# 1 Expanding Polynomials

The sum-product problem tells us that multiplicative and additive structure cannot coexist in a (finite) set A. One aspect of this phenomenon is the idea that the image of any non-degenerate polynomial (or function) must grow, that is, there exists some $c > 0$ such that

$$|f(a_1, ..., a_d) : a_i \in A| \gg |A|^{1+c}. \tag{1}$$

We are particularly interested in polynomials with two or three variables over either the reals or $\mathbb{F}_q$.

The *Elekes-Ronyai* problem over $\mathbb{R}$ classifies which polynomials expand: at a very high level, this says "either $f$ is degenerate (see below), or the $c$ in (1) satisfies $c > 1/3$".

For the Elekes-Ronyai problem, a degenerate polynomial is of the form

$$f(x, y) = h(x) + g(y) \text{ or } f(x, y) = h(x)g(y).$$

## 1.1 Part i - Expansion in $\mathbb{F}_q$

Only a few polynomials have been studied in $\mathbb{F}_q$ - can we find more? In this part of the project, the aim will be to find explicit functions over $\mathbb{F}_q$ (most likely bi or tri-variate), such that when applied to a Cartesian product, e.g $A \times A \times A$, the image set is significantly larger than the input set. A good place to start learning about this problem is [2]. A possible research question is: to what extent can you alter Theorem 1.1 in [2] to apply to rational functions?

## 1.2 Part ii - Growth for degenerate polynomials in $\mathbb{R}$ (and $\mathbb{C}$)

Some polynomials are degenerate in the sense of Elekes-Ronyai but still expand. For example $f(x, y) = x(y + 1)$ is degenerate, yet we have

$$|A(A + 1)| \gg |A|^{1+11/38}.$$

For which other degenerate polynomials do we get growth? What about rational functions?

In the bivariate case, convexity provides one family of such expanders: if f is of the form $f(x, y) = x + h(y)$ where $h$ is a convex function, then $f$ is degenerate but still expands. Can this be used to create polynomials in three variables that are degenerate in the sense of Elekes-Szabo? (see [1] for details on the Elekes Szabo problem)

### 1.3 Part iii - Explicit examples

Are there any explicit polynomials that beat the expansion promised by Elekes-Ronyai and Elekes-Szabo? That is, can we find $f(x, y)$ or $f(x, y, z)$ so that

$$|f(a, b) : a, b \in A| \gg |A|^{4/3+e}$$

$$|f(a, b, c) : a, b, c \in A| \gg |A|^{3/2+e}$$

for any set $A$? Similarly, do any polynomials match this expansion? That is, can we find $f(x, y)$ and a set $A$ so that

$$|f(a, b) : a, b \in A| \ll |A|^{4/3},$$

or $f(x, y, z)$ and a set $A$ so that

$$|f(a, b, c) : a, b, c \in A| \ll |A|^{3/2}.$$

For these questions we do not need to reach the exponents above - anything non-trivial could be interesting.

### 1.4 Part iv - Conditional expanders in $\mathbb{F}_p$

There are also results along the lines of ‘if $A$ has a specific structure, then $f(A, A)$ expands‘. For example, Mirzaei investigates $f(x, y)$ a ”non-degenerate” quadratic polynomial, and Tran and Van The look at $f(x, y) = g(x)[h(x) + y]$. Can you say anything about other types of polynomials?

**Reference List**


[1] M. Makhul, O. Roche-Newton, A. Warren and F. de Zeeuw, Constructions for the Elekes-Szabo and Elekes-Ronyai Problems
https://arxiv.org/pdf/1812.00654.pdf

*A good introduction into the pursuit of explicit constructions.*

[2] T. Pham, L.A. Vinh, and F. de Zeeuw, Three-Variable Expanding Polynomials and Higher-Dimensional Distinct Distances. Combinatorica 39
https://arxiv.org/abs/1612.09032

*This paper contains lots of information about expander bounds in finite fields - a good overview paper, very readable and short. A pseudo follow up paper is https://arxiv.org/abs/1809.06837 - more advanced, only certain parts are relevant.*

[3] O. Roche-Newton, M. Rudnev, and I.D. Shkredov, New sum-product type estimates over finite fields, Advances in Mathematics 293 (2016)
https://arxiv.org/abs/1408.0542

*A more advanced paper for further reading.*

[4] O. Raz, M. Sharir, J. Solymosi Polynomials vanishing on grids: The Elekes-Ronyai problem revisited https://arxiv.org/abs/1401.7419

*Read only Sections 1 for the results, and Section 5 for lots of cool applications.*

[5] S. Stevens, A. Warren, On sum sets of convex functions
https://arxiv.org/abs/2102.05446

*Does convexity help provide examples for three variable expanders? See in particular Section 2.*

[6] Stevens, S. and de Zeeuw, F. (2017), An improved point-line incidence bound over arbitrary fields. Bull. London Math. Soc., 49
https://arxiv.org/abs/1609.06284

*Contains the incidence bound used for most expanders - although we will most likely pursue other directions.*

[7] P.D. Tran, N. Van The https://arxiv.org/pdf/1911.10308.pdf and M. Mirzaei https://arxiv.org/pdf/1811.07454.pdf

*These are two nice papers that talk about conditional expansion in finite fields. Mirzaei's paper is perhaps the better one to read first.*

[8] F. de Zeeuw: A survey of Elekes-Ronyai-type problems https://arxiv.org/pdf/1601.06404.pdf

*Section 1 of this survey gives a great introduction to this investigation; Section 4.3 also gives a good overview of the finite field case.*