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Modelling Extreme Changes in Crude Oil Prices using Time Series and Extreme Value Theory

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The work contained in this thesis is my own work unless otherwise stated.

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Abstract

Financial institutions must maintain sufficient capital reserves to continue operating in times of extreme losses and for regulatory compliance. The ability to accurately model extreme losses or gains is essential for risk management and long-term planning. Specifically looking at crude oil prices, changes in crude oil prices can have far-reaching effects on the global economy. It is a key input in the production and transportation of goods and services. Accurate predictions of extreme price movements enables more informed decisions about investments, production levels and supply contracts.

This paper seeks to test and compare different models based on time series analysis and Extreme Value Theory (EVT) to model extreme losses in financial data, specifically using crude oil prices. This is done by back-testing the predictions from the methods proposed with historical prices.

1. Introduction

Changes in crude oil prices can have far-reaching effects on the global economy (Jiménez-Rodríguez, 2022). Given its significance, analysing extreme changes in crude oil movement can be crucial in future planning. The cost of goods and services can fluctuate in response to changes in oil prices and higher inflation is linked to increased oil price volatility (Oyuna and Yaobin, 2021). High volatility in oil prices can lead to large gains or losses in oil producing nations, oil dependent economies and independent investors (Salisu and Fasanya, 2013). There is evidence to suggest that the price of crude oil exhibits higher volatility during times of large supply chain disruptions or crisis, such as during the Gulf war and Asian financial crisis (Zavadska et al., 2020). Zavadska et al. (2020) used conditional heteroscedastic time series models to show that there is strong evidence of volatility clustering, in which periods of high volatility tend to be followed by periods of low volatility in crude oil price returns. Narayan and Narayan (2007) also used conditional heteroscedastic time series models to find that the behaviour of oil prices tends to change over a short period of time.

In mathematical terms, Value at Risk (VaR) is an estimate of the maximum potential loss within a specified probability, assuming that the data follows a particular probability distribution. Essentially, VaR corresponds to specific quantiles of this distribution. Expected Shortfall (ES), for a given VaR threshold, represents the expected loss, denoted as L, given that the loss exceeds the VaR threshold. Mathematically, this is expressed as $\mathrm{ES} = \mathbb{E}[L \mid L > \mathrm{VaR}]$. However, VaR and ES models can often fall short, especially when it comes to accurately predicting the magnitude of losses at the extreme ends of the financial data distribution.

In financial data there is considerable evidence that in many cases normality assumptions are too thin tailed to account for extreme values. Fully parametric methods of estimating VaR, that make assumptions about the distribution of the entire data set can have major problems. This is because the actual distribution of the data is unknown and can therefore lead to inaccurate estimates of VaR (Rocco, 2014).

To solve the problem of VaR estimations being too small or too large, EVT models have been used as a statistically sound parametric way to model extreme values in data. This is performed by fitting the extreme values to unique distributions found in EVT, specifically the Generalised Extreme Value (GEV) distribution and Generalised Pareto Distribution (GPD). Rocco (2014) provides a good overview of the existing literature on VaR and ES modelling using EVT.

Yi et al. (2014) and McNeil and Frey (2000) both combine conditional heteroscedastic time series models and EVT to estimate the VaR for higher quantiles with both concluding that there is evidence to support the claim that their EVT assumptions can provide better estimates for VaR at extreme values than assumptions of normality in the data.

The aim of this paper is to provide important insight into the behaviour of extreme price changes in the crude oil price. The findings of this study will leave a contribution to the broader field of EVT in financial data. This will be done by fitting GARCH models to crude oil data and forecasting the VaR and ES at future times. Expanding on the methods used in McNeil and Frey (2000), this paper seeks to use different time series models and attempts to predict the VaR forecasts at future time steps greater than 1 in a novel way. The methods explained can be used or modified to be used on many different time series data.

2. Methods

2.1. Motivation

EVT is a field of statistics based on extreme deviations from the median of a probability distribution. It provides valuable insights into future extreme events, potentially modelling occurrences even more severe than those previously observed. This capacity is especially important in domains like environmental science, economics and insurance, where knowledge of the tail end of the distribution is crucial. However, there are a number of challenges with EVT modelling, primarily because extreme events are inherently rare and it can be difficult to define what exactly qualifies as an "extreme value". Usually this is done by defining an extreme value as the largest value over a given time period or as the values over a sufficiently large threshold.

The origins of EVT are largely credited to Fisher and Tippett (1928). Pickands (1975) proved that for many common distributions, the Peaks Over a Threshold (POT) asymptotically tends towards a GPD as the threshold asymptotically increases towards a fixed point or infinity. Similarly, Gumbel (1958) proved that, for many common distribution functions, the block maxima asymptotically tends towards a GEV distribution as the size of the block maxima asymptotically increases. However, extreme value models can commonly rely on the assumption that data are i.i.d which is not the case for the data that are used in this paper (or financial data in general). The fact that EVT models are created from asymptotic reasoning presents a challenge, particularly when working with finite values which are by definition rarer and more scarce.

The methods below are based on taking points that exceed a certain quantile to be used to model the points over said quantile. This is in contrast to models that model the entire data set to find the upper quantiles.

2.2. Generalised Pareto Distribution

The Generalised Pareto Distribution (GPD) has the distributions function and probability density function (pdf):

$$G_{\xi,\beta}(y) = \begin{cases} 1 - \left(1 + \frac{\xi y}{\beta}\right)_+^{-\frac{1}{\xi}}, & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta}\right)_+, & \text{if } \xi = 0 \end{cases}$$

and

$$g_{\xi,\beta}(y) = \begin{cases} \frac{1}{\beta}\left(1 + \frac{\xi y}{\beta}\right)_+^{-\left(1+\frac{1}{\xi}\right)}, & \text{if } \xi \neq 0 \\ \frac{1}{\beta}\exp\left(-\frac{y}{\beta}\right)_+, & \text{if } \xi = 0 \end{cases}$$

respectively. Where $(x)_+ = \max(x, 0)$, $y \in \mathbb{R}^+$ if $\xi \geq 0$ and $y \in \left(0, \frac{-\beta}{\xi}\right]$ otherwise. The parameter β is referred to as the "scale" parameter and the parameter ξ is referred to as the "shape" parameter.

Pickands (1975) proved that for a large class of distributions F (particularly most "textbook" continuous distribution functions), there exists parameters $\xi \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ such that:

$$\lim_{u \to x_\infty} \inf_{0<\beta<\infty} \sup_{0 \leq y \leq \infty} |F_u(y) - G_{\xi,\beta}(y)| = 0 \tag{1}$$

where $x_\infty = \sup\{x : F(x) < 1\}$ and $F_u(y) = \frac{F(y+u)-F(u)}{1-F(u)}$ is satisfied.

The function $F_u(y)$ is equivalent to the distribution function for the Points Over a Threshold (POT) u from the distribution function F. This makes the result shown in Equation (1) significant because it makes it possible to model the POT by assuming the

POT asymptotically tend towards a GPD, without making many assumptions about the underlying distribution.

2.2.1. Goodness of Fit

For POT that follow a GPD for values over a threshold u_0, i.e. $X - u_0 | X > u_0$, with a scale parameter β_{u_0} and shape parameter ξ, then, for a new threshold $u > u_0$, it can be shown that the scale parameter β_u can be write as, $\beta_u = \beta_{u_0} + \xi(u - u_0)$ while ξ should not change (Coles, 2001, p. 83). For a data set $x_1, \ldots, x_n$, this result is used to test for numerical stability by plotting $\{u, \hat{\beta}_u\}$ and $\{u, \hat{\xi}_u\}$ with increasing values for u, where $\hat{\beta}_u$ and $\hat{\xi}_u$ are the MLEs for β and ξ for the data set $\{x_i - u : i = 1, \ldots, n, x_i > u\}$. If the data set follows a GPD well for values over a certain threshold then the MLE for the scale parameter is expected to be roughly linear with an increase in u and the shape parameter is expected to be roughly unchanging.

Following from this, if $X - u_0 | X > u_0$ follows a GPD with $\xi < 1$, then $E(X - u_0 | X > u_0) = \frac{\beta_{u_0}}{1-\xi}$ (Coles, 2001, p. 79) and for $u > u_0$,

$$E(X - u \mid X > u) = \frac{\beta_u}{1 - \xi} = \frac{\beta_{u_0} + \xi u}{1 - \xi}. \tag{2}$$

So, the mean of the exceeded values should be linear in u. This is used later to test for numerical stability by plotting the mean of POT minus the threshold against increasing thresholds.

