Properties of schemes

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1 Introduction

In the late 1950s, Grothendieck first introduced the concept of a scheme, a locally ringed space $(X, \mathcal{O}_X)$ such that X admits an open cover $\{U_i : i \in I\}$ such that the subspace $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme $\operatorname{Spec} A_i$ for some commutative ring A_i as a locally ringed space. The idea of schemes is to generalize varieties via the prime ideal spectrum of commutative rings. In an affine variety, points correspond to the maximal ideals of its coordinate ring, whereas in an affine scheme, points correspond to all prime ideals, with maximal ideals giving closed points and non-maximal prime ideals giving generic points representing irreducible closed subsets of the prime ideal spectrum.

A key innovation of schemes is that they allow *nilpotent elements* in coordinate rings, which makes concepts like intersection multiplicities precise. For example, if C and D are plane curves in $\mathbb{P}^2_k$ over a field k with no shared irreducible components, the "number of intersection points of C and D counted with multiplicity" can be defined as

$$\dim_k H^0(C \times_{\mathbb{P}^2} D, \mathcal{O}_{C \times_{\mathbb{P}^2} D}),$$

This allows Bézout's theorem to be stated rigorously. Another important feature of schemes is that algebraic geometry can be done over arbitrary rings, enabling, for example, the study of integral points on curves defined over number fields.

Now, we have already introduced schemes as a generalization of varieties, and it is essential that we investigate their basic properties. The fundamental properties of schemes can be defined as affine-local or global (i.e. if it suffices to check the properties on affine opens or it is necessary for us to consider the whole scheme). Similarly, we will explore the properties of morphisms of schemes, since many important geometric notions, such as projectivity, smoothness, and dimension—are properties of morphisms rather than of schemes in isolation. For instance, $\mathbb{P}^1_{\mathbb{C}}$ is projective over $\mathbb{C}$, but not over $\mathbb{Q}$; in the language of schemes, one says that the map $\mathbb{P}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is a projective morphism. Understanding morphisms thus allows us to study how schemes behave relative to each other and to their base.

2 Affine-Local Properties of schemes

If we want to construct global schemes by gluing affine pieces together, it is important to know how different intersections of affine opens look. For example, the projective line $\mathbb{P}^1_k$ can be obtained by gluing two affine schemes $U_1 = \operatorname{Spec} k[x]$ and $U_2 = \operatorname{Spec} k[y]$, identifying them on the overlap $U_1 \cap U_2$ by using the relation $y = 1/x$. This proposition tells us that the overlap of two affine open subschemes is again covered by affine opens that are distinguished in both subschemes.

Recall that for a commutative ring A and an element $f \in A$, the distinguished open subset of the zariski topological space $\operatorname{Spec} A$ associated to f is

$$D_A(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

It is not hard to prove that the distinguished open subset is naturally homeomorphic to $\operatorname{Spec} A_f$, where A_f is the localization of A at f. Also note that $\{D_A(f) : f \in A\}$ form a basis for the Zariski topology endowed on $\operatorname{Spec} A$.

Proposition 2.1. Let X be a scheme, let $\operatorname{Spec} A$ and $\operatorname{Spec} B$ be affine open subschemes in X. Then $\operatorname{Spec} A \cap \operatorname{Spec} B$ can be written as a union of open subsets that are simultaneously distinguished open subschemes in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

Proof. Pick an arbitrary point $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$. Since $\operatorname{Spec} A \cap \operatorname{Spec} B$ is open in $\operatorname{Spec} A$, there exists $a \in A$ such that

$$x \in D_A(a) \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B.$$

Similarly, since $D_A(a)$ is open in $\operatorname{Spec} B$, there exists $b \in B$ such that

$$x \in D_B(b) \subseteq D_A(a) \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B.$$

Now consider the ring homomorphisms induced by the inclusions of open subsets:

$$B \longrightarrow A_a \longrightarrow B_b.$$

Denote by $\widetilde{b} \in A_a$ the image of $b \in B$. Then we claim there is an isomorphism

$$A_{a\widetilde{b}} \simeq B_b.$$

In $A_{a\widetilde{b}}$, the image of b is a unit, and in B_b, the image of $\widetilde{b}$ is a unit. By the universal property of localization, the maps

$$B_b \longrightarrow A_{a\widetilde{b}} \quad \text{and} \quad A_{a\widetilde{b}} \longrightarrow B_b$$

are mutually inverse ring homomorphisms, giving the desired isomorphism.

