Problem:

Let G be a $k \times n$ random matrix with i.i.d standard normal entries. Further, let $v \in \mathbb{R}^n$ such that $||\nu||_2 = 1$. Find the distribution of $||Gv||_2^2$.

Solution:

Let

$$G := \begin{pmatrix} z_{11} & \dots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{k1} & \dots & z_{kn} \end{pmatrix}$$

and

$$v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then,

$$Gv = \begin{pmatrix} v_1 z_{11} + \cdots + v_n z_{1n} \\ \vdots \\ v_1 z_{k1} + \cdots + v_n z_{kn} \end{pmatrix} \overset{d}{=} \begin{pmatrix} N(0, ||v||_2^2) \\ \vdots \\ N(0, ||v||_2^2) \end{pmatrix} = \begin{pmatrix} N(0,1) \\ \vdots \\ N(0,1) \end{pmatrix} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \end{pmatrix}$$

where $Z_j \perp\!\!\!\perp Z_k \ \forall j \neq k$. Thus, Gv is a vector of independent standard normal random variables. This follows from the assumption that the entries of G are independent of one another and that linear combinations of independent normal random variables are normal.

It follows that

$$||Gv||_2^2 = Z_1^2 + \cdots + Z_k^2 \sim \chi_k^2$$

as the square of a standard normal random variable is χ_1^2 and sums of independent χ_1^2 random variables are chi-squared with degrees of freedom equal to the number of terms in the sum.