2.3. Value at Risk (VaR)

The return level, for a distribution function F, is the value z_p that satisfies the equation $F(z_p) = 1 - p$ (Coles, 2001, p. 49) or equivalently denoted as $z_p = \inf\{z \in R : F(x) \geq p\}$ as in McNeil and Frey (2000). z_p is the return level associated with the $1/p$ return period.

The value z_p is exceeded by the maximum in a given period with probability p or another way to interpret return levels is that the value z_p is expected to be exceeded roughly once in every $1/p$ periods.

If it is assumed that losses are the positive values from a time series X, then $\mathrm{VaR}_\alpha(X)$ can be defined as:

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

(Rocco, 2014). A return level of value z_q is equivalent to the VaR at an upper quantile q and so they are used interchangeably in this paper.

For an ordered data set of size n, $z_n \leq z_{n-1} \leq \ldots \leq z_1$, then for a value of $k < n$, fit a GPD to the data set $\{z_1, \ldots, z_k\}$ to get MLE estimates for β and ξ as $\hat{\beta}$ and $\hat{\xi}$ respectively for the distribution of $F_{z_{k+1}}(z)$. For a large enough value of x such that $x > z_{k+1}$, as paraphrased from McNeil and Frey (2000) and from Bayes theorem:

$$\Pr(Z > x) = \frac{\Pr(Z > x | Z > z_{k+1}) \Pr(Z > z_{k+1})}{\Pr(Z > z_{k+1} | Z > x)}$$

$$1 - F(x) = (1 - F(z_{k+1}))(1 - F_{z_{k+1}}(x - z_{k+1}))$$

Define $1 - \hat{F}(z_{k+1}) = \frac{\#\text{ of values greater than } z_{k+1}}{\#\text{ of values}} = \frac{k}{n}$, then

$$1 - \hat{F}(x) = \frac{k}{n}(1 - \hat{F}_{z_{k+1}}(x - z_{k+1})) \tag{3}$$

$$\hat{F}(x) = 1 - \frac{k}{n}\left(1 + \hat{\xi}\frac{x - z_{k+1}}{\hat{\beta}}\right)^{\frac{-1}{\hat{\xi}}}. \tag{4}$$

For $q > 1 - k/n$ the return level estimate $\hat{z}_q$, can be calculated by inverting Equation (4):

$$\hat{z}_q = z_{(k+1)} + \frac{\hat{\beta}}{\hat{\xi}} \left(\left(\frac{1-q}{k/n} \right)^{-\hat{\xi}} - 1 \right). \tag{5}$$

Equation (5) is used to estimate the value at risk percentiles in later sections of this paper.

2.4. Expected Shortfall (ES)

For values that exceed their expected upper quantile estimate $\hat{z}_p$, as calculated in Equation (5), the ES can be calculated as a way of testing how well the predictions for the distribution of the values over the VaR threshold are.

Recall as stated in Equation (2), that for a random variable Z that follows a GPD with $\xi < 1$ then,

$$E[Z|Z > z] = \frac{z + \beta}{1 - \xi}.$$

In this paper we assume that data past a certain threshold z_{k+1} follows a GPD and under this assumption the q return value $\hat{z}_q$ can be calculated as explained above in section 2.3. For $\hat{\xi} < 1$, the ES denoted as S_q, is estimated as:

$$\begin{aligned} \hat{S}_q &= E[Z \mid Z > \hat{z}_q] \\ &= E[Z - \hat{z}_q \mid Z > \hat{z}_q] + \hat{z}_q \\ &= \frac{\hat{z}_q}{1 - \hat{\xi}} + \frac{\hat{\beta} + \hat{\xi} z_{k+1}}{1 - \hat{\xi}}. \end{aligned} \tag{6}$$

This result is useful as it gives us an estimate of the expected values for the points over the VaR threshold that is congruent with the assumptions used to derive Equation (5).

2.5. Back-Testing

For a data set of length m, as $\boldsymbol{x} = \{x_1, \ldots, x_m\}$, the objective is to fit subsections of length n of $\boldsymbol{x}$ to estimate the VaR and ES value for a day that is h days past the last day in the subsection of $\boldsymbol{x}$.

Iteratively over each day i, a section of the n days before i, i.e. $\{x_{i-n}, \ldots, x_{i-1}\}$, is fit to different models and is used to see how well it predicts the 95% and 99% VaR threshold on day i, as described in Algorithm 1. If the VaR estimates are accurate at a quantile q then $100\% - q$ of the values in the set $\{x_{n+h} \ldots, x_m\}$ are expected to exceed their VaR threshold estimates.

For values n and h, the value for z_q is estimated at a time t using only the data points $\{x_{t-h}, \ldots, x_{t-h-n}\}$ and the ES is calculated the same way.

Algorithm 1 Predict VaR Thresholds

Let $x_1, x_2, \ldots, x_m$ be the dataset.
Let n be the number of days for the look-back period.
Let h be the future day for testing
Let model be a parametric model
Let q be a quantile
for $t = n + h$ to m **do**
    Let $\boldsymbol{X}_{(t-n-h+1):(t-h)} \leftarrow \{x_{t-n-h+1}, x_{t-n-h+2}, \ldots, x_{t-h}\}$
    Fit the model parameters $\hat{\theta}$ to $\boldsymbol{X}_{(t-n-h+1):(t-h)}$
    Predict the $\hat{z}_q^t$ for day t
end for
return $\{\hat{z}_q^t : t = n + h, \ldots, m\}$

3. Crude Oil Data

Figure 1 shows the West Texas Intermediate (WTI) crude oil weekday closing price and Figure 2 shows the negation of the log daily returns multiplied by a hundred. Figure 3 shows the weekday Brent crude oil closing price and Figure 4 the negation of the log daily returns multiplied by a hundred. WTI is a grade of crude oil that is commonly

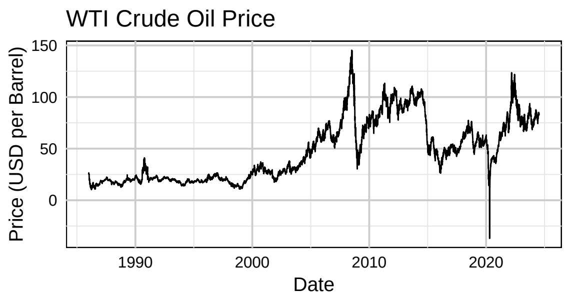

Figure 1.: WTI daily closing price from 02/01/1986 to 15/07/2024.

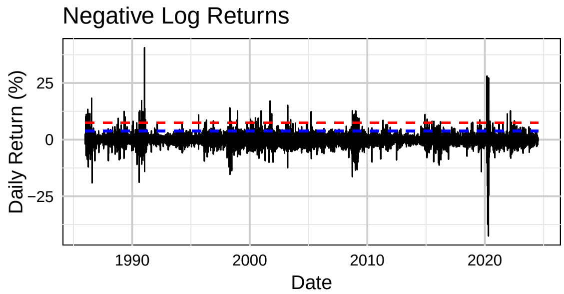

Figure 2.: WTI Crude Oil Negative log daily returns times 100 from 02/01/1986 to 15/07/2024.

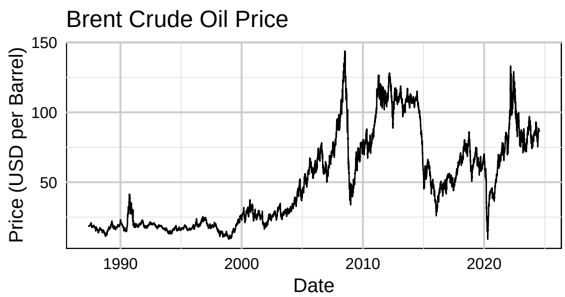

Figure 3.: Brent daily closing price from 20/05/1987 to 15/07/2024.

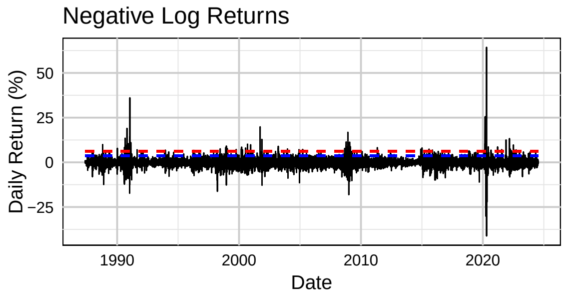

Figure 4.: Brent Crude Oil Negative log daily returns times 100 from 20/05/1987 to 15/07/2024.

used as the oil price benchmark. It is commonly used for gasoline refining. Brent crude oil is found in the North Sea in Europe as opposed to WTI oil which is found in North America. This means that it is more affected by events in Europe than WTI oil. Brent oil tends to be cheaper and it is popularly refined into diesel fuel and gasoline (U.S. Energy Information Administration, 2014).

