$\iff \exists B(x \in B \wedge y \in B \wedge B \in \mathscr{A})$ ($R_{\mathscr{A}}$ 定义)

$\iff \exists B(y \in B \wedge x \in B \wedge B \in \mathscr{A})$ (命题逻辑交换律)

$\iff \langle y, x\rangle \in R_{\mathscr{A}}$ ($R_{\mathscr{A}}$ 定义)

传递性：

$\forall x, y, z$

$\langle x, y\rangle \in R_{\mathscr{A}} \wedge \langle y, z\rangle \in R_{\mathscr{A}}$

$\iff \exists B(x \in B \wedge y \in B \wedge B \in \mathscr{A}) \wedge \exists B(y \in B \wedge z \in B \wedge B \in \mathscr{A})$ ($R_{\mathscr{A}}$ 定义)

$\implies x \in B_1 \wedge y \in B_1 \wedge B_1 \in \mathscr{A} \wedge y \in B_2 \wedge z \in B_2 \wedge B_2 \in \mathscr{A}$ ($\exists$消去)

$\implies x \in B_1 \wedge B_1 \in \mathscr{A} \wedge z \in B_2 \wedge B_2 \in \mathscr{A} \wedge y \in B_1 \wedge y \in B_2$ (命题逻辑交换律)

$\implies x \in B_1 \wedge B_1 \in \mathscr{A} \wedge z \in B_2 \wedge B_2 \in \mathscr{A} \wedge y \in B_1 \cap B_2$ (集合交定义)

$\implies x \in B_1 \wedge B_1 \in \mathscr{A} \wedge z \in B_2 \wedge B_2 \in \mathscr{A} \wedge B_1 \cap B_2 \neq \varnothing$ ($\varnothing$ 定义)

$\implies x \in B_1 \wedge B_1 \in \mathscr{A} \wedge z \in B_2 \wedge B_2 \in \mathscr{A} \wedge B_1 = B_2$ ($B_1 \cap B_2 \neq \varnothing \to B_1 = B_2$[2])

$\implies x \in B_1 \wedge z \in B_2 \wedge B_1 \in \mathscr{A}$ (外延原则)

$\implies \exists B(x \in B \wedge z \in B \wedge B \in \mathscr{A})$ ($\exists$引入)

$\iff \langle x, z\rangle \in R_{\mathscr{A}}$ ($R_{\mathscr{A}}$ 定义)

于是有 $R_{\mathscr{A}} \in \mathscr{E}$。

由 $R_{\mathscr{A}}$ 和商集定义立即得：$f(R_{\mathscr{A}}) = \mathscr{A}$。

故而 f 是满射的。

综合得，f 是双射的。 □

3.14

证明：先证：S 是自反的。

$\forall f$

$f \in \mathscr{A}$

$\implies \forall x(x \in [0,1] \to (f(x) - f(x)) = 0)$ ($f \in [0,1] \to \mathbb{R}$)

$\implies \forall x(x \in [0,1] \to (f(x) - f(x)) \geq 0)$ ($\geq$ 定义)

$\iff \langle f, f\rangle \in S$ (S 定义)

再证：S 是反对称的。

$\forall f, g$

$\langle f, g\rangle \in \mathscr{A} \wedge \langle g, f\rangle \in \mathscr{A}$

$\iff \forall x(x \in [0,1] \to (f(x) - g(x)) \geq 0) \wedge$
$\quad \forall x(x \in [0,1] \to (g(x) - f(x)) \geq 0)$ (S 定义)

$\iff \forall x((x \in [0,1] \to (f(x) - g(x)) \geq 0) \wedge$
$\quad (x \in [0,1] \to (g(x) - f(x)) \geq 0))$ (量词分配等值式)

$\iff \forall x((\neg x \in [0,1] \vee (f(x) - g(x)) \geq 0) \wedge$
$\quad (\neg x \in [0,1] \vee (g(x) - f(x)) \geq 0))$ (蕴涵等值式)

$\iff \forall x(\neg x \in [0,1] \vee ((f(x) - g(x)) \geq 0 \wedge (g(x) - f(x)) \geq 0))$ (命题逻辑分配律)

[2]这是划分定义第 (2) 项的逆否命题。由“假言易位”等值式知，它是永真命题。