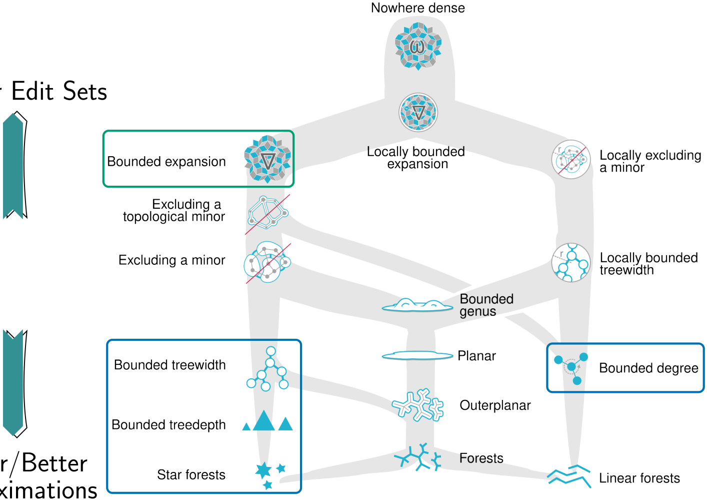

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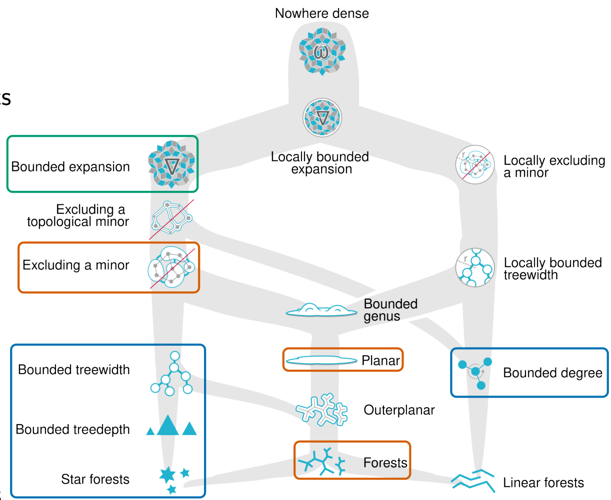

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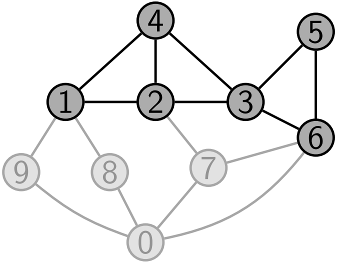
Generalized Coloring Numbers

- extend degeneracy to larger r-neighborhoods
- classes with bounded expansion have $\mathrm{col}_r(G) \leq f(r)$



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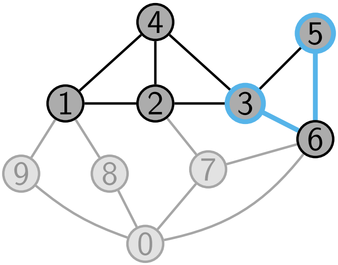

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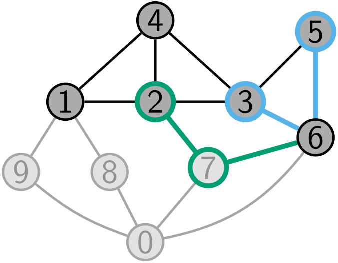

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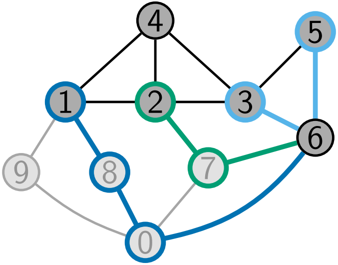

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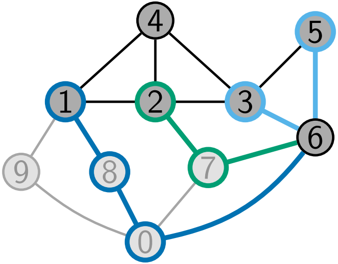

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$$\mathrm{col}_r(G) := \min_{\sigma \in \pi} \max_{u \in V} |\,\mathrm{R}_r(u, G_\sigma)|$$

Generalized Coloring Numbers

Theorem (Breen-McKay, **L**, Sullivan, 2022)

There is a greedy $(k-1)^{r-1}$ *approximation for* $\mathrm{col}_r(G)$ *in graphs with* r*-admissibility at most* k*.*

- $\mathrm{adm}_r(G) \leq \mathrm{col}_r(G)$

Theorem (Breen-McKay, **L**, Sullivan, 2022)

It is NP-hard to determine if $\mathrm{col}_r(G) \leq 6$ *when* $r \geq 2$*.*

Implications:

- No polynomial time $\frac{7}{6}$-approximation.
- No $O(n^{f(t)})$ time algorithm to check if $\mathrm{col}_r(G) \leq t$.

