The Gambler's Ruin Problem

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1 Problem Statement

In the Gambler's Ruin Problem, we have a gambler A and a casino B, who are playing a game against each other. The total combined fortune of the two is k dollars, with the gambler starting with i dollars, and the casino starting with $k - i$ dollars, where i and $k - i$ are known positive integers. On each play of the game, the probability that A will win one dollar from B is a known value $p \in (0, 1]$, and the probability that B will win one dollar from A is $1 - p$.

Suppose that the game is played repeatedly (and independently) until the fortune of either A or B is reduced to 0 dollars.

2 Problem Solution

Let a_i denote the probability that gambler A will reach k dollars before they reach 0 dollars, given that their initial fortune is i dollars. Since each play of the game is independent of the others, we can think of the problem essentially starts over on each play, with the only difference being that the "initial" fortunes of the gambler and casino have changed. The value of interest is a_i for $i \in \{0, 1, \ldots, k - 1, k\}$.

2.1 Solution for $i \in \{0, k\}$

The cases of $i = 0$ and $i = k$ are trivial. When A runs out of money they can no longer play, and thus $a_0 = 0$, and when A wins all k dollars, the casino can no longer play, and thus $a_k = 1$. Finding a_i when $i \notin \{0, k\}$ is nontrivial and is solved in Section 2.2 below.

2.2 Solution for $i \in \{1, 2, \ldots, k - 2, k - 1\}$

Define the following events:

- A_1 is the event in which the gambler wins one dollar (i.e., the casino loses one dollar) on the first play of the game,
- B_1 is the event in which the casino wins one dollar (i.e., the gambler loses one dollar) on the first play of the game, and
- W is the event in which the gambler wins all k dollars before reaching 0 dollars.

By the Law of Total Probability, we see that

$$\begin{aligned} \mathrm{P}(W) &= \mathrm{P}(A_1)\mathrm{P}(W|A_1) + \mathrm{P}(B_1)\mathrm{P}(W|B_1) \\ &= p\mathrm{P}(W|A_1) + (1-p)\mathrm{P}(W|B_1). \end{aligned} \tag{1}$$

Since the gambler starts with i dollars, we see that $\mathrm{P}(W) = a_i$, as defined earlier. If the gambler wins on the first play, they'd now have $i+1$ dollars, and by our assumption that each game play is independent, we see that $\mathrm{P}(W|A_1) = a_{i+1}$. Similarly, if the gambler loses on the first play they'd now have $i-1$ dollars, and therefore $\mathrm{P}(W|B_1) = a_{i-1}$.

By Equation 1, we find

$$a_i = pa_{i+1} + (1-p)a_{i-1}. \tag{2}$$

By plugging in $i = 1, 2, \ldots, k-2, k-1$ into Equation 2, we get $k-1$ equations

$$\begin{aligned} a_1 &= pa_2 + (1-p)a_0 = pa_2 \\ a_2 &= pa_3 + (1-p)a_1 \\ &\vdots \\ a_{k-2} &= pa_{k-1} + (1-p)a_{k-3} \\ a_{k-1} &= pa_k + (1-p)a_{k-2} = p + (1-p)a_{k-2}, \end{aligned} \tag{3}$$

where we can simplify our equations for a_1 and $a_k - 1$ by using $a_0 = 0$ and $a_k = 1$, as defined in Section 2.1.

We can rewrite these $k-1$ equations as

$$\begin{aligned}
a_2 - a_1 &= \frac{1-p}{p}(a_1 - a_0) = \left(\frac{1-p}{p}\right) a_1 \\
a_3 - a_2 &= \frac{1-p}{p}(a_2 - a_1) = \left(\frac{1-p}{p}\right)^2 a_1 \\
a_4 - a_3 &= \frac{1-p}{p}(a_3 - a_2) = \left(\frac{1-p}{p}\right)^3 a_1 \\
&\vdots \\
a_{k-1} - a_{k-2} &= \frac{1-p}{p}(a_{k-2} - a_{k-3}) = \left(\frac{1-p}{p}\right)^{k-2} a_1 \\
a_k - a_{k-1} = 1 - a_{k-1} &= \frac{1-p}{p}(a_{k-1} - a_{k-2}) = \left(\frac{1-p}{p}\right)^{k-1} a_1
\end{aligned} \tag{4}$$

and by equating the sum of the left sides of these equations with the sum of the right sides of these equations, we find

$$1 - a_1 = a_1 \sum_{i=1}^{k-1} \left(\frac{1-p}{p}\right)^i. \tag{5}$$

2.2.1 Solution When $p = 0.5$

When $p = 0.5$ (i.e., when the game is fair), we use Equation 5 to solve for a_1,

$$1 - a_1 = a_1 \sum_{i=1}^{k-1} \left(\frac{1-0.5}{0.5}\right)^i = a_1 \sum_{i=1}^{k-1} (1)^i = a_1 (k-1) \qquad \Rightarrow \qquad a_1 = \frac{1}{k}. \tag{6}$$

We can then use the recurrence relation in Equation 4 to recursively solve for $a_2, \ldots, a_{k-1}$, from which we find

$$a_i = \frac{i}{k} \quad \text{for} \quad i \in \{1, 2, \ldots, k-2, k-1\}. \tag{7}$$