As can be seen from Figure 2, there are several periods of high volatility in which there are large changes over a short period of time but with a general upwards trend. For instance, during the 2008 financial crisis, oil prices experienced large fluctuations (Zavadska et al., 2020). The price of WTI Crude Oil soared to an all-time high of over $140 per barrel in July 2008, only to plummet to around $30 per barrel by the end of that year. This period is clearly reflected in the graph as one of intense volatility. Another example is the oil price crash in 2020 due to the COVID-19 pandemic (Xu et al., 2024). This unprecedented event is also visible in Figures 2 and 4 as a sharp spike in volatility. Additionally, the Gulf War in 1990-1991 caused oil prices to surge dramatically (Zavadska et al., 2020), which is another period marked by high volatility in the graph. Each of these periods represents significant disruptions in the oil market, leading to rapid price changes.

4. Unconditional Extreme Value Theory

4.1. Implementation

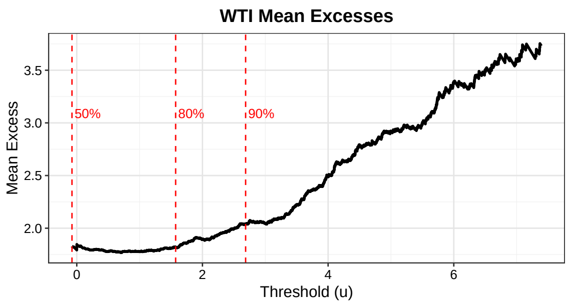

Figure 5.: Mean Excesses of WTI negative log daily returns for thresholds between the 50% and 99% quantiles.

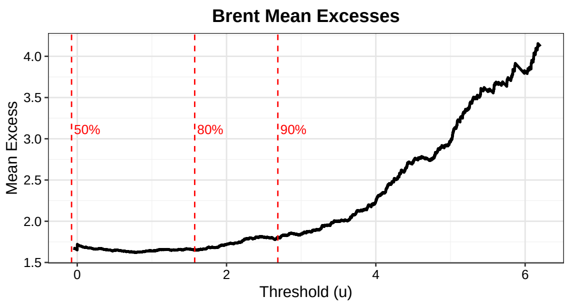

Figure 6.: Mean Excesses of Brent negative log daily returns for thresholds between the 50% and 99% quantiles.

As in Equation (6) for POT that follow a GPD the excesses over said threshold should be linear with an increasing threshold (u). As can be seen in Figures 5 and 6, the excesses over the 90% quantile to the 99% quantile seem to be far more linear than the values before them. There could be other threshold values in which there is a better case that there is more linearity in the graph. However, to model 95% VaR the quantile threshold needs to be notably smaller than that value. For this reason the models in this paper take $k = n(100\% - 90\%)$.

The segment length of 1000 days ($n = 1000$) is selected to ensure that the dataset is large enough to provide meaningful statistical analysis while focusing on a specific timeframe. This duration allows the model to capture the overall distribution of returns, including both typical and extreme movements, without introducing excessive noise or making the dataset so large that it becomes less sensitive to extreme events.

4.2. Results

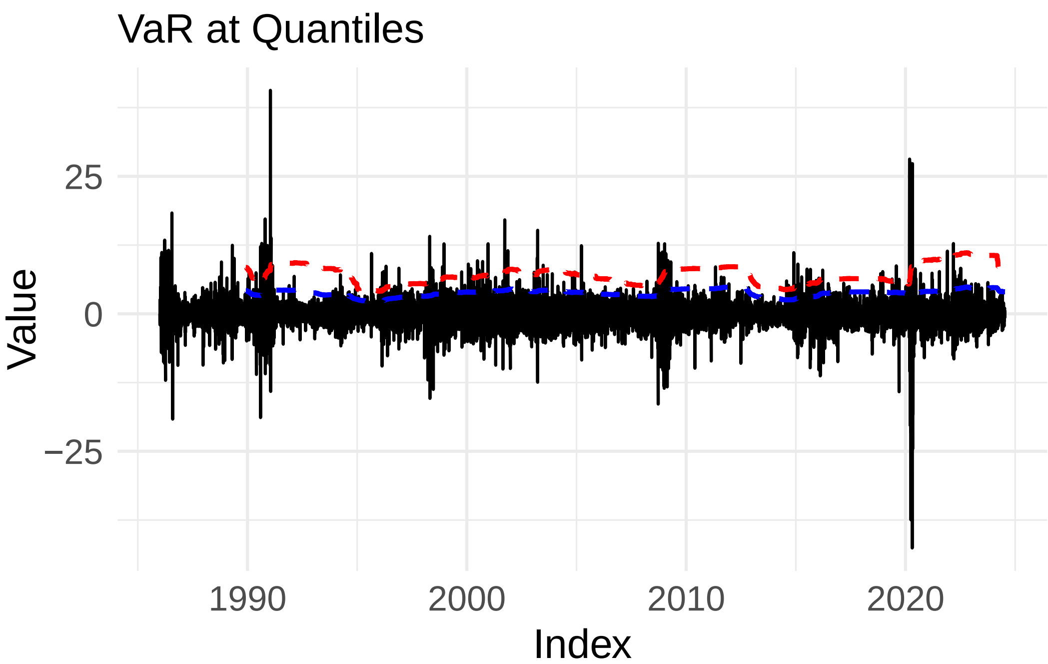

Figure 7.: Rolling VaR estimate with $n = 1000$ and $k = 100$. The dashed blue line is the 95% VaR quantile and the dashed red line is the 99% VaR quantile.

Figure 7 shows the unconditional EVT model rolling estimates of the 95% and 99% VaR estimates. To summarise how this method works:

- The largest 100 values are taken from a 1000 days segment of the data (i.e. $k = 100$ and $n = 1000$).
- A GPD is fit to the largest 100 days.
- Equation (5) is used to estimate the VaR at different quantiles for the day after the 1000 day segment.

	WTI Crude Oil			Brent Crude Oil		
Quantiles	**95%**	**99%**	**99.5%**	**95%**	**99%**	**99.5%**
1-step ahead forecast						
Breaches	468	111	56	439	92	54
Percentage Breaches	0.0538	0.0128	0.0064	0.0521	0.0109	0.0064
5-step ahead forecast						
Breaches	469	114	58	439	94	55
Percentage Breaches	0.0539	0.0131	0.0067	0.0521	0.0112	0.0065
10-step ahead forecast						
Breaches	470	116	63	437	95	57
Percentage Breaches	0.0541	0.0133	0.0072	0.0519	0.0113	0.0068
30-day ahead forecast						
Breaches	475	127	72	446	105	65
Percentage Breaches	0.0548	0.0146	0.0083	0.0531	0.0125	0.0077

Table 1.: Unconditional VaR breaches and percentage breaches for different quantiles and forecast horizons for WTI and Brent with $n = 1000$ and $k = 100$

The rolling VaR for h-steps forecast is the same as the 1-step ahead forecast shifted by $h - 1$ values. For example, 1-step ahead forecast for the VaR at time $t + 1$ then the training set is $\{x_t, \ldots, x_{t-n+1}\}$ which would be the same training set as the h-step ahead forecast at time $t + h$. For this reason, one would assume that smaller step forecasts are more accurate than for larger steps. The VaR estimates at different quantiles were estimated and the number of breaches and percentage of values breached for different h-step ahead were taken. This is shown in Table 1.

As can be seen in Figure 7, the back-test rolling predictions are very slow in changing its volatility. From 1990 to 1991 and from 2008 to 2009 the 99% VaR is breached frequently. However, from 2005 to 2007 it is only breached a small amount of times.

This graph shows that the EVT model is slow to respond to changing variance and tends to be breached multiple times in a row as it does not take into account the dependence of the data.

An example of this is in the 1-step ahead forecast where 13% of values, that come 1-step after values that breach their own 95% quantile, breach their own 95% quantile, while the average is 5.4%. As the step size increases, the number of breaches, especially at

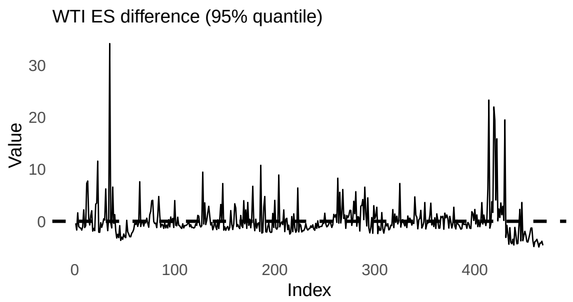

Figure 8.: Values over a 1-step 95% VaR minus the ES (mean = 0.1751 and variance = 12.3811).

