

3 | Hartree-Fock Theory

3.1 Hartree-Fock theory and the independent particle approximation

The simplest quantum chemistry method is the Hartree-Fock approach. This method relies on an independent particle approximation, where the wave function is expressed as a product of wave functions of single electron (orbitals). We will indicate orbitals with the lower-case Greek letter phi (ϕ). An orbital is a function of the coordinate vector $\mathbf{r} = (x, y, z)$ for an electron. To distinguish between different orbitals we will label them with an index and write them as

$$\phi_i(\mathbf{r}) \quad (3.1)$$

In the Hartree-Fock approach we approximate the wave function of N electrons as a product of the orbitals for each electron

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \approx \phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2) \cdots \phi_N(\mathbf{r}_N) \quad (3.2)$$

This approximation is convenient because it breaks down the problem of solving the Schrödinger for a wave function of $3N$ coordinates into a simpler problem where we solve N times for wave functions of 3 coordinates.¹ This simplification makes Hartree-Fock theory much more affordable than a direct solution of the Schrödinger equation!

3.2 Spin and the Pauli principle (antisymmetry)

The wave function form of Eq. (3.2) is still missing two important ingredients: it does not account for the spin of electrons and the Pauli exclusion principle.

Spin can be easily introduced by recognizing that each orbital must be accompanied by a spin function, that is, a label for the spin of the electron. As you have learned from previous classes, the electron can have two values of spin: up (alpha, α) and down (beta, β). So if we want to specify that an electron occupies orbital ϕ_i and has spin alpha we write $\phi_i(\mathbf{r})\alpha$, which is equivalent to the arrow notation to

$$\phi_i(\mathbf{r})\alpha \equiv \uparrow \quad (3.3)$$

¹For example, imagine that you represent the wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ using a grid that has 10 points along each dimension. For N electrons, this grid will have a total of 10^{3N} points since Ψ depends on $3N$ coordinates. In comparison, if we represent each orbital with the same grid, each orbital requires only 10^3 grid points, and the total number of grid points is $N10^3$. Even for as little as six electrons the exact wave function required already 10^{18} points, while the Hartree-Fock approach requires only 6000!

For an electron with beta spin we similarly write

$$\phi_i(\mathbf{r})\beta \equiv \downarrow \quad (3.4)$$

When we don not want to be specific about the spin of the electron (and thus allow for both options), we will use instead the **spinorbital** notation based on the Greek letter psi [$\psi(x)$] and use the variable $x = \{\mathbf{r}, \sigma\}$ to represent both the position vector (\mathbf{r}) and one spin coordinate (ω). For the moment, we can think of spin orbitals as always coming in alpha/beta pairs that share the same spatial orbital

$$\psi_i(x) = \phi_i(\mathbf{r})\alpha \quad (3.5)$$

$$\psi_{\bar{i}}(x) = \phi_i(\mathbf{r})\beta \quad (3.6)$$

Now we can be more precise and write the product wave function as

$$\Psi(x_1, x_2, \dots, x_N) \approx \psi_1(x_1)\psi_{\bar{1}}(x_2) \cdots \quad (3.7)$$

Equation (3.7) is still not fully correct. To make sure that the wave function satisfies the Pauli exclusion principle, we have to enforce that the sign of $\Psi(x_1, x_2, \dots, x_N)$ changes if we exchange the labels of any two particles. To illustrate this point, consider two electrons. The wave function $\Psi(x_1, x_2)$ depends on the position and spin of each electron (x_1, x_2), and it must change sign when we replace x_2 with x_1 and vice versa, which means

$$\Psi(x_1, x_2) = -\Psi(x_2, x_1) \quad (3.8)$$

It is easy to write down a wave function built from orbitals that satisfies this condition, it just requires making the product antisymmetric with respect to the coordinates. Such a wave function is called a **Slater determinant**² and for the two electron case is it written as³

$$\Psi_{SD}(x_1, x_2) = |\psi_i \psi_j\rangle = \frac{1}{\sqrt{2}} [\psi_i(x_1)\psi_j(x_2) - \psi_i(x_2)\psi_j(x_1)] \quad (3.9)$$

Slater determinants correspond exactly to the orbital diagrams with orbitals filled with electrons (represented by arrows). For example, the electron configuration with two electrons filling the same orbital corresponds to the following Slater determinant

$$|\psi_1 \psi_{\bar{1}}\rangle = |\phi_1 \alpha \phi_1 \beta\rangle \equiv \begin{array}{c} \text{---} 2 \\ \uparrow\downarrow 1 \end{array} \quad (3.10)$$

This electron arrangement is called a **closed shell**, because all the electrons are paired. It corresponds to a **singlet** electronic state.

