MAT137 PS6 2

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Claim 1. *Let $a, b \in \mathbb{R}$ with $a < b$ and let f be a bounded function on $[a, b]$. If $\forall\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U_p(f) - L_p(f) < \epsilon$, then f is integrable on $[a, b]$.*

Proof. We begin by assuming the hypothesis of the Claim 1 and declaring the variables in Claim 1. The requirement for f to be integrable on $[a, b]$ is $\underline{I_a^b} = \overline{I_a^b}$, where $\underline{I_a^b}$ is the supremum of all the lower sums of f for every partition of $[a, b]$, and $\overline{I_a^b}$ is the infimum of all the upper sums of f for every partition of $[a, b]$. Since f is a bounded function on $[a, b]$, f is bounded on every subinterval of P. This means that an infimum and a supremum of f on any subinterval of P exist and thus, the set of all $U_p(f)$ and $L_p(f)$ for any partition P is bounded. It follows that we are guaranteed $\underline{I_a^b}$ and $\overline{I_a^b}$ exist. We want to show that $\underline{I_a^b} = \overline{I_a^b}$.

By definition, we know that

$$L_p(f) \leq \underline{I_a^b} \leq \overline{I_a^b} \leq U_p(f) \tag{1}$$

for any partition P. Remembering the assumptions of Claim 1, we can derive the following inequality:

$$\begin{align}
\overline{I_a^b} &\leq U_p(f) \tag{2}\\
L_p(f) &\leq \underline{I_a^b} \tag{3}\\
\overline{I_a^b} + L_p(f) &\leq U_p(f) + \underline{I_a^b} \text{ (by (2) + (3))} \tag{4}\\
\overline{I_a^b} - \underline{I_a^b} &\leq U_p(f) - L_p(f) \text{ (after rearranging)} \tag{5}\\
&\implies \forall\epsilon > 0, \overline{I_a^b} - \underline{I_a^b} < \epsilon \text{ (by our assumption).} \tag{6}
\end{align}$$

Now, we want to show that $\underline{I_a^b} = \overline{I_a^b}$ by contradiction. Assume that $\underline{I_a^b} \neq \overline{I_a^b}$. One of the cases where this is true is when $\underline{I_a^b} > \overline{I_a^b}$. However, because of (1), we know this cannot be true. Now, all that is left is when $\underline{I_a^b} < \overline{I_a^b}$. This case would mean that $\exists L \in \mathbb{R}$ s.t. $L > 0 \wedge \overline{I_a^b} - \underline{I_a^b} = L$, a contradiction to (6). Thus, $\underline{I_a^b} > \overline{I_a^b}$ and $\underline{I_a^b} < \overline{I_a^b}$ both cannot be true so we conclude that indeed, $\underline{I_a^b} = \overline{I_a^b}$. This is also equivalent to proving that f, under our assumptions, is integrable on the interval $[a, b]$. $\square$