习题3.1

P174. 3. (10) $\int \frac{\cos 2x}{\sin^2 x} dx$

$= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x} dx$

$= \int (\cot^2 x - 1) dx = \int (\csc^2 x - 1 - 1) dx$

$= -\cot x - 2x + C.$

(14) $\int \frac{3x^4+3x^2+1}{x^2+1} dx = \int (3x^2 + \frac{1}{x^2+1}) dx$

$= x^3 + \arctan x + C$

(18) $\int \frac{\cos 2x}{\cos x + \sin x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx$

$= \int (\cos x - \sin x) dx = \sin x + \cos x + C.$

(20) $\int (\arcsin x + \arccos x) dx$

$= \int \frac{\pi}{2} dx = \frac{\pi}{2} x + C.$

4. $\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx = \int \frac{A (a \sin x + b \cos x)'}{a \sin x + b \cos x} dx$

$+ \int \frac{B (a \sin x + b \cos x)}{a \sin x + b \cos x} dx$

其中 $a_1 \sin x + b_1 \cos x = A(a \sin x + b \cos x)' + B(a \sin x + b \cos x)$

$= (Aa + Bb) \cos x + (-Ab + aB) \sin x$

$\therefore \begin{cases} a_1 = -Ab + aB \\ b_1 = Aa + Bb \end{cases}$ $\therefore A = \frac{ab_1 - a_1 b}{a^2+b^2}$, $B = \frac{aa_1 + bb_1}{a^2+b^2}$

$\therefore \int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx = A \ln|a \sin x + b \cos x| + Bx + C$

P191. 1. (11) $\int \frac{\sqrt{\tan x + 1}}{\cos^2 x} dx = \int \sqrt{\tan x + 1}\, d(\tan x + 1)$

$= \frac{2}{3} (\tan x + 1)^{\frac{3}{2}} + C$

(13) $\int \frac{\cos 2x}{(2+3\sin 2x)} dx = \int \frac{\frac{1}{6} d(2+3\sin 2x)}{2+3\sin 2x}$

$= \frac{1}{6} \ln|2+3\sin 2x| + C$

(18) $\int \tan^4 x\, dx = \int \tan^2 x (\sec^2 x - 1) dx$

$= \int \tan^2 x\, d(\tan x) - \int (\sec^2 x - 1) dx$

$= \frac{1}{3} \tan^3 x - \tan x + x + C.$

(21) $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx.$

令 $x = a \sin t$ $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $dx = a \cos t\, dt$

$\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \int \frac{a^2 \sin^2 t}{a \cos t} \cdot a \cos t\, dt$

$= a^2 \int \frac{1 - \cos 2t}{2} dt = a^2 [\frac{1}{2} t - \frac{1}{4} \sin 2t] + C$

$= a^2 [\frac{1}{2} \arcsin \frac{x}{a} - \frac{1}{4} \cdot 2 \cdot \frac{x}{a} \cdot \sqrt{1 - \frac{x^2}{a^2}}] + C$

$= \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{x}{2} \sqrt{a^2 - x^2} + C.$

(22) $\int \frac{\sqrt{x^2+a^2}}{x^2} dx$

令 $x = a \tan t$ $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\int \frac{\sqrt{x^2+a^2}}{x^2} dx = \int \frac{a \sec t}{a^2 \tan^2 t} \cdot a \sec^2 t\, dt$

$= \int \frac{1}{\cos^3 t} \cdot \frac{1}{\frac{\sin^2 t}{\cos^2 t}} dt = \int \frac{dt}{\cos t \sin^2 t}$

$= -\int \sec t\, d \cot t$

$= -\sec t \cdot \cot t + \int \cot t \sec t \cdot \tan t\, dt$

$= -\csc t + \ln|\sec t + \tan t| + C_1$

$= -\frac{\sqrt{x^2+a^2}}{x} + \ln|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}| + C_1$