However, we could also arrange the electrons in two different orbitals both with alpha spin in an **open-shell** configuration

$$|\psi_1 \psi_2\rangle = |\phi_1 \alpha \phi_2 \alpha\rangle \equiv \begin{array}{c} \uparrow 2 \\ \uparrow 1 \end{array} \quad (3.11)$$

² The Slater determinant is named for John C. Slater, who introduced the determinant in 1929 as a means of ensuring the antisymmetry of a many-electron wave function, although the wave function in the determinant form first appeared independently in Heisenberg's and Dirac's articles three years earlier. (Source: Wikipedia)



Fig. 3.1: John C. Slater.

³ Here I use the symbol $|\psi_i \psi_j \cdots\rangle$ to indicate a general Slater determinant.

In this state, the spin of the electrons add up and we get what is called a **triplet state**.

We have covered a lot of details in this section, so I want to make sure that the important message comes across: **in the Hartree-Fock method the wave function of many electrons is approximated with a single Slater determinant**. This approach satisfies the Pauli exclusion principle and it has a direct connection to the familiar orbital diagrams.

3.3 The Hartree-Fock equation

The main goal of the Hartree-Fock method is to find the orbitals that give the “best” possible wave function. Here by “best” we mean precisely the one that minimizes the expectation value of the energy, which is given by the integral

$$E_{\text{SD}} = \int \Psi_{\text{SD}}^* \hat{H} \Psi_{\text{SD}} \quad (3.12)$$

The **variational** theorem guarantees that the energy of an approximate wave function $\tilde{\Psi}$ is always greater than the exact energy, $\tilde{E} \geq E_{\text{exact}}$, and the closer we get to the exact energy E_{exact} , the better our wave function $\tilde{\Psi}$ approximate the exact solution Ψ_{exact} .

The orbitals that minimize the energy satisfy the **Hartree-Fock equations**

$$\hat{f}\psi_i(x) = \epsilon_i\psi_i(x), \quad \text{for all } \psi_i(x) \quad (3.13)$$

where \hat{f} is the **Fock operator**. This is an eigenvalue equation like the Schrödinger equation, but it is simpler because it only involves one orbital at the time (a functions of 3 coordinates only). However, it is important to remember that this equation is valid only in the Hartree-Fock approximation. The quantity ϵ_i is called the **orbital energy** and it is the eigenvalue corresponding to the orbital ψ_i .

The physical interpretation of the Hartree-Fock equation is that we can approximate the interaction (repulsion) of an electron in orbital ψ_i with all the remaining $N - 1$ electrons via an average potential. This potential is included in the Fock operator

$$\hat{f} = \underbrace{-\frac{1}{2}\nabla^2}_{\text{kinetic}} + \underbrace{-\sum_i^{\text{nuclei}} \frac{Z_i}{|\mathbf{r} - \mathbf{R}_i|}}_{\text{electron-nuclear}} + \underbrace{\int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}}_{\text{Coulomb interaction}} + \text{exchange interaction} \quad (3.14)$$

in the terms that depend on the density of electrons (ρ). The electron density is in turn given by the sum of the square of the occupied orbitals, which for a closed-shell molecule is given by

$$\rho(\mathbf{r}) = 2 \sum_i^{\text{occupied}} |\phi(\mathbf{r})|^2 \quad (3.15)$$

The Hartree-Fock equations and these last two equations tell us that to find the orbital we need to know \hat{f} , but to know \hat{f} we need to know the density, which is determined by the orbitals. In practice this means that the Hartree-Fock equations have to be solved via an iterative **self-consistent-field (SCF) procedure**.

