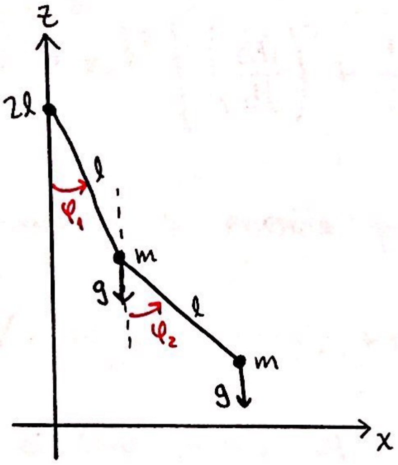
P6] a).



La posición de la masa 1 está descrita por:

$$\vec{r}_1 = l\sin\varphi_1\,\hat{\imath} + l(2-\cos\varphi_2)\,\hat{\jmath}$$

$$\Rightarrow \frac{d\vec{r}_1}{dt} = l\cos\varphi_1 \frac{d\varphi_1}{dt}\,\hat{\imath} + l\sin\varphi_1 \frac{d\varphi_1}{dt}\,\hat{\jmath}$$

$$\Rightarrow \left|\frac{d\vec{r}_1}{dt}\right|^2 = l^2\left[\frac{d\varphi_1}{dt}\right]^2$$

La posición de la masa 2 depende directamente de la masa 1:

$$\vec{r}_2 = l(\sin\varphi_2 + \sin\varphi_1)\,\hat{\imath} + l(2-\cos\varphi_2 - \cos\varphi_1)\,\hat{\jmath}$$

$$\Rightarrow \frac{d\vec{r}_2}{dt} = l\left(\cos\varphi_2\frac{d\varphi_2}{dt} + \cos\varphi_1\frac{d\varphi_1}{dt}\right)\hat{\imath} + l\left(\sin\varphi_2\frac{d\varphi_2}{dt} + \sin\varphi_1\frac{d\varphi_1}{dt}\right)\hat{\jmath}$$

$$\Rightarrow \left|\frac{d\vec{r}_2}{dt}\right|^2 = l^2\left[\left(\cos^2\varphi_2\left(\frac{d\varphi_2}{dt}\right)^2 + \cos^2\varphi_1\left(\frac{d\varphi_1}{dt}\right)^2 + 2\cos\varphi_2\cos\varphi_1\frac{d\varphi_2}{dt}\cdot\frac{d\varphi_1}{dt}\right)\right.$$
$$\left. + \left(\sin^2\varphi_2\left(\frac{d\varphi_2}{dt}\right)^2 + \sin^2\varphi_1\left(\frac{d\varphi_1}{dt}\right)^2 + 2\sin\varphi_2\sin\varphi_1\frac{d\varphi_2}{dt}\cdot\frac{d\varphi_1}{dt}\right)\right]$$
$$= l^2\left[\left(\frac{d\varphi_2}{dt}\right)^2 + \left(\frac{d\varphi_1}{dt}\right)^2 + 2\frac{d\varphi_1}{dt}\cdot\frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right]$$

Conociendo estas cantidades, podemos calcular la energía cinética del sistema:

$$K = \frac{1}{2}m\left(\left|\frac{d\vec{r}_1}{dt}\right|^2 + \left|\frac{d\vec{r}_2}{dt}\right|^2\right)$$
$$= \frac{ml^2}{\cancel{2}}\left[\cancel{2}\left(\frac{d\varphi_1}{dt}\right)^2 + \left(\frac{d\varphi_2}{dt}\right)^2 + \cancel{2}\frac{d\varphi_1}{dt}\cdot\frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right]$$

$$\Rightarrow \; K = ml^2\left[\left(\frac{d\varphi_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\varphi_2}{dt}\right)^2 + \frac{d\varphi_1}{dt}\cdot\frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right]$$

En cuanto a la energía potencial, ubicando $V=0$ en $z=0$:

$$V = mgl(2-\cos\varphi_1) + mgl(2-\cos\varphi_1-\cos\varphi_2).$$

Así, el lagrangiano del sistema queda como sigue:

$$\mathcal{L} = ml^2\left[\left(\frac{d\varphi_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\varphi_2}{dt}\right)^2 + \frac{d\varphi_1}{dt}\cdot\frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right] - 4mgl + 2mgl\cos\varphi_1$$

$$+ mgl\cos\varphi_2 .$$

Cuya derivada $\frac{d\mathcal{L}}{dt} = 0 \longrightarrow$ Hamiltoniano será la energía total del sistema.

