# MAT137 PS6 3a

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Let $a, b \in \mathbb{R}$ with $a < b$ and assume the following:

- $f$ is a function that is continuous on $[a, b]$
- $f$ is differentiable on $(a, b)$ and $f'$ is bounded on $(a, b)$
- $P = \{x_0, x_1, x_2, \ldots, x_n\}$ (where $n \in \mathbb{N}$) is a partition of $[a, b]$

Let $i \in \{1, 2, \ldots, n\}$ be given. The $i^{th}$ rectangle of the $P-$*upper* sum (or the $P-$*lower* sum) corresponds to the rectangle above the interval $[x_{i-1}, x_i]$.

**Claim 1.** *We claim that* $\exists C > 0$ *which does not depend on* $i$ *such that* $\forall s, t \in [x_{i-1}, x_i], |f(s) - f(t)| \leq C|s - t|$.

*Proof.* We begin by declaring all of the above variables and assuming they meet their outlined characteristics. Assume that $i$ has been given like above. We know that $f'$ is bounded on $(a, b) \implies f'$ has an infimum and supremum on $(a, b)$. Let

$$C = \max\{|\sup_{x \in (a,b)} f'(x)|, |\inf_{x \in (a,b)} f'(x)|, 1\}. \tag{1}$$

We will show that this is the required $C$. It is clear that $C > 0$ since the lowest value $C$ can take is 1. We know that given any interval $[x_{i-1}, x_i]$ from partition $P$, $[x_{i-1}, x_i] \subseteq [a, b]$ because $P$ is a partition of $[a, b]$. Since $f$ on $[a, b]$ satisfies the requirements of the Mean Value Theorem (as it is described above), so does $f$ on $[x_{i-1}, x_i]$. With the same logic, $f$ on all subintervals of $[x_{i-1}, x_i]$ satisfy the requirements of the Mean Value Theorem. Thus, by the Mean Value Theorem, $\forall s, t \in [x_{i-1}, x_i]$,

$$t > s \implies \exists c \in (s, t) \text{ s.t. } f'(c) = \frac{f(t) - f(s)}{t - s}. \tag{2}$$

The conclusion of (2) can be rewritten to show (using the same variables as above)

$$\exists c \in (s,t) \text{ s.t. } f'(c)(t-s) = f(t) - f(s) \tag{3}$$
$$\implies \exists c \in (s,t) \text{ s.t. } |f'(c)||t-s| = |f(t) - f(s)|. \tag{4}$$

We know that $c \in (s,t) \subseteq (a,b)$, thus, by definition, $\inf_{x\in(a,b)} f'(x) \leq f'(c) \leq \sup_{x\in(a,b)} f'(x)$. It follows that, by (1), $|f'(c)| \leq C$. Finally, rewriting all of (2) and using a modified version of (4) gives us: $\forall s,t \in [x_{i-1}, x_i], t > s$

$$\implies \exists c \in (s,t) \text{ s.t. } |f(t) - f(s)| = |f'(c)||t-s| \leq C|t-s| \tag{5}$$
$$\implies \exists c \in (s,t) \text{ s.t. } |f(t) - f(s)| \leq C|t-s|. \tag{6}$$

(5) holds true because $C > 0$ and moreover, $C \geq |f'(c)|$. (6) concludes that, $\forall s,t \in [x_{i-1}, x_i], t > s \implies |f(t)-f(s)| \leq C|t-s|$, which is a portion of what we wanted to show. Observe that $\forall s,t \in [x_{i-1}, x_i], s = t \implies |f(t)-f(s)| \leq C|t-s|$ as well. In this case, it is clear that $|f(t) - f(s)| = 0$ and $|t-s| = 0$ since $s = t$, so the conclusion results in $0 \leq 0$ which is true. Finally, using the same variables, we can rewrite (6) to show that

$$\exists c \in (s,t) \text{ s.t. } |-(-f(t) + f(s))| \leq C|-(-t+s)| \tag{7}$$
$$\implies \exists c \in (s,t) \text{ s.t. } |f(s) - f(t)| \leq C|s-t| \text{ (after simplifying).} \tag{8}$$

With (8), we have shown that $\forall s,t \in [x_{i-1}, x_i], t > s \implies |f(s) - f(t)| \leq C|s-t|$ as well. Combining all 3 cases from above, we have proved $\forall s,t \in [x_{i-1}, x_i], |f(s) - f(t)| \leq C|s-t|$ and indeed, $C$ is a constant that does not depend on $i$; $C = \max\{|\sup_{x\in(a,b)} f'(x)|, |\inf_{x\in(a,b)} f'(x)|, 1\}$. $\square$