STAT 40700

Assignment 1

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Question 1

For the time series $\{X_t\}_{t\in\mathbb{Z}}$

$$X_t = \beta_0 + \beta_1 t + \beta_2 t^2 + W_t$$

where $\{W_t\}_{t\in\mathbb{Z}} \sim \text{WN}(0, \sigma^2)$, and β_0, β_1, β_2 are real constants with $\beta_1 \neq 0$ and $\beta_2 \neq 0$.

A. Show that both X_t and ∇X_t are non-stationary.

B. Show that $\nabla^2 X_t$ is stationary.

First begin by writing the differences of the time series X_t using, $\nabla X_t = (1 - B)X_t$.

$$\begin{aligned}
X_t &= \beta_0 + \beta_1 t + \beta_2 t^2 + W_t \\
\nabla X_t &= (1 - B)(\beta_0 + \beta_1 t + \beta_2 t^2 + W_t) \\
\nabla X_t &= \beta_0 + \beta_1 t + \beta_2 t^2 + W_t - B(\beta_0 + \beta_1 t + \beta_2 t^2 + W_t) \\
\nabla X_t &= \beta_0 + \beta_1 t + \beta_2 t^2 + W_t - (B(\beta_0) + B(\beta_1 t) + B(\beta_2 t^2) + B(W_t)) \\
\nabla X_t &= \beta_0 + \beta_1 t + \beta_2 t^2 + W_t - \beta_0 - \beta_1(t-1) - \beta_2(t-1)^2 - W_{t-1} \\
\nabla X_t &= \beta_1 + \beta_2 t^2 + W_t - \beta_2(t^2 - 2t + 1) - W_{t-1} \\
\nabla X_t &= \beta_1 + W_t - \beta_2(-2t + 1) - W_{t-1} \\
\nabla X_t &= \beta_1 + \beta_2(2t - 1) + W_t - W_{t-1}
\end{aligned} \tag{1}$$

$$\begin{aligned}
\nabla X_t &= \beta_1 + \beta_2(2t - 1) + W_t - W_{t-1} \\
\nabla^2 X_t &= (1 - B)(\beta_1 + \beta_2(2t - 1) + W_t - W_{t-1}) \\
\nabla^2 X_t &= \beta_1 + \beta_2(2t - 1) + W_t - W_{t-1} - B(\beta_1 + \beta_2(2t - 1) + W_t - W_{t-1}) \\
\nabla^2 X_t &= \beta_1 + \beta_2(2t - 1) + W_t - W_{t-1} - B(\beta_1) - B(\beta_2(2t - 1)) - B(W_t) + B(W_{t-1}) \\
\nabla^2 X_t &= \beta_1 + \beta_2(2t - 1) + W_t - W_{t-1} - \beta_1 - \beta_2(2(t - 1) - 1)) - W_{t-1} + W_{t-2} \\
\nabla^2 X_t &= \beta_2(2t - 1) + W_t - W_{t-1} - \beta_2(2t - 3) - W_{t-1} + W_{t-2} \\
\nabla^2 X_t &= 2\beta_2 + W_t - 2W_{t-1} + W_{t-2}
\end{aligned} \tag{2}$$

To be stationary a time series must show the following 3 properties:

i. $E(X_t^2) < \infty \quad \forall t \in \mathbb{Z}$

ii. $E(X_t) = \mu \quad \forall t \in \mathbb{Z}$ where μ is a constant

iii. $\gamma(r, s) = \gamma(r + t, s + t) \quad \forall r, s, t \in \mathbb{Z}$

A.) For part A we will show that X_t and ∇X_t are not stationary using property [ii.] as it will become apparent that the expected value of both is not constant but rather depends on t. Using the property $E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$ [1] we obtain eq. (3) for X_t.

$$\begin{aligned}
E(X_t) &= E(\beta_0 + \beta_1 t + \beta_2 t^2 + W_t) \\
&= E(\beta_0) + E(\beta_1 t) + E(\beta_2 t^2) + E(W_t) \\
&= \beta_0 + \beta_1 E(t) + \beta_2 E(t^2) + E(W_t)
\end{aligned} \tag{3}$$

$E(W_t) = 0$ as WN $(0, \sigma^2)$ so has mean 0. The expected value of $E(t) = t$ and $E(t^2) = t^2$ as t is deterministic. Giving the final result of eq. (4):

$$E(X_t) = \beta_0 + \beta_1 t + \beta_2 t^2 \tag{4}$$

Here we see that $E(X_t)$ is dependant on t and not a constant, therefore X_t is not stationary.

For ∇X_t we obtain eq. (5) after using a similar expansion as above.

$$\begin{aligned} E(\nabla X_t) &= E(\beta_1 + \beta_2(2t-1) + W_t - W_{t-1}) \\ &= E(\beta_1) + E(\beta_2(2t-1)) + E(W_t) - E(W_{t-1}) \\ &= \beta_1 + \beta_2 E(2t-1) + E(W_t) - E(W_{t-1}) \end{aligned} \tag{5}$$

White noise has expected value 0 so $E(W_t) = 0$. Similarly to as before the expected value of $E(t) = t$ as t is deterministic giving eq. (6).

$$E(\nabla X_t) = \beta_1 + \beta_2(2t-1) \tag{6}$$

Again we see that $E(\nabla X_t)$ is dependant on t and not a constant, therefore X_t is not stationary by property [ii.].

B.) For part B we will show that $\nabla^2 X_t$ is stationary as it obeys all 3 properties.

i. To show that the variance is finite we can compute $E((\nabla^2 X_t)^2)$ and show that it is finite i.e. a constant.

   First we compute $(\nabla^2 X_t)^2$

$$\begin{aligned} (\nabla^2 X_t)^2 &= (2\beta_2 + W_t - 2W_{t-1} + W_{t-2})^2 \\ &= (2\beta_2)^2 + W_t^2 + (-2W_{t-1})^2 + W_{t-2}^2 \\ &\quad + 2(2\beta_2 W_t) + 2(2\beta_2)(-2W_{t-1}) + 2(2\beta_2 W_{t-2}) \\ &\quad + 2(W_t)(-2W_{t-1}) + 2(W_t)(W_{t-2}) + 2(-2W_{t-1})(W_{t-2}) \\ &= 4\beta_2^2 + W_t^2 + 4W_{t-1}^2 + W_{t-2}^2 \\ &\quad + 4\beta_2 W_t - 8\beta_2 W_{t-1} + 2\beta_2 W_{t-2} \\ &- 2W_t W_{t-1} + 2W_t W_{t-2} - 4W_{t-1}W_{t-2} \end{aligned} \tag{7}$$

   Since Var$(Y_t) = E(Y_t^2) - [E(Y_2)]^2$ then we know Var$(Y_t) + [E(Y_2)]^2 = E(Y_t^2)$. If we examine each of the terms below eq. (8) we can see that none of the terms will exceed $a \times$ Var$(Y_t) + [E(Y_2)]^2$ where a is just some constant. So we can establish that the variance of X_t is finite.

