Rotation and Transformation Matricies 1,2 & 3

demonstrated

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For tutoring purposes | Entry level Robotics | 11/23/2022

```
clc; clear; close all
```

Motivation

Express a frame of refrence, orientation and position, with respect to some origin, or space, frame

Background

Some details are provided for convenience. Familiarity with these topics is reccomended.

Matricies and Matrix Multiplication

Configuration Space: Chapter 2 of Modern Robotics, by Kevin Lynch and Frank C Park

A. Degrees of Freedom

  1. Grublers formula

B. Implicit vs Explicit representations

  1. Explicit representations - use one variable for each degree of freedom, DOF, to represent a state

     a. Suffer from singularities, use minimal coordinates

     b. Singularities occur during a discontinuity or unproportionate change in coordinate representation where the associated movement is continuous

     c. Latitude and Longitude form an Explicit represenation, but suffer from a singularity at and near the poles

  2. Implicit representations - use a space embedded in a higher dimension subject to constraints

     a. Avoid singularities at the cost of storing more numbers for formatting purposes

C. Topology vs coordinate representation

  1. topological equivelancy

Visualization Strategy

in Matlab, quiver3(X,Y,Z,U,V,W) plots arrows with directional components U, V, and W at the Cartesian coordinates specified by X, Y, and Z

Matrix row and column extraction

- matrix rows or columns in a matrix A can be extracted by multiplying A by some other matrix

multiplying matrix A, 2x2 with [1 0]'

```
A = [1 2 3; 4 5 6; 7 8 9];
column = [1 0 0]';
A*column %extracting the column requires postmultiplying A and a 3x1 matrix
```

```
ans =
     1
     4
     7
```

Rotation Matricies

Similarly, axis can be extracted and rotated with rotation matricies

An orientation of a new frame, expressed in a space frame can be represented with three unit vectors.This Implicit representation implies constraints requiring the x and y axis to be 90 degrees from eachother and the z axis being orthoganal to the xy plane.

a 3D rotation matrix is written as a set of three vectors where the new axis are defined by each vector, X Y Z, or expressed in terms of an angle

For Example

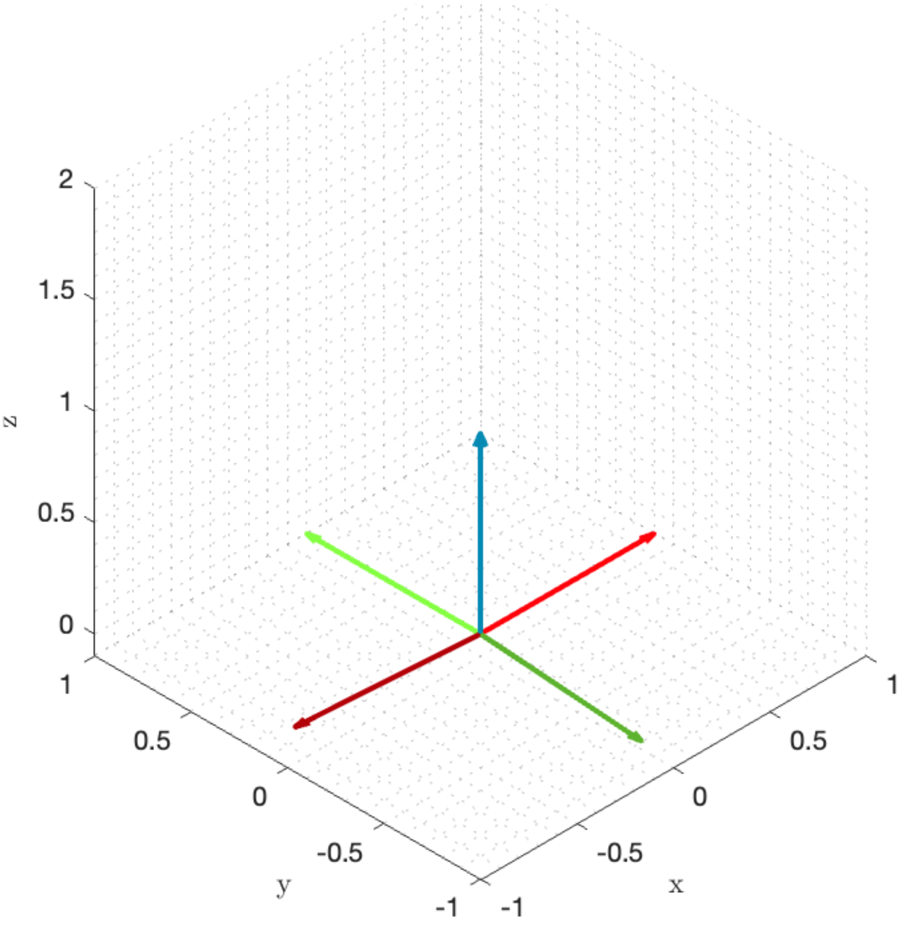
```
syms angle; rotz = rotZ(angle)
```

rotz =

$$\begin{pmatrix} \cos(\mathrm{angle}) & -\sin(\mathrm{angle}) & 0 \\ \sin(\mathrm{angle}) & \cos(\mathrm{angle}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
% Space Frame axis
X = [1 0 0]'; Y = [0 1 0]'; Z = [0 0 1]';
location = [0 0 0];
```

```
axisLimits = initializeAxisLimits();
endAngle = 360;

% Create each frame of the animation
loop = 1;
for i = 0.0:8.0:endAngle
    hold on

    % New orientation
    orientation = [X Y Z]*rotZ(i*pi/180);

    % New axis colors
    darkness = i/endAngle/1.15;

    % Draw the space frame axis
    axisLimits = drawAxis(location,eye(3),0,axisLimits);

    % Draw the new axis
    axisLimits = drawAxis(location,orientation,darkness,axisLimits);

    % Include the current angle in the title
    title(sprintf('Rotation about the Z Axis by %0.2f degrees',i),'Interpreter','Latex');
    xlim([-1 1]); ylim([-1 1]); zlim([-0.1 2]); grid minor

