一.

1. $\lim\limits_{x\to 0} \dfrac{3\sin x + x^2 \sin\frac{1}{x^2}}{(1+\cos x)\ln(1+x)}$

$= \lim\limits_{x\to 0} \dfrac{3\sin x + x^2\sin\frac{1}{x^2}}{2\cdot x}$

$= \lim\limits_{x\to 0} \left(\frac{3}{2}\frac{\sin x}{x} + \frac{1}{2}x\sin\frac{1}{x^2}\right)$

$= \frac{3}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \underline{\frac{3}{2}}$

2. $\lim\limits_{n\to\infty} [\arctan(1+n^2)]^{\frac{1}{n}}$

$= \lim\limits_{x\to+\infty} [\arctan(1+x^2)]^{\frac{1}{x}}$

$= \lim\limits_{x\to+\infty} e^{\frac{\ln\arctan(1+x^2)}{x}}$

$= e^0 = \underline{1}$

3. $f(x) = |(x+1)(x+2)|\sin(x+1)$

$\lim\limits_{x\to-1} \dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to-1} \dfrac{|(x+1)(x+2)|\sin(x+1)}{x+1}$

$= 0$

$\lim\limits_{x\to-2} \dfrac{f(x)-f(-2)}{x-(-2)} = \lim\limits_{x\to-2} \dfrac{|(x+1)(x+2)|\sin(x+1)}{x+2}$

不存在.

$\therefore x=-2$ 是不可导点.

$\therefore$ $\underline{k=1}$

4. $-8 = \lim\limits_{x\to 0} \dfrac{f(1+\sin x) - f(1-3x)}{\sqrt{1-2x}-1}$

$= \lim\limits_{x\to 0} \dfrac{f(1+\sin x) - f(1-3x)}{\frac{1}{2}(-2x)}$

$= \lim\limits_{x\to 0}\left[\dfrac{f(1+\sin x)-f(1)}{\sin x}\cdot\dfrac{\sin x}{-x} + \dfrac{f(1)-f(1-3x)}{3x}\cdot\dfrac{3x}{-x}\right]$

$= -f'(1) - 3f'(1) = -4f'(1)$

$\therefore f'(1) = 2$

$\because f'(1) = f'(6)$ $\quad\therefore f'(6) = \underline{2}$

5. $f(x) = x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$

$f'(x) = (e^{\frac{1}{x}\ln x})' = x^{\frac{1}{x}}\cdot(\frac{1}{x}\ln x)'$

$= x^{\frac{1}{x}}\ \dfrac{\frac{1}{x}\cdot x - \ln x\cdot 1}{x^2}$

$= x^{\frac{1}{x}}\cdot\dfrac{1-\ln x}{x^2}$

令 $f'(x)=0$ $\quad x=e$ 唯一驻点

当 $x<e$ 时. $f'(x)>0$, $\therefore f(x)$ 单增

当 $x>e$ 时. $f'(x)<0$, $\therefore f(x)$ 单减

$\therefore f(e) = \underline{e^{\frac{1}{e}}}$ 是极大值, 也是最大值.

6. $\varphi'(y) = \dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}} = \dfrac{1}{f'(x)}$

$\varphi''(y) = \dfrac{d^2x}{dy^2} = \dfrac{d}{dy}\left(\dfrac{1}{f'(x)}\right)$

$= \dfrac{d}{dx}\left(\dfrac{1}{f'(x)}\right)\cdot\dfrac{dx}{dy}$

$= \dfrac{-f''(x)}{(f'(x))^2}\cdot\dfrac{1}{f'(x)}$

$= \dfrac{-f''(x)}{(f'(x))^3}$

$\varphi(-2)=0$. $\quad\therefore \varphi''(-2) = \dfrac{-f''(0)}{(f'(0))^3}$

$= \dfrac{2}{(-2)^3} = \underline{\dfrac{1}{-4}}$

7. $f(x) = xe^x$ $\quad u(x)=e^x$, $v(x)=x$

$f^{(n)}(x) = C_n^0 e^x \cdot x + C_n^1 e^x \cdot 1$

$= e^x \cdot x + ne^x$

$= e^x(x+n)$

$f^{(100)}(0) = e^0(0+100) = \underline{100}$

二. 1. $\underline{D.}$ $y=x^4$ 在$x=0$有极小值.

