Brian Murphy Macro PS 1

1. $\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$ s.t. $c_t + k_{t+1} - (1-\delta)k_t = Ak_t^{\alpha}$

a. Dynamic Program:

$v(k) = \max_{c_t, k_{t+1}} [\ln(c_t) + \beta v(k')]$

s.t. $c_t + k_{t+1} - (1-\delta)k_t = Ak_t^{\alpha}$

We can sub in our constraint. Rewriting,

$v(k) = \max_{k'} [\ln(Ak_t^{\alpha} + (1-\delta)k_t - k_{t+1}) + \beta v(k')]$

Now, find Euler.

FOC: $\frac{1}{c_t} \cdot (-1) + \beta v'(k') = 0 \Rightarrow \frac{1}{c_t} = \beta v'(k')$

Apply Envelope Thm.

$v'(k) = \frac{1}{c_t}[\alpha A k_t^{\alpha-1} + (1-\delta)]$

Now, use Ben. Sch. Thm.

$v'(k') = \frac{1}{c_{t+1}}[\alpha A k_t^{\alpha-1} + (1-\delta)]$. Now, plug into FOC.

$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}}[\alpha A k_t^{\alpha-1} + (1-\delta)]$. We can rewrite:

$\frac{c_{t+1}}{c_t} = \beta[\alpha A k_t^{\alpha-1} + (1-\delta)] \Rightarrow$ E.E.

b. Code attached

In steady state, $\frac{c_{t+1}}{c_t} = 1$. Thus,

$1 = \beta[\alpha A k_{ss}^{\alpha-1} + (1-\delta)]$

$\frac{1}{\beta} = \alpha A k_{ss}^{\alpha-1} + (1-\delta)$

$\frac{1}{\beta} - (1-\delta) = \alpha A k_{ss}^{\alpha-1}$

$\left(\frac{1/\beta - (1-\delta)}{\alpha A}\right)^{1/\alpha-1} = k_{ss}$

or $\left(\frac{\alpha A}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} = k_{ss}$

The rest of this problem is attached seperately.

2. Preferences: $\frac{n_t^{1-\sigma}}{1-\sigma} + \frac{c_{t+1}^{1-\sigma}}{1-\sigma}$

$h_{t+1} = \theta(1-n_t)e_t^{\gamma} h_t^{\delta}$

Assume balanced budget; $E_t = \tau Y_t$

initial old generation has h_0 human capital

youth has choice of leisure or learn

1 unit HC = 1 unit of consumption

$\max_{n_t, c_{t+1}} \frac{n_t^{1-\sigma}}{1-\sigma} + \frac{c_{t+1}^{1-\sigma}}{1-\sigma}$ s.t. $h_{t+1} = \theta(1-n_t)e_t^{\gamma} h_t^{\delta}$

& $E_t = \tau Y_t$

get E_t in percapita terms: $\frac{E_t}{L_t} = \tau \frac{Y_t}{L_t}$

Since population is constant: $e_t = \tau y_t$

Additionally, $Y_t = h_t$, so $e_t = \tau h_t$

Further, $C_{t+1} = A(1-\tau)h_{t+1}$

$\uparrow$ technology parameter

So, final problem:

$\max_{n_t, c_{t+1}} \frac{n_t^{1-\sigma}}{1-\sigma} + \frac{c_{t+1}^{1-\sigma}}{1-\sigma}$ s.t. $h_{t+1} = \theta(1-n_t)e_t^{\gamma} h_t^{\delta}$

$C_{t+1} = A(1-\tau)h_{t+1}$

$e_t = \tau h_t$

Plug in: $C_{t+1} = A(1-\tau)\theta(1-n_t)\tau^{\gamma} h_t^{\delta+\gamma}$

Substitute & maximize.