Coloring Number Deletion

r-Coloring Number Deletion

Input: a graph $G = (V, E)$ and a target t.

Problem: find a set of vertices $X \subseteq V$ of minimum size such that $\mathrm{col}_r(G \setminus X) \leq t$.

Theorem (Breen-McKay, Kloster, **L**, Sullivan, 2019)

There exists a constant c such that it is NP-hard to approximate r-Coloring Number Deletion within a factor of $c \cdot \log \frac{n}{r+t}$.

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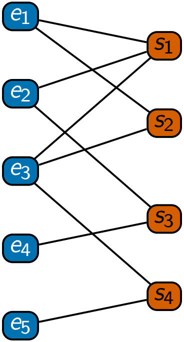
Set Cover

Input: a universe of elements $\mathcal{U}$ and a family of sets $\mathcal{F}$.
Problem: find a set $S \subseteq \mathcal{F}$ of minimum size such that $\bigcup_{s_i \in S} s_i = \mathcal{U}$.

Coloring Number Deletion

$$\mathcal{U} = \{e_1, e_2, e_3, e_4, e_5\} \; \mathcal{F} = \Big\{(e_1, e_2, e_3), (e_1, e_3), (e_2, e_4), (e_3, e_5)\Big\}$$



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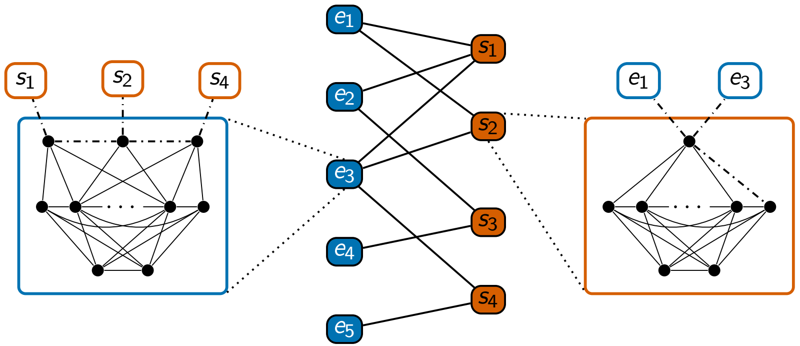

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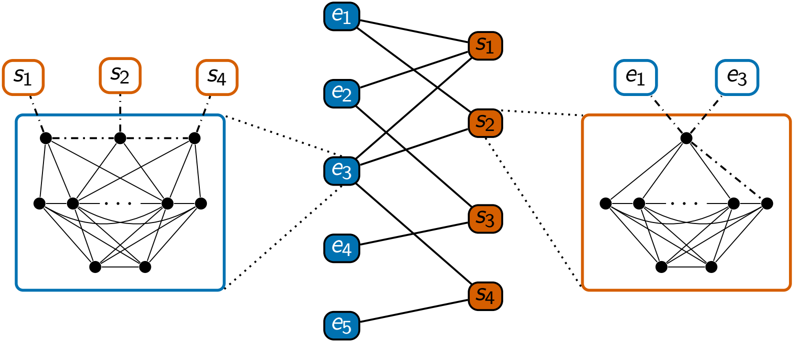

- Implies Degeneracy Deletion is $o(\log \frac{n}{t})$ inapproximable.

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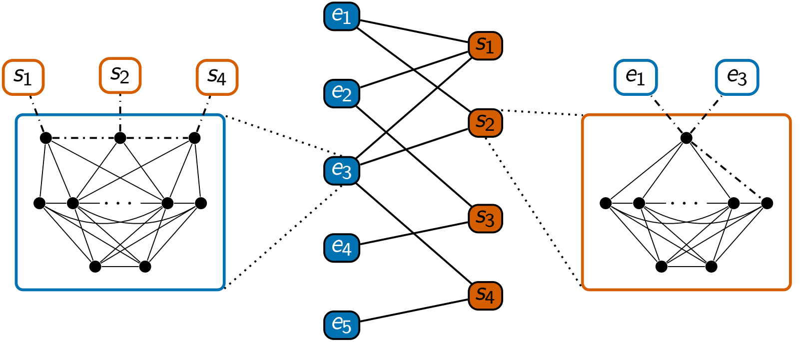

Caveat: The NP-hardness of computing $\mathrm{col}_r(G)$ implies that no multiplicative approximation exists.