$y''|_{x=0} = 0$.

2. $\lim\limits_{x\to 0} \dfrac{f(x)}{\frac{1}{2}x^2} = -1 < 0$

$\therefore \exists \mathring{U}(0)$ $\quad \forall x \in \mathring{U}(0)$ 有 $\dfrac{f(x)}{\frac{1}{2}x^2} < 0$

$\therefore f(x) < 0$ $\quad$ 而 $f(0) = \lim\limits_{x\to 0} f(x) = 0$

$\therefore f(x) < f(0)$ $\quad \forall x \in \mathring{U}(0)$

$\therefore f(0)$是极大值. $\quad \underline{A.}$

3. $F'(0) = \lim\limits_{x\to 0} \dfrac{F(x)-F(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{f(x)(1+|\sin x|) - f(0)}{x}$ 存在.

~~$F(0) = \lim\limits_{x\to 0} F(x) = \lim\limits_{x\to 0} f(x)(1+|\sin x|) = \lim\limits_{x\to 0} f(x)$~~

~~$= f(0)$~~

~~$\lim\limits_{x\to 0} (f(x)(1+|\sin x|) - f(0)) = 0$~~

~~$\lim\limits_{x\to 0} f(x) = f(0)$ $\therefore f(x)$在$x=0$处连续~~

$\therefore F'_-(0) = \lim\limits_{x\to 0^-} \dfrac{F(x)-F(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{f(x)(1+|\sin x|)-f(0)}{x-0}$

$= \lim\limits_{x\to 0^-} \dfrac{f(x)(1-\sin x) - f(0)}{x}$

$= \lim\limits_{x\to 0^-} \left[\dfrac{f(x)-f(0)}{x} - f(x)\dfrac{\sin x}{x}\right]$

$= f'_-(0) - f(0)$

$= f'(0) - f(0)$

$F'_+(0) = \lim\limits_{x\to 0^+} \dfrac{F(x)-F(0)}{x-0}$

$= \lim\limits_{x\to 0^+} \left[\dfrac{f(x)-f(0)}{x} + f(x)\dfrac{\sin x}{x}\right]$

$= f'_+(0) + f(0)$

$= f'(0) + f(0)$

$F'(0)$存在 $\Longleftrightarrow F'_-(0) = F'_+(0)$

$\therefore f'(0) - f(0) = f'(0) + f(0)$

$\therefore f(0) = 0.$ $\quad \therefore \underline{A}.$

三、

1. $f'(x)=(x^x)'=(e^{x\ln x})'$

$=x^x\cdot(x\ln x)'$

$=x^x[\ln x+1]$

$f''(x)=(x^x[\ln x+1))'$

$=(x^x)'(\ln x+1)+x^x\cdot\frac{1}{x}$

$=x^x(\ln x+1)^2+x^{x-1}$

2. $\lim\limits_{x\to0}\frac{1-\sqrt{\cos x}}{x\sin(\sin x)}$

$=\lim\limits_{x\to0}\frac{1-\cos x}{x\cdot\sin x\,(1+\sqrt{\cos x})}=\lim\limits_{x\to0}\frac{\frac{1}{2}x^2}{x\cdot x\cdot 2}$

$=\frac{1}{4}$.

3. $\lim\limits_{x\to+\infty}x^2[\arctan(x+1)-\arctan x]$

$=\lim\limits_{x\to+\infty}\frac{\arctan(x+1)-\arctan x}{x^{-2}}$

$=\lim\limits_{x\to+\infty}\frac{\frac{1}{1+(x+1)^2}-\frac{1}{1+x^2}}{-2x^{-3}}$

$=\lim\limits_{x\to+\infty}\frac{1+x^2-1-(x+1)^2}{[1+(x+1)^2][1+x^2]}\cdot\frac{x^3}{-2}$

$=1$.