$= -\frac{\sqrt{x^2+a^2}}{x} + \ln|x + \sqrt{x^2+a^2}| + C$

$\sqrt{x^2+a^2}$, x, t, a

$C = C_1 - \ln a$

习题3.1

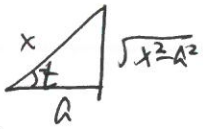
1.(23) $\int \frac{\sqrt{x^2-a^2}}{x}dx$

令 $x=a\sec t \qquad 0<t<\frac{\pi}{2}$

$\int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{a\tan t}{a\sec t}\cdot a\sec t\cdot \tan t\,dt$

$= a\int \tan^2 t\,dt = a\int(\sec^2 t-1)\,dt$

$= a\tan t - at + c$



$= a\cdot\frac{\sqrt{x^2-a^2}}{a} - a\cdot\arccos\frac{a}{x}+c$

$= \sqrt{x^2-a^2} - a\arccos\frac{a}{x}+c$

(24) $\int x^5(2-5x^3)^{\frac{2}{3}}dx$

$=\frac{-1}{15}\int\frac{-1}{5}[(2-5x^3)-2](2-5x^3)^{\frac{2}{3}}d(2-5x^3)$

$=\frac{1}{75}\int[(2-5x^3)^{\frac{5}{3}}-2(2-5x^3)^{\frac{2}{3}}]d(2-5x^3)$

$=\frac{1}{75}\cdot\frac{(2-5x^3)^{\frac{8}{3}}}{\frac{8}{3}}-\frac{2}{75}\cdot\frac{(2-5x^3)^{\frac{5}{3}}}{\frac{5}{3}}+c$

$=\frac{8}{25}(2-5x^3)^{\frac{8}{3}}-\frac{2}{125}(2-5x^3)^{\frac{5}{3}}+c.$

(27) $\int\frac{x+1}{x(1+xe^x)}dx$

$=\int\frac{e^x(x+1)}{xe^x(1+xe^x)}dx$

$=\int\frac{d(\cancel{1+}xe^x)}{xe^x(1+xe^x)}$

$=\int\left(\frac{1}{xe^x}-\frac{1}{1+xe^x}\right)d(xe^x)$

$=\ln\frac{xe^x}{1+xe^x}+c.$

2(3) $\int x\sin x\cos x\,dx = \int x\cdot\frac{1}{2}\sin 2x\,dx$

$=\int x\cdot\frac{-1}{4}d(\cos 2x) = -\frac{x}{4}\cos 2x+\int\cos 2x\cdot\frac{1}{4}dx$

$=-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x+c.$

(5) $\int\frac{x\cos x}{\sin^2 x}dx = \int x\cot x\cdot\csc x\,dx$

$=\int x\,d(-\csc x) = -x\csc x+\int\csc x\,dx$

$=-x\csc x+\ln|\csc x-\cot x|+c.$

(9) $\int x^2\ln x\,dx = \int\ln x\,d\left(\frac{x^3}{3}\right)$

$=\ln x\cdot\frac{x^3}{3}-\int\frac{x^3}{3}\cdot\frac{1}{x}dx$

$=\frac{x^3}{3}\ln x-\frac{1}{9}x^3+c$

(10) $\int\ln(x+\sqrt{1+x^2})\,dx$

$=x\ln(x+\sqrt{1+x^2})-\int x\cdot\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}dx$

$=x\ln(x+\sqrt{1+x^2})-\int\frac{x}{\sqrt{1+x^2}}dx$

$=x\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int(1+x^2)^{-\frac{1}{2}}d(1+x^2)$

$=x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C$

3.(2) $\int\frac{dx}{\sqrt{1+e^{2x}}}=\int\frac{dx}{e^x\sqrt{1+e^{-2x}}}$

$=\int\frac{-d(e^{-x})}{\sqrt{(e^{-x})^2+1^2}}=-\ln|e^{-x}+\sqrt{(e^{-x})^2+1}|+c$

(4) $\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\int\frac{\sin x\cos x\,dx}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}$

