15.23

(1)

证明: 记 V_1 的载体为 A，则 I_A 显然是 V_1 到 V_1 的同态且为双射，所以有 $V_1 \cong V_1$。 □

(2)

证明: 记 $V_1 = \langle A, \circ_1, \circ_2, \ldots, \circ_k \rangle, V_2 = \langle B, \bar{\circ}_1, \bar{\circ}_2, \ldots, \bar{\circ}_k \rangle$。

由同构定义知，存在双射 $\varphi : A \to B$，使得对所有的运算 $\circ_i, \bar{\circ}_i$ 都有：

$$\varphi(\circ_i(x_1, x_2, \ldots, x_{k_i})) = \bar{\circ}_i(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{k_i})), \quad \forall x_1, x_2, \ldots, x_{k_i} \in A$$

从而由教材定理 3.9 知 $\varphi^{-1} : B \to A$ 也是双射的，且对所有的运算 $\circ_i, \bar{\circ}_i$ 都有：

$\forall y_1, y_2, \ldots, y_{k_i} \in B$，

$$\begin{aligned}
&\varphi^{-1}(\bar{\circ}_i(y_1, y_2, \ldots, y_{k_i})) \\
=&\varphi^{-1}(\bar{\circ}_i(\varphi(\varphi^{-1}(y_1)), \varphi(\varphi^{-1}(y_2)), \ldots, \varphi(\varphi^{-1}(y_{k_i})))) && (\varphi \circ \varphi^{-1} = I_B) \\
=&\varphi^{-1}(\varphi(\circ_i(\varphi^{-1}(y_1), \varphi^{-1}(y_2), \ldots, \varphi^{-1}(y_{k_i})))) && (\varphi \text{ 是 } V_1 \text{ 到 } V_2 \text{ 的同态}) \\
=& \circ_i (\varphi^{-1}(y_1), \varphi^{-1}(y_2), \ldots, \varphi^{-1}(y_{k_i})) && (\varphi^{-1} \circ \varphi = I_A)
\end{aligned}$$

从而证明了 φ^{-1} 是 V_2 到 V_1 同态。又由于 φ^{-1} 是双射，所以就有 $V_2 \cong V_1$。 □

(3)

证明: 记 $V_1 = \langle A, \circ_1, \circ_2, \ldots, \circ_k \rangle, V_2 = \langle B, \bar{\circ}_1, \bar{\circ}_2, \ldots, \bar{\circ}_k \rangle, V_3 = \langle C, \circ'_1, \circ'_2, \ldots, \circ'_k \rangle$。

由同构定义知，存在双射 $\varphi_1 : A \to B$ 和 $\varphi_2 : B \to C$，使得对所有的运算 $\circ_i, \bar{\circ}_i, \circ'_i$ 都有：

$$\varphi(\circ_i(x_1, x_2, \ldots, x_{k_i})) = \bar{\circ}_i(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{k_i}), \quad \forall x_1, x_2, \ldots, x_{k_i}) \in A$$

$$\bar{\circ}_i(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_{k_i}) = \varphi(\circ'_i(y_1, y_2, \ldots, y_{k_i})), \quad \forall y_1, y_2, \ldots, y_{k_i}) \in B$$

从而由教材定理 3.4(3) 知，$\varphi_2 \circ \varphi_1 : A \to C$ 也是双射的，且对所有的运算 $\circ_i, \bar{\circ}_i, \circ'_i$ 都有：

$\forall x_1, x_2, \ldots, x_{k_i} \in A$，

$$\begin{aligned}
&\varphi_2 \circ \varphi_1(\circ_i(x_1, x_2, \ldots, x_{k_i})) \\
=&\varphi_2(\varphi_1(\circ_i(x_1, x_2, \ldots, x_{k_i}))) && (\text{教材定理 } 3.3) \\
=&\varphi_2(\bar{\circ}_i(\varphi_1(x_1), \varphi_1(x_2), \ldots, \varphi_1(x_{k_i}))) && (\varphi_1 \text{ 是 } V_1 \text{ 到 } V_2 \text{ 的同态}) \\
=& \circ'_i (\varphi_2(\varphi_1(x_1)), \varphi_2(\varphi_1(x_2)), \ldots, \varphi_2(\varphi_1(x_{k_i}))) && (\varphi_2 \text{ 是 } V_2 \text{ 到 } V_3 \text{ 的同态}) \\
=& \circ'_i (\varphi_2 \circ \varphi_1(x_1), \varphi_2 \circ \varphi_1(x_2), \ldots, \varphi_2 \circ \varphi_1(x_{k_i})) && (\text{教材定理 } 3.3)
\end{aligned}$$

从而证明了 $\varphi_2 \circ \varphi_1$ 是 V_1 到 V_3 同态。又由于 $\varphi_2 \circ \varphi_1$ 是双射，所以就有 $V_1 \cong V_3$。 □

15.24

(1) 不是同态。

证明: 由于 $1 \in \mathbb{C}$，但 $\varphi(1 \cdot 1) = \varphi(1) = |1| + 1 = 2$，而 $\varphi(1) \cdot \varphi(1) = 2 \cdot 2 = 4$，从而 $\varphi(1 \cdot 1) \neq \varphi(1) \cdot \varphi(1)$。这就证明了 φ 不是同态。 □

(2) 是同态，同态像是 $\langle \mathbb{R} - \mathbb{R}^-, \cdot \rangle$。

证明: $\forall a_1 e^{i\theta_1}, a_2 e^{i\theta_2} \in \mathbb{C}$，

$$\begin{aligned}
\varphi(a_1 e^{i\theta_1} \cdot a_2 e^{i\theta_2}) &= \varphi(a_1 a_2 e^{i(\theta_1 + \theta_2)}) && (\text{复数乘法定义}) \\
&= |a_1 a_2 e^{i(\theta_1 + \theta_2)}| && (\varphi \text{ 定义}) \\
&= a_1 a_2 && (\text{模运算定义}) \\
&= |a_1 e^{i\theta_1}| \cdot |a_2 e^{i\theta_2}| && (\text{模运算定义}) \\
&= \varphi(a_1 e^{i\theta_1}) \cdot \varphi(a_2 e^{i\theta_2}) e && (\varphi \text{ 定义})
\end{aligned}$$