Thus, the distinguished opens satisfy

$$D_A(a\widetilde{b}) = D_B(b),$$

and each $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$ lies in such a distinguished open. Therefore, we have a covering

$$\operatorname{Spec} A \cap \operatorname{Spec} B = \bigcup_x U_x,$$

where $U_x = D_A(a_x \widetilde{b}_x) = D_B(b_x)$ is simultaneously a distinguished open in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

Finally, note that

$$D_A(a_x \widetilde{b}_x) \simeq \operatorname{Spec} A_{a_x \widetilde{b}_x}, \quad D_B(b_x) \simeq \operatorname{Spec} B_{b_x}.$$

This completes the proof. □

That means that the intersection of any two arbitrary affine open subschemes of a scheme X can be written as $\bigcup_x U_x$ where $U_x = D_A(a_x) = D_B(b_x)$ for some suitably chosen $a_x \in A$ and $b_x \in B$. Therefore, we know that overlaps of affine opens are again covered by affine opens in a natural way, which is crucial for gluing affine schemes to construct global schemes. And now it's time to introduce the affine-local property:

Definition 2.2. We say a property P of a ring A is *affine-local* if:

1. (Locality) If A satisfies P, then A_f satisfies P for any $f \in A$
2. (Gluing condition) If there exists $\{f_1, \dots, f_r\} \subseteq A$ such that $(f_1, \dots, f_r) = A$, and A_{f_i} satisfy P for all i, then A satisfies P.

That is, the affine-local property is preserved under localization of the ring, and if the localization of the ring at every element in a finite generating set has the property, then the entire ring also has the property.

Example 2.3. Being reduced is a affine-local property of rings:

(1) If a ring A is reduced then A_f is reduced for any $f \in A$: if $x \in A_f$ such that $x^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$ in A_f, then write $x = a/f^m$ for some $m \in \mathbb{Z}_{\geq 0}$ with $a \in A$. Hence we have $0 = x^n = (a/f^m)^n = a^n/f^{mn}$, so there exists $k \in \mathbb{Z}_{\geq 0}$ such that $f^k(1 \cdot a^n - 0 \cdot f^{mn}) = 0$, so $f^k a^n = 0$, $f^{nk} a^n = 0$, and $(f^k a)^n = 0$, since A has no nilpotents, we know that $f^k a = 0$ in A, hence $a/1 = 0$ in A_f, then $x = a/f^m = 0$.

(2) If there exists $\{f_1, \dots, f_r\} \in A$ such that $(f_1, \dots, f_r) = A$, and A_{f_i} is reduced for all i, then A is reduced: If we have $a \in A$ such that $a^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$ in A. Note the image of a under $\ell_{S_{f_i}} : A \to A_{f_i}, \quad a \mapsto a/1$ is nilpotent, hence $a/1 = 0$ by the reducedness of A_{f_i}, so there exists $n_i \in \mathbb{Z}_{\geq 0}$ such that ${f_i}^{n_i} = 0$ in A for every i. Define $N = \sum_{i=1}^r n_i$, since $A = (f_1, \dots, f_r)$, there exists $g_1, \dots, g_r \in A$ such that $1 = \sum_{i=1}^r g_i f_i$, note that $(\sum_{i=1}^r g_i f_i)^N = \sum_{i=1}^r g_i' f_i^N = \sum_{i=1}^r g_i' f_i^{N-n_i}(f_i^{n_i})$ for some suitable $g_i' \in A$ by partition of unity, and thus we have $a = 1 \cdot a = (\sum_{i=1}^r g_i f_i)^N \cdot a = 0$ since ${f_i}^{n_i} = 0$ in A for every i

Some other typical affine-local properties of rings are noetherianity, irreducibility, and coherence, they can be checked locally on affine open subsets.

Lemma 2.4 ([Tag01OO])**.** Let X be a scheme and P be an affine-local property of rings, then the followings are equivalent:

1. For every affine open subset $\operatorname{Spec} A \subseteq X$, A satisfies P
2. There exists some affine open cover $\{\operatorname{Spec} A_i : i \in I\}$ of X such that A_i has property P for all $i \in I$