$=\int\frac{\frac{1}{2}\sin 2x\,dx}{1-\frac{1}{2}\sin^2 2x}=\int\frac{-\frac{1}{4}d(\cos 2x)}{\frac{1}{2}-\frac{1}{2}\cos^2 2x}$

$=\left(-\frac{\sqrt{2}}{4}\right)\int\frac{d(\cos 2x\cdot\frac{1}{\sqrt{2}})}{(\frac{1}{\sqrt{2}})^2-(\frac{1}{\sqrt{2}}\cos 2x)^2}$

$=-\frac{\sqrt{2}}{4}\cdot\frac{1}{2\cdot\frac{1}{\sqrt{2}}}\ln\left|\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos 2x}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos 2x}\right|+c$

习题3.2

$$= -\frac{1}{4}\ln\left|\frac{1+\cos 2x}{1-\cos 2x}\right| + C$$

3(6) $\int x^3 \sqrt[3]{1-x}\, dx$

$u = \sqrt[3]{1-x}$ 则 $x = 1-u^3$

$$\int x^3\sqrt[3]{1-x}\, dx = \int (1-u^3)^3 \cdot u \cdot (-3u^2)\, du$$

$$= \int (1-3u^3+3u^6-u^9)(-3u^3)\, du$$

$$= \int (-3u^3+9u^6-9u^9+3u^{12})\, du$$

$$= -\frac{3}{4}u^4 + \frac{9}{7}u^7 - \frac{9}{10}u^{10} + \frac{3}{13}u^{13} + C$$

其中 $u = \sqrt[3]{1-x}$

(8) $\int x \ln\frac{1+x}{1-x}\, dx$

$$= \int x\ln(1+x)\, dx - \int x\ln(1-x)\, dx$$

$$= \int \ln(1+x)\, d\left(\frac{x^2}{2}\right) - \int x\ln(1-x)\, dx$$

$$= \frac{x^2}{2}\ln(1+x) - \int \frac{x^2}{2}\cdot\frac{1}{1+x}\, dx$$

$$- \ln(1-x)\cdot\frac{x^2}{2} + \int \frac{x^2}{2}\cdot\frac{-1}{1-x}\, dx$$

$$= \frac{x^2}{2}\ln\frac{1+x}{1-x} + \frac{-1}{2}\int \frac{x^2(1-x)+x^2(1+x)}{1-x^2}\, dx$$

$$= \frac{x^2}{2}\ln\frac{1+x}{1-x} - \int \frac{x^2}{1-x^2}\, dx$$

$$= \frac{x^2}{2}\ln\frac{1+x}{1-x} + \int \frac{(1-x^2)-1}{1-x^2}\, dx$$

$$= \frac{x^2}{2}\ln\frac{1+x}{1-x} + x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

(10) $\int \frac{x\ln(1+\sqrt{1+x^2})}{\sqrt{1+x^2}}\, dx$

$$= \int \ln(1+\sqrt{1+x^2})\, d(1+\sqrt{1+x^2})$$

$$= \ln(1+\sqrt{1+x^2})(1+\sqrt{1+x^2}) - \int (1+\sqrt{1+x^2})\cdot\frac{\frac{x}{\sqrt{1+x^2}}}{1+\sqrt{1+x^2}}\, dx$$

$$= (1+\sqrt{1+x^2})\ln(1+\sqrt{1+x^2}) - \frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}\, d(1+x^2)$$

$$= (1+\sqrt{1+x^2})\ln(1+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

(12) $\int e^{-|x|}\, dx$

当 $x \geq 0$ 时, $\int e^{-|x|}dx = \int e^{-x}dx = -e^{-x} + C_1$

当 $x < 0$ 时, $\int e^{-|x|}dx = \int e^{x}dx = e^{x} + C_2$

$\because e^{-|x|}$ 在 $(-\infty, +\infty)$ 连续 $\therefore \int e^{-|x|}dx$ 在 $x=0$ 处连续. $\therefore \lim\limits_{x\to 0^+}(-e^{-x}+C_1) = \lim\limits_{x\to 0}(e^{x}+C_2)$

