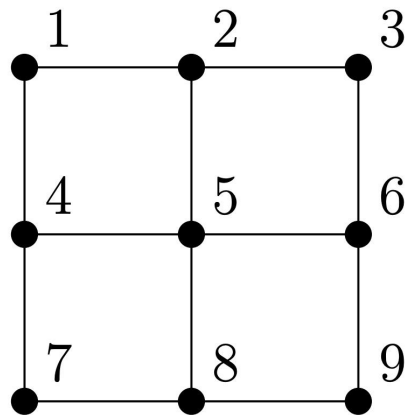# Nauty Explained

Initialization

The algorithm starts with an input graph G with n vertices and an initial colouring (partition) π_0.

If no initial colouring is provided, nauty and Traces typically begin with the unit partition (μ), where all vertices are in a single part.

Begin with unit partition on 9 vertices:
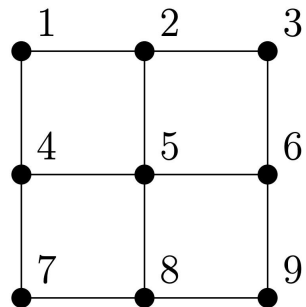[9] = (123456789)

Equitable Refinement

Equitable refinement refines the initial colouring based on vertex degrees and connectivity patterns.

A colouring is equitable if any two vertices within the same part (cell) have the same number of neighbours in each of the other parts.

(123456789) can be refined as:

- Degree 2: 1 3 7 9
- Degree 3: 2 4 6 8
- Degree 4: 5

We obtain (1379 | 2468 | 5)
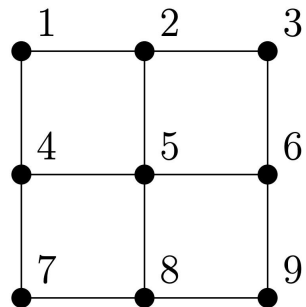
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- Degree 2: 1 3 7 9
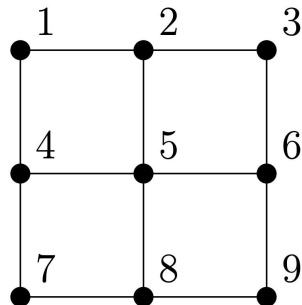- Degree 3: 2 4 6 8
- Degree 4: 5

We obtain (1379 | 2468 | 5)
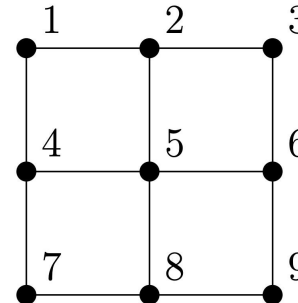
Search Tree Generation

Construction:
- Root: The root of the search tree corresponds to the initial equitable partition.
- Branching: Branches in the search tree are created by selecting a "target cell" (a non-singleton part of the partition) and then "individualising" each vertex within that cell, one at a time. This creates a new branch for each vertex in the target cell.
- Refinement: After each individualisation, the equitable refinement procedure is applied to further refine the colouring.