$$\begin{aligned} E(\nabla^2 X_t)^2 &= E(4\beta_2^2) + E(W_t^2) + E(4W_{t-1}^2) + E(W_{t-2}^2) \\ &\quad + E(4\beta_2 W_t) - E(8\beta_2 W_{t-1}) + E(2\beta_2 W_{t-2}) \\ &- E(2W_t W_{t-1}) + E(2W_t W_{t-2}) - E(4W_{t-1}W_{t-2}) \end{aligned} \tag{8}$$

ii. As $E(B^{-j} W_t) = E(W_t) = 0, \quad \forall t \in \mathbb{Z}$ and $E(W_t) = 0$ as WN$(0, \sigma^2)$ we find eq. (9) where the expected value is a constant.

$$\begin{aligned} E(\nabla^2 X_t) &= E(2\beta_2 + W_t - 2W_{t-1} + W_{t-2}) \\ E(\nabla^2 X_t) &= E(2\beta_2) + E(W_t) - 2E(W_{t-1}) + E(W_{t-2}) \\ E(\nabla^2 X_t) &= 2\beta_2 \end{aligned} \tag{9}$$

iii. To show the autocovariance is only dependant on the lag we first write out the autocovariance and expand it with all the cross terms eq. (11).

$$
\begin{aligned}
&\mathsf{Cov}(\nabla^2 X_t, \nabla^2 X_{t+h}) \\
&= \mathsf{Cov}(2\beta_2 + W_t - 2W_{t-1} + W_{t-2}, 2\beta_2 + W_{t+h} - 2W_{t+h-1} + W_{t+h-2}) \\
&= \mathsf{Cov}(2\beta_2, 2\beta_2) + \mathsf{Cov}(2\beta_2, W_{t+h}) - 2\mathsf{Cov}(2\beta_2, W_{t+h-1}) + \mathsf{Cov}(2\beta_2, W_{t+h-2}) \\
&\quad + \mathsf{Cov}(W_t, 2\beta_2) + \mathsf{Cov}(W_t, W_{t+h}) - 2\mathsf{Cov}(W_t, W_{t+h-1}) + \mathsf{Cov}(W_t, W_{t+h-2}) \\
&\quad - 2\mathsf{Cov}(W_{t-1}, 2\beta_2) - 2\mathsf{Cov}(W_{t-1}, W_{t+h}) + 4\mathsf{Cov}(W_{t-1}, W_{t+h-1}) - 2\mathsf{Cov}(W_{t-1}, W_{t+h-2}) \\
&\quad + \mathsf{Cov}(W_{t-2}, 2\beta_2) + \mathsf{Cov}(W_{t-2}, W_{t+h}) - 2\mathsf{Cov}(W_{t-2}, W_{t+h-1}) + \mathsf{Cov}(W_{t-2}, W_{t+h-2})
\end{aligned}
\tag{10}
$$

All the cross terms with constant terms eg ($\mathsf{Cov}(2\beta_2, 2\beta_2)$ and $\mathsf{Cov}(2\beta_2, W_{t+h})$) are $= 0$ as constants aren't dependant on the series or each other.

$$
\begin{aligned}
&\mathsf{Cov}(\nabla^2 X_t, \nabla^2 X_{t+h}) \\
&= \mathsf{Cov}(W_t, W_{t+h}) - 2\mathsf{Cov}(W_t, W_{t+h-1}) + \mathsf{Cov}(W_t, W_{t+h-2}) \\
&- 2\mathsf{Cov}(W_{t-1}, W_{t+h}) + 4\mathsf{Cov}(W_{t-1}, W_{t+h-1}) - 2\mathsf{Cov}(W_{t-1}, W_{t+h-2}) \\
&+ \mathsf{Cov}(W_{t-2}, W_{t+h}) - 2\mathsf{Cov}(W_{t-2}, W_{t+h-1}) + \mathsf{Cov}(W_{t-2}, W_{t+h-2})
\end{aligned}
\tag{11}
$$

We then examine all the different cases of h to identify that the autocovariance is only dependant on the lag (h). For $\mathsf{Cov}(W_x, W_y)$ $x = y$ otherwise $\mathsf{Cov}(W_x, W_y) = 0$. Also for white noise $\sim \mathsf{WN}(0, \sigma^2)$ then $\mathsf{Var}(B^{-j} W_t) = \mathsf{Var}(W_t) = \sigma_\omega^2, \quad \forall t \in \mathbb{Z}$

<u>**Case h = 0**</u>

$$
\begin{aligned}
&\mathsf{Cov}(\nabla^2 X_t, \nabla^2 X_{t+0}) \\
&= \mathsf{Cov}(W_t, W_{t+0}) - \cancel{2\mathsf{Cov}(W_t, W_{t+0-1})} + \cancel{\mathsf{Cov}(W_t, W_{t+0-2})} \\
&- \cancel{2\mathsf{Cov}(W_{t-1}, W_{t+0})} + 4\mathsf{Cov}(W_{t-1}, W_{t+0-1}) - \cancel{2\mathsf{Cov}(W_{t-1}, W_{t+0-2})} \\
&+ \cancel{\mathsf{Cov}(W_{t-2}, W_{t+0})} - \cancel{2\mathsf{Cov}(W_{t-2}, W_{t+0-1})} + \mathsf{Cov}(W_{t-2}, W_{t+0-2}) \\
&= \mathsf{Cov}(W_t, W_{t+0}) + 4\mathsf{Cov}(W_{t-1}, W_{t+0-1}) + \mathsf{Cov}(W_{t-2}, W_{t+0-2}) \\
&= \mathsf{Var}(W_t) + 4\mathsf{Var}(W_{t-1}) + \mathsf{Var}(W_{t-2}) \\
&= \sigma_\omega^2 + 4\sigma_\omega^2 + \sigma_\omega^2 = 6\sigma_\omega^2
\end{aligned}
\tag{12}
$$

<u>**Case h = 1**</u>

$$
\begin{aligned}
&\mathsf{Cov}(\nabla^2 X_t, \nabla^2 X_{t+1}) \\
&= \cancel{\mathsf{Cov}(W_t, W_{t+1})} - 2\mathsf{Cov}(W_t, W_{t+1-1}) + \cancel{\mathsf{Cov}(W_t, W_{t+1-2})} \\
&- \cancel{2\mathsf{Cov}(W_{t-1}, W_{t+1})} + \cancel{4\mathsf{Cov}(W_{t-1}, W_{t+1-1})} - 2\mathsf{Cov}(W_{t-1}, W_{t+1-2}) \\
&+ \cancel{\mathsf{Cov}(W_{t-2}, W_{t+1})} - \cancel{2\mathsf{Cov}(W_{t-2}, W_{t+1-1})} + \cancel{\mathsf{Cov}(W_{t-2}, W_{t+1-2})} \\
&= -2\mathsf{Cov}(W_t, W_{t+1-1}) - 2\mathsf{Cov}(W_{t-1}, W_{t+1-2}) \\
&= -2\mathsf{Var}(W_t) - 2\mathsf{Var}(W_{t-1}) \\
&= -2\sigma_\omega^2 - 2\sigma_\omega^2 = -4\sigma_\omega^2
\end{aligned}
\tag{13}
$$