4. $f'(x)=e^{\frac{\pi}{2}+\arctan x}+(x-1)e^{\frac{\pi}{2}+\arctan x}\cdot\frac{1}{1+x^2}$

令$f'(x)=0$　$x=0$，$x=-1$

当$x<-1$时，$f'(x)>0$，当$-1<x<0$，$f'(x)<0$

当$x>0$时　$f'(x)>0$

$f(-1)=-2e^{\frac{\pi}{4}}$ 极大值

$f(0)=-e^{\frac{\pi}{2}}$　极小值

5. $f'(x)=\ln(x+\sqrt{1+x^2})+x\cdot\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}-\frac{x}{\sqrt{1+x^2}}$

$=\ln(x+\sqrt{1+x^2})$

令$f'(x)=0$　得 $x=0$ 唯一驻点

当$x<0$时，$f'(x)<0$

当$x>0$时，$f'(x)>0$

∴由极值第一充分条件得 $f(0)=-1$是极小值，也是最小值

$\lim\limits_{x\to-\infty}[x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}]$　偶函数.

$=\lim\limits_{x\to+\infty}[x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}]$

$=+\infty$.

无最大值.

∴ $f(x)$有最小值-1，无最大值。

6. 当$x\neq0$时，$f'(x)=\arctan\frac{1}{x^2}+x\cdot\frac{-2x^{-3}}{1+\frac{1}{x^4}}$

$=\arctan\frac{1}{x^2}-\frac{2x^2}{1+x^4}$

当$x=0$时．$\lim\limits_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\arctan\frac{1}{x^2}$

$=\frac{\pi}{2}$

∴ $f'(0)=\frac{\pi}{2}$.

∴ $f'(x)=\begin{cases}\arctan\frac{1}{x^2}-\frac{2x^2}{1+x^4}, & x\neq0\\ \frac{\pi}{2}, & x=0\end{cases}$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \arctan\frac{1}{x^2} - \frac{2x^2}{1+x^4}$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} = f'(0)$$

$\therefore f'(x)$ 在 $x=0$ 处连续.

7. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{\frac{1}{t}\cdot t - \ln t}{t^2}}{\ln t + t\cdot\frac{1}{t}}$$

$$= \frac{1-\ln t}{t^2(\ln t+1)}$$

令 $\frac{dy}{dx}=0$ 得 $t=e$

当 $t=0, \frac{1}{e}$ 时, y'_x 不存在.

$\therefore t>1$ 内, 仅有 $t=e$ 即驻点 $x=e$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1-\ln t}{t^2(\ln t+1)}\right)\cdot\frac{dt}{dx}$$

$$= \frac{-\frac{1}{t}t^2(\ln t+1) - (1-\ln t)\left[2t(\ln t+1)+t^2\cdot\frac{1}{t}\right]}{t^4(\ln t+1)^2}\cdot\frac{1}{\ln t+1}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-2e}{e^4\cdot 2^2}\cdot\frac{1}{2}$$

$$= -\frac{1}{4e^3} < 0.$$

$\therefore y|_{x=e} = \frac{1}{e}$ 是极大值.

四. 1. 证: 令 $F(x) = e^x(f(x)-x)$

$F(x)$ 在 $[0,1]$ 上连续, 且在 $(0,1)$ 内可导

$F(0) = f(0) = 0 \qquad F(1) = e(f(1)-1) = 0$

$F(0) = F(1)$, $F(x)$ 满足 Rolle 定理条件,

则 $\exists \xi\in(0,1)$. s.t. $F'(\xi)=0$

$$F'(x) = (f'(x)-1)e^x + (f(x)-x)e^x$$

$$\therefore (f'(\xi)-1)e^\xi + (f(\xi)-\xi)e^\xi = 0$$

$\because e^\xi > 0 \quad \therefore f'(\xi) - 1 + f(\xi) - \xi = 0$

即 $f'(\xi) + (f(\xi)-\xi) = 1$

2. 证明: $x^3 - 3xy^2 + 2y^3 = 32$ 两边对 x

求导得. $3x^2 - 3y^2 - 3x\cdot 2y\cdot y' + 6y^2\cdot y' = 0$

$$\therefore y' = \frac{(x-y)(x+y)}{2y(x-y)} = \frac{x+y}{2y}$$

$x=y$ 舍.

令 $y'=0$. $y=-x$

代入 $x^3-3xy^2+2y^3=32$

得. $x=-2$, $y=2$.

$$y'' = \frac{(1+y')2y - (x+y)\cdot 2y'}{4y^2}$$

$$\therefore y''|_{x=-2} = \frac{1}{4} > 0.$$

$\therefore y=2$ 是极小值.