CS 4150

Wednesday, October 7, 2020

Today's topic: Duality

- Duality
  - upper bounds
  - strong/weak duality
- Bipartite matching

Finding Bounds

optimal solution x^*

$$2x_1^* + 3x_2^* \le p$$

maximize $2x_1 + 3x_2$

s.t.

$$\begin{cases} 4x_1 + 8x_2 \le 12 \\ 2x_1 + x_2 \le 3 \\ 3x_1 + 2x_2 \le 4 \end{cases}$$

$$x_1, x_2 \ge 0$$

minimize $12y_1 + 3y_2 + 4y_3$

s.t.

$$4y_1 + 2y_2 + 3y_3 \ge 2$$

$$8y_1 + y_2 + 2y_3 \ge 3$$

$$y_1, y_2, y_3 \ge 0$$

- $2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$
- $2x_1 + 3x_2 \le 2x_1 + x_2 + 3x_1 + 2x_2 = 5x_1 + 3x_2 \le 7$
- $2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le 6$

$$\boxed{2x_1 + 3x_2 \le y_1(4x_1 + 8x_2) + y_2(2x_1 + x_2) + y_3(3x_1 + 2x_2)} \le$$

$$(4y_1 + 2y_2 + 3y_3)x_1 \qquad 12y_1 + 3y_2 + 4y_3$$

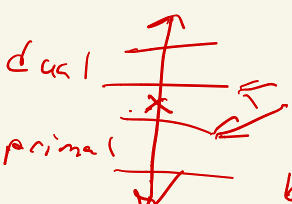
Duality

Primal ← original		Dual $\amalg$
Π: max $c \cdot x$ s.t. $Ax \le b$, $x \ge 0$	$\Longleftrightarrow$	min $y \cdot b$ s.t. $yA \ge c$, $y \ge 0$

Dual of the dual = primal

Strong duality: If x^* is an <u>optimal</u> solution to a canonical linear program Π, then there is an <u>optimal</u> solution y^* to the dual $\amalg$ s.t. $c \cdot x^* = y^* \cdot b$

Weak duality: If x is a <u>feasible</u> solution to a canonical LP Π and y is a <u>feasible</u> solution to the dual $\amalg$, then $c \cdot x \le y \cdot b$.



the same by strong duality

Dual Practice

maximize $4x_1 + x_2 + 3x_3$

s.t.

$$x_1 + 4x_2 \leq 2$$

$$3x_1 - x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

minimize $2y_1 + 4y_2$

$$y_1 + 3y_2 \geq 4$$

$$4y_1 - y_2 \geq 1$$

$$y_2 \geq 3$$

$$y_1, y_2 \geq 0$$

$$-x - 4x_2 \geq -2$$

$$4x_1 + x_2 + 3x_3 \leq y_1(x_1 + 4x_2) + y_2(3x_1 - x_2 + x_3) \leq 2y_1 + 4y_2$$

$$(y_1 + 3y_1)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3$$

maximize $x_1 + 3x_2$

s.t.

$$2x_1 + x_2 \le 5$$

$$x_1 + 3x_2 \le 7$$

$$2x_1 + 3x_2 \le 10$$

$$x_1, x_2 \ge 0$$

minimize $5a + 7b + 10c$

$$1 \le 2a + b + 2c$$

$$3 \le a + 3b + 3c$$

not $\ge$!

$$a, b, c \ge 0$$

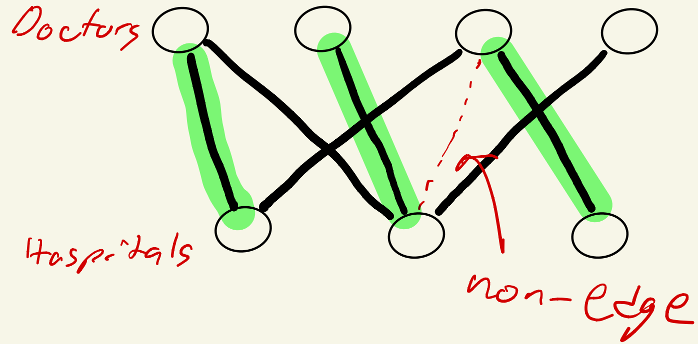
we want this to be true

$$1x_1 + 3x_2 \boxed{\le}$$

$$a(2x_1 + 1x_2) + b(x_1 + 3x_2) + c(2x_1 + 3x_2) \le 5a + 7b + 10c$$

$$(2a + b + 2c)x_1 + (a + 3b + 3c)x_2$$

Bipartite Matching



Input

- disjoint sets A and B
- set of edges $E \subseteq A \times B$

Definition

$M \subseteq E$ is a matching if for every $ab \in M$,

$\nexists\, a'b' \in M$ such that $a = a'$ or $b = b'$.

Goal

find the largest possible matching

Breakout! Write an LP for max matching

Hint!

- edges → variables
- vertices → constraints

Bipartite Matching LP

Let $x_{ab} = \begin{cases} 1 & \text{if } a \text{ is matched to } b \\ 0 & \text{otherwise} \end{cases}$

variable for edge ab

maximize $\sum_{ab} x_{ab}$

s.t.

$\forall a \quad \sum_{b \in N(a)} x_{ab} \le 1$

$\forall b \quad \sum_{a \in N(b)} x_{ab} \le 1$

$\forall ab \quad 0 \le x_{ab} \le 1$

allows for fractional solutions

* In general, it's hard to get integral solutions.

But, for max bip matching, lemma $\Rightarrow \exists$ integral optimal solution

Bipartite Matching Dual

Primal

maximize $\sum_{ab} x_{ab}$

s.t.

1 per vertex

$\forall a \quad \sum_{b \in N(a)} x_{ab} \leq 1$

$\forall b \quad \sum_{a \in N(b)} x_{ab} \leq 1$

$0 \leq x_{ab} \leq 1$

variables → constraints

constraints → variables

Dual

minimize $\sum_{u} x_u$

$\forall uv \quad x_u + x_v \geq 1$

$\forall u \quad 0 \leq x_u \leq 1$

when does x_{uv} appear in a constraint?