

*Problem:*

Let  $G$  be a  $k \times n$  random matrix with i.i.d standard normal entries. Further, let  $v \in \mathbb{R}^n$  such that  $\|v\|_2 = 1$ . Find the distribution of  $\|Gv\|_2^2$ .

*Solution:*

Let

$$G := \begin{pmatrix} z_{11} & \dots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{k1} & \dots & z_{kn} \end{pmatrix}$$

and

$$v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then,

$$Gv = \begin{pmatrix} v_1 z_{11} + \dots + v_n z_{1n} \\ \vdots \\ v_1 z_{k1} + \dots + v_n z_{kn} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} N(0, \|v\|_2^2) \\ \vdots \\ N(0, \|v\|_2^2) \end{pmatrix} = \begin{pmatrix} N(0, 1) \\ \vdots \\ N(0, 1) \end{pmatrix} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \end{pmatrix}$$

where  $Z_j \perp Z_k \forall j \neq k$ . Thus,  $Gv$  is a vector of independent standard normal random variables. This follows from the assumption that the entries of  $G$  are independent of one another and that linear combinations of independent normal random variables are normal.

It follows that

$$\|Gv\|_2^2 = Z_1^2 + \dots + Z_k^2 \sim \chi_k^2$$

as the square of a standard normal random variable is  $\chi_1^2$  and sums of independent  $\chi_1^2$  random variables are chi-squared with degrees of freedom equal to the number of terms in the sum.