   % Capture the current frame
    frames(loop) = getframe(gcf);
    clf

    loop = loop + 1;
end

% play the frames inside the script
figure
ax = gca; ax.Visible = 'off';
```



```
movie(frames,1)

% Save the result as a gif
saveGif('output.gif', frames);
```

Similarly x and y can be described by

```
rotx = rotX(angle)
```

rotx =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathrm{angle}) & -\sin(\mathrm{angle}) \\ 0 & \sin(\mathrm{angle}) & \cos(\mathrm{angle}) \end{pmatrix}$$

```
roty = rotY(angle)
```

roty =

$$\begin{pmatrix} \cos(\mathrm{angle}) & 0 & \sin(\mathrm{angle}) \\ 0 & 1 & 0 \\ -\sin(\mathrm{angle}) & 0 & \cos(\mathrm{angle}) \end{pmatrix}$$

Furthermore, a rotation about an arbitrary axis can be described with Rodrigues Formula

Transformation Matricies

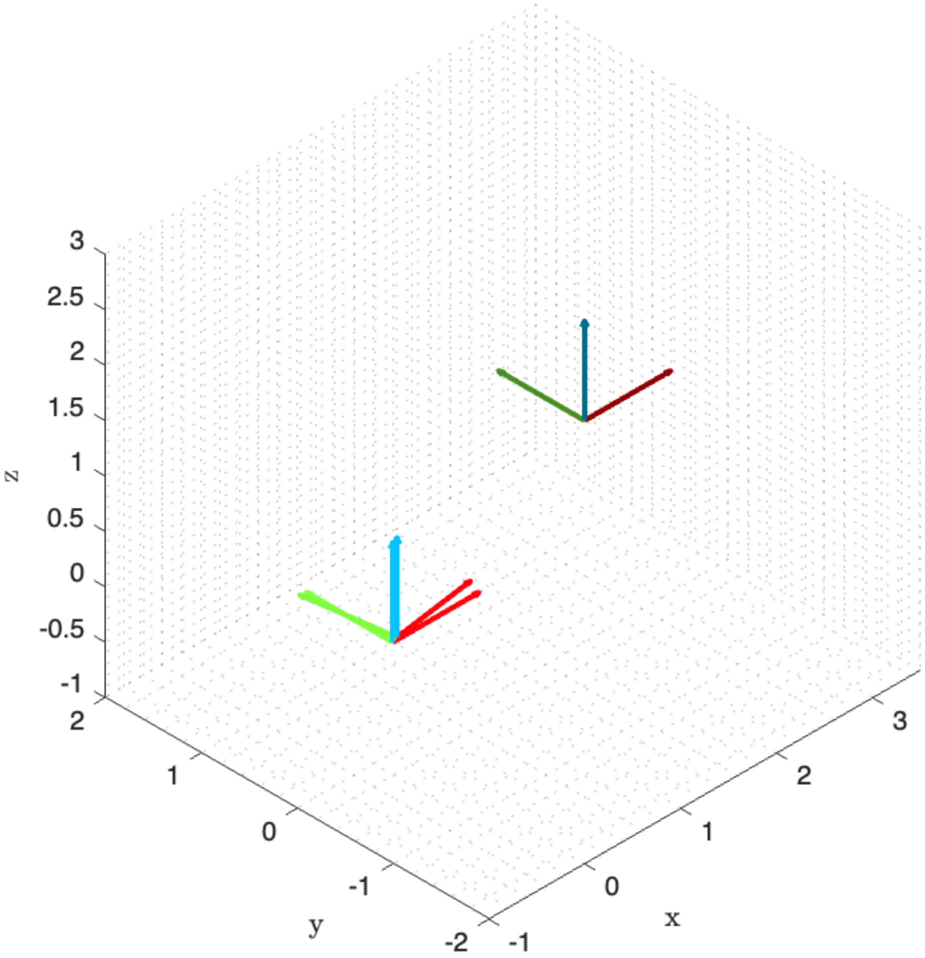
translation and change in rotation

T = [R 0 ; 0 1]

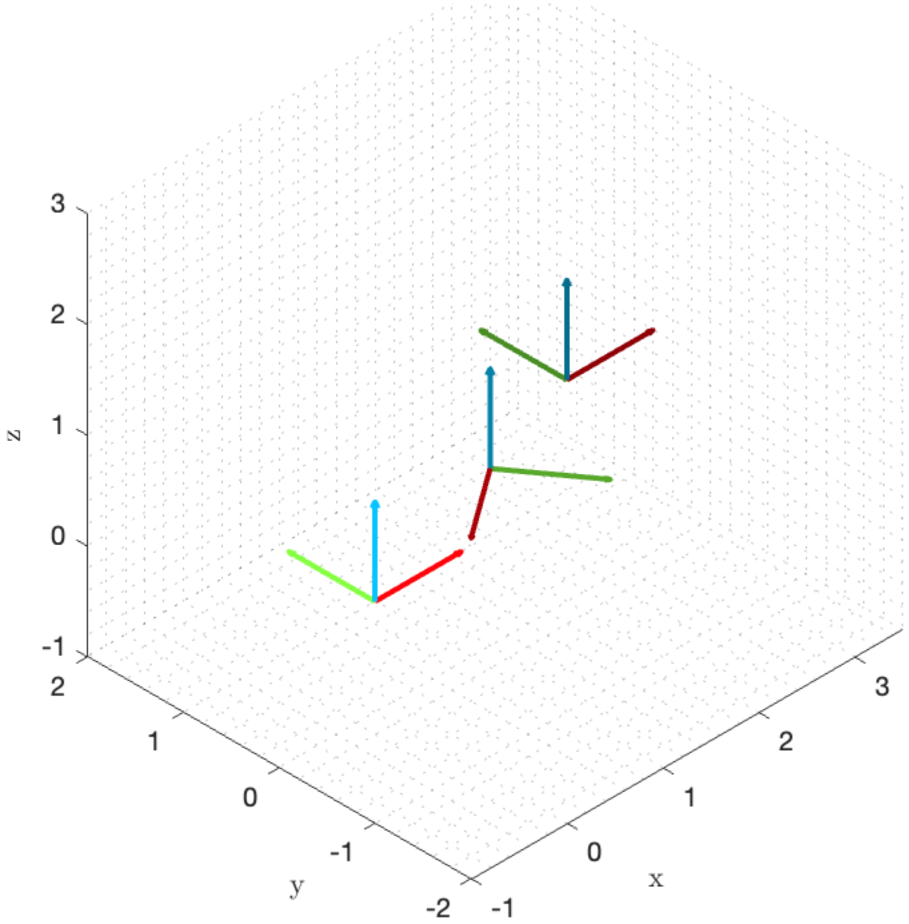
A tranformation matrix conveys a rotation and translation