$\max_{n_t} \frac{n_t^{1-\sigma}}{1-\sigma} + \frac{(A(1-\tau)\theta(1-n_t)\tau^{\gamma} h_t^{\delta+\gamma})^{1-\sigma}}{1-\sigma}$

FOC: $\frac{1}{n_t^{\sigma}} = \frac{(C_{t+1})^{1-\sigma}}{(1-n_t)^{\sigma}}$ $\leftarrow$ check derivative

$(1-n_t)^{\sigma} = (C_{t+1})^{1-\sigma} n_t^{\sigma}$

$1-n_t$ is time spent learning.

$$\frac{1-n_t}{n_t} = (C_{t+1})^{1-\sigma/\sigma}$$

$\Uparrow$ $\quad = A(1-\tau)\theta(1-n)\tau^{\gamma} h_t^{\delta+\gamma}$

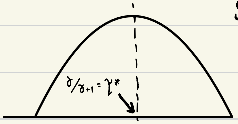
tradeoff between learning & leisure

b. $\frac{1-n_t}{n_t} = A(1-\tau)\theta(1-n)\tau^{\gamma} h_t^{\delta+\gamma}$

$T = (1-\tau)\tau^{\gamma}$, $E = \frac{1-n_t}{n_t}$

$\partial E/\partial T > 0 \Rightarrow \frac{\partial E}{\partial T} \cdot \frac{\partial T}{\partial \tau}$

Thus, $\tau \in [0,1]$, but max: $\tau = \gamma/\gamma+1$



So, there is an optimal tax level at τ^*. So, as $\tau \to \tau^*$, public expenditure per young is higher, so time allocated to learning increases. But after, time decreases.

c) Log Pref when $\sigma \to 1$.

$\max_{n_t} \ln(n_t) + \ln(A(1-\tau)\theta(1-n_t)\tau^{\gamma} h_t)$

FOC: $\frac{1}{n_t} = \frac{1}{1-n_t}$

$1-n_t = n_t \quad 1 = 2n_t \Rightarrow \frac{1}{2} = n_t$

So in the log case, the agent will spend half of his time in leisure.

The agent will always choose this learning to leisure ratio.

3. a. Habit Persistence: $u(C_t, C_{t-1})$

s.t. a budget constraint, i.e. $C_t + K_{t+1} - (1-\delta)K_t \leq f(K_t)$

C_{t-1} is already maximized.

$$v(C_{t-1}, K) = \max_{C_t, K'} \left[u\left(f(K_t + (1-\delta)K_t - K_{t+1}, C_{t-1}\right) + \beta v(K_{t+1}) \right]$$

b. $\sum_{t=0}^{\infty} \beta^t u(b_t)$

s.t. $b_t + S_{t+1} = RS_t + w$

$$v(S_t) = \max_{S'} \left[u\left(RS_t\right) + w - S_{t+1} \right) + \beta v(S_{t+1}) \Big]$$

c. $\sum_{t=0}^{\infty} \beta^t u(C_t)$ s.t. $C_t + K_{t+1} - (1-\delta)K_t + V_{t+1} - V_t = f(K_t)$

$f'(K) \to 0$ (steady state) $K \to \infty$

$$v(K, V) = \max_{K', V'} \left[u\left[f(K_t) + V_t - V' + (1-\delta)K_t - K' \right] + \beta v(K', V') \right]$$

derive optimal conditions. Take FOCs

K': $u'(C_t)(-1) + \beta v_1'(K', V')$

V': $u'(C_t)(-1) + \beta v_2'(K', V')$

Apply envelope thm.

$v_1'(K, V) = u'(C_t)(f'(K_t) + (1-\delta))$

$v_2'(K, V) = u'(C_t)(1)$

Use Ben. Sch. to push forward.

$v_1'(K', V') = u'(C_{t+1})(f'(K') + (1-\delta))$

$v_2'(K', V') = u'(C_{t+1})$.

Plug in to FOC.

$u'(C_t) = \beta u'(C_{t+1})(f'(K') + (1-\delta))$

$u'(C_t) = \beta u'(C_{t+1})$

So E.E. for K': $u'(C_t) = \beta u'(C_{t+1})(f'(K') + (1-\delta))$

& E.E for V': $u'(C_t) = \beta u'(C_{t+1})$