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# Quantitative/qualitative region-change uncertainty/certainty in attribute reduction: Comparative region-change analyses based on granular computing

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ABSTRACT

Attribute reduction is a fundamental research theme in rough sets and granular computing (GrC). Its scientific construction originally depends on the region-change law. At present, only region-change non-monotonicity/monotonicity is mined in the quantitative/qualitative model. The in-depth region-change truth and its GrC mechanism have significance, especially for follow-up attribute reduction. This paper commences probing region-change essence, mainly from a novel uncertainty/certainty viewpoint. Concretely, we make comparative region-change analyses based on GrC, by resorting to the qualitative Pawlak-Model and quantitative DTRS-Model (the decision-theoretic rough set model). (1) Knowledge-coarsening is investigated to describe attribute deletion. (2) Granule-merging and its region-distribution are studied to probe region-change functions. (3) Region-change is analyzed in Pawlak-Model to mine qualitative region-change certainty and its relevant properties. (4) Region-change is analyzed in DTRS-Model to mine quantitative region-change uncertainty and its relevant properties. (5) Comparative region-change analyses are summarized, and further experiment verification is provided. Knowledge-coarsening and granule-merging establish GrC mechanisms for extensive region-change analyses. Quantitative/qualitative region-change uncertainty/certainty and relevant principles are discovered via DTRS-Model/Pawlak-Model. By virtue of the GrC technology and comparative strategy, this study reveals region-change uncertainty/certainty to deepen region-change non-monotonicity/monotonicity; furthermore, it underlies attribute reduction, especially with regard to quantitative models.



## 1. Introduction

*Rough set theory (RS-Theory)* [32] is a fundamental uncertainty analysis theory that is designed to process uncertain, imprecise, and incomplete data information. Its basis is *the rough set model (RS-Model)*. The initial RS-Model, Pawlak-Model [31], originates from qualitative definitions for regions and thus acts as only a qualitative model. Because it lacks quantitative mechanisms, Pawlak-Model cannot effectively tackle extensive quantitative problems that concern fault-tolerance or robustness. Uncertainty measure-based quantitative models, which are partly unified by subsethood measures [54], exhibit improvements and

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applications. As a classical uncertainty measure, the probability was introduced into RS-Theory to construct *the probabilistic rough set (PRS)* [1,28,45,51,52,59,73]. At present, PRS relies on measurability, generality, and flexibility to generate mainstream models of multiple types. With regard to the probability and threshold, *the decision-theoretic rough set (DTRS)* uses a pair of thresholds [57], the variable precision rough set uses a single threshold [75], the game-theoretic rough set determines the threshold via game theory [1], the Bayesian rough set compares the prior and posterior probabilities [42], and the parameterized rough set uses two thresholds to describe rough membership [6]. Note that the probability in PRS is only a type of relative measure with a statistical concentration. In contrast, an absolute measure that involves vivid intuition also exists, such as the grade. Accordingly, the grade measure is utilized to construct the graded rough set, which is a type of quantitative model that has absoluteness [24,55]. Furthermore, the relative and absolute measures systematically construct double-quantitative models to manifest diversity and completeness [16,65–67].

DTRS, a fundamental quantitative model, is appropriately introduced for relevant use. DTRS adopts the conditional probability and Bayesian risk decision to establish threshold-quantitative semantics and three-way decisions [57]. As a result, DTRS improves on some basic RS-Models and provides a quantitative exploration platform. For relevant studies, three-way decision superiority was analyzed in [52], model development and threshold calculation were studied in [5,16–19,37,38,43,70], attribute reduction was discussed in [10,14,29,58,68,69,71], and model applications on clustering and regression were researched in [12,20,25,60]. In particular, three-way decisions have been promoted to the three-way decision theory with extensive research [5,7,13,18–20,26,61,63,74].

Attribute reduction serves as an essential subject in RS-Theory. It utilizes optimization and generalization to exhibit effective applications in knowledge discovery and data mining [11,27,29,30,35,44–47,58,66]. In the classical pattern, attribute reduction is accompanied by the RS-Model and, thus, depends on *the region-change (Rg-Change)* law that is mined in the RS-Model. In the qualitative Pawlak-Model, *the classification-positive region (Cl-POS)* change has monotonicity. This benign feature, which essentially originates from general Rg-Change monotonicity, naturally causes the Cl-POS preservation criterion to construct Pawlak-Reduction [31,32]. In the quantitative model, Cl-POS change exhibits non-monotonicity, which has been verified [10,29,30,45,46,58,66]. This new phenomenon, which in fact comes from Rg-Change non-monotonicity, is not sufficient to inspire a reasonable reduction criterion. Therefore, quantitative reduction transcends qualitative Pawlak-Reduction, and some reduction anomalies emerge [30,46,58]. At present, the quantitative reduction construction becomes not only a focus but also a difficulty. As far as quantitative DTRS-Reduction is concerned, basic reducts with regard to the Cl-POS preservation, Cl-POS measure, minimum cost, region distribution, and structure hierarchy are constructed in [71], [14], [10], [29], and [68,69], respectively; moreover, general reducts are established via multiple measures [58].