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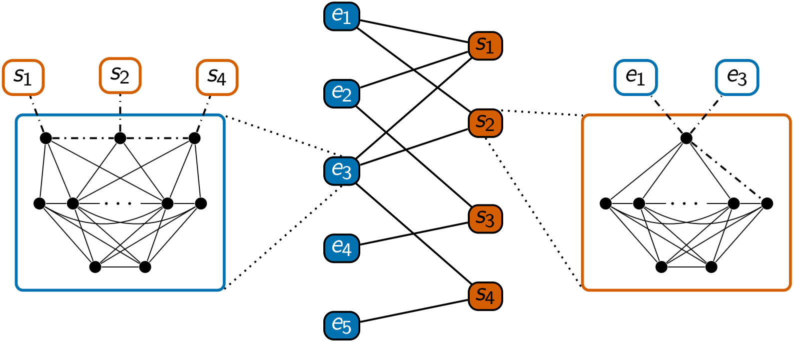


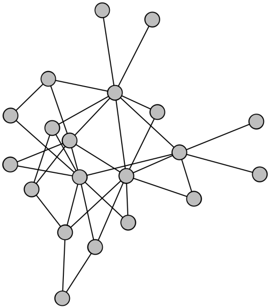
Optimistic view: Still applies when $\mathrm{col}_r(G) > t$ is guaranteed.

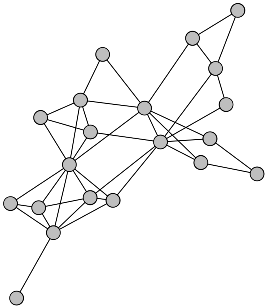
Structural Rounding in Practice

- Vertex Cover in Near-Bipartite Graphs
- Dominating Set in Near-Bounded Treewidth Graphs

Graph Corpuses

- `vc-synth`: 1,620 graphs generated with preferential attachment restricted by random labels (E, L, R)
- `ds-synth`: 750 graphs generated by taking a random subgraph of a random *k*-tree





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corpus	vertices	edges
`vc-synth`	10K	39K - 4M
`ds-synth`	4K - 10K	6K - 4M
`pace`	153 - 138K	1K - 454K
`networks`	9 - 935K	16 - 17M

Structural Rounding in Practice

- Vertex Cover in Near-Bipartite Graphs
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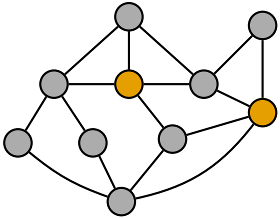
Vertex Cover in Near-Bipartite Graphs

Edit: Odd Cycle Transversal

- `greedy-edit`: builds from maximal cut.
- `indset-edit`: builds from two maximal independent sets.
- `bfs-edit`: builds from layers of BFS tree.



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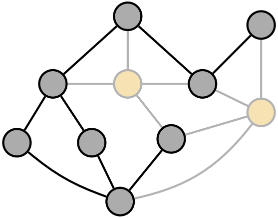

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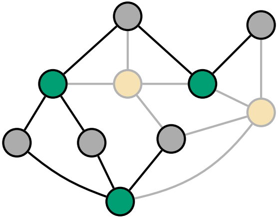

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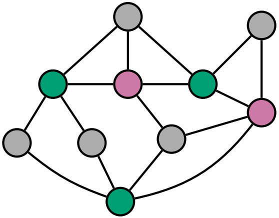

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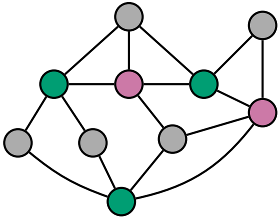

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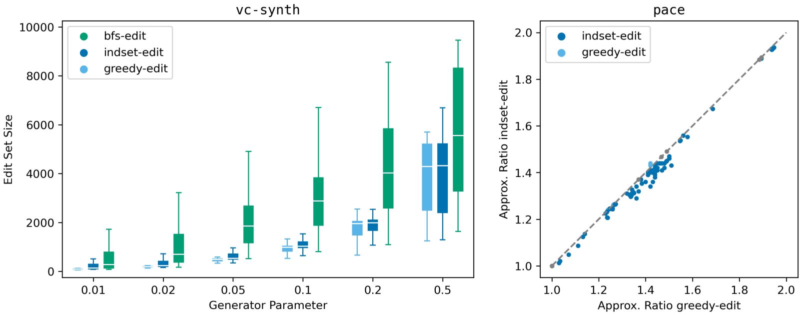
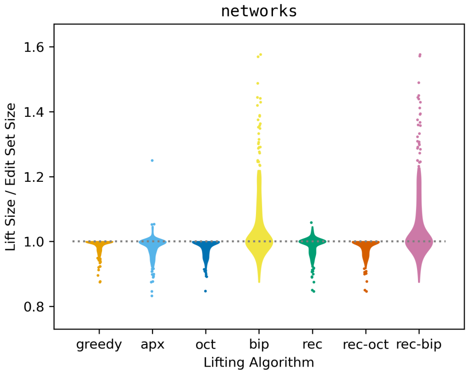