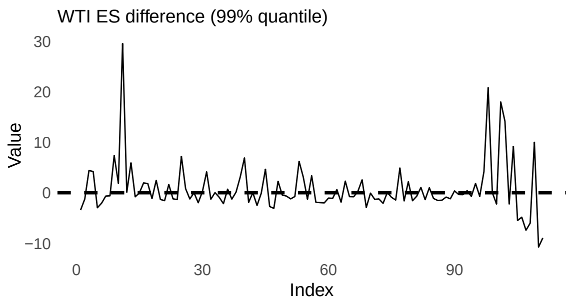

Figure 9.: Values over a 1-step 99% VaR minus the ES (mean = 0.6631 and variance = 26.1364).

higher quantiles, increases.

The difference between the actual values and the estimated ESs for values that breach a given VaR quantile are listed as $\{x_t - \hat{S}_q^t : t \in \{n+h, \ldots, m\}, x_t > \hat{z}_q^t\}$. This is plotted in Figures 8 and 9 with $n = 1000$ and $k = 100$. In these graphs it can be seen that, even though the variance is quite high, the mean values are close to what they are expected to be at 0. Near the end of Figures 8 and 9 there as several values that are, in succession, significantly larger then the mean. This shows that, as with the VaR estimates, the ES estimates are slow to address the changing variance.

5. Time Series Models

5.1. Motivation

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are used in time series analysis to model and forecast financial volatility. They are an extension of the Autoregressive Conditional Heteroskedasticity (ARCH) models developed by Engle (1982). The GARCH model was introduced by Bollerslev (1986) to address some of the limitations of the ARCH model, particularly the need for a large number of parameters in long-memory time series.

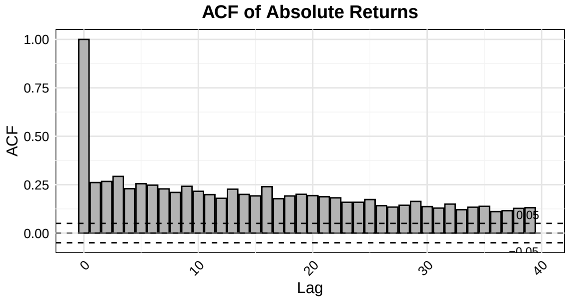

Figure 10.: acf of Absolute Log Daily Returns for WTI Crude Oil.

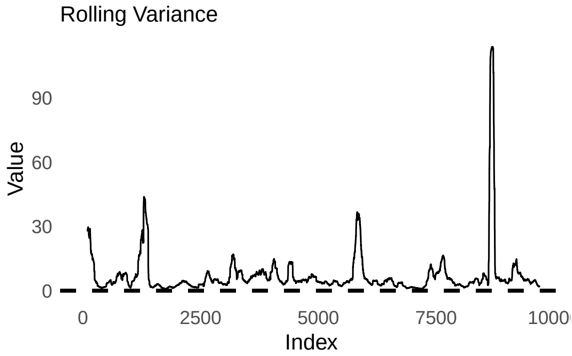

Figure 11.: 100 day window Rolling Variance for Log Daily Returns for WTI Crude Oil.

Conditional heteroscedastic time series models are introduced to address the inadequacies of only an EVT approach. As can be seen in the AutoCorrelation Function (acf) Figure 10, the absolute values of the log daily returns are highly correlated showing strong correlation in the volatility. Figure 11 shows that the variance can be extremely different with a maximum value of 114 and its minimum value of 0.76. Similar results are found for Brent crude oil.

This supports the results in Zavadska et al. (2020) that there is volatility clustering. Conditional heteroscedastic time series models can remove dependence in the data because these models are specifically designed to account for the changing variance (heteroscedasticity) in the data by modeling the volatility as a function of past errors or shocks. By doing so, they can effectively remove or reduce the dependence in the data caused by volatility clustering, leading to more accurate forecasts and a better understanding of the underlying volatility dynamics in financial data.

5.2. Overview

Assume a time series:

$$X_t = \mu_t + \sigma_t Z_t \tag{7}$$

in which $Z_t \sim \mathcal{N}(0,1)$ is an i.i.d white noise process and μ_t and σ_t are calculable using $\mathcal{F}_{t-1}$, where $\mathcal{F}_{t-1}$ is all the information up to $t-1$.

For an initial AR(p)-GARCH(q,r) model μ_t is calculable using to the parameters for the AR part of the model as:

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i}$$

where ϕ_i are the parameters of the AR(p) model and ϕ_0 is a constant term.

For the GARCH(q, r) model, the conditional variance σ_t^2 is specified as:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i = 1, \ldots, q$ and $\beta_i > 0$ for $i = 1, \ldots, r$ are the parameters and $\epsilon_t = X_t - \mu_t$.

5.3. Stationarity

A stationary time series is one in which the unconditional joint probability distribution does not change when shifted in time. Assumptions of stationarity are used in time series analysis and modelling because they ensure that the statistical properties of the time series remain consistent over time.

In mathematical terms, if a time series is stationary it must the following 3 criteria: (i) finite mean and variance i.e. $\mathrm{E}[X_t] < \infty$ and $\mathrm{var}[X_t] < \infty$, (ii) constant mean, $\mathrm{E}[X_{t_1}] = \mathrm{E}[X_{t_2}]$, for all possible values for t_1 and t_2 and (iii) the covariance between two indexes t_1 and t_2 is only dependent on the absolute difference for all possible values of t_1 and t_2 i.e. $\mathrm{E}[(X_{t_1} - \mathrm{E}[X_{t_1}])(X_{t_2} - \mathrm{E}[X_{t_2}])] = \mathrm{E}[(X_{|t_1-t_2|} - \mathrm{E}[X_{|t_1-t_2|}])(X_0 - \mathrm{E}[X_0])]$.

An AR mean model is stationary if the root(s) of $1 - \sum_{i=1}^{p} \phi_i z^i = 0$ are all within the unit circle. For more details see (Hamilton, 1994, pp. 54-77).

A GARCH model is stationary if the root(s) of $\sum_{j=1}^{\max(q,r)} (\alpha_j + \beta_j) < 1$ where $\alpha_j = 0$ for $j > q$ and $\beta_j = 0$ for $j > r$.

5.4. Parameter Estimates

While estimating the parameters it is assumed that Z_t is normally distributed. The conditional likelihood function for each observation then is:

$$f(X_t|\mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{(X_t - \mu_t)^2}{2\sigma_t^2}\right),$$

where $\mathcal{F}_{t-1}$ is the information set available at time $t-1$.

The MLE for $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_p)$ is calculated as

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \sum_{t=p+1}^{T} \left(X_t - \phi_0 - \sum_{i=1}^{p} \phi_i X_{t-i}\right)^2.$$

The log-likelihood function for the entire sample is:

$$l(\theta) = \sum_{t=\max(p,q,r)+1}^{m} \left[\log(f(X_t|\mathcal{F}_{t-1}))\right],$$

$$l(\theta) = \sum_{t=\max(p,q,r)+1}^{m} \left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma_t^2) - \frac{(X_t - \hat{\mu}_t)^2}{2\sigma_t^2}\right],$$

$$\hat{\theta} = \arg\max_{\theta} l(\theta) \tag{8}$$

where $\theta = (\alpha_0, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_r)$ is the vector of parameters and $\hat{\theta} = (\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_q, \hat{\beta}_1, \ldots, \hat{\beta}_r)$ is the MLE and $\hat{\mu}_t = \hat{\phi}_0 + \sum_{i=1}^{p} \hat{\phi}_i X_{t-i}$.

There is no closed form solution to Equation (8) and so it is calculated using numerical optimisation methods. For more details on the stationarity of a GARCH model or how the parameters are estimated see Tsay. (2010).

5.5. Forecasting

The mean value forecast, denoted as $\hat{\mu}_t(h)$, for μ_{t+h}, given time series values $X_1, \ldots, X_t$, can be calculated recursively in which:

$$\hat{\mu}_t(h) = \hat{\phi}_0 + \sum_{i=1}^{p} \hat{\phi}_i \hat{\mu}_t(h-i)$$

where $\hat{\mu}_t(h-i) = X_{t+h-i}$ if $h \leq i$.

