CH08-320201
Algorithms and Data Structures

Lecture 17 — 17 Apr 2018

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3.6 Hash Tables

Direct Access Table

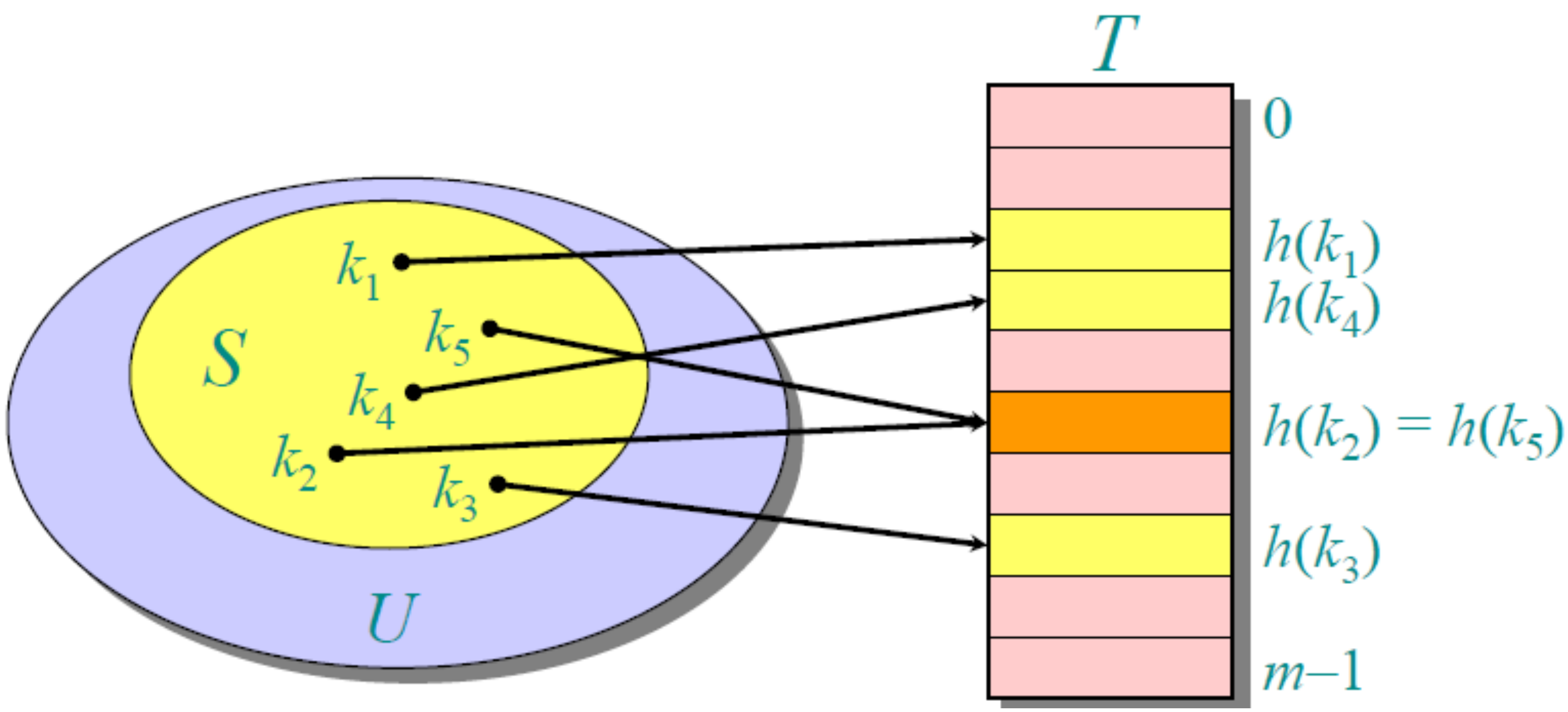
- The idea of a direct access table is that objects are directly accessed via their key.
- Assuming that keys are out of U = {0,1,…,m-1}.
- Moreover, assume that keys are distinct.
- Then, we can set up an array T[0..m-1] with

$$T[k] = \begin{cases} x & \text{if } x \in K \text{ and } key[x] = k \\ \text{NIL} & \text{otherwise.} \end{cases}$$

- Running time:
  With this set-up, we can have the dynamic-set operations (Search, Insert, Delete, …) in Θ(1).
- Problem: m is often large.
  For example, for 64-bit numbers we have 18,446,744,073,709,551,616 different keys.