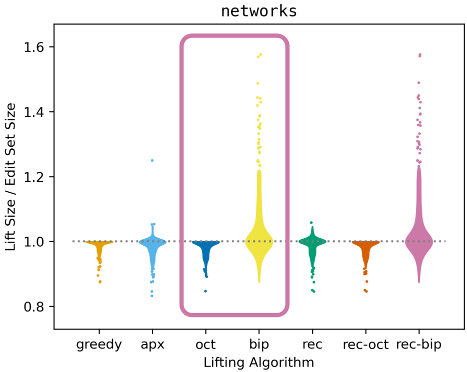
- `indset-edit` results in better solutions even though it finds larger edit sets.

Vertex Cover in Near-Bipartite Graphs



- Guaranteeing the lifting constant matters.

Vertex Cover in Near-Bipartite Graphs



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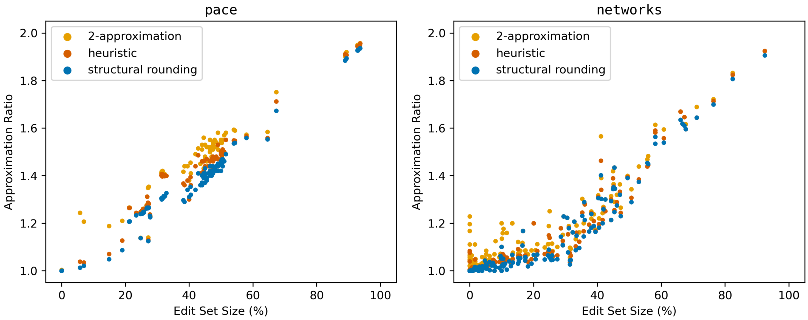

- Better performance even with large edit sets ($> 40\%$).
- About 1.8x slower than standard approximations/heuristics.
- Real-world networks aren't exactly *close* to bipartite.

Structural Rounding in Practice

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- Dominating Set in Near-Bounded Treewidth Graphs

Dominating Set in Near-Treewidth Graphs

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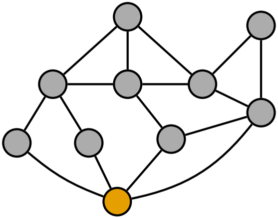
- `root-edit`: removes vertex separators until $\mathrm{tw}(G) \leq t$.



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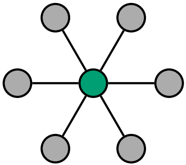

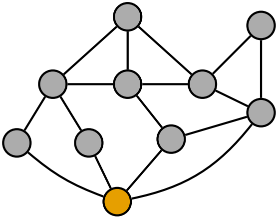

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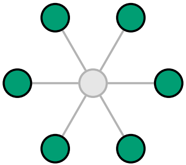

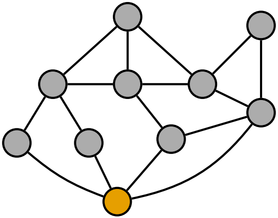

Dominating Set in Near-Treewidth Graphs

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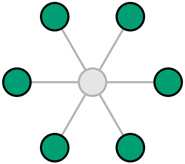

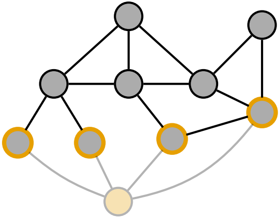

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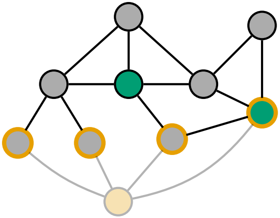

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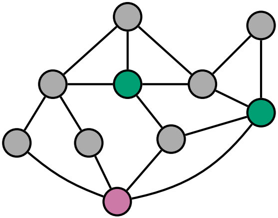