Calculamos los momentos de cada ángulo:

$$P_1 = \frac{\partial\mathcal{L}}{d\dot{\varphi}_1} = ml^2\left[2\frac{d\varphi_1}{dt} + \frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right]$$

$$P_2 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_2} = ml^2\left[\frac{d\varphi_2}{dt} + \frac{d\varphi_1}{dt}\cos(\varphi_1-\varphi_2)\right]$$

A lo que despejando:

$$\frac{d\varphi_1}{dt} = \frac{P_1 - P_2\cos(\varphi_1-\varphi_2)}{ml^2(1+\sin^2(\varphi_1-\varphi_2))}$$

$$\frac{d\varphi_2}{dt} = \frac{2P_2 - P_1\cos(\varphi_1-\varphi_2)}{ml^2(1+\sin^2(\varphi_1-\varphi_2))}. \qquad (*)$$

Siendo el Hamiltoniano del sistema $\mathcal{H} = K + V$:

$$\Rightarrow \mathcal{H} = ml^2\left[\left(\frac{d\varphi_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\varphi_2}{dt}\right)^2 + \frac{d\varphi_1}{dt}\cdot\frac{d\varphi_2}{dt}\cos(\varphi_1-\varphi_2)\right]$$

$$+ mgl(2-\cos\varphi_1) + mgl(2-\cos\varphi_1-\cos\varphi_2).$$

Y reemplazando lo obtenido en (*):

$$\Rightarrow \mathcal{H} = ml^2\left[\left(\frac{p_1 - p_2\cos(\varphi_1-\varphi_2)}{ml^2(1+\sin^2(\varphi_1-\varphi_2))}\right)^2 + \frac{1}{2}\left(\frac{2p_2 - p_1\cos(\varphi_1-\varphi_2)}{ml^2(1+\sin^2(\varphi_1-\varphi_2))}\right)^2\right.$$

$$\left. + \frac{(p_1-p_2\cos(\varphi_1-\varphi_2))(2p_2-p_1\cos(\varphi_1-\varphi_2))}{[ml^2(1+\sin^2(\varphi_1-\varphi_2))]^2}\cos(\varphi_1-\varphi_2)\right]$$

$$+ 4mgl - 2mgl\cos\varphi_1 - mgl\cos\varphi_2$$

$$\Rightarrow \mathcal{H} = ml^2\left[\left(\frac{p_1^2 - 2p_1p_2\cos(\varphi_1-\varphi_2) + p_2^2\cos^2(\varphi_1-\varphi_2)}{[ml^2(1+\sin^2(\varphi_1-\varphi_2))]^2}\right)\right.$$

$$+\frac{1}{2}\left(\frac{4p_2^2 - 4p_2p_1\cos(\varphi_1-\varphi_2) + p_1^2\cos^2(\varphi_1-\varphi_2)}{[ml^2(1+\sin^2(\varphi_1-\varphi_2))]^2}\right)$$

$$\left. + \left(\frac{2p_1p_2\cos(\varphi_1-\varphi_2) + p_1^2\cos^2(\varphi_1-\varphi_2) - 2p_2^2\cos^2(\varphi_1-\varphi_2) + p_1p_2\cos^3(\varphi_1-\varphi_2)}{[ml^2(1+\sin^2(\varphi_1-\varphi_2))]^2}\right)\right]$$

$$+ mgl(4-2\cos\varphi_1-\cos\varphi_2).$$

Después de mucha álgebra...

$$\therefore \mathcal{H} = \frac{1}{2ml^2} \cdot \frac{p_1^2 + 2p_2^2 - 2p_1p_2\cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}$$

$$+ mgl(4 - 2\cos\varphi_1 - \cos\varphi_2). \quad \blacksquare$$