Case h = -1

$$\begin{aligned}
&\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t+1}) \\
&= \cancel{\text{Cov}(W_t, W_{t-1})} - 2\cancel{\text{Cov}(W_t, W_{t-1-1})} + \cancel{\text{Cov}(W_t, W_{t-1-2})} \\
&- 2\text{Cov}(W_{t-1}, W_{t-1}) + 4\cancel{\text{Cov}(W_{t-1}, W_{t-1-1})} - 2\cancel{\text{Cov}(W_{t-1}, W_{t-1-2})} \\
&+ \cancel{\text{Cov}(W_{t-2}, W_{t-1})} - 2\text{Cov}(W_{t-2}, W_{t-1-1}) + \cancel{\text{Cov}(W_{t-2}, W_{t-1-2})} \\
&= -2\text{Cov}(W_{t-1}, W_{t-1}) - 2\text{Cov}(W_{t-2}, W_{t-2}) \\
&= -2\text{Var}(W_{t-1}) - 2\text{Var}(W_{t-2}) \\
&= -2\sigma_\omega^2 - 2\sigma_\omega^2 = -4\sigma_\omega^2
\end{aligned} \tag{14}$$

Case h = 2

$$\begin{aligned}
&\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t+2}) \\
&= \cancel{\text{Cov}(W_t, W_{t+2})} - 2\cancel{\text{Cov}(W_t, W_{t+2-1})} + \text{Cov}(W_t, W_{t+2-2}) \\
&- 2\cancel{\text{Cov}(W_{t-1}, W_{t+2})} + 4\cancel{\text{Cov}(W_{t-1}, W_{t+2-1})} - 2\cancel{\text{Cov}(W_{t-1}, W_{t+2-2})} \\
&+ \cancel{\text{Cov}(W_{t-2}, W_{t+2})} - 2\cancel{\text{Cov}(W_{t-2}, W_{t+2-1})} + \cancel{\text{Cov}(W_{t-2}, W_{t+2-2})} \\
&= \text{Cov}(W_t, W_{t+2-2}) \\
&= \text{Var}(W_t) \\
&= \sigma_\omega^2
\end{aligned} \tag{15}$$

Case h = -2

$$\begin{aligned}
&\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t-2}) \\
&= \cancel{\text{Cov}(W_t, W_{t-2})} - 2\cancel{\text{Cov}(W_t, W_{t-2-1})} + \cancel{\text{Cov}(W_t, W_{t-2-2})} \\
&- 2\cancel{\text{Cov}(W_{t-1}, W_{t-2})} + 4\cancel{\text{Cov}(W_{t-1}, W_{t-2-1})} - 2\cancel{\text{Cov}(W_{t-1}, W_{t-2-2})} \\
&+ \text{Cov}(W_{t-2}, W_{t-2}) - 2\cancel{\text{Cov}(W_{t-2}, W_{t-2-1})} + \cancel{\text{Cov}(W_{t-2}, W_{t-2-2})} \\
&= \text{Cov}(W_{t-2}, W_{t-2}) \\
&= \text{Var}(W_{t-2}) \\
&= \sigma_\omega^2
\end{aligned} \tag{16}$$

Case h $\geq$ 3

$$\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t+1}) = 0 \tag{17}$$

Finally we arrange these answers to show that $\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t+1})$ is only dependant on the lag (h) and not t.

$$\text{Cov}(\nabla^2 X_t, \nabla^2 X_{t+h}) = \begin{cases} 6\sigma_\omega^2 & \text{for } |h| = 0 \\ -4\sigma_\omega^2 & \text{for } |h| = 1 \\ \sigma_\omega^2 & \text{for } |h| = 2 \\ 0 & \text{for } |h| \geq 3 \end{cases} \tag{18}$$

Finally we establish that $\nabla^2 X_t$ is stationary as it obeys all three properties.

Question 2

Assume that $\{X_t\}$ satisfies the equation

$$(1 - \phi B)X_t = W_t,$$

where $|\phi| < 1$ and $\{W_t\}$ is a white noise process with a normal distribution, mean 0, and variance 4.

(a) Generate $n = 300$ observations from the process with $\phi = 0.35$. Plot the simulated time series and its ACF. Provide your R code. Does this generated time series look stationary? Justify your answer.

(b) Repeat part (a) with $\phi = 0.999$. Plot the simulated time series and its ACF. Does this generated time series look stationary? Justify your answer.

We begin be expanding the equation and applying the backshift operator to obtain the following eq. (19).

$$\begin{aligned}
(1 - \phi B)X_t &= W_t \\
X_t - \phi B(X_t) &= W_t \\
X_t - \phi X_{t-1} &= W_t \\
X_t &= W_t + \phi X_{t-1}
\end{aligned} \tag{19}$$

We then define the code which will plot the process. As the process depends on the previous value of X_t we define the first step as 0 in the array.

R code for plotting time series and acf of AR(1) process. ϕ is 0.35 and 0.999 for parts a and b

```r
library(astsa) # import lib

n<-300 # sample size
sigma2<-4 # variance of the WN
phi = 0.35 # Constant value of phi as in AR(1) process

# Create a white noise array
WN<-rnorm(n, mean = 0, sd = sqrt(sigma2))

# create an empty array for the time series X_t
Xt <- numeric(n)

# loops from second element to end of array.
for (t in 2:n) {
  Xt[t] <- phi * Xt[t - 1] + WN[t]
}

#plot time series and acf
plot.ts(Xt, ylab="X_t") # Plot X_t
acf(Xt) # Plot ACF
```

A.) - For $\phi = 0.35$

For $\phi = 0.35$ we examine both the time series plot and the ACF plot below. From time series plot, fig. 1, we see that the time series has mean about 0 and the variance seems reasonably stable over time through visual inspection. When examining the ACF plot, fig. 2, we see that the function quickly drops (exponentially) indicating that the series is lowly correlated after even a couple of lags. Both of these indicate that the time series is **stationary**.

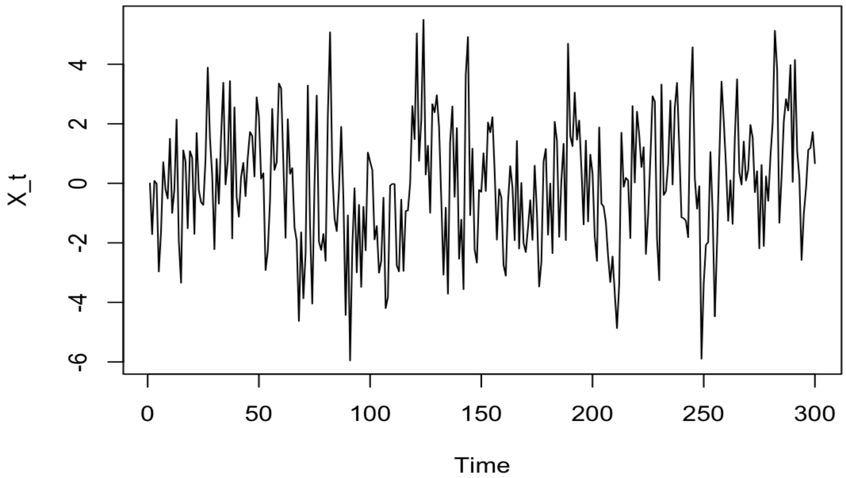


Figure 1: Time series for AR(1) process with $\phi = 0.35$. Process is seen to have mean oscillating around 0 and variance seems stable over the time series