Hash function

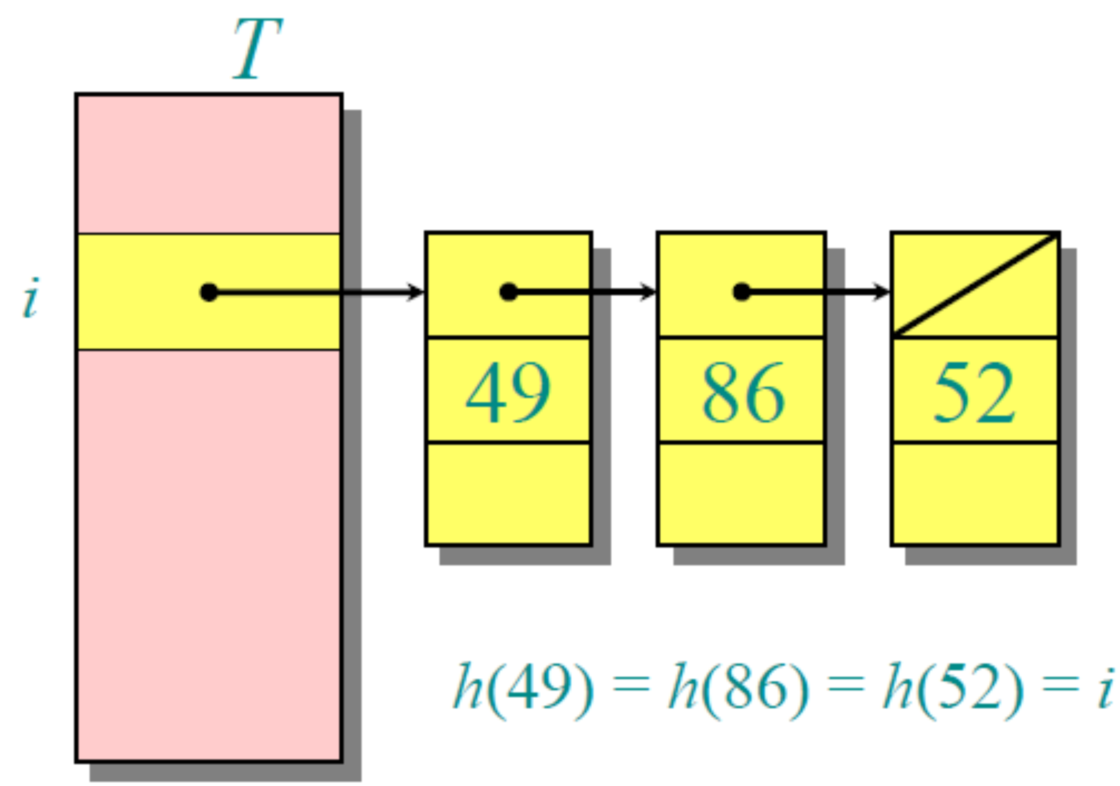
- Use a function h that maps U to a smaller set {0, 1,…, m-1}.



- Such a function is called a hash function.
- The table T is called a hash table.
- If two keys are mapped to the same location, we have a collision.

Resolving collisions

- Collisions can be resolved by storing the colliding mappings in a (singly-)linked list.



- Worst case:
  All keys are mapped to the same location. Then, access time is Θ(n).

Average case analysis

- Assumption (simple uniform hashing): Each key is equally likely to be hashed to any slot of the table, independent of where other keys are hashed.
- Let n be the number of keys.
- Let m be the number of slots.
- The load factor $\alpha = n/m$ represents the average number of keys per slot.

Average case analysis

Theorem:

- In a hash table in which collisions are resolved by chaining, an unsuccessful search takes average-case time $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

Proof:

- Any key k not already stored in the table is equally likely to hash to any of the m slots.
- The expected time to search unsuccessfully for a key k is the expected time to search to the end of list T[h(k)]
- Expected length of the list is $E[n_{h(k)}] = \alpha$
- Time for computing $h(k) = O(1) \Rightarrow$ overall time $\Theta(1+\alpha)$

Average case analysis

- Runtime for unsuccessful search:

  The expected time for an unsuccessful search is $\Theta(1+\alpha)$ including applying the hash function and accessing the slot and searching the list.