2.2.2 Solution When $p \neq 0.5$

When $p \neq 0.5$ (i.e., when the odds of the game are skewed in favor of either the gambler or the casino), we can rewrite Equation 5 as

$$1 - a_1 = a_1 \frac{\left(\frac{1-p}{p}\right)^k - \left(\frac{1-p}{p}\right)}{\left(\frac{1-p}{p}\right) - 1}, \tag{8}$$

and then solving for a_1 we find

$$a_1 = \frac{\left(\frac{1-p}{p}\right) - 1}{\left(\frac{1-p}{p}\right)^k - 1}. \tag{9}$$

In an identical method to that used in Section 2.2.1, we use the recurrence relation in Equation 4 to recursively solve for $a_2, \ldots, a_{k-1}$, from which we find

$$a_i = \frac{\left(\frac{1-p}{p}\right)^i - 1}{\left(\frac{1-p}{p}\right)^k - 1} \quad \text{for} \quad i \in \{1, 2, \ldots, k-2, k-1\}. \tag{10}$$

3 Solution Summary

Given the probability $p \in (0, 1]$ that the gambler will win each play of the game, and given the initial fortune i of the gambler and the initial fortune $k - i$ of the casino, we can calculate the probability that each of them would the the first to run out of money.

The probability a_i that gambler will reach k dollars before they reach 0 dollars, given that they start with i dollars is given by

$$a_i = \begin{cases} \frac{i}{k} & p = 0.5 \\ \frac{\left(\frac{1-p}{p}\right)^i - 1}{\left(\frac{1-p}{p}\right)^k - 1} & p \neq 0.5 \end{cases} \quad \text{for} \quad i \in \{0, 1, \ldots, k-1, k\}. \tag{11}$$

For select values of k, plots of a_i as a function of i and p are shown in Figure 1.

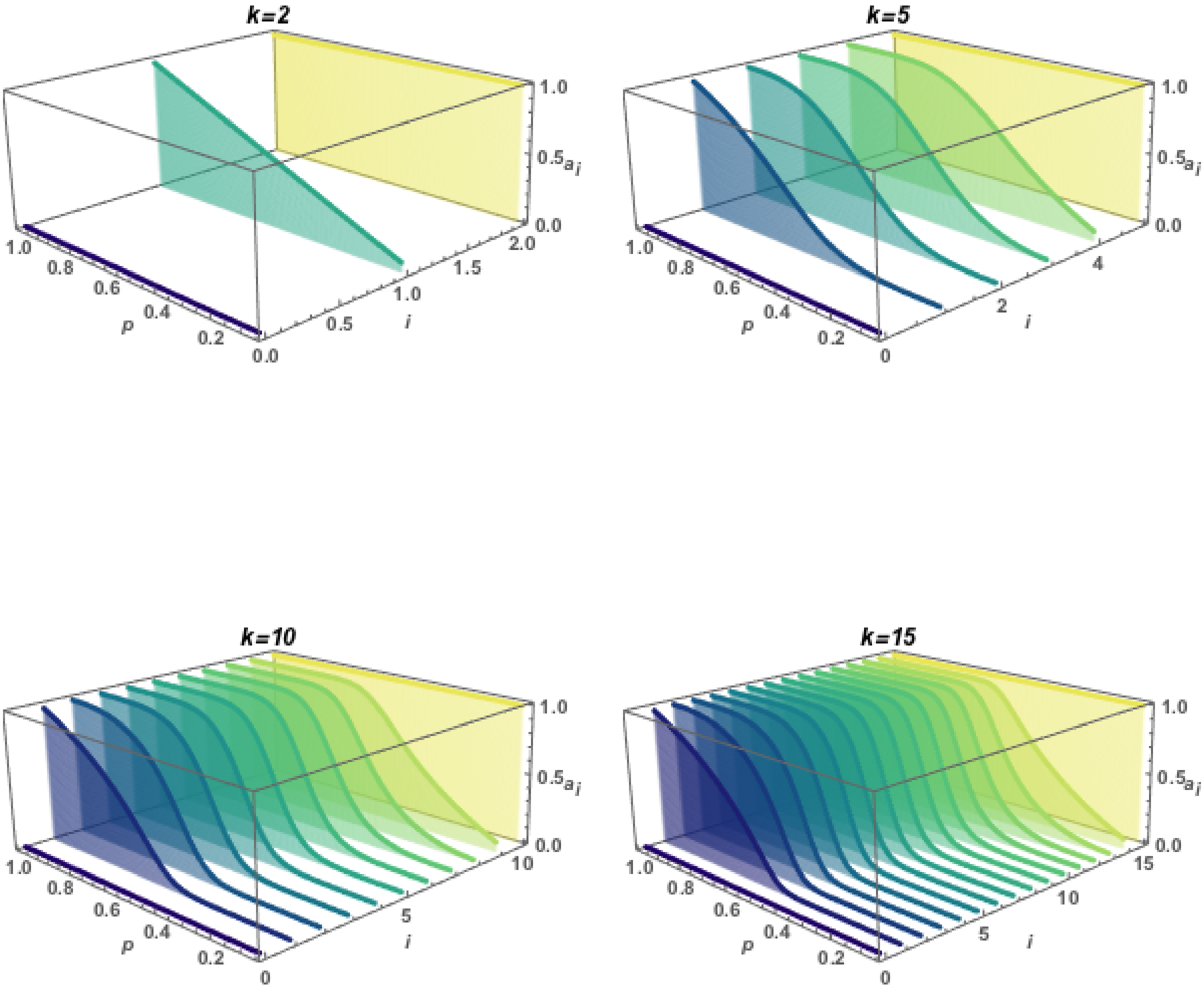

Figure 1: Plots of a_i as a function of i and p for select values of k.