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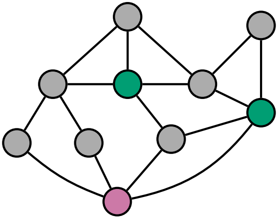
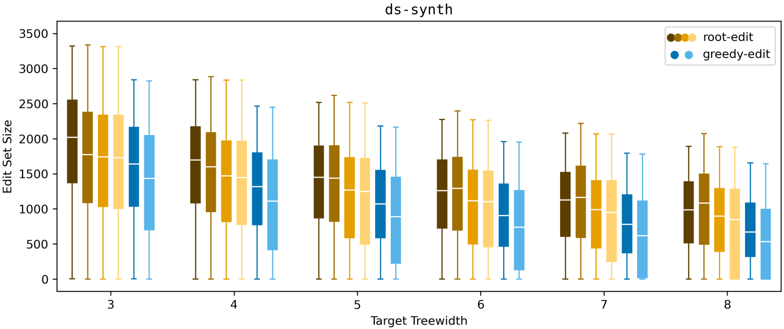


Dominating Set in Near-Bounded Treewidth Graphs



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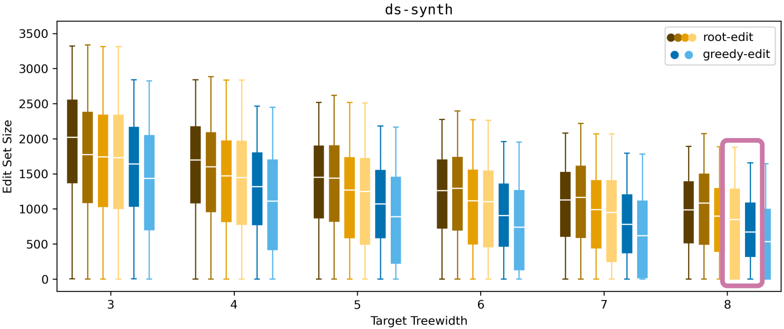

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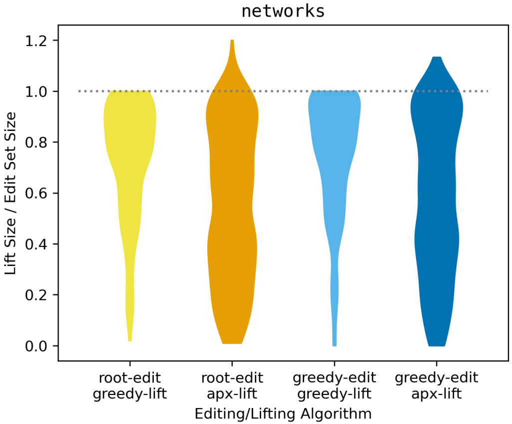
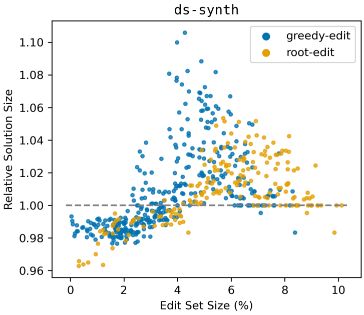
- final algorithm has runtime quadratic in treewidth of input.

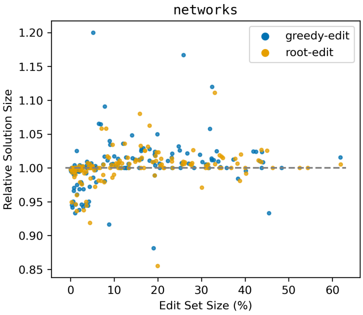
Dominating Set in Near-Bounded Treewidth Graphs



- Guaranteeing the lifting constant still matters.

Dominating Set in Near-Bounded Treewidth Graphs





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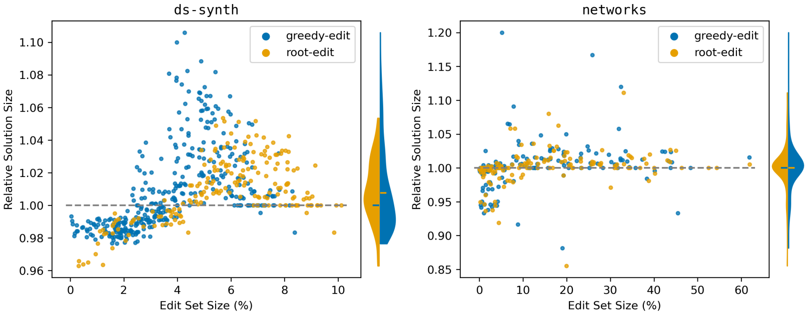

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- Bicriteria inapproximability
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Thanks!

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