MAT137 PS6 4

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Claim 1. *Let $a, b \in \mathbb{R}$ with $a < b$. If f is a function that is continuous on $[a, b]$ and differentiable on (a, b) and f' is bounded on (a, b), then f is integrable on $[a, b]$.*

Proof. Assume the hypothesis of Claim 1. We want to show that

$$\forall \epsilon > 0, \text{ there exists a partition } P \text{ of } [a, b] \text{ s.t. } U_p(f) - L_p(f) < \epsilon. \tag{1}$$

By our assumption, f is bounded on $[a, b]$. Therefore, f meets the requirement for (1) to mean that f is integrable on $[a, b]$ (as we proved in PS6 2).

As well, with our assumptions, we know that f is continuous on $[a, b]$, differentiable on (a, b), and f' is bounded on (a, b) (it meets the requirements for our claims in PS6 3). Since f' is bounded on (a, b), f' has an infimum and a supremum on (a, b). It follows that we can take

$$C = \max\{|\sup_{x \in (a,b)} f'(x)|, |\inf_{x \in (a,b)} f'(x)|, 1\}. \tag{2}$$

Let P be an arbitrary partition of $[a, b], P = \{x_0, x_1, \ldots, x_n\},$ where $n \in \mathbb{N}$. Let $i \in \{1, 2, \ldots, n\}$. We know that our choice of C makes it so $\forall s, t \in [x_{i-1}, x_i], |f(s) - f(t)| < C|s - t|$ (as we proved in PS6 3a). A consequence of this is that the difference between the area of the i^{th} rectangle of the $P - \textit{upper}$ sum and the i^{th} rectangle of the $P - \textit{lower}$ sum is at most $C\Delta x_i^2$ (as we proved in PS6 3b). Put formally, let $U_i, L_i \in [x_{i-1}, x_i]$. If $f(U_i)$ and $f(L_i)$ are the heights of the the i^{th} rectangle of the $P - \textit{upper}$ sum and $P - \textit{lower}$ sum, respectively, then

$$f(U_i)\Delta x_i - f(L_i)\Delta x_i \leq C\Delta x_i^2. \tag{3}$$

Now, $\forall \epsilon > 0$, let

$$L = \frac{b-a}{\left\lceil \frac{(b-a)^2 C}{\epsilon} + 1 \right\rceil}. \tag{4}$$

We know from (2) that $C > 0$. Likewise, $(b > a \implies b - a > 0) \land \epsilon > 0$ so our choice of L is valid (the denominator is greater than 0). Let $n = \left\lceil \frac{b-a}{L} \right\rceil = \left\lceil \frac{(b-a)^2 C}{\epsilon} + 1 \right\rceil$. Define the partition P of $[a, b]$ to be $P = \{x_0, x_1, x_2, \dots, x_n\} = \{a, a+L, a+2L, \dots, b\}$ and let $\Delta x_i = x_i - x_{i-1},\ \text{where } i \in \{1, 2, \dots, n\}$. We know that $\Delta x_i = L$ for all i from 1 through n because $\frac{b-a}{L}$ is equal to a natural number (thus the interval $[a, b]$ is split up into sections of the same length, L, in P). Since (3) holds true for our partition P (as discussed before (3)), it can be used to show

$$\begin{aligned}
& f(U_i)\Delta x_i - f(L_i)\Delta x_i \leq C\Delta x_i^2 \text{ (the original inequality)} && (5)\\
& f(U_i)\Delta x_i - f(L_i)\Delta x_i \leq CL^2 \text{ (since } \Delta x_i = L) && (6)\\
\implies & \sum_{i=1}^{n} [f(U_i)\Delta x_i - f(L_i)\Delta x_i] \leq \sum_{i=1}^{n} CL^2 \text{ (since (3) still holds true)} && (7)\\
\implies & \sum_{i=1}^{n} f(U_i)\Delta x_i - \sum_{i=1}^{n} f(L_i)\Delta x_i \leq nCL^2 \text{ (expanding/simplifying)} && (8)\\
\implies & U_p(f) - L_p(f) \leq nCL^2. && (9)
\end{aligned}$$

Now, we will show that $nCL^2 < \epsilon$.

$$\begin{aligned}
nCL^2 &= \left\lceil \frac{(b-a)^2 C}{\epsilon} + 1 \right\rceil \left[\frac{b-a}{\left\lceil \frac{(b-a)^2 C}{\epsilon} + 1 \right\rceil} \right]^2 C \text{ (subbing in variables)} && (10)\\
&= \frac{(b-a)^2}{\left\lceil \frac{(b-a)^2 C}{\epsilon} + 1 \right\rceil} C \text{ (after simplifying).} && (11)
\end{aligned}$$

We know that the following inequality must be true because we are adding

1 and using the ceil function on the right side:

$$\frac{(b-a)^2C}{\epsilon} < \left\lceil \frac{(b-a)^2C}{\epsilon} + 1 \right\rceil \tag{12}$$

$$\implies nCL^2 = \frac{(b-a)^2}{\left\lceil \frac{(b-a)^2C}{\epsilon} + 1 \right\rceil} C < \epsilon \tag{13}$$

$$\implies nCL^2 < \epsilon. \tag{14}$$

We got to (13) after multiplying (12) by ϵ and dividing (12) by $\left\lceil \frac{(b-a)^2C}{\epsilon} + 1 \right\rceil$. This is valid since they are both greater than 0. Finally, using (9) with (14) we can construct the final inequality,

$$U_p(f) - L_p(f) \leq nCL^2 < \epsilon \tag{15}$$

$$\implies U_p(f) - L_p(f) < \epsilon \tag{16}$$

which applies to all $\epsilon > 0$. Using our defined P, we've proven (1) and this is equivalent to proving, under our assumptions, f is integrable on $[a, b]$ (as we've shown in PS6 2). $\square$