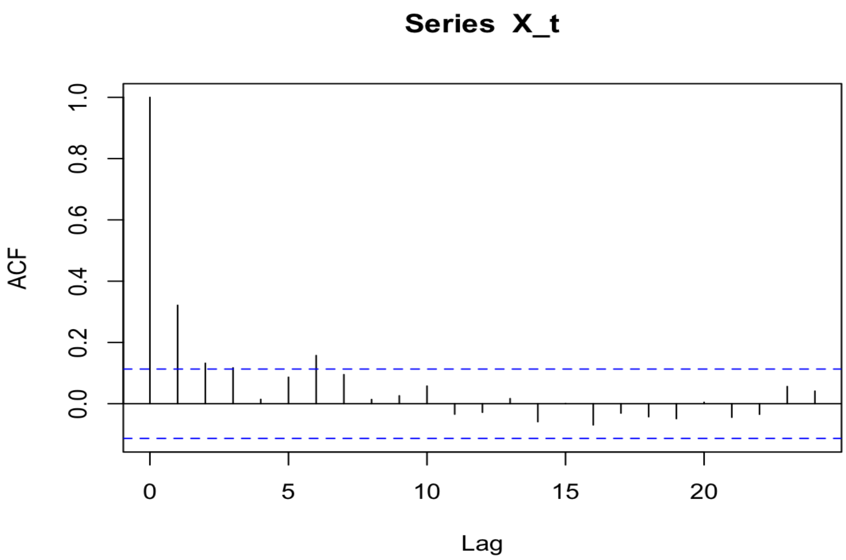


Figure 2: ACF for AR(1) process with $\phi = 0.35$. ACF is seen to quickly diminish indicating stationarity

B.) - For $\phi = 0.999$

For $\phi = 0.999$ we examine both the time series plot and the ACF plot below. From time series plot, fig. 3, we see that the time series does not have a stable mean and the variance seems to change over time through visual inspection. When examining the ACF plot, fig. 4, we see that the function maintains a high value indicating that the series is correlated to some degree and does not converge. Both of these indicate that the time series is **not stationary**.

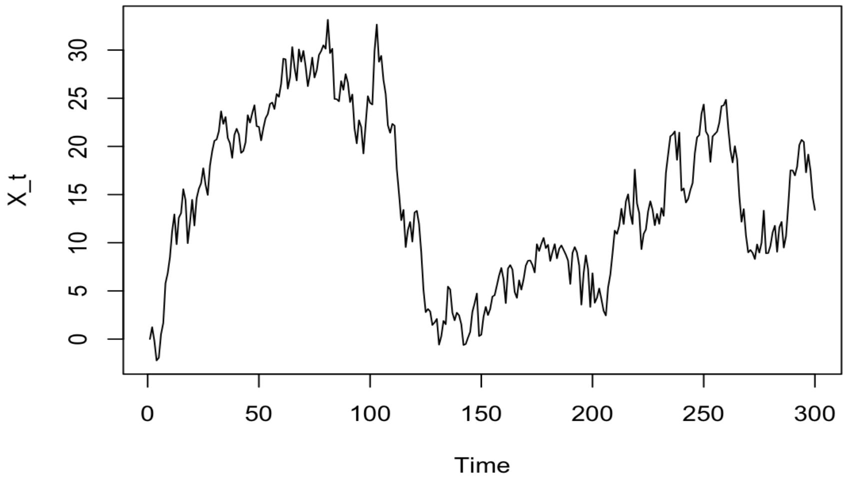


Figure 3: Time series for AR(1) process with $\phi = 0.999$. Process mean is NOT around 0 and variance seems large over the different sections of the time series

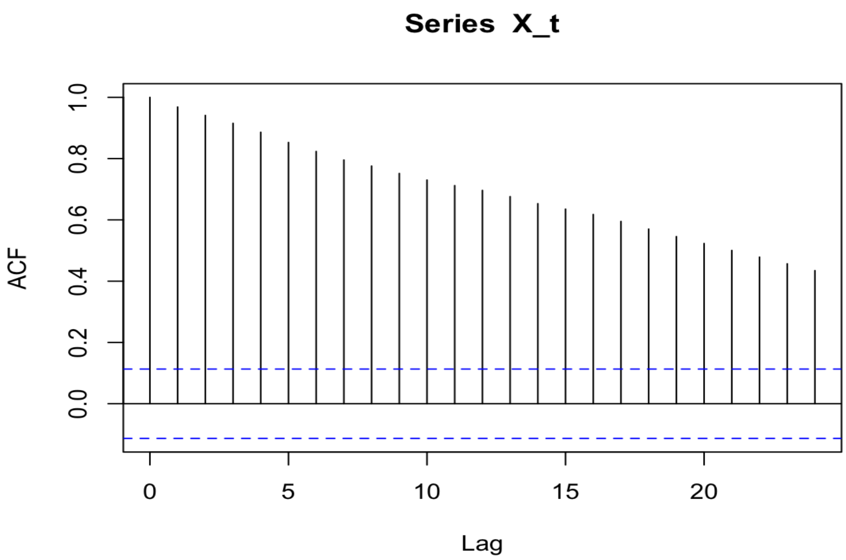


Figure 4: ACF for AR(1) process with $\phi = 0.999$. ACF is seen to maintain amplitude indicating the time series is not stationary

Question 3

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series satisfying the representation

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}, \quad t \in \mathbb{Z},$$

for $|\phi| > 1$, where $\{W_t\}_{t\in\mathbb{Z}} \sim \text{WN}(0, \sigma^2)$.

(a) Write Equation (2) in terms of backshift operators and compute the mean and variance functions of X_t.

Hint: $B^{-j}Y_t = Y_{t-(-j)} = Y_{t+j}$.

(b) Compute the autocovariance of $\{X_t\}_{t\in\mathbb{Z}}$ and argue that this process is stationary.

A.)

We begin by writing the desired equation in terms of the backshift operator, using $B^{-j}Y_t = Y_{t-(-j)} = Y_{t+j}$. Rewriting this as $W_{t+j} = B^{-j}W_t$ and substituting gives eq. (20).

$$\begin{aligned} X_t &= -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j} \\ X_t &= -\sum_{j=1}^{\infty} \phi^{-j} B^{-j} W_t \end{aligned} \tag{20}$$

Next we compute the expected value of X_t. Similarly we use $E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$ to bring the expected value operator inside the summation.

$$\begin{aligned} E(X_t) &= E(-\sum_{j=1}^{\infty} \phi^{-j} B^{-j} W_t) \\ E(X_t) &= -\sum_{j=1}^{\infty} \phi^{-j} E(B^{-j} W_t) \end{aligned} \tag{21}$$

For white noise $\sim \text{WN}(0, \sigma^2)$ then $E(B^{-j}W_t) = E(W_t) = 0, \quad \forall t \in \mathbb{Z}$, giving eq. (22).

$$\begin{aligned} E(X_t) &= E(-\sum_{j=1}^{\infty} \phi^{-j} B^{-j} W_t) \\ E(X_t) &= -\sum_{j=1}^{\infty} \phi^{-j} (0)) \\ E(X_t) &= 0 \end{aligned} \tag{22}$$