```
axis = [0 0 1];
clear

%Space Frame axis
X = [1 0 0]'; Y = [0 1 0]'; Z = [0 0 1]';
locationStart = [0 0 0];
locationEnd = [1 -1 2];

axisLimits = initializeAxisLimits();
endAngle = 360;

grid minor
loop = 1;
for i = 0:8:endAngle
        grid minor
        hold on

        % New axis
        orientation = [X Y Z]*rotZ(i*pi/180);

        % New axis colors
        darkness = i/endAngle/1.15;

        % Draw the space frame axis
        axisLimits = drawAxis(locationStart,eye(3),0,axisLimits);

        % Draw the End axis
        drawAxis(locationEnd,[X Y Z]*rotZ(endAngle*pi/180),.8,axisLimits);   %location,orientation,darkness,axisLimits)

        % Draw the new axis
        locationProgress = i/endAngle*locationEnd;
        axisLimits = drawAxis(locationProgress,orientation,darkness,axisLimits);

        % Include the current angle in the title
        lp(1) = locationProgress(1); lp(2) = locationProgress(2); lp(3) = locationProgress(3);
        title(sprintf('Rotation about the Z Axis by %0.2f degrees\n Translation at %0.2f,%0.2f,%0.2f',i,lp(1),lp(2),lp(3)),'Interp
        xlim([-1 3.5]); ylim([-2 2]); zlim([-1 3]);
        grid minor

        % Capture the current frame
        frames(loop) = getframe(gcf);
        clf

        loop = loop + 1;
end
```

Rotation about the Z Axis by 8.00 degrees
Translation at 0.02,-0.02,0.04



```
clf
% play the frames inside the script
figure
ax = gca; ax.Visible = 'off';
```

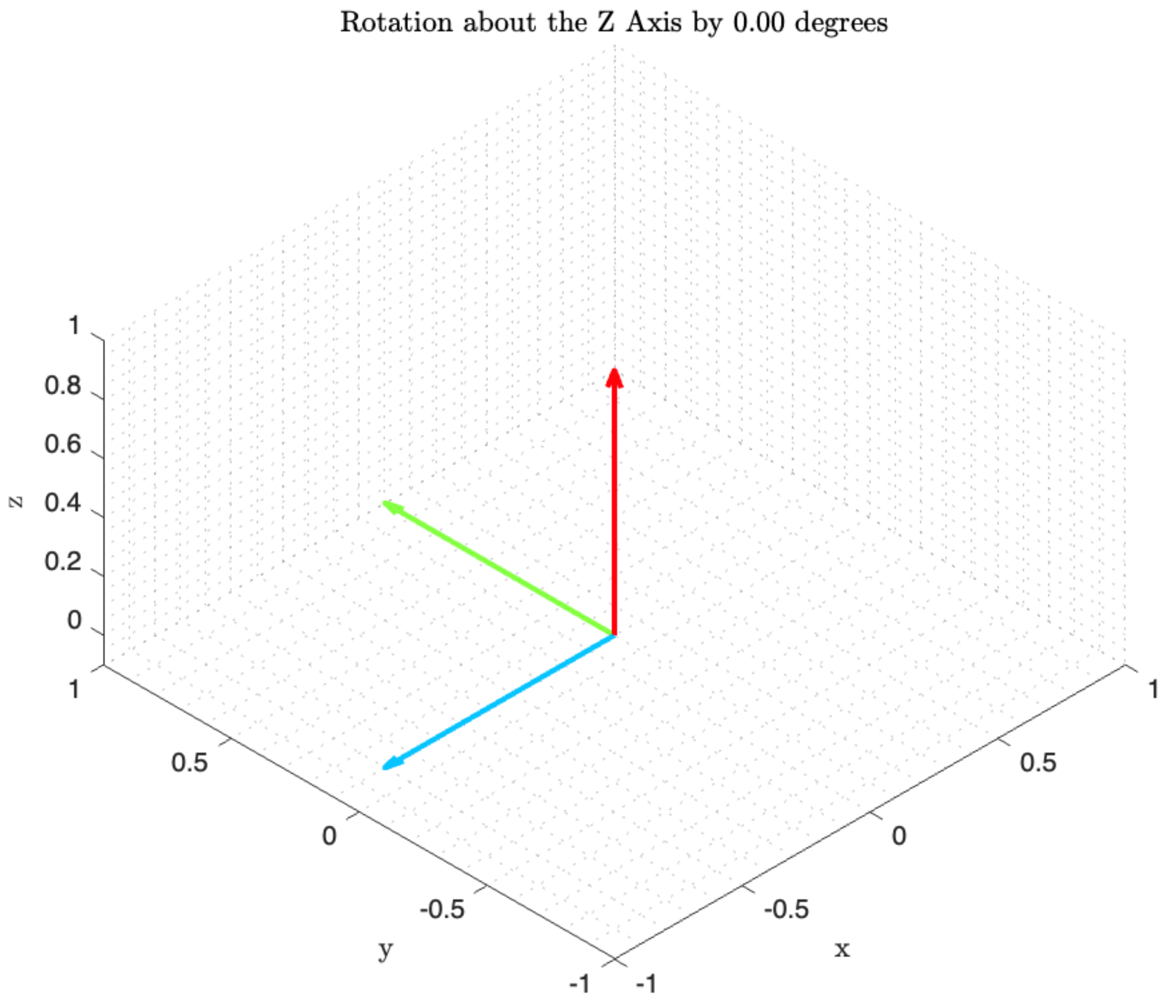