Proof. The necessity is obvious and the sufficiency can be easily seen by the definition of the affine-local properties of rings: if we have an affine open cover $\{U_i \simeq \operatorname{Spec} A_i : i \in I\}$ of X such that every A_i has property P, then any other affine open $V = \operatorname{Spec} B \subseteq X$ can be covered by $\{U_i \cap V : i \in I\}$, and since $\operatorname{Spec} B$ is quasi-compact, it admits a finite affine open subcover $\{U_{i_1} \cup V, \ldots, U_{i_n} \cup V\}$ with $i_j \in I$ for all $j \in \{1, \ldots, n\}$. Note the Proposition 2.1 tells us that $U_{i_j} \cap V = D_B(g_{i_j}) \simeq \operatorname{Spec} B_{g_{i_j}}$ for some suitable $g_{i_j} \in B$ hence B is generated by the set $\{g_{i_1}, \ldots, g_{i_n}\}$. From locality, we know B_{g_i} satisfies P for all $i_j \in I$, and since $(g_{i_1}, \ldots, g_{i_n}) = B$, B itself must also satisfy P by the gluing condition. $\square$

This lemma tells us that the coordinate ring of an affine open cover brings the property to all coordinate rings of affine open subsets of X.

So far, we have seen that intersections of affine schemes can be covered by distinguished affine opens (Theorem 2.1), which tells us that many properties of a global scheme can be verified by looking at an affine open cover of the scheme, like being reduced, integral, etc, but we also want to investigate how the schemes are correlated to each other by studying the morphisms of schemes

3 Affine-Local Properties of Morphisms of Schemes

Recall that we actually have a bijection between two sets of morphisms:

$$Hom_{Rings}(A, B) \simeq Hom_{Sch}(\operatorname{Spec} B, \operatorname{Spec} A)$$

This tells us that some properties of a morphism of affine schemes are determined by the corresponding properties of the underlying ring homomorphism, which we call them affine-local properties, and now it is time to introduce the definition.

Definition 3.1. [JH24] We say a property P of the ring homomorphism $A \to B$ is *affine-local* if:

1. (Stable on the source) If $A \to B$ satisfies P, then for every $f \in A$, the localized map $A_f \to B_{f'}$ satisfies P, where f' is the image of f in B.
2. (Stable on the target) If $A \to B$ satisfies P, then for every $g \in B$, the map $A \to B_g$ satisfies P.

3. (Local on the source) If there exists $f_1, \dots, f_r \in A$ with $(f_1, \dots, f_r) = A$ and each localized map $A_{f_i} \to B_{f'}$ satisfies P where f_i' is the image of f_i in B, then $A \to B$ satisfies P.

4. (Local on the target) If there exists $g_1, \dots, g_s \in B$ with $(g_1, \dots, g_s) = B$ and each localized map $A \to B_{g_j}$ satisfies P, then $A \to B$ satisfies P.

Generally speaking, an affine-local property of a ring homomorphism is preserved under restriction to distinguished affine opens via localization and can be verified locally on affine charts.

Example 3.2 (Flatness as an affine-local property)**.** Recall that being flat means for any injective homomorphism of A-modules $N_1 \hookrightarrow N_2$, the induced homomorphism $N_1 \otimes_A M \to N_2 \otimes_A M$ is also injective.

A morphism $f : X \to Y$ is flat if over every point of Y, the fiber of X behaves like it is varying continuously in a geometric sense.

Let $A = k[x]$ and $B = k[x,y]/(y^2 - x)$ with the natural map $A \to B$, $x \mapsto x$. Then B is flat over A since B is free as an A-module with basis $\{1, y\}$.

1. **Stable on the source:** For $f \in A$, e.g., $f = x$, localizing gives

$$A_f = k[x]_x \quad \text{and} \quad B_f = (k[x,y]/(y^2 - x))_x,$$

which is flat over A_f.

2. **Stable on the target:** For $g \in B$, e.g., $g = y$, the localized map $A \to B_g$ is flat.

3. **Local on the source:** If $f_1, \dots, f_r \in A$ generates A and each localized map $A_{f_i} \to B_{f_i'}$ is flat, then $A \to B$ is flat.

4. **Local on the target:** If $g_1, \dots, g_s \in B$ generates B and each map $A \to B_{g_j}$ is flat, then $A \to B$ is flat.

The localization of a ring A at an element $f \in A$ is equivalent to restricting the Spec A to the distinguished affine open subset $D_A(f)$. If we have a finite generating set of unit ideal of A, say $(f_1, \dots, f_n) = A$, then $\{D_A(f_1), \dots, D_A(f_n)\}$ will form a finite open cover of $\operatorname{Spec} A$. Therefore, if a ring homomorphism becomes an isomorphism after localizing at each target, then geometrically it will induce an isomorphism on an affine open cover, so we have the following theorem.