Next we compute the variance of X_t. We use the property $\text{Var}(aX + b) = a^2\text{Var}(X)$ [2]to give eq. (23).

$$\begin{aligned}
\text{Var}(X_t) &= \text{Var}(-\sum_{j=1}^{\infty}\phi^{-j}B^{-j}W_t) \\
\text{Var}(X_t) &= (-)^2\sum_{j=1}^{\infty}(\phi^{-j})^2\text{Var}(B^{-j}W_t) \\
\text{Var}(X_t) &= \sum_{j=1}^{\infty}(\phi^{-2j})\text{Var}(B^{-j}W_t)
\end{aligned} \tag{23}$$

For white noise $\sim \text{WN}(0,\sigma^2)$ then $\text{Var}(B^{-j}W_t) = \text{Var}(W_t) = \sigma_\omega^2, \quad \forall t \in \mathbb{Z}$ giving eq. (24).

$$\begin{aligned}
\text{Var}(X_t) &= \sum_{j=1}^{\infty}(\phi^{-2j})\text{Var}(B^{-j}W_t) \\
\text{Var}(X_t) &= \sum_{j=1}^{\infty}(\phi^{-2j})\sigma_\omega^2 \\
\text{Var}(X_t) &= \sigma_\omega^2\sum_{j=1}^{\infty}(\phi^{-2j})
\end{aligned} \tag{24}$$

As $|\phi|$>1, then $|\phi^{-x}| < 1$ allowing us to use geometric series:

$$\sum_{j=1}^{\infty}\phi^{-2j} = \frac{\phi^{-2}}{1-\phi^{-2}}$$

$$\frac{\phi^{-2}}{1-\phi^{-2}}\frac{\phi^2}{\phi^2} = \frac{1}{\phi^2-1}$$

To finally give the variance as eq. (25)

$$\begin{aligned}
\text{Var}(X_t) &= \sigma_\omega^2\sum_{j=1}^{\infty}(\phi^{-2j}) \\
\text{Var}(X_t) &= \frac{\sigma_\omega^2}{\phi^2-1}
\end{aligned} \tag{25}$$

B.)

Next we compute the autocovariance of X_t. Add h into backshift operator, just to keep consistency.

$$\begin{aligned}
\text{Cov}(X_t, X_{t+h}) &= \text{Cov}(-\sum_{j=1}^{\infty}\phi^{-j}B^{-j}W_t, -\sum_{k=1}^{\infty}\phi^{-k}B^{-k}W_{t+h}) \\
\text{Cov}(X_t, X_{t+h}) &= \text{Cov}(-\sum_{j=1}^{\infty}\phi^{-j}B^{-j}W_t, -\sum_{k=1}^{\infty}\phi^{-k}B^{-k-h}W_t) \\
\text{Cov}(X_t, X_{t+h}) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\phi^{-j}\phi^{-k}\text{Cov}(B^{-j}W_t, B^{-k-h}W_t)
\end{aligned} \tag{26}$$

The value of $\text{Cov}(B^{-j}W_t, -B^{-k-h}W_t) = 0 \quad \forall j \neq k+h$, So let $j = k+h$. Substitute this and combine summations to obtain eq. (27).

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+h}) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\phi^{-j}\phi^{-k}\mathrm{Cov}(B^{-j}W_t, B^{-k-h}W_t) \\
\mathrm{Cov}(X_t, X_{t+h}) &= \sum_{k=1}^{\infty}\phi^{-k-h}\phi^{-k}\mathrm{Cov}(B^{-k-h}W_t, B^{-k-h}W_t) \\
\mathrm{Cov}(X_t, X_{t+h}) &= \sum_{k=1}^{\infty}\phi^{-2k-h}\mathrm{Cov}(B^{-k-h}W_t, B^{-k-h}W_t) \\
\mathrm{Cov}(X_t, X_{t+h}) &= \phi^{-h}\sum_{k=1}^{\infty}\phi^{-2k}\mathrm{Cov}(B^{-k-h}W_t, B^{-k-h}W_t)
\end{aligned} \tag{27}$$

$\mathrm{Cov}(B^{-k-h}W_t, B^{-k-h}W_t) = \mathrm{Var}(B^{-k-h}W_t) = \mathrm{Var}(W_t) = \sigma_\omega^2$, adding this gives eq. (28)

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+h}) &= \phi^{-h}\sum_{k=1}^{\infty}\phi^{-2k}\mathrm{Cov}(B^{-k-h}W_t, B^{-k-h}W_t) \\
\mathrm{Cov}(X_t, X_{t+h}) &= \phi^{-h}\sigma_\omega^2\sum_{k=1}^{\infty}\phi^{-2k}
\end{aligned} \tag{28}$$

Again use $\sum_{j=1}^{\infty}\phi^{-2j} = \frac{\phi^{-2}}{1-\phi^{-2}} = \frac{1}{\phi^2-1}$ to simplifying and give our final answer eq. (29)

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+h}) &= \phi^{-h}\sigma_\omega^2\sum_{k=1}^{\infty}\phi^{-2k} \\
\mathrm{Cov}(X_t, X_{t+h}) &= \frac{\phi^{-h}\sigma_\omega^2}{\phi^2-1}
\end{aligned} \tag{29}$$

Question 4

Let $\{Y_t\}_{t\in\mathbb{Z}}$ be a stationary process with mean function μ and autocovariance function $\mathrm{cov}(Y_{t+h}, Y_t) = \gamma_Y(h)$, for $h \in \mathbb{Z}$. Define

$$X_t = \nabla^d Y_t = Y_t - Y_{t-d}, \quad t \in \mathbb{Z},$$

where $d \in \mathbb{N}$. Compute the mean function and the autocovariance function of $\{X_t\}_{t\in\mathbb{Z}}$ in terms of $\gamma_Y(\cdot)$. Is $\{X_t\}_{t\in\mathbb{Z}}$ stationary? Justify your answer.

Begin by computing the mean using $E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$ and $E(Y_t) = E(Y_{t-d}) = \mu$ as the process is stationary to give eq. (30).

$$\begin{aligned}
\mathrm{E}(X_t) &= E(Y_t - Y_{t-d}) \\
\mathrm{E}(X_t) &= E(Y_t) - E(Y_{t-d}) \\
\mathrm{E}(X_t) &= \mu - \mu = 0
\end{aligned} \tag{30}$$

Then we compute the autocovariance of X_t. Begin by expanding the cross terms of the covariance.