- What does this mean?
  - $m \sim n$, i.e., if $n=O(m) \Rightarrow \alpha = n/m = O(m)/m = O(1)$
  - Thus, search time is $O(1)$
- A successful search has the same asymptotic bound.

Choosing a hash function

- What makes a good hash function?
  - The goal for creating a hash function is to distribute the keys as uniformly as possible to the slots.
- **Division method**.
  - Define hashing function $h(k) = k \bmod m$.
  - Deficiency: Don‘t pick an m that has a small divisor d, as a prevalence of keys with *the same modulo d* can negatively effect uniformity.
    - Example: if m is a power of 2, the hash function only depends on a few bits: If k = 1011000111011010 and $m = 2^6$, then h(k) = 011010.
  - Common choice: Pick m to be a prime not too close to a power of 2 or 10 and not otherwise prominently used in computing environments.
    - Example: n = 2000; we are OK with avg. 3 elements in our collision chain —> m = 701 (a prime number close to 2000/3); $h(k) = k \bmod 701$.

Choosing a hash function

- **Multiplication method**:
  - Assume all keys are integers, $m = 2^r$, and the computer uses w-bit words.
  - Define hash function $h(k) = (A \cdot k \bmod 2^w) >> (w-r)$, where „>>“ is the right bit-shift operator and A is an odd integer with $2^{w-1} < A < 2^w$.
  - Note that these operations are faster than divisions.
  - Example: $m = 2^3 = 8$ and $w = 7$.

$$
\begin{array}{r l}
1011001 & = A \\
\times \quad 1101011 & = k \\
\hline
1001010\underbrace{011}_{h(k)}0011 &
\end{array}
$$

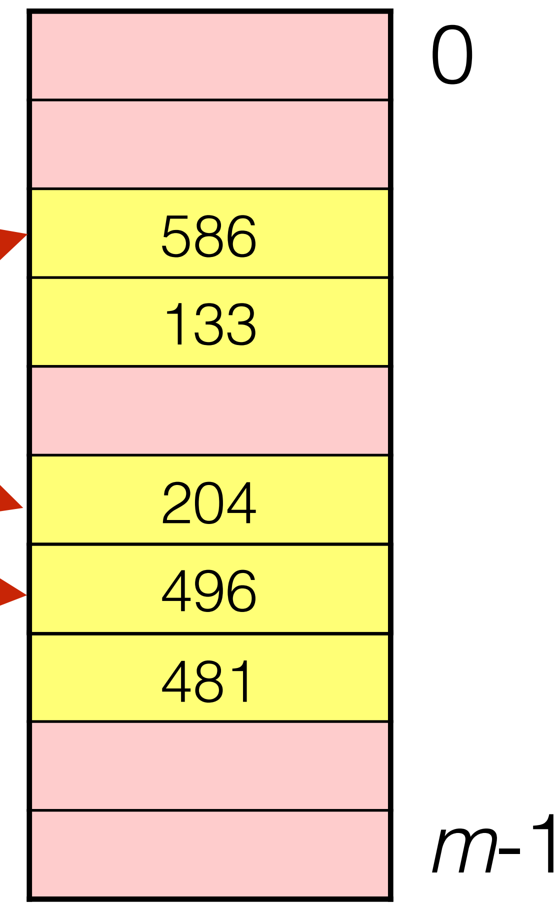
Resolving collisions by **open addressing**

- No additional storage is used.
- Only store one element per slot.
- Insertion probes the table systematically until an empty slot is found.
- The hash function depends on the key and the probe number, i.e., h : U x {0,1,…,m-1} -> {0,1,…,m-1}.
- The probe sequence <h(k,0), h(k,1), …, h(k,m-1)> should be a permutation of {0,1,…,m-1}.

