MAT137 PS6 3b

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Let $a, b \in \mathbb{R}$ with $a < b$ and assume the following:

- f is a function that is continuous on $[a, b]$
- f is differentiable on (a, b) and f' is bounded on (a, b)
- $P = \{x_0, x_1, x_2, \ldots, x_n\}$ (where $n \in \mathbb{N}$) is a partition of $[a, b]$

Let $i \in \{1, 2, \ldots, n\}$ be given. The i^{th} rectangle of the $P - upper$ sum (or the $P - lower$ sum) corresponds to the rectangle above the interval $[x_{i-1}, x_i]$.

Claim 1. *We claim that the difference between the area of the i^{th} rectangle of the $P - upper$ sum and the i^{th} rectangle of the $P - lower$ sum is at most $C\Delta x_i^2$.*

Proof. Assume all of the variables and functions have been declared like above. Let $C = \max\{|\sup_{x\in(a,b)} f'(x)|, |\inf_{x\in(a,b)} f'(x)|, 1\}$ and let i be given as described above. We know from PS6 3a that $\forall s, t \in [x_{i-1}, x_i], |f(s) - f(t)| \leq C|s-t|$. Declare Δx_i to be $x_i - x_{i-1}$. As well, let $L_i, U_i \in [x_{i-1}, x_i]$ and suppose that $f(L_i)$ represents the height of the i^{th} rectangle of the $P - lower$ sum and $f(U_i)$ represents the height of the i^{th} rectangle of the $P - upper$ sum. Since $L_i, U_i \in [x_{i-1}, x_i]$, we know that

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\begin{aligned}
|f(U_i) - f(L_i)| &\leq C|U_i - L_i| && (1)\\
\implies f(U_i) - f(L_i) &\leq C|U_i - L_i| && (2)\\
\implies f(U_i)\Delta x_i - f(L_i)\Delta x_i &\leq C|U_i - L_i|\Delta x_i. && (3)
\end{aligned}
$$

We got to (3) from (2) after multiplying (2) by Δx_i. (3) holds true because $x_i > x_{i-1} \implies \Delta x_i > 0$. Observe that $f(U_i)\Delta x_i - f(L_i)\Delta x_i$ is the difference between the area of the i^{th} rectangle of the $P-upper$ sum and the i^{th} rectangle of the $P - lower$ sum and it is being bounded from above by $C|U_i - L_i|\Delta x_i$.

It is clear that that value of U_i and L_i from $[x_{i-1}, x_i]$ affect the inequality's right hand side. In particular, when $(U_i = x_i \wedge L_i = x_{i-1}) \vee (U_i = x_{i-1} \wedge L_i = x_i)$, the right hand side is equal to $C|x_i - x_{i-1}|\Delta x_i = C|x_{i-1} - x_i|\Delta x_i = C\Delta x_i^2$. We will show that this is the largest value it can take. We know that $\forall U_i, L_i \in [x_{i-1}, x_i], |U_i - L_i| \leq \Delta x_i = x_i - x_{i-1}$. Therefore, the largest value $|U_i - L_i|$ can take is $|U_i - L_i| = \Delta x_i$. Since in $C|U_i - L_i|\Delta x_i$ for any interval $[x_{i-1}, x_i]$, C and Δx_i are fixed, its value is maximized when $|U_i - L_i| = \Delta x_i$ (in other words, when $|U_i - L_i|$ is maximized). Therefore, in the cases described above, when $(U_i = x_i \wedge L_i = x_{i-1}) \vee (U_i = x_{i-1} \wedge L_i = x_i)$,

$$f(U_i)\Delta x_i - f(L_i)\Delta x_i \leq C\Delta x_i^2. \tag{4}$$

Since (5) shows that when the difference between the area of the i^{th} rectangle of the $P-upper$ sum and the i^{th} rectangle of the $P-lower$ sum is maximized, it is being bounded from above by $C\Delta x_i^2$, we have proven Claim 1. $\square$