The Gambler's Ruin Problem

Brian Weinstein - bmw2148

Probability & Statistics (STAT-W4700), Columbia University, Fall 2015

December 15, 2015

1 Problem Statement

In the Gambler's Ruin Problem, we have a gambler A and a casino B, who are playing a game against each other. The total combined fortune of the two is k dollars, with the gambler starting with i dollars, and the casino starting with $k-i$ dollars, where i and $k-i$ are known positive integers. On each play of the game, the probability that A will win one dollar from B is p, where $0 < p < 1$, and the probability that B will win one dollar from A is $1-p$.

Suppose that the game is played repeatedly (and independently) until the fortune of either A or B is reduced to 0 dollars.

2 Problem Solution

Let a_i denote the probability that gambler A will reach k dollars before it reaches 0 dollars, given that their initial fortune is i dollars. Since each play of the game is independent of the others, we can think of the problem essentially starts over on each play, with the only difference being that the "initial" fortunes of the gambler and casino have changed. The value of interest is a_i for $i \in \{0, 1, \ldots, k-1, k\}$.

2.1 Solution for $i \in \{0, k\}$

The cases of $i = 0$ and $i = k$ are trivial. When A runs out of money they can no longer play, and thus $a_0 = 0$, and when A wins all k dollars, the casino can no longer play, and thus $a_k = 1$. Finding a_i when $i \notin \{0, k\}$ is nontrivial and is solved in Section 2.2 below.

2.2 Solution for $i \in \{1, 2, \ldots, k-2, k-1\}$

Define the following events:

- A_1 is the event in which the gambler wins one dollar (i.e., the casino loses one dollar) on the first play of the game,
- B_1 is the event in which the casino wins one dollar (i.e., the gambler loses one dollar) on the first play of the game, and
- W is the event in which the gambler wins all k dollars before reaching 0 dollars.

By the Law of Total Probability, we see that

$$\begin{aligned} \mathrm{P}(W) &= \mathrm{P}(A_1)\mathrm{P}(W|A_1) + \mathrm{P}(B_1)\mathrm{P}(W|B_1) \\ &= p\mathrm{P}(W|A_1) + (1-p)\mathrm{P}(W|B_1). \end{aligned} \tag{1}$$

Since the gambler starts with i dollars, we see that $\mathrm{P}(W) = a_i$, as defined earlier. If the gambler wins on the first play, they'd now have $i+1$ dollars, and by our assumption that each game play is independent, we see that $\mathrm{P}(W|A_1) = a_{i+1}$. Similarly, if the gambler loses on the first play they'd now have $i-1$ dollars, and therefore $\mathrm{P}(W|B_1) = a_{i-1}$.

By Equation 1, we find

$$a_i = pa_{i+1} + (1-p)a_{i-1}. \tag{2}$$

By plugging in $i = 1, 2, \ldots, k-2, k-1$ into Equation 2, we get $k-1$ equations

$$\begin{aligned} a_1 &= pa_2 + (1-p)a_0 = pa_2 \\ a_2 &= pa_3 + (1-p)a_1 \\ &\vdots \\ a_{k-2} &= pa_{k-1} + (1-p)a_{k-3} \\ a_{k-1} &= pa_k + (1-p)a_{k-2} = p + (1-p)a_{k-2}, \end{aligned} \tag{3}$$

where we can simplify our equations for a_1 and $a_k - 1$ by using $a_0 = 0$ and $a_k = 1$, as defined in Section 2.1.

We can rewrite these $k-1$ equations as

$$asdf \tag{4}$$

and by equating the sum of the left sides of the equations with the sum of the right sides of the equations in Equation 4, we find

$$asdf \tag{5}$$