综上所述，有 $\mathscr{A}$ 是划分。

下面证明 $\mathscr{A}$ 既是 π_1 的加细又是 π_2 的加细。

对任意 $A_i \cap B_j \in \mathscr{A}$，由引理 1.2 知，有 $A_i \cap B_j \subseteq A_i$ 和 $A_i \cap B_j \subseteq B_j$。即，$\mathscr{A}$ 中的每一个划分块都含于 π_1 和 π_2 的某个划分块中。由加细定义知，$\mathscr{A}$ 既是 π_1 的加细又是 π_2 的加细。 □

2.39

(1) $R_\pi = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle\} \cup I_A$；

$A/R_\pi = \pi = \{\{1,2,3\},\{4\}\}$；

(2) $\pi_1 = A/R_{\pi_1} = \{\{1\},\{2\},\{3\},\{4\}\}$；

$R_{\pi_1} = I_A$；

$\pi_2 = A/R_{\pi_2} = \{\{1,2\},\{3\},\{4\}\}$；

$R_{\pi_2} = \{\langle 1,2\rangle, \langle 2,1\rangle\} \cup I_A$；

$\pi_3 = A/R_{\pi_3} = \{\{1,3\},\{2\},\{4\}\}$；

$R_{\pi_3} = \{\langle 1,3\rangle, \langle 3,1\rangle\} \cup I_A$；

$\pi_4 = A/R_{\pi_4} = \{\{1\},\{2,3\},\{4\}\}$；

$R_{\pi_4} = \{\langle 2,3\rangle, \langle 3,2\rangle\} \cup I_A$；

$\pi_5 = A/R_{\pi_5} = \pi$；

$R_{\pi_5} = R_\pi = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle\} \cup I_A$。

2.40

证明： 必要性：

由加细定义有，$\forall \mathscr{A}(\mathscr{A} \in A/R_1 \to \exists \mathscr{B}(\mathscr{B} \in A/R_2 \wedge \mathscr{A} \subseteq \mathscr{B}))$，故有：

$\forall x, y \in A$

$$
\begin{aligned}
& \langle x,y\rangle \in R_1 \\
\iff & \exists \mathscr{A}(\mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A}) && (\text{商集定义}) \\
\implies & \mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} && (\exists\text{消去}) \\
\iff & \mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \wedge \exists \mathscr{B}(\mathscr{B} \in A/R_2 \wedge \mathscr{A} \subseteq \mathscr{B}) && (\text{前提}) \\
\iff & \exists \mathscr{B}(\mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \wedge \mathscr{B} \in A/R_2 \wedge \mathscr{A} \subseteq \mathscr{B}) && (\text{量词辖域扩张等值式}) \\
\implies & \mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \wedge \mathscr{B} \in A/R_2 \wedge \mathscr{A} \subseteq \mathscr{B} && (\exists\text{消去}) \\
\implies & \mathscr{B} \in A/R_2 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \wedge \mathscr{A} \subseteq \mathscr{B} && (\text{命题逻辑化简律、交换律}) \\
\implies & \mathscr{B} \in A/R_2 \wedge x \in \mathscr{B} \wedge y \in \mathscr{B} && (\text{子集关系定义}) \\
\implies & \exists \mathscr{B}(B \in A/R_2 \wedge x \in \mathscr{B} \wedge y \in \mathscr{B}) && (\exists\text{引入}) \\
\iff & \langle x,y\rangle \in R_2 && (\text{商集定义})
\end{aligned}
$$

充分性：

只需证明 $\forall \mathscr{A} \forall x \forall y(\mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \to \exists \mathscr{B}(\mathscr{B} \in A/R_2 \wedge x \in \mathscr{B} \wedge y \in \mathscr{B}))$。

$\forall \mathscr{A}, x, y$

$$
\begin{aligned}
& \mathscr{A} \in A/R_1 \wedge x \in \mathscr{A} \wedge y \in \mathscr{A} \\
\iff & \langle x,y\rangle \in R_1 && (\text{商集定义}) \\
\iff & \langle x,y\rangle \in R_1 \wedge R_1 \subseteq R_2 && (\text{前提}) \\
\implies & \langle x,y\rangle \in R_2 && (\text{子集关系定义}) \\
\iff & \exists \mathscr{B}(\mathscr{B} \in A/R_2 \wedge x \in \mathscr{B} \wedge y \in \mathscr{B}) && (\text{商集定义})
\end{aligned}
$$