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+h}) &= \mathrm{Cov}(Y_t - Y_{t-d}, Y_{t+h} - Y_{t+h-d}) \\
\mathrm{Cov}(X_t, X_{t+h}) &= \underline{\mathrm{Cov}(Y_t, Y_{t+h})}_1 - \underline{\mathrm{Cov}(Y_{t-d}, Y_{t+h})}_2 \\
&\quad + \underline{\mathrm{Cov}(Y_{t-d}, Y_{t+h-d})}_3 - \underline{\mathrm{Cov}(Y_t, Y_{t+h-d})}_4
\end{aligned} \tag{31}$$

Then deal with each of the terms separately By definition in the question

$$\mathrm{Cov}(Y_t, Y_{t+h}) = \gamma_Y(h) \tag{32}$$

As the lag h is the difference between the two indices $t+h$ ans $t-d$. Lag = t+h - (t-d) = h+d

$$\text{Cov}(Y_{t-d}, Y_{t+h}) = \gamma_Y(h+d) \tag{33}$$

Let t = t-d to show that this is the same as eq. (32).

$$\text{Cov}(Y_{t-d}, Y_{t+h-d}) = \gamma_Y(h) \tag{34}$$

The difference between the two indices $t+h-d$ and t: Lag = t+h-d - (t) = h-d

$$\text{Cov}(Y_t, Y_{t+h-d}) = \gamma_Y(h-d) \tag{35}$$

Substituting these into eq. (31) gives eq. (36)

$$\begin{aligned}
\text{Cov}(X_t, X_{t+h}) &= \text{Cov}(Y_t, Y_{t+h}) - \text{Cov}(Y_{t-d}, Y_{t+h}) \\
&\quad + \text{Cov}(Y_{t-d}, Y_{t+h-d}) - \text{Cov}(Y_t, Y_{t+h-d}) \\
\text{Cov}(X_t, X_{t+h}) &= \gamma_Y(h) - \gamma_Y(h+d) + \gamma_Y(h) - \gamma_Y(h-d) \\
\text{Cov}(X_t, X_{t+h}) &= 2\gamma_Y(h) - \gamma_Y(h+d) - \gamma_Y(h-d)
\end{aligned} \tag{36}$$

$E = 0 \quad \forall t \in \mathbb{Z}$ and the autocovariance is only dependant on the lag $\gamma(h)$, meaning that the process X_t is stationary.

Question 5

Let $\{W_t\}_{t\in\mathbb{Z}} \sim \text{WN}(0, \sigma^2)$. Identify the AR and MA polynomials and determine whether the following ARMA processes are causal and/or invertible. Also, watch out for parameter redundancy.

(a) $X_t = -0.2X_{t-1} + 0.48X_{t-2} + W_t$.

(b) $X_t = -1.9X_{t-1} - 0.88X_{t-2} + W_t + 0.2W_{t-1} + 0.7W_{t-2}$.

A) We begin by rearranging process (a) to gather X_t and W_t processes on same side. Then factorise to reveal the AR and MA polynomials eq. (37).

$$\begin{aligned}
&X_t = -0.2X_{t-1} + 0.48X_{t-2} + W_t \\
&X_t + 0.2X_{t-1} - 0.48X_{t-2} = W_t \\
&X_t + 0.2BX_t - 0.48B^2X_t = W_t \\
&(1 + 0.2B - 0.48B^2)X_t = W_t
\end{aligned} \tag{37}$$

Next we identify the AR polynomial $\phi(B)$ and the MA polynomial $\Theta(B)$

$$\begin{aligned}
\Phi(B)X_t &= (1 + 0.2B - 0.48B^2)X_t \\
\Theta(B)W_t &= (1)W_t \\
\Phi(B) &= (1 + 0.2B - 0.48B^2) \\
\Theta(B) &= (1)
\end{aligned} \tag{38}$$

As $\Theta(B) = 1$ there's trivially no parameter redundancy.

Next we check for causality by finding the roots of $\Phi(B)$ using -b formula eq. (39).

$$\begin{gathered}(0.48B^2 - 0.2B - 1) = 0 \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \frac{0.2 \pm \sqrt{(0.2)^2 - 4(0.48)(-1)}}{2(0.48)} = -1.25, 1.67\end{gathered} \tag{39}$$

As both roots lie outside of the unit circle we determine that the process is causal.

Next we check if the process is invertible. For an invertible process W_t can be represented as eq. (40).

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B) X_t \tag{40}$$

As eq. (38) already represents W_t in this form we know the process is invertible. As in this case $\pi(B) = \phi(B)$ so we already have that representation.

So this process is causal and invertible and there is no parameter redundancy.

B) Again begin by rearranging process (b) to gather X_t and W_t processes on same side. Then factorise to reveal the AR and MA polynomials eq. (41).

$$\begin{aligned}X_t &= -1.9X_{t-1} - 0.88X_{t-2} + W_t + 0.2W_{t-1} + 0.7W_{t-2} \\ X_t + 1.9X_{t-1} + 0.88X_{t-2} &= W_t + 0.2W_{t-1} + 0.7W_{t-2} \\ X_t + 1.9BX_t + 0.88B^2X_t &= W_t + 0.2BW_t + 0.7B^2W_t \\ (1 + 1.9B + 0.88B^2)X_t &= (1 + 0.2B + 0.7B^2)W_t\end{aligned} \tag{41}$$

I had many attempts at finding parameter redundancy, see appendix for handwritten solutions and attempts. But it appears that there is no parameter redundancy.

Next check for causality by finding the roots of $\Phi(B)$ using -b formula eq. (42).

$$\begin{gathered}(1 + 1.9B + 0.88B^2) = 0 \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \frac{-1.9 \pm \sqrt{(1.9)^2 - 4(0.88)(1)}}{2(0.88)} = -0.91, -1.25\end{gathered} \tag{42}$$

As one of the roots lies inside the unit circle we determine that the process is not causal.

Next check for invertibility by finding the roots of $\Theta(B)$ using -b formula eq. (43).

$$\begin{gathered}(1 + 0.2B + 0.7B^2) = 0 \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{gathered}$$

$$\begin{aligned}
&\frac{-0.2 \pm \sqrt{(0.2)^2 - 4(0.7)(1)}}{2(0.7)} \\
&\frac{-0.2 \pm \sqrt{0.04 - 2.8}}{1.4} \\
&\frac{-0.2 \pm \sqrt{-2.76}}{1.4} \\
&\frac{-0.2 \pm \sqrt{-1}\sqrt{2.76}}{1.4}
\end{aligned} \tag{43}$$

Finally we compute the roots eq. (44) and identify that as the roots are outside the unit circle (imaginary part is greater than 1 so the magnitude is already >1) the process is invertible.

$$\frac{-0.2 \pm 1.66\mathbf{i}}{1.4} = -0.2 - 1.187\mathbf{i}, -0.2 + 1.187\mathbf{i} \tag{44}$$

So this process is not causal, is invertible and there is no parameter redundancy.

References

[1] D. W. Barreto-Souza, *Slide 8 - time series - module details 2024-25 - time series analysis stat30010/stat40700*, Accessed 14/10/24. [Online]. Available: %5Curl%7Bhttps://brightspace.ucd.ie/d2l/le/content/275957/viewContent/3134803/View%7D.

[2] D. W. Barreto-Souza, *Slide 9 - time series - module details 2024-25 - time series analysis stat30010/stat40700*, Accessed 14/10/24. [Online]. Available: %5Curl%7Bhttps://brightspace.ucd.ie/d2l/le/content/275957/viewContent/3134803/View%7D.

Appendix

This appendix contains the handwritten form of the solutions. Just including for completeness and that you can see my attempts at factorising Q5 and other small bits ect. in case you wanted to see.