$C_1 - 1 = C_2 + 1 \quad \therefore C_1 = C_2 + 2$

令 $C_2 = C$, 则 $C_1 = C+2$.

$$\therefore \int e^{-|x|}dx = \begin{cases} -e^{-x} + C + 2, & x \geq 0 \\ e^{x} + C, & x < 0 \end{cases}$$

习题3.3

P201 1.(1) $\int \frac{x^3-1}{4x^3-x}\, dx$

$$4x^3 - x = x(4x^2-1) = x(2x-1)(2x+1)$$

$$= x\cdot 4\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$$

$$\frac{x^3-1}{4x^3-x} = \frac{1}{4}\left[\frac{A}{x} + \frac{B}{x-\frac{1}{2}} + \frac{C}{x+\frac{1}{2}}\right]$$

$$= \frac{1}{4}\cdot\frac{A(x-\frac{1}{2})(x+\frac{1}{2}) + Bx(x+\frac{1}{2}) + Cx(x-\frac{1}{2})}{x(x-\frac{1}{2})(x+\frac{1}{2})}$$

$$\therefore x^3 - 1 = A\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) + Bx\left(x+\frac{1}{2}\right) + Cx\left(x-\frac{1}{2}\right)$$

令 $x=0$ 得 $A=4$

令 $x=\frac{1}{2}$ 得 $B=-\frac{7}{4}$

令 $x=-\frac{1}{2}$ 得 $C=\frac{9}{4}$

$$\therefore \int \frac{x^3-1}{4x^3-x}\, dx = \frac{1}{4}\int\left(\frac{4}{x} + \frac{-\frac{7}{4}}{x-\frac{1}{2}} + \frac{\frac{9}{4}}{x+\frac{1}{2}}\right)dx$$

$$= \frac{1}{4}\left[4\ln|x| - \frac{7}{4}\ln\left|x-\frac{1}{2}\right| + \frac{9}{4}\ln\left|x+\frac{1}{2}\right|\right] + C$$

习题3.3

1.(6) $\int \frac{3x+4}{(x^2+1)^2}dx$

$= \int \frac{\frac{3}{2}(x^2+1)'+4}{(x^2+1)^2}dx$

$= \frac{3}{2}\int \frac{(x^2+1)'dx}{(x^2+1)^2} + \frac{4}{2}\int \frac{x^2+1-x^2+1}{(x^2+1)^2}dx$

$= \frac{3}{2}(-1)\cdot\frac{1}{x^2+1} + 2\int\frac{1}{x^2+1}dx + 2\int\frac{1-x^2}{(x^2+1)^2}dx$

$= -\frac{3}{2}\frac{1}{x^2+1} + 2\cdot\arctan x + 2\int\left(\frac{x}{x^2+1}\right)'dx$

$= -\frac{3}{2}\cdot\frac{1}{x^2+1} + 2\arctan x + 2\frac{x}{x^2+1} + C.$

2.(1) $\int \cos^4 x\sin^3 x\,dx$

$= -\int\cos^4 x\sin^2 x\,d(\cos x)$

$= \int \cos^4 x(\cos^2 x-1)\,d(\cos x)$

$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

(7) $\int\frac{dx}{4-5\sin x}$

令 $u=\tan\frac{x}{2}$ 则 $x = 2\arctan u$

$dx = \frac{2}{1+u^2}du \quad \sin x = \frac{2u}{1+u^2}$

$\int\frac{dx}{4-5\sin x} = \int\frac{1}{4-\frac{10u}{1+u^2}}\frac{2}{1+u^2}du$

$= \int\frac{1}{2u^2-5u+2}du = \int\frac{1}{(2u-1)(u-2)}du$

$= \int\frac{1}{2}\cdot\frac{du}{(u-\frac{1}{2})(u-2)} = \frac{1}{2}\int\left[\frac{-\frac{2}{3}}{u-\frac{1}{2}}+\frac{\frac{2}{3}}{u-2}\right]du$

