Embedding the SET deck in de Bruijn tori

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1 Background and Definitions

Definition 1.1. *A **de Bruijn sequence** is a cyclic sequence of a given alphabet A with size k, for which every possible subsequence of length n in A appears as a sequence of consecutive characters exactly once. We denote a de Bruijn sequence as $B(k, n)$.*

Definition 1.2. *A **de Bruijn torus** denoted as $(k^r, k^s, m, n)_k$ is a k-ary $k^r \times k^s$ toroidal array such that every k-ary $m \times n$ matrix appears exactly once contiguously on the torus.*

2 Generating multiple $(9, 9, 2, 2)_3$ de Bruijn tori

Let a be a $B(3, 2)$, $b = (012345678)$ be a $B(9, 1)$. To generate a de Bruijn torus A, let the first column of A be a^T, the second column be the first shifted cyclically by the first digit of b, which is 0. Let the third column be the second shifted cyclically by the second digit of b, which is 1. We can continue this process until we obtain a 9×9 matrix, namely a $(9, 9, 2, 2)_3$ de Bruijn torus. By using this process, different de Bruijn tori are generated for different $B(3, 2)$. [1]

We want to justify that the construction above works given any $a = B(3, 2)$. Since a contains all possible substrings of length 2 with alphabet $\{0, 1, 2\}$, if we lay two a^T side by side and shift the second column by $0, 1, ..., 8$, we will obtain all possible 2×2 submatrices containing $\{0, 1, 2\}$ with $3^4 = 81$ of them in total. The above construction ensures that all possible 2×2 matrices appear exactly once in a 9×9 matrix contiguously, since all possible shifts from 0 to 8 are included exactly once.

3 Embedding the SET deck

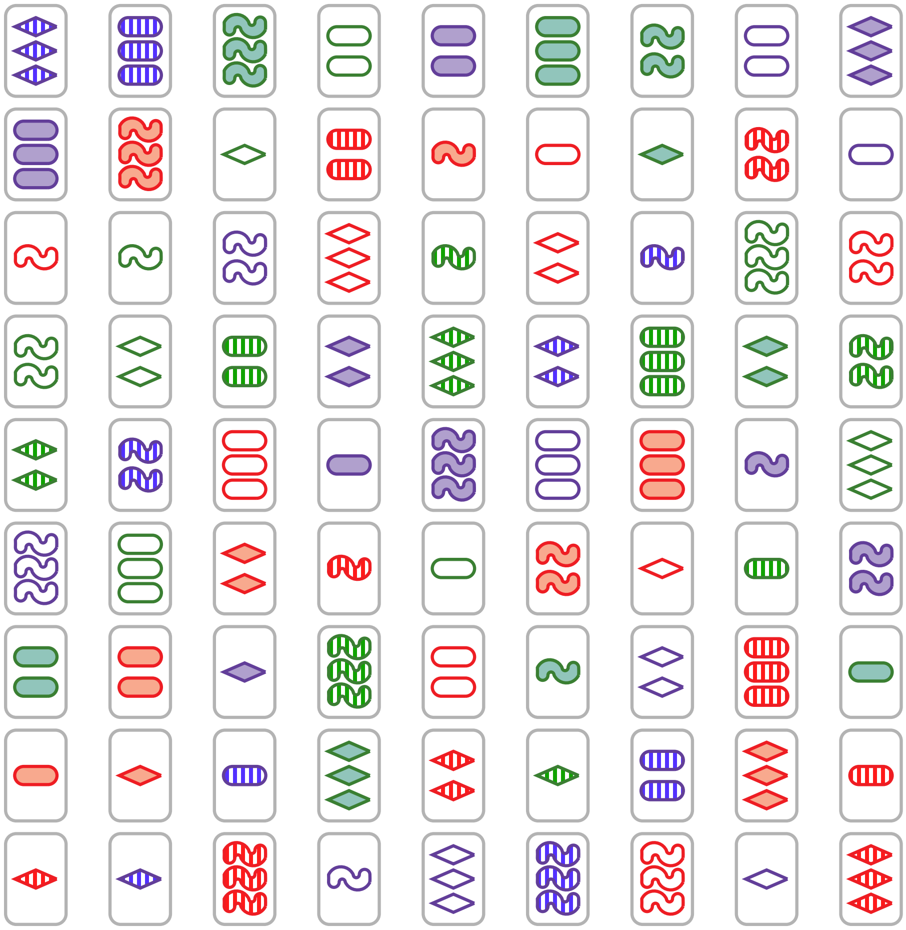
Given a $B = (9, 9, 2, 2)_3$ de Bruijn torus, let $(B(x, y), B(x + 1, y), B(x, y + 1), B(x + 1, y + 1)) = (a, b, c, d)$, where x, y represents the rows and columns, a, b, c, d corresponds to number, color, filling, and shape of a SET card. We will use the value assignment showcased in Table 1, thus each 2×2 submatrix can be used to represent a specific SET card.

This is an interesting extension of "In TetraCycles", as instead of embedding a SET deck arrangement in a two-dimensional sequence, now we can embed a SET deck in a three-dimensional matrix. The advantage of such embedding is that more partial information can be deducted given full information of one card. For the sequence embedding, full information of one card provides partial information to 6 other cards, while for the torus embedding, full information of one card provides

Table 1. SET cards features encoded by elements of the alphabet $A = \{0, 1, 2\}$				
A	Number	Color	Filling	Shape
0	three	purple	striped	diamond
1	one	red	solid	oval
2	two	green	unfilled	squiggle

partial information to 8 other cards. This seemly small difference could be significant to a magician. The current challenge of such construction is that magicians are enable to deduct more information given some partial information without memorization. The reason is that the current de Bruijn torus construction does not provide any recursive properties, unlike the de Bruijn sequence in "In Tetracycles". Below we have produced one of the de Bruijn tori and the SET deck arrangement corresponding to it:

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 0 \\
0 & 0 & 1 & 2 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 2 \\
2 & 2 & 2 & 2 & 0 & 2 & 0 & 2 & 2 \\
2 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\
2 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}$$



References

[1] Glenn Hurlbert and Garth Isaak. On the de bruijn torus problem. *Journal of Combinatorial Theory, Series A*, 64(1):50–62, 1993.