Q1

$$\nabla X_t = (1-B)X_t = X_t - X_{t-1}$$

$$\nabla^2 X_t = (1-B)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$$

Q1.

$$\nabla X_t = (1-B)X_t$$

$$(1-B)(\beta_0 + \beta_1 t + \beta_2 t^2 + W_t)$$

- $\cancel{\beta_0} + \cancel{\beta_1 t} + \beta_2 t^2 + W_t - (\cancel{\beta_0} + \beta_1(\cancel{t}-1) + \beta_2(t-1)^2 + W_{t-1})$

$= + \beta_2 \cancel{t^2} + W_t + \beta_1 - \beta_2(\cancel{t^2} - 2t + 1) - W_{t-1}$

$W_t + \beta_1 + 2t - 1 + W_{t-1}$

$= \beta_1 + 2t - 1 + W_t - W_{t-1}$

$E(X_t) = E(\beta_1 + 2t - 1 + W_t - W_{t-1})$

$= E(\beta_1) + E(2t) - E(1) + E(W_t) - E(W_{t-1})$

- Depends on t

$(t-1)^2$

$(t^2 - 2t + 1)$

ii) $(1-B)(\beta_1 + 2t - 1 + W_t - W_{t-1})$

$\beta_1 + 2t - 1 + W_t - W_{t-1}$

$-(B(\beta_1) + B(2t) - B(1) + B(W_t) - B(W_{t-1}))$

$\cancel{\beta_1} + 2\cancel{t} - \cancel{1} + W_t - W_{t-1}$

$-(\cancel{\beta_1} + 2(t-1) - \cancel{1} + W_{t-1} - W_{t-2})$

$= +1 + W_t - 2W_{t-1} + W_{t-2}$ $\quad E(\ ) = 0$

$$\cancel{\nabla^2 X_t = 2\beta + 2t - 1}$$

$$\nabla^2 X_t = 2\beta_2 + W_t - 2W_{t-1} + W_{t-2}$$

$\gamma(Y_t\ Y_{t+h})$ $\quad \nabla^2 X_{t+h} = 2\beta_2 + W_{t+h} - 2W_{t+h-1} + W_{t+h-2}$

$$Cov(\nabla^2 X_t, \nabla^2 X_{t+h}) =$$

$$E(X_1 X_2) - \underbrace{E(X_1)}_{0}\underbrace{E(X_2)}_{0}$$

$E($ $\quad Cov(\beta_2, W_t \ldots) = 0$

$$= cov(W_t, W_{t+h}) + cov(W_t, -2W_{t+h-1}) + cov(W_t, W_{t+h-2})$$

$$+ cov(-2W_{t-1}, W_{t+h}) + Cov(-2W_{t-1}, -2W_{t+h-1}) + Cov(-2W_{t-1}, W_{t+h-2})$$

$$+ Cov(W_{t-2}, W_{t+h}) + Cov(W_{t-2}, -2W_{t+h-1}) + cov(W_{t-2}, W_{t+h-2})$$

Case $h = 0$

$$Var(W_t) + 4Var(\cancel{4}W_{t-1}) + Var(W_{t-2})$$

Case $h = 1$

$$-2\,Var(W_t)\ \cancel{4Cov} - 2\,var(W_{t-1})$$

$Cov(W_t, W_{t+h}) - 2Cov(W_t, W_{t+h-1}) + Cov(W_t, W_{t+h-2}))$

$-2Cov(W_{t-1}, W_{t+h}) + 4Cov(W_{t-1}, W_{t+h-1}) - 2Cov(W_{t-1}, W_{t+h-2})$

$+Cov(W_{t-2}, W_{t+h}) - 2Cov(W_{t-2}, W_{t+h-1}) + Cov(W_{t-2}, W_{t+h-2})$

$\underline{h=0}$

$Cov(W_t, W_t) \cancel{-2Cov(W_t, W_{t-1}} + 4Cov(W_{t-1}, W_{t-1})$

$+Cov(W_{t-2}, W_{t-2})$

$Var(W_t) + 4Var(W_{t-1}) + \cancel{Cov} Var(W_{t-2})$

$= 6Var(W_t) = 6\sigma_w^2$

$\underline{h=1}$

$-2Cov(W_t, W_t) - 2Cov(W_{t-1}, W_{t-1})$

$= -2Var(W_t) - 2Var(W_{t-1}) = -4Var(W_t) = -4\sigma_w^2$

$\underline{h=2}$

$Cov(W_t, W_t) \cancel{+Cov(W_{t-2}, W_{t+2})} = Var(W_t) = \sigma_w^2$

$\underline{h=-1}$

$-2Cov(W_{t-1}, W_{t-1}) - 2Cov(W_{t-2}, W_{t-2}) = -4\sigma_w^2$

$\underline{h=-2}$

$Cov(W_{t-2}) = \sigma_w^2$

else 0

Q2

Time series | Q2

$\boxed{Q2}$

$$\left.(1-\phi B)X_t = W_t\right\} \; WN(0, 4)$$

$X_t = (1-\phi B)^{-1} W_t$ is invertible.

~~redo Q1~~

$Y_t = (1-\phi B)X_t$ $\boxed{\text{20 Mins}}$

$n = 300$

$$X_t = (1-\phi B)^{-1} W_t$$

$$X_t - \phi X_{t-1} = W_t$$

$$X_t = (1-\phi B)^{-1} W_t$$

$$=$$

$$X_t = \frac{1}{1-\phi B} = \sum_{j=0}^{\infty} (-B)^j z^j$$

$$X_t = \sum_{j=0}^{\infty} \phi^j B^j W_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

$$X_t - \phi X_{t-1} = W_t$$

$$\left. X_t = W_t + \phi X_{t-1} \right\} \Rightarrow$$

Q3

Scet

Q3 $X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j} \qquad t \in \mathbb{Z}$

$|\phi| > 1$

Write in terms of backshift operator

$X_t = -\sum \phi^{-j} B^{-j} W_t$

$E(X_t) = E\left(-\sum \phi^{-j} B^{-j} W_t\right)$

$X_t = -\left(\phi^{-1} W_{t+1} + \phi^{-2} W_{t+2} + \phi^{-3} W_{t+3} + \cdots\right.$

$X_t = -\left(\phi^{-1} B W_t + \phi^{-2} B\right.$

$\left(\phi^{-1} B W_t +\right.$

$\cancel{\phi_1 X} \quad \phi_1 W_{t-1} + \phi_2 W_{t-2} + \phi_3 W_{t-3}$

$\phi_1 B W_t + \phi_2 B^2 W_t + \phi_3 B^3 W_t$

$\phi^{-1}(B) = 1 + \phi B + \phi^2 B^2$

$E(X_t) = E\left(-\sum_{j=1}^{\infty} \phi^{-j} B^{j} W_t\right)$

$E\left(-(W_t)\left(\sum_{j=1}^{\infty} \phi^{-j} B^{-j}\right)\right) \qquad \left(\phi^{-1} B^{1} + \phi^{-2} B^{2}\right) W_t$