This procedure consists of the following steps:

1. Forming an initial guess. At beginning of an SCF procedure we need to start from orbitals that are close to the optimal ones. Common ways to guess the orbitals include neglecting electron repulsion or forming the average potential from a superposition of atomic densities.
2. Updating the Fock matrix. Using the current set of orbitals, the average potential and the Fock operator are built.
3. Determining the orbitals and orbital energies. Using the current Fock matrix, the Hartree-Fock equations are solved to obtain one set of orbitals and orbital energies and an updated value for the total energy.
4. Convergence check. The program checks if the change in energy and orbitals with respect to the previous iteration is less than a predefined convergence threshold. If the computation has converged, we stop. Otherwise, we go back to step 2. and use the new set of orbital to compute a new Fock operator.

The following is the output of a Hartree-Fock computation on water. In this example, the iterations start with a guess of the density given by the superposition of atomic densities (SAD)⁴. As the iterations progress, the energy becomes lower and lower (in accordance with the variational principle) and we reach convergence in eight iterations to the final value $-76.02665366185 E_h$. Note that in the last step the energy is converged to $-5.4 \times 10^{-10} E_h$.⁵

⁴Computational chemists love acronyms. With time you will become conversant with this language.

⁵Quantum chemistry computations are usually converged to high accuracy. It is not uncommon to see the energy optimized to less than 10^{-7} – $10^{-8} E_h$, and in certain cases down to even $10^{-12} E_h$.

The quantity labeled “RMS |[F,P]|” measures the derivative of the energy, and it is another way to monitor the convergence (remember that at a minimum, the gradient is zero). The label “DIIS”, indicates that the code is using the direct inversion of the iterative subspace (DIIS) algorithm to accelerate the convergence.

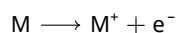
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Water/Hartree-Fock
==> Iterations <==
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	Total Energy	Delta E	RMS [F,P]
@RHF iter SAD:	-75.50773245935	-7.55077e+01	0.00000e+00
@RHF iter 1:	-75.95378451304	-4.46052e-01	3.03072e-02 DIIS
@RHF iter 2:	-76.00708060915	-5.32961e-02	1.73589e-02 DIIS
@RHF iter 3:	-76.02624491762	-1.91643e-02	1.58426e-03 DIIS
@RHF iter 4:	-76.02663310976	-3.88192e-04	3.65265e-04 DIIS
@RHF iter 5:	-76.02665270339	-1.95936e-05	6.70876e-05 DIIS
@RHF iter 6:	-76.02665363515	-9.31757e-07	1.06688e-05 DIIS
@RHF iter 7:	-76.02665366131	-2.61654e-08	1.50538e-06 DIIS
@RHF iter 8:	-76.02665366185	-5.39330e-10	3.50179e-07 DIIS

Energy and wave function converged.

3.4 Interpretation of the orbital energies

It is important to understand the precise meaning of the orbital energies ϵ_i because it is slightly different than our intuition as chemists. **Koopmans' theorem** shows that the orbital energies are related to the energy necessary to remove or add electrons to an orbital. More precisely, for orbitals that are occupied by electrons, Koopmans' theorem says that $-\epsilon_i$ correspond to the energy necessary to remove one electron (ionize) from the system

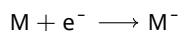


Koopmans' theorem can be stated as

$$\text{Ionization potential for electron in } \psi_i = IP_i = E^{N-1} - E^N = -\epsilon_i. \quad (3.16)$$

If an occupied orbital has a negative energy, then we have to put energy into a molecule to remove that electron (the IP is positive).

Similarly, for orbitals that are not occupied, the orbital energy ϵ_a corresponds to the energy released when an electron is added to form an anion (if the starting molecule is neutral)



In this case, Koopmans' theorem helps us quantify the electron affinity

$$\text{Electron affinity for electron in } \psi_a = EA_a = E^N - E^{N+1} = -\epsilon_a. \quad (3.17)$$

Therefore, if an unoccupied orbital has a negative energy (positive EA), a molecule will attach an electron and form a stable negative ion. If instead ϵ_a is positive, a molecule will not accept an electron, and its anion will be unstable with respect to self ionization (the anion will spontaneously loose an electron in a finite amount of time).