- Root: (1379 | 2468 | 5)
- Choose non-singleton cell "1379"
- Individualize vertex 1
- Obtain (1 | 379 | 2468 | 5)
- Apply refinement so that any vertex in the same cell share identical degree distribution
- (1 | 379 | 2468 | 5)
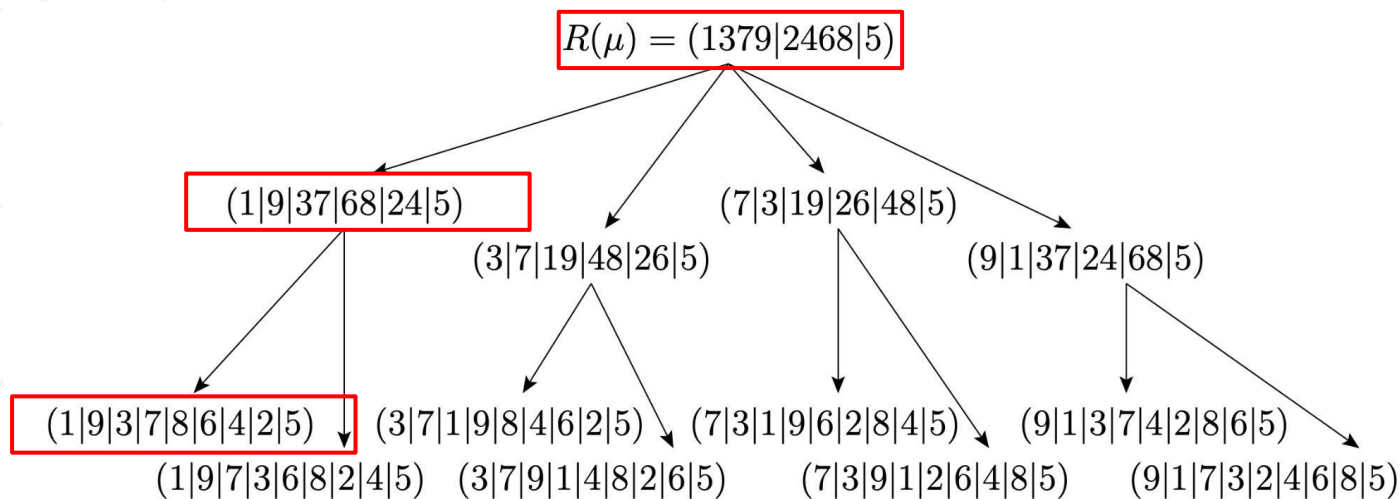- (1 | 9 | 37 | 68 | 24 | 5)
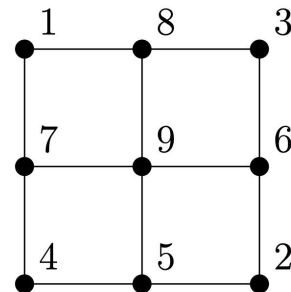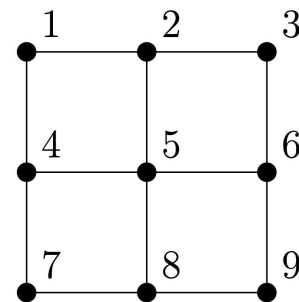
Search Tree Generation - Continue

Construction:
- Root: The root of the search tree corresponds to the initial equitable partition.
- Branching: Branches in the search tree are created by selecting a "target cell" (a non-singleton part of the partition) and then "individualising" each vertex within that cell, one at a time. This creates a new branch for each vertex in the target cell.
- Refinement: After each individualisation, the equitable refinement procedure is applied to further refine the colouring.

- Node: $(1 \mid 9 \mid 37 \mid 68 \mid 24 \mid 5)$
- Choose non-singleton cell "37"
- Individualize vertex 3
- Obtain $(1 \mid 9 \mid 3 \mid 7 \mid 68 \mid 24 \mid 5)$
- Apply refinement so that any vertex in the same cell share identical degree distribution
- $(1 \mid 9 \mid 3 \mid 7 \mid 8 \mid 6 \mid 4 \mid 2 \mid 5)$

Canonical Labeling

$R(\mu) = (1379|2468|5)$

Original

Canonical

$(1|9|37|68|24|5)$

$(3|7|19|48|26|5)$

$(7|3|19|26|48|5)$

$(9|1|37|24|68|5)$

$(1|9|3|7|8|6|4|2|5)$

$(1|9|7|3|6|8|2|4|5)$

$(3|7|1|9|8|4|6|2|5)$

$(3|7|9|1|4|8|2|6|5)$

$(7|3|1|9|6|2|8|4|5)$

$(7|3|9|1|2|6|4|8|5)$

$(9|1|3|7|4|2|8|6|5)$

$(9|1|7|3|2|4|6|8|5)$

The node invariants associated with the leaf nodes are compared. In nauty, the canonical label is determined by the node invariant, which is computed based on a sequence of quotient graphs generated during the refinement process.

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