Theorem 3.3. (Local isomorphisms gluing) Let $\phi : A \to B$ be a ring homomorphism and there exist $\{f_1, \dots, f_n\} \subseteq A$ such that $(f_1, \dots, f_n) = A$. If $\phi : A_{f_i} \to B_{\phi(f_i)}, \frac{a}{f_i^n} \mapsto \frac{\phi(a)}{\phi(f_i)^n}$ is an isomorphism for all $i \in \{1, \dots, n\}$, then ϕ is an isomorphism.

Proof. For injectivity, if $a \in \ker(\phi)$ then $\phi_{f_i}(\frac{a}{1}) = \frac{\phi(a)}{1} = 0$ in $B_{\phi(f_i)}$, so there exists N_i such that $f_i^{N_i} a = 0$ in A. Define $N = \max_i N_i$, then $f_i^N a = 0$ in A for all

i. Since $(f_1, \ldots, f_n) = A$, by partition of unity there exists $\{r_1, \ldots, r_n\}$ such that $1 = \sum_{i=1}^n r_i f_i = (\sum_{i=1}^n r_i f_i)^N = \sum_{i=1}^n r_i' f_i^N$ for some suitable $\{r_1', \ldots, r_n'\} \subseteq A$ so $a = a \cdot 1 = \sum_{i=1}^n r_i'(f_i^N a) = 0$, so $\ker(\phi) = 0$. For surjectivity, let $b \in B$, for each i, by the surjectivity of ϕ_{f_i}, we know there exists $a_i \in A$ and $m_i \in \mathbb{Z}_{\geq 0}$ such that $\phi_{f_i}(\frac{a_i}{f_i^{m_i}}) = \frac{b}{1}$ and $\frac{\phi(a_i)}{\phi(f_i)^{m_i}} = \frac{b}{1}$ for all i, so there exists some $t_i \in \mathbb{Z}_{\geq 0}$ such that $\phi(f_i)^{m_i+t_i} b = \phi(f_i^{t_i} a_i)$, define $N_i = m_i + t_i \in \mathbb{Z}_{\geq 0}$ and $c_i = f_i^{t_i} a_i \in A$, so $\phi(f_i)^{N_i} b = \phi(c_i)$ in B for all i, then define $N = \max_i N_i$, so $\phi(f_i)^N b = \phi(c_i')$, where $c_i' = f_i^{N-N_i} c_i \in A$, and since $1 = \sum_{i=1}^n s_i' f_i^N$ for some suitable $s_i' \in A$ by partition of unity, we have $b = \phi(\sum_{i=1}^n s_i' f_i^N) \cdot b = \sum_{i=1}^n \phi(s_i')\phi(f_i)^N b = \sum_{i=1}^n \phi(s_i')\phi(c_i') = \phi(\sum_{i=1}^n s_i' c_i') \in \operatorname{im}(\phi)$ □

The following proposition shows that the property P holds uniformly on all affine open subsets of Y and their preimages in X if and only if on this particular affine open cover of Y, every affine open subset of the preimage has property P on the corresponding ring map.

Once an affine-local property is verified on a particular affine open cover of the target, they automatically hold on all affine open subsets of the target

Proposition 3.4. [RV25] Let P be an affine-local property of ring homomorphism and $f : X \to Y$ be a morphism of scheme. Then the followings are equivalent:

1. There exists an open cover of $\{U_i \simeq \operatorname{Spec} A_i : i \in I\}$ such that $f^{-1}(U_i) = \bigcup_{j \in J_i} \operatorname{Spec} B_{ij}$ and the ring homomorphisms $A_i \to B_{ij}$ satisfy P
2. For any affine open $\operatorname{Spec} A \subseteq Y$ and $\operatorname{Spec} B \subseteq f^{-1}(\operatorname{Spec} A)$, the induced ring homomorphism $A \to B$ satisfies P.

Proof. Similarly to Lemma 2.4, by using the affine communication lemma and quasi-compactness of affine open subschemes and ending up with the definition of affine-local properties of ring homomorphisms. □

Example 3.5 (Projective space as a non-affine scheme)**.** The projective space $\mathbb{P}_k^n = (\mathbb{A}_k^{n+1} - 0)/\sim$ over a field k is a classical example of a non-affine scheme. For $n \geq 1$, $\mathbb{P}_k^n$ cannot be covered by a single affine open, yet it is a scheme because it can be covered by the standard affine opens (see [Har77] projective variety for more details)

$$U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\} \simeq \operatorname{Spec} k\Big[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\Big], \quad i = 0, \ldots, n.$$

Each U_i is affine, isomorphic to $\mathbb{A}_k^n$, and forms an open cover of $\mathbb{P}_k^n$. This allows us to check affine-local properties on $\mathbb{P}_k^n$ by examining the U_i.