The sigma forecast, denoted as $\hat{\sigma}_t(h)$, for σ_{t+h}, can be calculated recursively in which:

$$\hat{\sigma}_t^2(h) = \hat{\alpha}_0 + \sum_{i=1}^{q} \hat{\alpha}_i \hat{\epsilon}_{t+h-i}^2 + \sum_{j=1}^{r} \hat{\beta}_j \hat{\sigma}_t^2(h-j)$$

where $\hat{\sigma}_t^2(h-j) = \hat{\sigma}_{t+h-j}$ if $h \leq j$ and

$$\hat{\epsilon}_\tau^2 = \begin{cases} (X_\tau - \hat{\mu}_\tau)^2, & \text{if } \tau \leq t \\ \hat{\sigma}_\tau^2, & \text{otherwise.} \end{cases}$$

The value forecast, $\hat{X}_t(h)$, can be calculated recursively, where $\hat{X}_t(h) = \hat{\mu}_t(h) + \hat{\sigma}_t(h)\hat{Z}_{t+h}$. However since Z_{t+h} is a white noise process, it is has an expected value of 0 and so $\hat{X}_t(h) = \hat{\mu}_t(h)$.

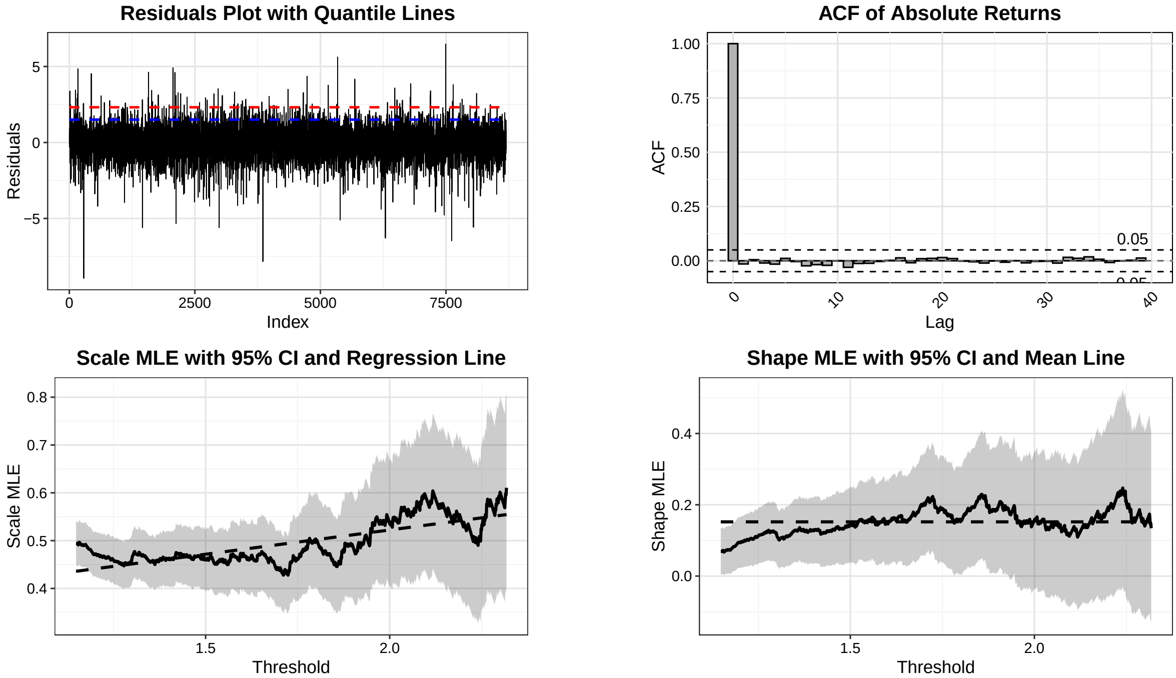

Figure 12.: The top left plot is the 1-step ahead residual estimates over time with a top line at the 99% quantile and the bottom line at the 95% quantile. The top right shows the auto correlation of the absolute values of the residuals. The bottom plots shows the MLE estimate for the scale (β) and shape (ξ) parameters with a linear regression line and mean value line.

5.6. Estimates for the Residuals

Parameter	Estimate	Std. Error	p-value
$\hat{\phi}_0$	-0.037	0.019	0.048
$\hat{\phi}_1$	-0.013	0.011	0.225
$\hat{\alpha}_0$	0.085	0.013	0.000
$\hat{\alpha}_1$	0.106	0.007	0.000
$\hat{\beta}_1$	0.886	0.007	0.000

Table 2.: MLE for an AR(1)-GARCH(1,1) model fitted to the WTI log daily returns data

The initial analysis was done using an AR(1)-GARCH(1,1) model because it had the smallest BIC value for the models tested with an auto-regressive mean component when fit to the entire data set and for simplicity. The BIC and AIC values were taken for several different models and the results are in Table 4 in the supplementary material.

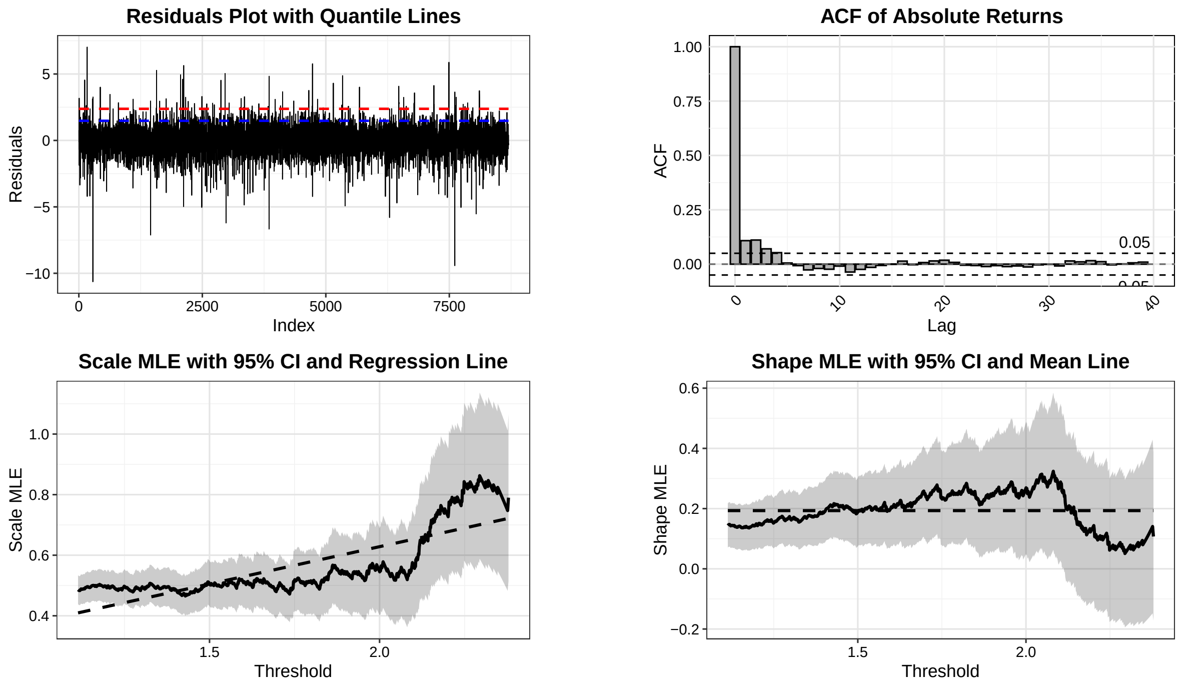

Figure 13.: The top left plot is the 5-step ahead residual estimates over time with a top line at the 99% quantile and the bottom line at the 95% quantile. The top right shows the auto correlation of the absolute values of the residuals. The bottom plots shows the MLE estimate for the scale (β) and shape (ξ) parameters with a linear regression line and mean value line.

The entire data set was fit to an AR(1)-GARCH(1,1) model and the residuals are estimated as $(\hat{z}_m, \ldots, \hat{z}_{n+1}) = \left(\frac{x_m - \hat{\mu}_{m-1}(1)}{\hat{\sigma}_{m-1}(1)}, \ldots, \frac{x_{n+1} - \hat{\mu}_n(1)}{\hat{\sigma}_n(1)}\right)$, in which $\hat{\mu}_{p-1}(1)$ and $\hat{\sigma}_{p-1}(1)$ are the one step ahead forecasts given values from time $p - n$ to time $p - 1$ using the MLE for each parameter as shown in Table 2.