The Rg-Change law in the RS-Model underlies the attribute reduction, especially in the usual regional approach. Rg-Change non-monotonicity/monotonicity is currently acquired in the quantitative/qualitative model. For the quantitative models, quantitative expansion brings some complexity that transcends the qualitative model. In particular, Rg-Change non-monotonicity brings some reduction confusions and hampers further quantitative reduction. What is the Rg-Change truth hidden behind the non-monotonicity phenomenon? What is the relevant formation mechanism? Both of these questions become important for the quantitative models' applications, especially for quantitative reduction development. However, there are rarely in-depth reports on the quantitative Rg-Change essence, especially from an uncertainty viewpoint. Based on an extensive survey, we have discovered that Rg-Change uncertainty acts as the objective truth and broad context for Rg-Change non-monotonicity. Thereby, Rg-Change uncertainty analyses become a novel and valuable approach, and relevant uncertainty mechanisms underlie quantitative reduction. Note that uncertainty is an essential feature and a critical subject in intelligent information processing. In RS-Theory, some uncertainty aspects were explored in [1,4,8,21,36,39,40,53,64], and the relevant uncertainty was specifically measured by information entropy [28,29,44,45]. In contrast, for the qualitative model, we discover that Rg-Change certainty becomes general to underlie Rg-Change monotonicity; thus, relevant Rg-Change certainty analyses are required as well.

For uncertainty/certainty analyses, *Granular computing (GrC)* [23,62] is one of the most effective technologies, and GrC-based uncertainty/certainty analyses have become a challenging focus. In fact, GrC is a powerful structural methodology for effectively processing hierarchical information. It highlights trialistic characteristics with regard to multiple granules, levels, and perspectives. GrC exhibits extensive research [13,15,33,34,47,49,50], which is especially based on information granulation. In particular, GrC is addressed concretely in RS-Theory [9,17,21,22,37,38,41,48,56,66]. Note that attribute reduction closely adheres to GrC. Attribute reduction mainly depends on knowledge granulation, which is a type of hierarchical transformation of knowledge structures, and knowledge granulation leads to structural coarsening, granular merging, and further regional change. Therefore, knowledge granulation becomes the root cause of presentative uncertainty/certainty, and the relevant uncertainty/certainty mainly depends on information transformation among different knowledge-granular hierarchies. In short, attribute reduction depends on knowledge granulation to closely follow GrC, while the GrC approach can thoroughly analyze attribute reduction, especially its Rg-Change uncertainty/certainty.

Against the above background, this paper begins to probe the Rg-Change essence in attribute reduction, mainly by adopting a novel uncertainty/certainty viewpoint. To reveal the relevant mechanisms and rules, qualitative Pawlak-Model and quantitative DTRS-Model are concretely utilized to generate comparative Rg-Change (certainty and uncertainty) analyses based on GrC. Five gradual parts are produced, as follows: (1) Knowledge-coarsening is studied to describe attribute deletion. (2) Granule-merging and its region-distribution are studied to probe Rg-Change functions. (3) Rg-Change is analyzed in the Pawlak-Model to mine qualitative Rg-Change certainty and its relevant properties. (4) Rg-Change is analyzed in the DTRS-Model to mine quantitative Rg-Change uncertainty and its relevant properties. (5) Comparative Rg-Change analyses are summarized, and further experiment verification is provided. By virtue of the GrC technology and comparative strategy, this study makes systematic Rg-Change analyses to perform uncertainty/certainty mining. As a result, it has three fundamental contributions.

(1) Knowledge-coarsening and granule-merging establish GrC mechanisms for Rg-Change analyses. Relevant results apply in general to extensive RS-Models.
(2) Quantitative/qualitative Rg-Change uncertainty/certainty and its relevant principles are mined to reveal the quantitative/qualitative Rg-Change essence. They deepen and explain quantitative/qualitative Rg-Change non-monotonicity/monotonicity.
(3) For quantitative models, the Rg-Change law and corresponding in-depth mechanism can largely eliminate current Rg-Change confusions. They further underlie the rational construction of quantitative attribute reduction.

The remainder of this paper is organized as follows. Section 2 reviews the Pawlak-Model and DTRS-Model. Section 3 studies knowledge coarsening. Section 4 explores granule-merging and its region-distribution, Rg-Change functions. Section 5 produces Rg-Change certainty analyses in the qualitative Pawlak-Model. Section 6 produces Rg-Change uncertainty analyses in the quantitative DTRS-Model. Section 7 provides comparative Rg-Change analyses and further experiment verification. Finally, Section 8 concludes this paper.