$E(0) \qquad E(W_t)$

Autocov

$$Cov\left(-\sum \phi^{-j}B^{-j}W_t, -\sum \phi^{-j}B^{-j}W_{t+h}\right)$$

add h into backshift

$$cov\left(-\sum_{j=0}^{\infty}\phi^{-j}B^{-j}W_t, -\sum_{k=0}^{\infty}\phi^{-k}B^{-k-h}W_t\right)$$

$$= +\sum_k\sum_j \phi^{-j}\phi^{-k}\, Cov\left(B^{-j}W_t, B^{-k-h}W_t\right)$$

$Cov = 0 \quad \forall \quad j \neq k+h$

$j = k+h$

$$\sum_{k=0}^{\infty}\phi^{-2k-h}\, Cov\left(B^{-k-h}W_t, B^{-k-h}W_t\right)$$

$$= Var\left(B^{-k-h}W_t\right) = Var(W_t)$$

$$= \sigma_w^2$$

$$= \phi^{-h}\sum \phi^{-2k}\sigma_w^2 \qquad \sum \phi^{-2k} = \frac{1}{\phi^2-1}$$

$$= \frac{\phi^{-2}}{1-\phi^{-2}}$$

$$\cancel{= \phi^{-h}\sigma_w^2\frac{1}{1-\phi^2} \quad h \to 0 \quad \phi^0\frac{\sigma^2}{1-\phi^2} \quad h=0}$$

$$= \frac{1}{\phi^2-1}$$

$$\sigma_w^2\phi^{-h}\sum\phi^{-2k} = \phi^{-h}\frac{\sigma_w^2}{\phi^2-1} \qquad \phi^{-2k} < 1$$

so converge

$$\gamma(h) = \frac{\sigma_w^2\phi^{-h}}{\phi^2-1}$$

Q4

Q4 - Y_t $\quad Cov(Y_{t+h}, Y_t) = \gamma_Y(h)$

$X_t = \nabla_d Y_t = Y_t - Y_{t-d}$

$E(X_t) = E(Y_t - Y_{t-d}) = E(Y_t) - E(Y_{t-d})$

Y is stationary so $E(Y_t) = E(Y_{t-d}) = \mu \ \forall d$

$E(X_t) = \mu - \mu = 0$

$\cancel{\gamma(h)}$ $Cov(X_t, X_{t+h}) = Cov(Y_t - Y_{t-d}, Y_{t+h} - Y_{t-d+h})$

$= Cov(Y_t, Y_{t+h}) - Cov(Y_{t-d}, Y_{t+h}) = \gamma_Y(h-d)$

$+ Cov(Y_{t-d}, Y_{t-d+h}) - Cov(Y_t, Y_{t-d+h})$

$\cancel{h=0}$ $\quad \gamma_Y(h+d)$

$\cancel{= Var(Y_t)} \quad = \gamma_Y(h)$

$+ \gamma_Y(h)$??

$= 2\gamma_Y(h) - \gamma_Y(h+d) - \gamma_Y(h-d)$

$t+h \quad t-d$

is stationary as $E = 0$ & $\gamma(h)$ not on t.

Q5

Q5.

a) $X_t = -0.2X_{t-1} + 0.48X_{t-2} + W_t$

$X_t + 0.2BX_t - 0.48B^2X_t = W_t$

$(1 + 0.2B - 0.48B^2)X_t = W_t$

$-1 \quad 0.1 \quad B$

$-1 \quad 0.1 \quad B$

$10 + 19B + 0$

$100 + 190B + 88B^2$

$100 \quad 20B + 70B^2$

b) $X_t = -1.9X_{t-1} - 0.88X_{t-2} + W_t + 0.2W_{t-1} + 0.7W_{t-2}$

$X_t + 1.9X_{t-1} + 0.88X_{t-2} = W_t + 0.2W_{t-1} + 0.7W_{t-2}$

$X_t + 1.9BX_t + 0.88B^2X_t = W_t + 0.2BW_t + 0.7B^2W_t$

$(1 + 1.9B + 0.88B^2)X_t = (1 + 0.2B + 0.7B^2)W_t$

$1 \quad B$

$1 \quad B$

$(1 + 0.8B)(1 + 1.1B)$

$(1 + 0.2x + 0.7x^2)$

$1 \quad 0.875x$

$1 \quad 0.8x$

$\cancel{0.48x^2 - 0.2x - 1}$

$1 + 1.9x + 0.88x^2$

$0.44 \quad 8 \quad 0.8 \quad 1 \quad 0.5$

$0.11 \quad 1.1 \quad 0.7 \quad 1.4$

1

$1 \quad 0.8^2 x$

$1 \quad 1.1x$

$\cancel{Q4}$ $0.48B^2 - 0.2B - 1 = 0$

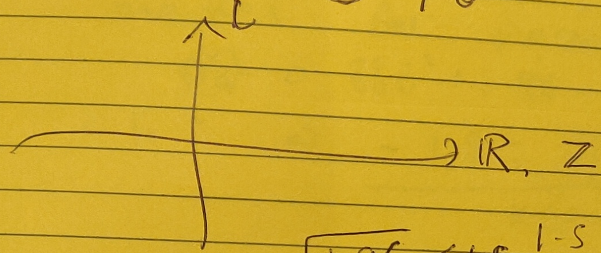
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{0.2 \pm \sqrt{(0.2)^2 - 4(0.48)(-1)}}{2(0.48)}$$

$\sqrt{0.04 + 4(0.4)}$

$$\begin{array}{r} 48 \\ \times\ 4 \\ \hline 32 \\ 160 \\ 192 \end{array}$$

$\sqrt{1.96}$

$$\frac{0.2 \pm \sqrt{1.96}}{0.96}$$ inside unit circle.



$\sqrt{1.96} < 1.5$

$$\begin{array}{r} 1.5 \\ \times 1.5 \\ \hline 1.5 \\ +0.25 \end{array}$$

$$\frac{0.2 \pm 1.5}{0.96} =$$

$$\frac{0.2 \pm 1.4}{0.96} = \frac{1.6}{0.96}, \frac{-1.2}{0.96}$$

$\frac{5}{3}$ -1.25

$\Theta(B) = 1$

Both outside unit circle so causal.

$W_t = \boxed{\ \ }$

Not Stat

Q5 B)

$$X_t = -1.9X_{t-1} - 0.88X_{t-2} + W_t + 0.2W_{t-1} + 0.7W_{t-2}$$

$$0.88B^2X_t + 1.9BX_t + X_t = W_t + 0.2BW_t + 0.7B^2W_t$$

$$(0.88B^2 + 1.9B + 1)X_t = (0.7B^2 + 0.2B + 1)W_t$$

checked for parameter redundancy.

$$\phi(B) = 0.88B^2 + 1.9B + 1$$

$$-b \pm \Rightarrow \frac{-1.9 \pm \sqrt{(1.9)^2 - 4(0.88)}}{2(0.88)}$$

$$= \frac{-1.9 + \frac{3}{10}}{1.76} = -0.9090$$

$$= \frac{-1.9 - 0.3}{1.76} = -1.25$$

Not causal

$$\Theta(B) = 0.7B^2 + 0.2B + 1$$

$$\frac{-0.2 \pm \sqrt{(0.2)^2 - 4(0.7)(1)}}{2(0.7)} = \frac{-0.2 \pm \sqrt{0.04 - 2.8}}{1.4}$$

$$\frac{-0.2 \pm \sqrt{-2.76}}{1.4} \qquad \sqrt{-2.76} = \sqrt{-1}\sqrt{2.76} = 1.66i$$

$$= -0.2 + 1.187i, \ -0.2 - 1.187i$$

is invertible.