```
movie(frames,1)
% Save the result as a gif
[imind,cm] = rgb2ind(frames(1).cdata,256);
imwrite(imind,cm,'animation.gif','gif','Loopcount',inf);
for i = 2:length(frames)
    [imind,cm] = rgb2ind(frames(i).cdata,256);
    imwrite(imind,cm,'animation.gif','gif','WriteMode','append');
end
```

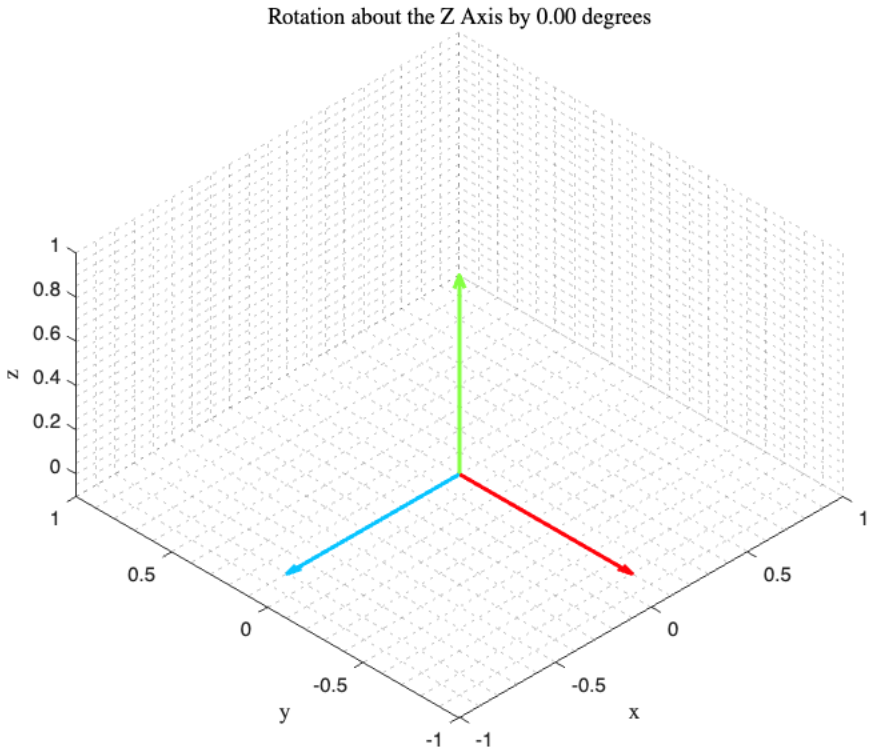
Rolling Cube Animation

```
clc; clear; close all
%Space Frame axis
%locationStart and orientationStart
p = [0 0 0]'; R = eye(3);
%TransformationMatrix Start
Tstart = [[R p];[0 0 0 1]];
endAngle = -90; time = 1;
    j = 1;
    for angle = 0:-5:endAngle
        % New axis
        Rcurrent = rotY(angle*pi/180);
        Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
        Tnow = Tstart*Tlast_to_now;

        % New axis colors
        darkness = angle/endAngle/1.15;

        % Draw the current axis
        drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)

        % Include the current angle in the title
        title(sprintf('Rotation about the Y Axis by %0.2f degrees',angle),'Interpreter','Latex');
```

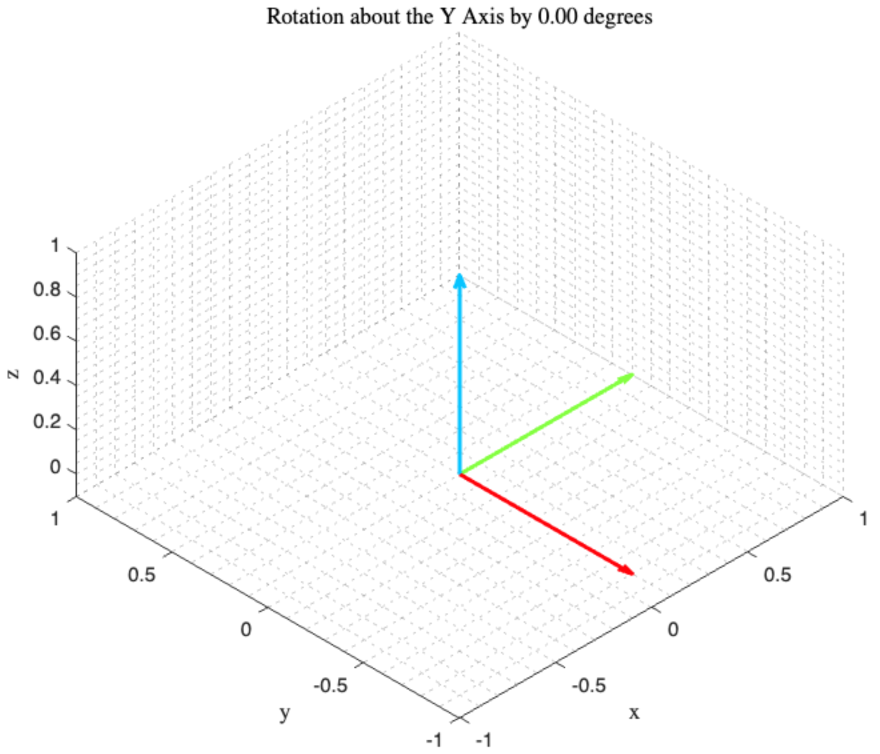
```
            axis equal
    xlim([-1 1]); ylim([-1 1]); zlim([-0.1 1]);