Koopmans' theorem assumes that during the electron ionization/attachment the orbitals do not change, which means that it ignores relaxation effects to the addition or removal of electrons. Koopmans' theorem also neglects electron correlation, so the IP and EA will deviate from experiment. This means that in practice Koopmans' theorem is not a very accurate way to compute the IP and EA of molecules. Nevertheless, it gives physical meaning to the orbital energies and it can be used to qualitatively estimate the IP and EA of molecules.

The importance of Koopmans' theorem is that it build a bridge to experiments that measure the IP and EA of molecules. For example, photoelectron spectroscopy, can measure the ionization potential (also called the binding energy) of electrons in different orbitals. In this spectroscopy, a molecule is ionized with a high energy photon and one electron is released with kinetic energy equal to the energy of the photons ($h\nu$) minus the binding energy, $h\nu - IP$. Therefore, it allows to measure the binding energy of different orbitals. An example of the photoelectron spectrum of water in the gas phase is given in Fig. 3.3.

The example below shows the Hartree-Fock orbital energies of the occupied and unoccupied orbitals of an isolated water molecule in the gas phase.

Water/Hartree-Fock

Orbital Energies [Eh]

Doubly Occupied:

1A1	-20.550918	2A1	-1.335304	1B2	-0.697799
3A1	-0.566090	1B1	-0.492954		

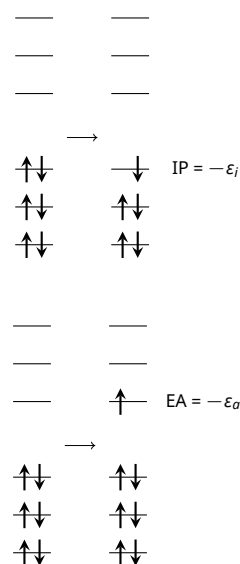


Fig. 3.2: Illustration of Koopmans' theorem.

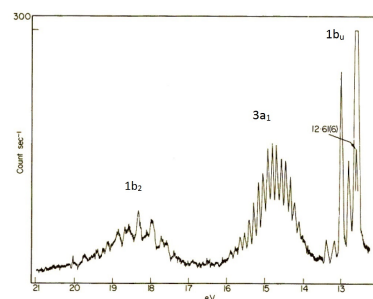


Fig. 3.3: Photoionization spectrum of water in the gas phase (in eV). Source: Wikimedia Commons.

Virtual:

4A1	0.185103	2B2	0.255850	3B2	0.787301
5A1	0.851798	6A1	1.163709	2B1	1.200353
4B2	1.253480	7A1	1.444918	1A2	1.475588
3B1	1.674083	8A1	1.867861	5B2	1.931955
6B2	2.446380	9A1	2.483524	4B1	3.283306
2A2	3.336170	10A1	3.506961	11A1	3.862825
7B2	4.144454				

Final Occupation by Irrep:

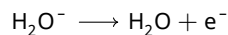
	A1	A2	B1	B2
DOCC [3,	0,	1,	1]

@RHF Final Energy: -76.02665366185380

=> Energetics <=

Nuclear Repulsion Energy =	9.1681932964243487
One-Electron Energy =	-123.1035625229022514
Two-Electron Energy =	37.9087155646241101
Total Energy =	-76.0266536618537998

The lowest ionization potential of water is estimated according to Koopmans' theorem from the energy of the highest occupied MO (HOMO, labeled $1b_1$), which has an energy of $-0.493 E_h$. The corresponding IP is equal to $0.493 E_h = 13.4$ eV, a value that compares well with the experimental gas-phase IP (ca. 12.6 eV, indicated in the figure). The ionization potential for the second orbital (labeled $3a_1$) is $0.566 E_h = 15.4$ eV. This corresponds to the middle broad band in the spectrum centered around 15 eV. Note that Koopmans' theorem applied to the Hartree-Fock solution also predicts that the anion H_2O^- is unstable with respect to the process ionization process



since the energy of the first unoccupied orbital (virtual) is positive ($0.185 E_h$).