For instance, properties such as being reduced, integral, Noetherian can be verified on each U_i. Since if these properties hold on the affine opens U_i, they will follow globally on $\mathbb{P}_k^n$.

Thus, $\mathbb{P}_k^n$ demonstrates that even non-affine schemes can have affine-local properties.

4 Global properties of morphisms of schemes

So far we have studied affine-local properties, which can be checked on affine charts. In this section we will look at some global properties of morphisms of schemes, obtained by gluing these affine-local conditions together acorss some open covers of the target. Let's go through some definitions:

Definition 4.1. A morphism of schemes $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a closed immersion if $f : X \to Y$ is a homeomorphism onto a closed subset of $|Y|$ and the map on structure sheaves $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is surjective.

Corollary 4.2. Every closed immersion is quasi-compact.

Proof. Let $f : Z \to X$ be a closed immersion. Firstly, we take an arbitrary affine open subset $U = \operatorname{Spec} A \subseteq X$, since $f_U^{\#} : \Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U, f_*\mathcal{O}_Z)$ is surjective, and $\Gamma(U, \mathcal{O}_X) = A$, we have an isomorphism $A/I \simeq \Gamma(f^{-1}(U), \mathcal{O}_Z)$ for some ideal $I \subseteq A$, and hence $f^{-1}(U) \simeq \operatorname{Spec} A/I$ is affine. (Actually this tells us every closed immersion is affine, we will see later). Secondly, let V be a quasi-compact open subset of X, and take an affine open cover $\{U_i : i \in I\}$ of Y, so we have $V = \bigcup_{i\in I}(U_i \cap V)$, since each $U_i \cap V$ is open in Y and U_i is affine open in Y, we know each $U_i \cap V$ is affine open in V, and by the quasi-compactness it admits an finite open subcover, so we have $V = \bigcup_{j=1}^{n}(U_{i_j} \cap V)$ so it can be written as a union of finite affine open subsets, where each $U_{i_j} \cap V$ is affine and quasi-compact. Therefore we know $f^{-1}(V) \simeq \bigcup_{i=1}^{n} f^{-1}(U_{i_j} \cap V)$ is also quasi-compact. $\square$

Definition 4.3. A morphism of schemes $f : X \to Y$ is separated if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. Note that every separated morphism of schemes is quasi-separated by the previous corollary (Recall that a morphism of schemes is quasi-separated if its diagonal is quasi-compact.)

Example 4.4. Given two morphisms of schemes $f : X \to Y$ and $g : Y \to Z$ where g is separated, then the induced graph $\Gamma_f : X \to X \times_Z Y$, $x \mapsto (x, f(x))$ is a closed immersion.

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[r] \arrow[d, "F_f"] & X \arrow[d] \\
Y \arrow[r, "\Delta_{Y/Z}"'] \arrow[dr, "id"] & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
 & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

By the construction of fiber products, it is not hard to verify that the top left square is Cartesian. That is, if there exists a scheme T and morphisms $u : T \to X \times_Z Y, v : T \to Y$ such that $F_f \circ u = \Delta_{Y/Z} \circ v$, then there exists a unique morphism $\phi : T \to X$ such that $\Gamma_f \circ \phi = u, f \circ \phi = v$.

Definition 4.5. A morphism of schemes $f : X \to Y$ is an affine morphism if there exists an affine open cover $\{V_i \simeq \operatorname{Spec} A_i : i \in I\}$ of Y such that $f^{-1}(V_i) \simeq U_i$ where $U_i \simeq \operatorname{Spec} B_i$ is affine.

Proposition 4.6. If a morphism of schemes $f : X \to Y$ is an affine morphism then every preimage of affine open subschemes under f is an affine scheme.

In simple terms, this proposition tells us that affineness of the preimage is determined by just one affine cover of Y.

5 Conclusion

In this article, we investigated how local properties can be extended to global properties in the theory of schemes. we began by illustrating saw some affine-local properties which can be checked on affine cases and then glued to yield global results, this tells us some theorems about schemes and their morphisms are controlled by their local pieces. Finally, we discussed with some global properties of morphisms to see how local-to-global methods provide an analogue for understanding both structure and behaviour in algebraic geometry. Goodbye!

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