    % Capture the current frame
    frames(time) = getframe(gcf);

    time = time + 1;
    clf
end
Rcurrent = rotY(-90*pi/180);
Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
j=2;
for angle = 0:-5:endAngle
    Rcurrent = rotZ(angle*pi/180);
    Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    Tnow = Tstart*Tlast_to_now;
    darkness = angle/endAngle/1.15;
    drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
    title(sprintf('Rotation about the Z Axis by %0.2f degrees',angle),'Interpreter','Latex');
            axis equal
    xlim([-1 1]); ylim([-1 1]); zlim([-0.1 1]);
    % Capture the current frame
    frames(time) = getframe(gcf);
    time = time + 1;
    clf
end
```



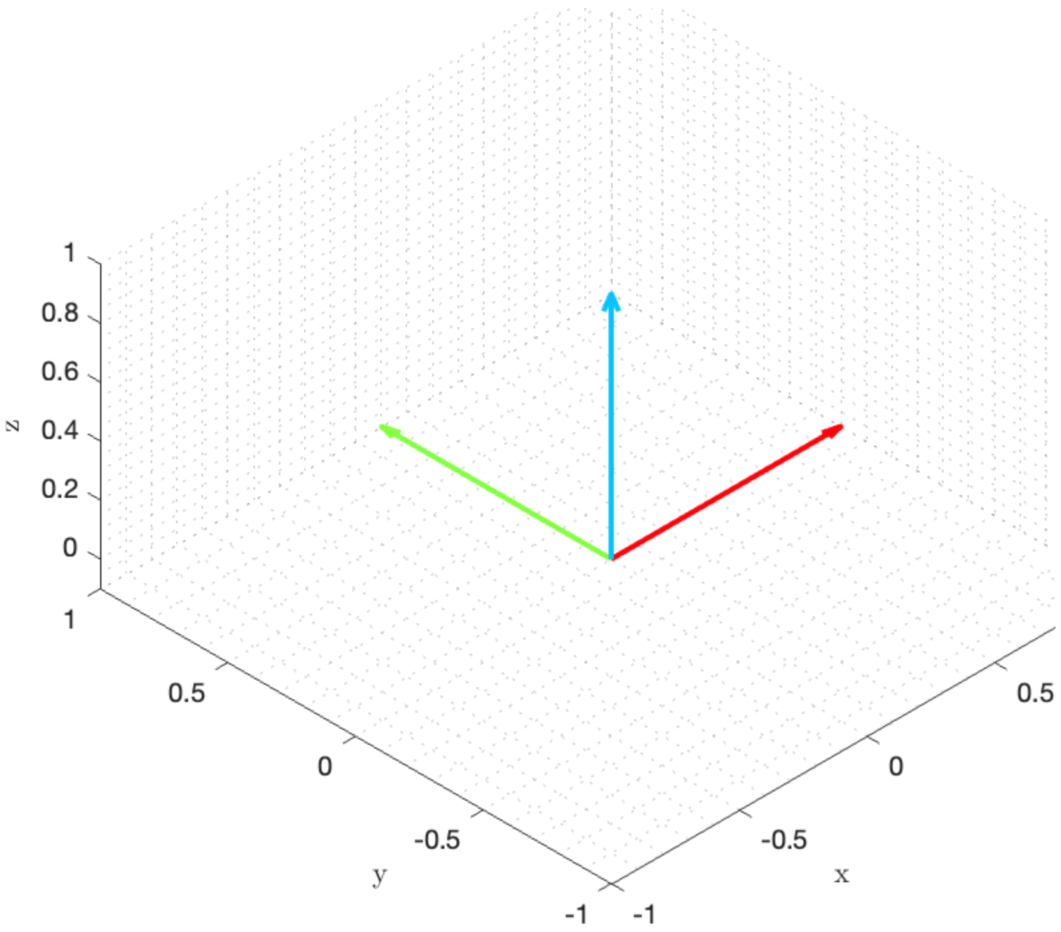
```
Rcurrent = rotY(-90*pi/180)*rotZ(-90*pi/180);
Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
j=2;
for angle = 0:-5:endAngle
    Rcurrent = rotX(angle*pi/180);
    Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    Tnow = Tstart*Tlast_to_now;
    darkness = angle/endAngle/1.15;
    drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
    title(sprintf('Rotation about the Z Axis by %0.2f degrees',angle),'Interpreter','Latex');
            axis equal
    xlim([-1 1]); ylim([-1 1]); zlim([-0.1 1]);
    % Capture the current frame
    frames(time) = getframe(gcf);
    time = time + 1;
    clf
end
```


Rotation about the Z Axis by 0.00 degrees