The 5 step ahead forecast (i.e. $h = 5$) for $\hat{\mu}_t(h)$ and $\hat{\sigma}_t(h)$ are calculated by using only the values $\{x_t, \ldots, x_{t-n+1}\}$ and the MLEs for the parameters seen in Table 2 as $(z_m, \ldots, z_{n+h}) = \left(\frac{x_m - \hat{\mu}_{m-h}(h)}{\hat{\sigma}_{m-h}(h)}, \ldots, \frac{x_{n+h} - \hat{\mu}_n(h)}{\hat{\sigma}_n(h)}\right)$.

Figures 12 and 13 show the estimated 1 step and 5 steps ahead residuals respectively for the WTI data and associated diagnostic plots for the residuals. In both Figures the acf for the absolute residuals is very low. In Figure 12 all the acf values are below 0.05 and in Figure 13 all but 4 of the acf values are below 0.05. The acf plots of residuals show that they are less correlated than than the WTI log daily returns seen in Figure 10.

In Figures 12 and 13 the bottom 2 plots show the MLE for the scale and shape parameters for increasing thresholds. The MLE values for the scale parameter should increase linearly with the increasing threshold. As can be seen in both scale parameter plots, the linear regression line is within the 95% confidence interval, except for small parts at the start of each plot. The MLE for the shape parameter should stay roughly constant with increasing thresholds. As can be seen in the plots the shape parameter stays relatively constant.

Since the residuals are relatively uncorrelated and the MLE estimates of the GPD behave as expected with increases in the threshold this is evidence that the residuals are a good fit to a GPD. The assumption that the residuals follow a GPD is the foundation of future VaR and ES prediction methods used later in this paper.

5.7. Value at Risk and Expected shortfalls

In the unconditional EVT model, that does not take into consideration any time series modelling, the k largest values, from a window of n consecutive log daily returns, are used to fit a GPD which is then used to estimate values of upper quantiles. Expanding on this using time series models, a window of n consecutive negative log daily returns are used to estimate the model parameters and n residual values. Then the k largest values, from the window of n residual values, are used to fit a GPD which is then used to estimate values of upper quantiles of the residuals. Using the estimates of the upper quantile of the residuals, the forecasted mean $\hat{\mu}_t(h)$ and forecasted variance $\hat{\sigma}_t(h)$ the upper quantile is estimated for a future value h-step ahead of the window of n consecutive values.

Recall Equation (5) in section 2.3, for the residuals of a GARCH model the estimated quantiles at time $t+h$, given values up to time t, are calculated as:

$$\hat{x}_q^t(h) = \hat{\mu}_t(h) + \hat{\sigma}_t(h)\hat{z}_q$$
$$= \hat{\mu}_t(h) + \hat{\sigma}_t(h)\left(z_{(k+1)} + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{1-q}{k/n}\right)^{-\hat{\xi}} - 1\right)\right) \tag{9}$$

for $q > 1 - \frac{k}{n}$.

Recall Equation (6) in section 2.4, in this paper it is assumed that the residuals above a threshold z_{k+1} follow a GPD and under this assumption the q upper quantile estimate $\hat{z}_q$ can be calculated as

$$E[Z|Z > z_q] = \frac{z_q}{1-\xi} + \frac{\beta + \xi z_{k+1}}{1-\xi}$$

.

For the combination of EVT and GARCH models the expected shortfall is estimated as

$$\hat{S}_q^t(h) = \hat{\mu}_t(h) + \hat{\sigma}_t(h)E[Z|Z > \hat{z}_q]$$
$$= \hat{\mu}_t(h) + \hat{\sigma}_t(h)\left(\frac{\hat{z}_q}{1-\hat{\xi}} + \frac{\hat{\beta} + \hat{\xi}z_{k+1}}{1-\hat{\xi}}\right) \tag{10}$$

for $q > 1 - \frac{k}{n}$. Equations (9) and (10) are expansions of equations explained in McNeil and Frey (2000), which are generalised to include forecasts greater than 1.

To test how well our model performs then for events that exceed a threshold $\hat{z}_q$, the values in the data set $\left\{\frac{x_{t+s} - \hat{S}_q^t(h)}{\hat{\sigma}_t(h)} : t \in \{n, \ldots, m-h\}, x_{t+h} > \hat{x}_q^t(h)\right\}$ is taken and plot in Figure 15, for the 1-step ahead 95% and 99% predictions, using $n = 1000$, $k = 100$ and an AR(1)-GARCH(1,1) for the WTI and Brent crude oil data.

This method is not theoretically sound for VaR and ES predictions for periods greater than one step as it only takes into account situations in which the μ_{t+h} and σ_{t+h} values

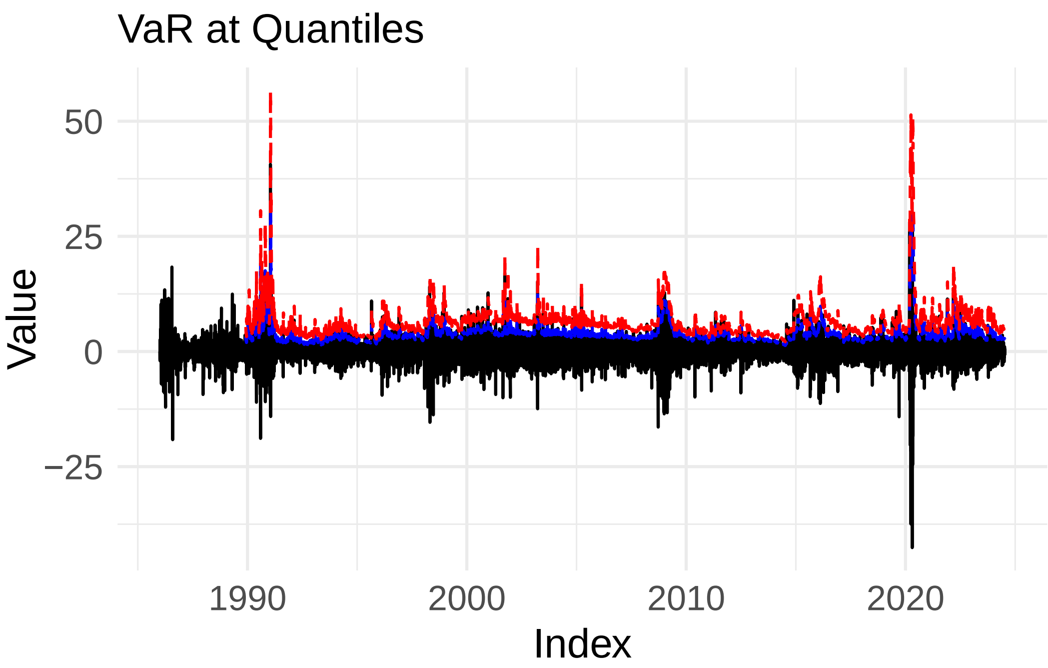

Figure 14.: 1-step ahead rolling prediction using a AR(1)-GARCH(1,1) model, $n = 1000$ and $k = 100$ for the WTI log daily returns. The blue line is the 95% quantile estimate and the red line is the 99% quantile estimate.

are the values that are forecasted and that the only way a given quantile can be breached is if the white noise value Z_t is above a given quantile in it's own distribution.

One could easily imagine a situation in which μ_{t+h} and/or σ_{t+h} are sufficiently large, such that the VaR threshold is breached without the white noise value Z_{t+h} being above a given quantile in it's own distribution. The opposite could also occur in which μ_{t+h} and/or σ_{t+h} are sufficiently small, such that the VaR threshold is not breached even though the white noise value Z_{t+h} is above a given quantile in it's own distribution.