Example

- Insert key k = 496:

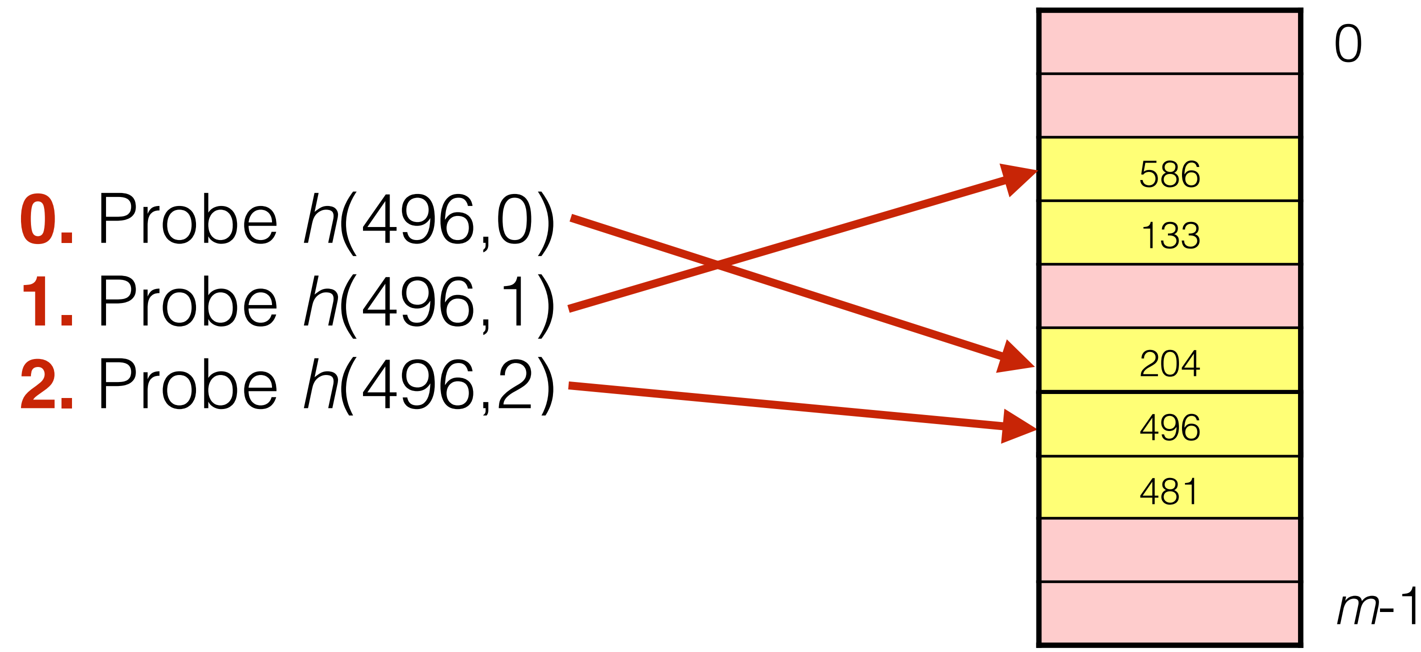
0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$




```
HASH-INSERT(T, k)
i = 0
repeat
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
    else i = i + 1
until i == m
error "hash table overflow"
```

Example

```
HASH-SEARCH(T, k)
i = 0
repeat
    j = h(k, i)
    if T[j] == k
        return j
    i = i + 1
until T[j] == NIL or i == m
return NIL
```



- Search key k = 496
  - Search uses the same probe sequence, terminating successfully if it finds the key and unsuccessfully if it encounters an empty slot (or made it all the way through the list)
- What about delete?
  - Have a special node type: DELETED
  - Note though: search times no longer depend on load factor α
  - Chaining more commonly used when keys must also be deleted

Probing strategies

Linear probing:

- Given an ordinary hash function h'(k), linear probing uses the hash function $h(k,i) = (h'(k) + i) \bmod m$.
- This is a simple computation.
- However, it may suffer from <u>primary clustering</u>, where long runs of occupied slots build up and tend to get longer.
    - empty slot preceded by i full slots gets filled next with probability $(i+1)/m$

Probing strategies

Quadratic probing:

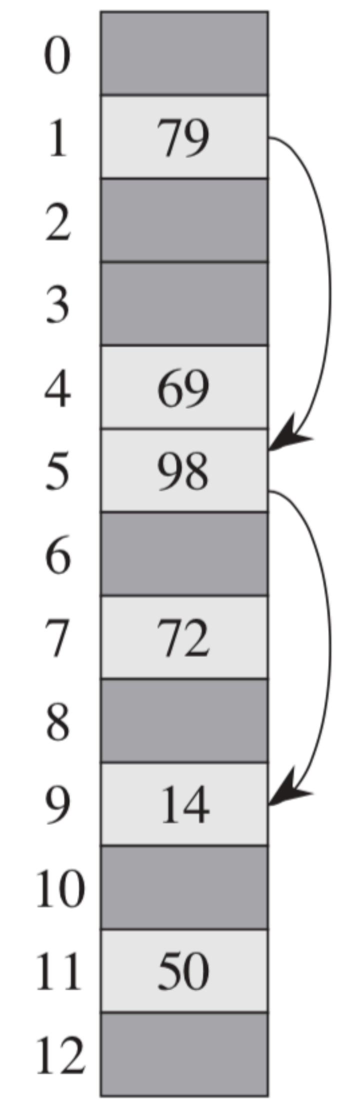
- Quadratic probing uses the hash function $h(k,i) = (h'(k) + c_1 \cdot i + c_2 \cdot i^2) \bmod m$.
- Offset by amount that depends on quadratic manner, works much better than linear probing
- But, it may still suffer from secondary clustering: If two keys have initially the same value, then they also have the same probe sequence
- In addition c_1, c_2, and m need to be constrained to make full use of the hash table

Probing strategies

Double hashing:

- Given two ordinary hash functions $h_1(k)$ and $h_2(k)$, double hashing uses the hash function $h(k,i) = (h_1(k) + i \cdot h_2(k)) \bmod m$.
- The initial probe goes to position $T[h_1(k)]$; successive probe positions are offset by $h_2(k)$ —> the initial probe position, the offset, or both, may vary
- This method generates excellent results, if $h_2(k)$ "relatively prime" to the hash-table size *m*,
  - e.g., by making *m* a power of 2 and design $h_2(k)$ to only produce odd numbers.
  - or let *m* be prime and to design h_2 so that it always returns a positive integer less than *m*, e.g. let *m'* be slightly less than *m*:

$$
\begin{aligned}
h_1(k) &= k \bmod m , \\
h_2(k) &= 1 + (k \bmod m')
\end{aligned}
$$



$h_1(k) = k \bmod 13$
$h_2(k) = 1 + (k \bmod 11)$

—> $k=14$; $h_1(k)=1$, $h_2(k)=4$

—> $k=27$; $h_1(k)=1$, $h_2(k)=6$

Analysis of open addressing

Theorem:

- Assume uniform hashing, i.e., each key is likely to have any one of the m! permutations as its probe sequence.
- Given an open-addressed hash table with load factor $\alpha = n/m < 1$.
- The expected number of probes in an unsuccessful search is, at most, $1/(1-\alpha)$.

Analysis of open addressing

Proof:

- At least, one probe is always necessary.
- With probability n/m, the first probe hits an occupied slot, i.e., a second probe is necessary.
- With probability (n-1)/(m-1), the second probe hits an occupied slot, i.e., a third probe is necessary.
- With probability (n-2)/(m-2), the third probe hits an occupied slot, i.e., a fourth probe is necessary.
- …

Analysis of open addressing

Given that $\frac{n-i}{m-i} < \frac{n}{m} = \alpha$ for $i = 1, 2, \ldots, n$.

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\left(1+\frac{1}{m-n+1}\right)\cdots\right)\right)\right)$$

$$\leq 1+\alpha(1+\alpha(1+\alpha(\cdots(1+\alpha)\cdots)))$$

$$\leq 1+\alpha+\alpha^2+\alpha^3+\cdots$$

$$= \sum_{i=0}^{\infty}\alpha^i$$

$$= \frac{1}{1-\alpha}.$$

Analysis of open addressing

- The successful search takes less number of probes [expected number is $1/\alpha \ln (1/(1-\alpha))$].
- We conclude that if α is constant, then accessing an open-addressed hash table takes constant time.
- For example, if the table is half full, the expected number of probes is $1/(1-0.5) = 2$.
- Or, if the table is 90% full, the expected number of probes is $1/(1-0.9) = 10$.

3.7 Summary

Summary

- Dynamic sets with queries and modifying operations.
- Array: Random access, search in O(lg n), but modifying operations O(n).
- Stack: LIFO only. Operations in O(1).
- Queue: FIFO only. Operations in O(1).
- Linked List: Modifying operations in O(1), but search O(n).
- BST: All operations in O(h).
- Red-black Trees: All operations in O(lg n).
- Heap: All operations in O(lg n).
- Hash Tables: Operations in O(1), but additional storage space.