```
Rcurrent = rotY(-90*pi/180)*rotZ(-90*pi/180)*rotX(-90*pi/180);
Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
for angle = 0:-5:endAngle
    Rcurrent = rotY(angle*pi/180);
    Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    Tnow = Tstart*Tlast_to_now;
    darkness = angle/endAngle/1.15;
    drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
    title(sprintf('Rotation about the Y Axis by %0.2f degrees',angle),'Interpreter','Latex');
            axis equal
    xlim([-1 1]); ylim([-1 1]); zlim([-0.1 1]);
    % Capture the current frame
    frames(time) = getframe(gcf);
    time = time + 1;
    clf
end
```



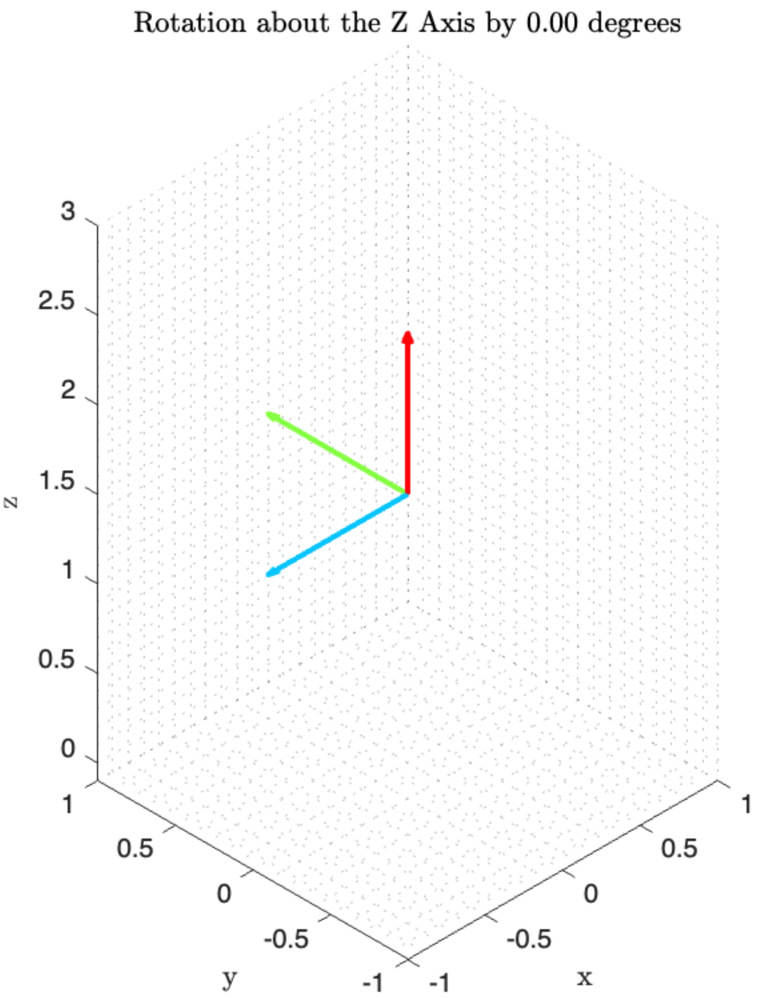
```
close all; figure
ax = gca; ax.Visible = 'off';
```



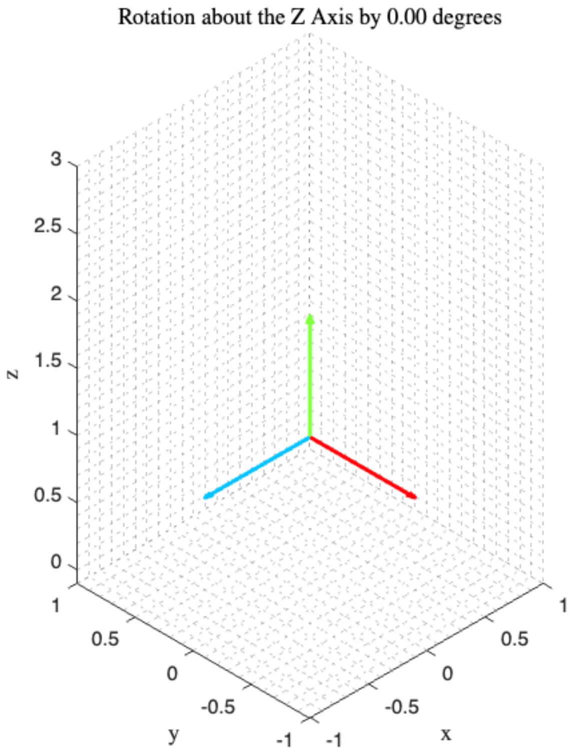
```
movie(frames,1)
```

Hovering Cube Animation

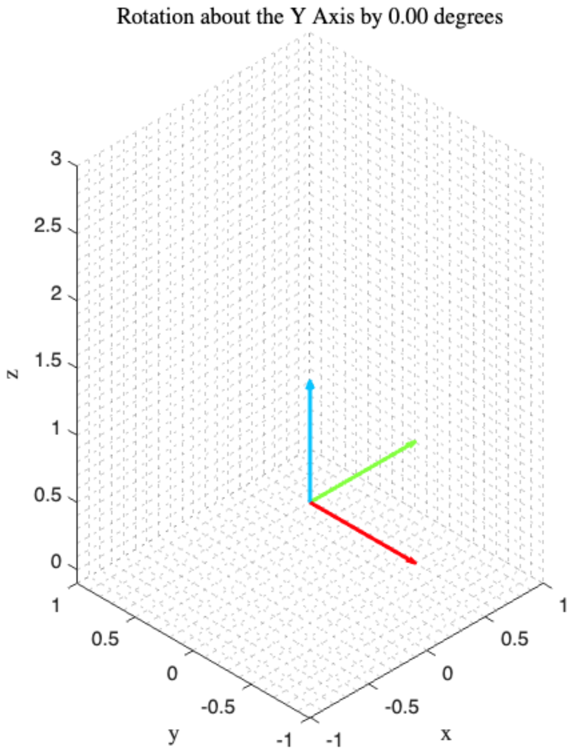
```
clc; clear; close all
%Space Frame axis
%locationStart and orientationStart
p = [0 0 0]'; R = eye(3);
%TransformationMatrix Start
Tstart = [[R p];[0 0 0 1]];
endAngle = -90; time = 1;
stepSize=3;
totalSteps = -endAngle/stepSize ∗ 4 + 4 ;
path = sin(linspace(0,2∗pi,totalSteps))/2 + 1;
    for angle = 0:-stepSize:endAngle
        % New axis
        Rcurrent = rotY(angle∗pi/180);
        z = path(time);
        Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
        Tnow = Tstart∗Tlast_to_now;
        Tnow(1:3,4) = [0 0 z]';

        % New axis colors
        darkness = angle/endAngle/1.15;

        % Draw the current axis
        drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)