5.8. Results

As can be seen in Figure 14, the model's quantile predictions quickly change with the immediate volatility of the time period which leads to less instances in which a quantile is breached multiple times in a row when compared to the unconditional VaR estimates seen in Figure 7. The max value for the 99% quantile estimate in Figure 14 is close to

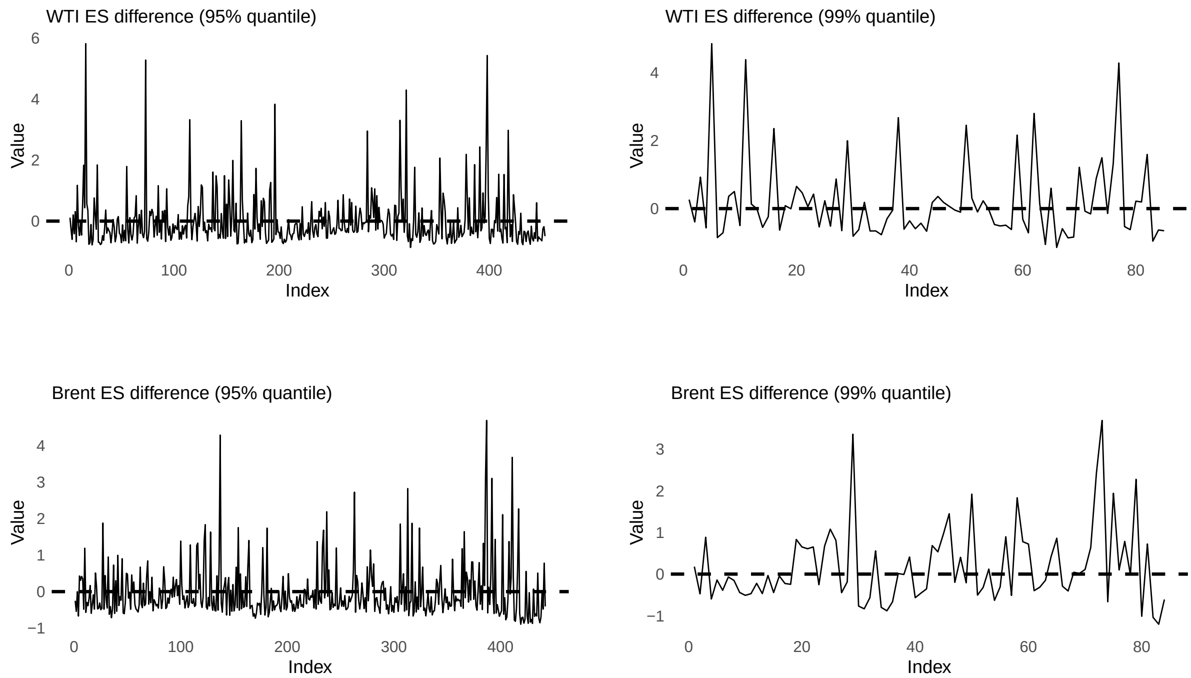

Figure 15.: Combined figures showing the values over 1-step quantiles and VaR minus the ES for different confidence levels: (a) WTI Values over a 1-step 95% quantile minus the ES (mean = −0.008436376 and variance = 0.7427909), (b) WTI Values over a 1-step 99% quantile minus the ES (mean = 0.2084888 and variance = 1.434531), (c) Brent Values over a 1-step 95% VaR minus the ES (mean = −0.02648052 and variance = 0.5434432) and (d) Brent Values over a 1-step 99% VaR minus the ES (mean = 0.1728784 and variance = 0.8488096).

55 while in Figure 7 the max value for the 99% quantile estimate is roughly 11. This is because within a GARCH model the future variance forecast is largely influenced by the recent volatility in the data. This means that in places of high volatility, the value for $\hat{\sigma}_t(h)$ is likely to be larger then in places of low volatility, therefore the quantile values are likely to be larger in places of high volatility. For larger steps and quantile values, breaches are less common compared to the unconditional VaR estimates and closer to the expected number of breaches.

Figure 15 is the POT minus it's ES. The means are close to zero for each value which is evidence that the ES estimate is a relatively good guess prediction of the empirical ES. The acf for all of the data plotted in Figure 15 were all less than 0.2 which shows

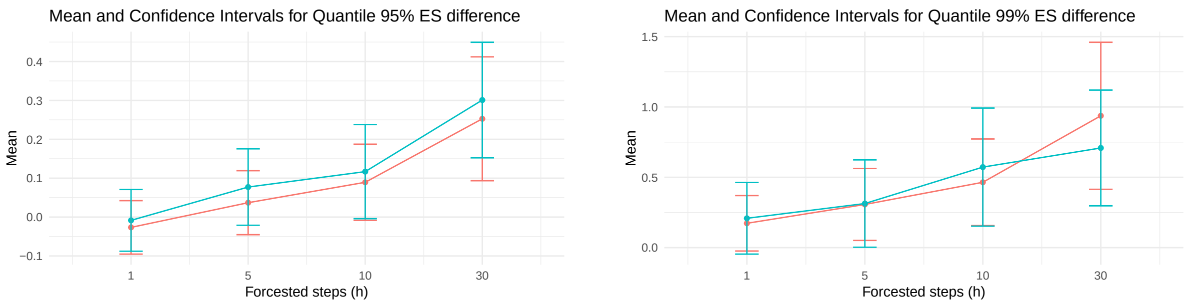

Figure 16.: Mean and 95% Confidence Intervals for the Difference in ES of Residuals for 95% and 99% VaR using a AR(1)-GARCH(1,1) model, $n = 1000$ and $k = 100$: The left plot represents the 95% VaR and the right plot represents the 99% VaR. The blue line denotes WTI crude oil residuals, the red line denotes Brent crude oil residuals.

independence in the points over the threshold. Figure 18 shows the average difference between the ES and the actual values over said ES's associated VaR, for 95% and 99% VaRs, using the WTI and Brent crude oil residuals.

In both plots in Figure 18, the mean increases from the value 0 as the forecast steps (h-value) increase, which is evidence that the accuracy of the prediction decreases as the forecast steps increase. The mean also increases away from 0 for the higher quantiles, this is seen in the 99% quantile plot, where the mean is higher for every forecast step. For forecasts greater than 1, 0 is outside the 95% confidence interval which is evidence that ES estimates consistently underestimates the imperial ES for both datasets. For an AR(0)-GARCH(1,1) model the results were found to be almost identical. Table 6 is the number of VaR estimates that are below the residuals and the percentage of all the residuals tested. Both Figure 18 and Table 6 provide evidence that the VaR and ES estimates preform worse as the steps ahead increase.

Comparing Table 6 and Table 1, the estimates used in Table 6 have a percentage breach closer to the expected percentage breach in 20 out of 24 values. This is evidence that the combination of time series and EVT model estimates are imperially better than the unconditional EVT estimates. In both tables the difference between the percentage breach and the expected percentage breach become further apart as the steps ahead increase. The parameters $n = 500$ and $k = 50$ were tested and the results are in Table 5

	WTI Crude Oil			Brent Crude Oil		
Quantiles	**95%**	**99%**	**99.5%**	**95%**	**99%**	**99.5%**
1-step ahead forecast						
Breaches	453	85	48	442	84	46
Percentage Breaches	0.0520	0.0098	0.0055	0.0524	0.0100	0.0055
5-step ahead forecast						
Breaches	420	91	49	408	84	48
Percentage Breaches	0.0483	0.0105	0.0056	0.0484	0.0100	0.0057
10-step ahead forecast						
Breaches	411	87	51	398	88	55
Percentage Breaches	0.0473	0.0100	0.0059	0.0473	0.0105	0.0065
30-step ahead forecast						
Breaches	377	104	62	387	97	61
Percentage Breaches	0.0435	0.0120	0.0071	0.0461	0.0115	0.0073

Table 3.: VaR breaches and percentage breaches for WTI and Brent crude oil with $n = 1000$, $k = 100$ and AR(1)-GARCH(1,1)

in the Supplementary material. Imperially the results were consistently worse for most time steps and quantile values.

6. Conclusion

This analysis found no significant evidence that incorporating an AR(1) mean component enhances the model's accuracy compared to using a constant mean, i.e. an AR(0) mean component. This suggests that the simpler constant mean model is sufficient for capturing the dynamics of crude oil prices within the scope of this paper.

Empirically, the number of breaches observed is generally in line with the expected number, indicating that the VaR estimates for future values are relatively accurate across different data sets. However, it is note worthy that the accuracy of these estimates tends to decrease as the forecasting horizon extends. This decline in accuracy over longer prediction periods suggests that, while the model performs well in the short term, additional considerations may be needed for more reliable long-term predictions.

Overall, the findings underscore the effectiveness of the GARCH(1,1) model in estimating VaR for crude oil prices, with results remaining consistent despite different data sets and periods. However, the diminishing accuracy over longer horizons highlights an area of potential further study in future modelling efforts.

The ES estimates are consistently less than the empirical shortfalls and the number of breaches at highest quantile (99.5%) are consistently breached more than expected, which may suggest that the estimates are too conservative for very high quantiles.

The ES estimates are consistently lower than the observed empirical shortfalls and the number of breaches at the highest quantile (99.5%) estimates consistently exceeds expectations. This pattern suggests that the ES estimates may be too conservative for very high quantiles, potentially underestimating the risk in these extreme scenarios.

7. Discussion

The VaR estimates are consistently breached close to the expected number of breaches. In the conclusion of McNeil and Frey (2000) they concluded that VaR estimates they used, for longer time horizons of 5 or 10 days did not perform well in practice, particularly for stock market returns. Instead they propose using Monte Carlo methods. This paper found that the VaR estimates for time horizons of 5, 10 and 30 days preformed relatively well imperially using a different method for estimation, however generally it does not perform as well at time horizons of 1.