$= -\frac{1}{3}\ln\left|u-\frac{1}{2}\right| + \frac{1}{3}\ln|u-2| + C$

$= \frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}-2}{\tan\frac{x}{2}-\frac{1}{2}}\right| + C.$

3(4) $\int\frac{dx}{\sqrt{1-2x-3x^2}}$

$= \int\frac{dx}{\sqrt{(x+1)(1-3x)}} = \int\frac{2\,d(\sqrt{x+1})}{\sqrt{1-3x}}$

$= 2\int\frac{\frac{1}{\sqrt{3}}d(\sqrt{3}\sqrt{x+1})}{\sqrt{4-(\sqrt{3(x+1)})^2}}$

$= \frac{2}{\sqrt{3}}\int\frac{d(\sqrt{3(x+1)})}{\sqrt{2^2-(\sqrt{3(x+1)})^2}}$

$= \frac{2}{\sqrt{3}}\cdot\arcsin\frac{\sqrt{3(x+1)}}{2} + C.$

(6) $\int\frac{x+3}{\sqrt{4x^2+4x+3}}dx$

$= \int\frac{\frac{1}{8}[(2x+1)^2]'+\frac{5}{2}}{\sqrt{(2x+1)^2+2}}$

$= \frac{1}{8}\int\frac{d[(2x+1)^2+2]}{\sqrt{(2x+1)^2+2}} + \frac{5}{2}\int\frac{\frac{1}{2}d(2x+1)}{\sqrt{(2x+1)^2+(\sqrt{2})^2}}$

$= \frac{1}{8}\cdot\frac{\sqrt{(2x+1)^2+2}}{\frac{1}{2}} + \frac{5}{4}\ln\left|2x+1+\sqrt{(2x+1)^2+(\sqrt{2})^2}\right|+C$

$= \frac{1}{4}\sqrt{4x^2+4x+3} + \frac{5}{4}\ln\left|2x+1+\sqrt{4x^2+4x+3}\right|+C$

4(2) $\int\frac{x}{1-\cos x}dx = \int\frac{x}{2\sin^2\frac{x}{2}}dx$

$= \int x\,d(-\cot\frac{x}{2})$

$= -x\cot\frac{x}{2} + \int\cot\frac{x}{2}\,dx$

$= -x\cot\frac{x}{2} + 2\ln\left|\sin\frac{x}{2}\right| + C.$

(4) $\int\frac{\ln x-1}{\ln^2 x}dx$

$= \int\frac{1}{\ln x}dx - \int\frac{1}{\ln^2 x}dx$

$= \frac{x}{\ln x} - \int x\cdot\frac{-\frac{1}{x}}{\ln^2 x}dx - \int\frac{1}{\ln^2 x}dx$

$= \frac{x}{\ln x} + C.$

习题3.4

P202 4(6) $\int \frac{x^2-1}{x^4+1} dx$

$= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$

$= \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-(\sqrt{2})^2}$

$= \frac{1}{2\sqrt{2}} \ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$

(8) $\int \frac{1+\sin x}{1+\cos x} e^x dx$

$= \int \frac{e^x}{1+\cos x} dx + \int \frac{\sin x \cdot e^x}{1+\cos x} dx$

$= \int \frac{e^x}{1+\cos x} dx + \int \frac{\sin x}{1+\cos x} d(e^x)$

$= \int \frac{e^x}{1+\cos x} dx + \frac{\sin x}{1+\cos x} \cdot e^x - \int e^x \cdot \frac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2} dx$

$= \int \frac{e^x}{1+\cos x} dx + \frac{\sin x \cdot e^x}{1+\cos x} - \int \frac{e^x}{1+\cos x} dx$

$= \frac{\sin x \cdot e^x}{1+\cos x} + C$