        % Include the current angle in the title
        title(sprintf('Rotation about the Y Axis by %0.2f degrees',angle),'Interpreter','Latex');
        axis equal
        xlim([-1 1]); ylim([-1 1]); zlim([-0.1 3]);
        % Capture the current frame
        frames(time) = getframe(gcf);
        time = time + 1;
        clf
    end
    Rcurrent = rotY(-90*pi/180);
    Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    j=2;
    for angle = 0:-stepSize:endAngle
        Rcurrent = rotZ(angle*pi/180);
        z = path(time);
        Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
        Tnow = Tstart*Tlast_to_now;
        Tnow(1:3,4) = [0 0 z]';
        darkness = angle/endAngle/1.15;
        drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
        title(sprintf('Rotation about the Z Axis by %0.2f degrees',angle),'Interpreter','Latex');
        axis equal
        xlim([-1 1]); ylim([-1 1]); zlim([-0.1 3]);
        % Capture the current frame
        frames(time) = getframe(gcf);
        time = time + 1;
        clf
    end
```



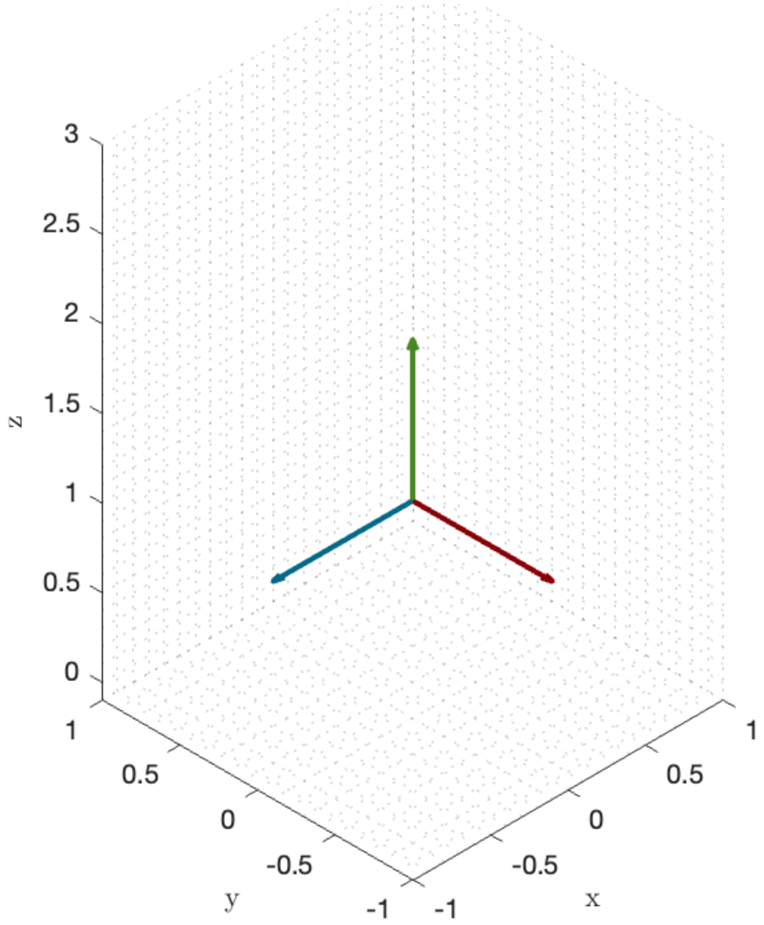
```
    Rcurrent = rotY(-90*pi/180)*rotZ(-90*pi/180);
    Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    j=2;
    for angle = 0:-stepSize:endAngle
        Rcurrent = rotX(angle*pi/180);
        z = path(time);
        Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
        Tnow = Tstart*Tlast_to_now;
        Tnow(1:3,4) = [0 0 z]';
        darkness = angle/endAngle/1.15;
        drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
        title(sprintf('Rotation about the Z Axis by %0.2f degrees',angle),'Interpreter','Latex');
        axis equal
        xlim([-1 1]); ylim([-1 1]); zlim([-0.1 3]);
        % Capture the current frame
        frames(time) = getframe(gcf);
        time = time + 1;
        clf
    end
```



```
    Rcurrent = rotY(-90*pi/180)*rotZ(-90*pi/180)*rotX(-90*pi/180);
    Tstart = [[Rcurrent [0 0 0]'];[0 0 0 1]];
    for angle = 0:-stepSize:endAngle
        Rcurrent = rotY(angle*pi/180);
        z = path(time);
        Tlast_to_now = [[Rcurrent [0 0 0]'];[0 0 0 1]];
        Tnow = Tstart*Tlast_to_now;
        Tnow(1:3,4) = [0 0 z]';
        darkness = angle/endAngle/1.15;
        drawAxis2(Tnow,darkness);   %location,orientation,darkness,axisLimits)
        title(sprintf('Rotation about the Y Axis by %0.2f degrees',angle),'Interpreter','Latex');
        axis equal
        xlim([-1 1]); ylim([-1 1]); zlim([-0.1 3]);
        % Capture the current frame
        frames(time) = getframe(gcf);
        time = time + 1;
        clf
    end
```



```
close all; figure
ax = gca; ax.Visible = 'off';
```



```
movie(frames,1)
```

Functions

Draw Axis Function

```
function axisLimits = drawAxis(location,orientation,darkness,axisLimits)
%Axis Colors
f = 1+darkness; %Darkness should be from 0 to 1
    color(1,:) = [182, 2, 8]/182/f;   % red
    color(2,:) = [59, 114, 29]/114/f; % green
    color(3,:) = [4, 110, 143]/143/f; % blue

%plotting 3 vectors
    Isometric = [-45 35.264];
    hold on
    for i = 1:1:3
        vec = orientation(:,i);
        quiver3(location(1),location(2),location(3),vec(1),vec(2),vec(3),'LineWidth',2,'Color',color(i,:));
        axisLimits = checkAxisLimits(vec,axisLimits);
    end
    hold off
%Viewing and Plot Settings
    axis equal
    view(Isometric)
    grid minor
    xlabel('x', 'Interpreter', 'Latex'),ylabel('y', 'Interpreter', 'Latex'),zlabel('z', 'Interpreter', 'Latex')
end
```

Draw Axis 2

```
function drawAxis2(T,darkness)
Isometric = [-45 35.264];
```

```
%Axis Colors
f = 1+darkness; %Darkness should be from 0 to 1
color = [ [182, 2, 8]/182/f ; [59, 114, 29]/114/f ; [4, 110, 143]/143/f ];

location = T(1:3,4);
orientation = T(1:3,1:3);

%plotting 3 vectors
    hold on
    for i = 1:1:3
        vec = orientation(:,i);
        quiver3(location(1),location(2),location(3),vec(1),vec(2),vec(3),'LineWidth',2,'Color',color(i,:));
    end
    hold off

%Viewing and Plot Settings
    view(Isometric)
    grid minor
    xlabel('x', 'Interpreter', 'Latex'),ylabel('y', 'Interpreter', 'Latex'),zlabel('z', 'Interpreter', 'Latex')
end
```

Axis Limits Functions