The average difference between the values that breach their VaR thresholds minus their ES estimates are close to zero for both data sets for small time steps. This agrees with the conclusion in McNeil and Frey (2000) that ES is a risk measure with good theoretical and empirical properties. However the difference between the ES for larger time steps was constantly smaller on average than the actual values, perhaps corrective measures could be taken increases the accuracy of the ES estimates.

Further research can be done on the theoretical properties of the methods explained. Understanding the expected accuracy for the methods explained for different time series with different distribution assumptions is crucial to understanding the soundness of the theoretical foundation, which is not fully explored in this paper. Further research can also be done on the imperial results of the methods explained using different time series data to be compared to already existing methods.

End Matter

Data Availability Statement

The data utilised in this study are publicly available from the USA's Energy Information Administration (EIA) website. Specifically:

- **West Texas Intermediate (WTI) Crude Oil Price Data:** Daily historical data for WTI crude oil prices are available from the EIA at the following link: https://www.eia.gov/dnav/pet/hist/RWTCD.htm. The data were downloaded on 17/07/2024.
- **Brent Crude Oil Price Data:** Daily historical data for Brent crude oil prices are available from the EIA at the following link: https://www.eia.gov/dnav/pet/hist/RBRTED.htm. The data were downloaded on 23/07/2024.

Code Availability Statement

The code used in this paper is publicly available on GitHub via the following link:

https://github.com/BrianMacCarvill/Modelling-Extreme-Changes-in-Crude-Oil-Prices

This repository contains all data and functions used to produce the results in this paper. The code is licensed under the CC-BY-4.0 license. Crucial R packages used in the code were `rugarch`, `ismev` and `evd`.

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A. Supplementary Material

A.1. Time Series Parameter Testing

q	r	p	AIC (WTI)	BIC (WTI)	AIC (Brent)	BIC (Brent)
1	1	0	4.411656	4.414616	4.276778	4.280570
1	2	0	4.411403	4.415103	4.277160	4.281712
1	3	0	4.411226	4.415666	4.277514	4.282824
2	1	0	4.411817	4.415517	4.276850	4.281401
2	2	0	4.411609	4.416049	4.277062	4.282372
2	3	0	4.411432	4.416612	4.277425	4.283493
3	1	0	4.411992	4.416432	4.277130	4.282439
3	2	0	4.411782	4.416962	4.276964	4.283032
3	3	0	4.411638	4.417558	4.275666	4.282493
1	1	1	4.411710	4.415410	4.277944	4.280978
1	2	1	4.411444	4.415884	4.278326	4.282119
1	3	1	4.411263	4.416443	4.278680	4.283231
2	1	1	4.411871	4.416311	4.278084	4.281877
2	2	1	4.411650	4.416830	4.278296	4.282848
2	3	1	4.411469	4.417389	4.278657	4.283967
3	1	1	4.412046	4.417226	4.278348	4.282899
3	2	1	4.411823	4.417743	4.278168	4.283478
3	3	1	4.411675	4.418335	4.276902	4.282971

Table 4.: Combined AR-GARCH Model Selection Criteria using WTI and Brent Crude Oil

A.2. EVT Parameter Testing

	AR(1)-GARCH(1,1)			AR(0)-GARCH(1,1)		
Quantiles	**95%**	**99%**	**99.5%**	**95%**	**99%**	**99.5%**
1-step ahead forecast						
Breaches	481	99	66	475	96	63
Percentage Breaches	0.0523	0.0108	0.0072	0.0516	0.0104	0.0068
5-step ahead forecast						
Breaches	453	102	65	458	103	66
Percentage Breaches	0.0492	0.0111	0.0071	0.0498	0.0112	0.0072
10-step ahead forecast						
Breaches	436	97	63	434	97	61
Percentage Breaches	0.0474	0.0106	0.0069	0.0472	0.0106	0.0066
30-step ahead forecast						
Breaches	412	127	86	416	125	86
Percentage Breaches	0.0449	0.0138	0.0094	0.0453	0.0136	0.0094

Table 5.: VaR breaches and percentage breaches for WTI crude oil with $n = 500$, $k = 50$

	WTI Crude Oil			Brent Crude Oil		
Quantiles	**95%**	**99%**	**99.5%**	**95%**	**99%**	**99.5%**
1-step ahead forecast						
Breaches	452	88	47	442	81	45
Percentage Breaches	0.0519	0.0101	0.0054	0.0524	0.0096	0.0053
5-step ahead forecast						
Breaches	420	94	49	402	83	48
Percentage Breaches	0.0483	0.0108	0.0056	0.0477	0.0099	0.0057
10-step ahead forecast						
Breaches	414	87	51	394	88	54
Percentage Breaches	0.0476	0.0100	0.0059	0.0468	0.0105	0.0064
30-step ahead forecast						
Breaches	380	106	62	386	97	61
Percentage Breaches	0.0438	0.0122	0.0071	0.0460	0.0115	0.0073

Table 6.: VaR breaches and percentage breaches for WTI and Brent crude oil with $n = 1000$, $k = 100$ and AR(0)-GARCH(1,1)

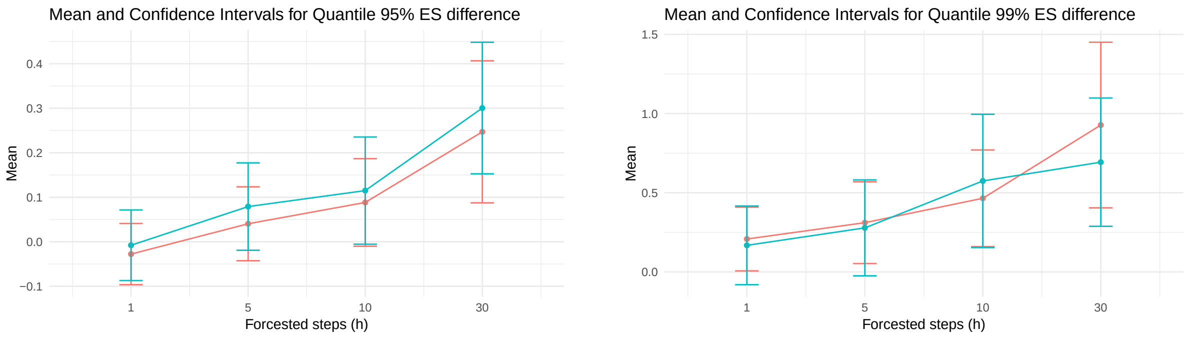

Figure 17.: Mean and 95% Confidence Intervals for the Difference in ES of Residuals for 95% and 99% VaR using a AR(0)-GARCH(1,1) model, $n = 1000$ and $k = 100$: The left plot represents the 95% VaR and the right plot represents the 99% VaR. The blue line denotes WTI crude oil residuals, the red line denotes Brent crude oil residuals.

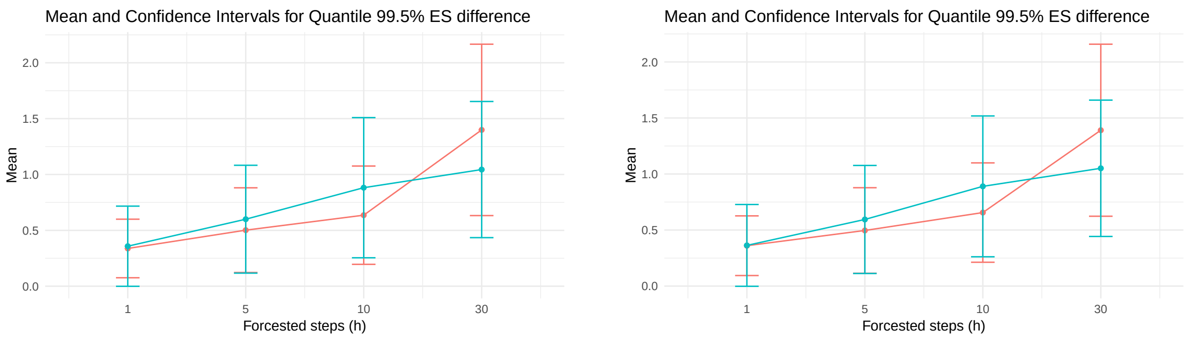

Figure 18.: Mean and 95% Confidence Intervals for the Difference in ES of Residuals at 99.5% VaR for a AR(1)-GARCH(1,1) model (left) and AR(0)-GARCH(1,1) model (right) with $n = 1000$ and $k = 100$. The blue line denotes WTI crude oil residuals, the red line denotes Brent crude oil residuals.