```
function axisLimits = initializeAxisLimits()
    %Initialize axis limits
    axisLimits.xmin=-0.01; axisLimits.xmax=0.1; axisLimits.ymin=-0.01; axisLimits.ymax=0.1; axisLimits.zmin=-0.01; axisLimits.zmax=
end
function [axisLimits] = checkAxisLimits(p_now,axisLimits);
x = p_now(1); y = p_now(2) ; z = p_now(3);
%Check the current point in 3D space
%Goal of the function is to get the maximum and minimum x y z for all time
%For setting axis limits
        if x > axisLimits.xmax
            axisLimits.xmax = x;
        elseif x < axisLimits.xmin
            axisLimits.xmin = x;
        end
        if y > axisLimits.ymax
            axisLimits.ymax = y;
        elseif y < axisLimits.ymin
            axisLimits.ymin = y;
        end
        if z > axisLimits.zmax
            axisLimits.zmax = z;
        elseif z < axisLimits.zmin
            axisLimits.zmin = z;
        end
end

function [axisLimits] = increaseAxisLimits(axisLimits,inc)
%Simply to shorten the code above
    axisLimits.xmin = inc*axisLimits.xmin;
    axisLimits.xmax = inc*axisLimits.xmax;
    axisLimits.ymin = inc*axisLimits.ymin;
    axisLimits.ymax = inc*axisLimits.ymax;
    axisLimits.zmin = inc*axisLimits.zmin;
    axisLimits.zmax = inc*axisLimits.zmax;
end

function setAxisLimits(axisLimits)
    xl = [axisLimits.xmin axisLimits.xmax]; yl = [axisLimits.ymin axisLimits.ymax]; zl = [axisLimits.zmin axisLimits.zmax];
    xlim(xl); ylim(yl); zlim(zl);
end
```

Save Gif Function

```
% Save Gif Function
function saveGif(filename, frames1)

    for i = 1:1:length(frames1)
        frames{i} = uint8(frames1(i).cdata);
    end

    % Add a unique timestamp to the filename
    timestamp = datetime('now', 'Format', 'yyyyMMdd_HHmmss');
    new_filename = [filename(1:end-4) '_' char(timestamp) '.gif'];

    % Write the first frame and get the colormap
    [imind, cmap] = rgb2ind(frames{1}, 256);
    imwrite(imind, cmap, new_filename, 'gif', 'LoopCount', Inf, 'DelayTime', 0.05);
    % Append the remaining frames
    numFrames = length(frames);
    for i = 2:numFrames
        imind = rgb2ind(frames{i}, cmap);
        imwrite(imind, cmap, new_filename, 'gif', 'WriteMode', 'append', 'DelayTime', 0.05);
    end
end
```

Script Specific Functions

```
function axisLimits = formatPlot(background)
    %Formatting the plot
    set(gca,'Color',background)
    axisLimits = initializeAxisLimits();
end
```

Animate rodrigues Rotation

```
function animateRotation(axis,endAngle)
    %Space Frame axis
    X = [1 0 0]'; Y = [0 1 0]'; Z = [0 0 1]';
    location = [0 0 0];
    axisLimits = initializeAxisLimits();
    endAngle = -360;
    for i = 0:6:endAngle
        hold on
        % New axis
        orientation = [X Y Z]*rot(axis,i*pi/180);
        % New axis colors
        darkness = i/endAngle/1.15;
        % Draw the space frame axis
        axisLimits = drawAxis(location,eye(3),0,axisLimits);
        % Draw the new axis
        axisLimits = drawAxis(location,orientation,darkness,axisLimits);
        % Draw the rotation axis
        hold on
        l = location;
        quiver3([l(1) -l(1)],[l(2) -l(2)],[l(3) -l(3)],[axis(1) -axis(1)],[axis(2) -axis(2)],[axis(3) -axis(3)],"--"), hold on
        % Include the current angle in the title
        title(sprintf('Rotation about the Z Axis by %0.2f degrees',i),'Interpreter','Latex');
        xlim([-1 1]); ylim([-1 1]); zlim([-1 1]);
        grid minor
        saveFrame('animation.gif',i)
        if i< endAngle
            clf
        end
    end
end
```

Functions

Create a Skew Symmetric Matrix

```
function so3mat = VecToso3(omg)
% useful for chapter 3: rigid-body motions
% takes a 3-vector (angular velocity)
% returns the skew symmetric matrix in so(3)
    so3mat = [0, -omg(3), omg(2); omg(3), 0, -omg(1); -omg(2), omg(1), 0];
end
function rot = rotX(a)
% useful for rigid-body motions
% takes an angle of rotation (radians)
% returns the corresponding rotation matrix, about the Z axis
    rot = [1,0,0;0,cos(a),-sin(a);0,sin(a),cos(a)];
end

function rot = rotY(b)
% useful for rigid-body motions
% takes an angle of rotation (radians)
% returns the corresponding rotation matrix, about the Y axis
    rot = [cos(b),0,sin(b);0,1,0;-sin(b),0,cos(b)];
end

function rot = rotZ(g)
% useful for rigid-body motions
% takes an angle of rotation (radians)
% returns the corresponding rotation matrix, about the Z axis
    rot= [cos(g),-sin(g),0;sin(g),cos(g),0;0,0,1];
end

function rot = rot(wHat,theta)
```

```
% useful for rigid-body motions
% takes a unit vector and an angle of rotation (radians)
% returns the corresponding rotatin matrix 3 by 3
    w1 = wHat(1); w2=wHat(2); w3=wHat(3);
    c = cos(theta); s=sin(theta); omc=(1-c);
    row1 = [c+w1^2*omc w1*w2*omc*-w3*s w1*w3*omc*w2*s];
    row2 = [w1*w2*omc+w3*s c+w2^2*omc w2*w3*omc-w1*s];
    row3 = [w1*w3*omc-w2*s w2*w3*omc+w1*s c+w3^2*omc];
    rot = [row1;row2;row3];
end
```