In this paper, the terms *qualitative* and *quantitative* adhere more to the Pawlak-Model and DTRS-Model, respectively, and the term *region* usually refers to the set. Herein, the abbreviations are introduced for simplification. The alphabet-based replacement includes the following: *Decision* → Dc, *Knowledge* → Kn, *Granule* → Gr, and *Region* → Rg. Thus, following phrases become clear, i.e., *Dc-Table, Kn-Coarsening, Kn-Preservation, Kn-Non-Preservation, Gr-Preservation, Gr-Merging, Rg-Preservation, Rg-Change, Rg-Changed/Rg-Unchanged Granule/Type, Rg-Distribution, Rg-Belonging, Rg-Certainty, and Rg-Uncertainty*. *POS*, *NEG*, and *BND* denote the set's positive, negative, and boundary regions, respectively, while *Cl-POS* denotes the classification's positive region. Moreover, $\mathbf{R}^{0+}$ labels the real number set with 0, and *iff* means "if and only if".

## 2. Pawlak-Model and DTRS-Model

As a preliminary, this section reviews Pawlak-Model and DTRS-Model by Refs. [31,32] and [52,57], respectively. In essence, both RS-Models exhibit qualitative and quantitative features, respectively, and their major semantic differences are presented in [73].

### 2.1. Pawlak-Model

RS-Theory focuses on data represented in an information table:

$$(U, AT, \{V_a : a \in AT\}, \{I_a : a \in AT\}),$$

where $U$ is the universe with finite objects, $AT$ is the finite attribute set, $V_a$ is the value domain for $a \in AT$, and $I_a$: $U \to V_a$ is an information function. Each object $x$ takes a value $I_a(x)$ on attribute $a$. *A decision table (Dc-Table)* is a special type of information table with $AT = C \cup D$ and $C \cap D \neq \emptyset$, where $C$ and $D$ are the condition and decision attribute sets, respectively. In this paper, Dc-Table acts as a main framework and is simply denoted as $(U, C \cup D)$; moreover, suppose $A \subseteq C$, $C' \subseteq C$, $c \in C$, $2^C = \{A : A \subseteq C\}$.

$$IND(A) = \{(x, y) \in U \times U : \forall a \in A, I_a(x) = I_a(y)\}$$

serves as an equivalence relation to cause equivalence class $[x]_A$. The classified structure $U/IND(A) = \{[x]_A : x \in U\}$ radically means knowledge, and the knowledge $R$ is also directly called/used without confusions. The basic granule $[x]_A$ is called the knowledge granule, and $|U/IND(A)|$ denotes the knowledge-granular cardinality. If $A_2 \subseteq A_1 \subseteq C$, then $IND(A_1) \subseteq IND(A_2)$; thus, knowledge $A_1$ deduces knowledge $A_2$ (or knowledge $A_2$ depends on knowledge $A_1$), which is denoted as $U/IND(A_1) \Longrightarrow U/IND(A_2)$ or the simpler $A_1 \Longrightarrow A_2$. Note that the knowledge deducibility/dependency serves as an essential property that underlies rough reasoning.

Set $X \subseteq U$ is also called a concept, its complementarity uses style $\neg X$. It produces the following *set-positive, set-negative, and set-boundary regions (POS, NEG, and BND)*:

$$\begin{cases} \mathrm{POS}_A(X) = \{x : [x]_A \subseteq X\}, \\ \mathrm{NEG}_A(X) = \{x : [x]_A \subseteq \neg X\}, \\ \mathrm{BND}_A(X) = \{x : [x]_A \cap X, \neg X \neq \emptyset\}. \end{cases} \tag{1}$$

POS/NEG and BND contain knowledge-based certainty and uncertainty for a concept, respectively. Moreover, the three-way regions exhibit a sort of qualitative description, and they determine Pawlak-Model and its qualitative feature. In other words, Pawlak-Model is essentially a qualitative model.

$D$ can also motivate equivalence relation $IND(D)$ and classification $U/IND(D)$, where $X \in U/IND(D)$ refers to the decision class. By combining all decision classes' POS, *the classification-positive region (Cl-POS)* of $D$ on $A$ is defined as follows:

$$\mathrm{POS}_A(D) = \bigcup_{X \in U/IND(D)} \mathrm{POS}_A(X). \tag{2}$$

$$\gamma_A(D) = \frac{|\mathrm{POS}_A(D)|}{|U|} \tag{3}$$

further constructs the dependency degree to evaluate the classification quality.

If $A_1 \Longrightarrow A_2$, then

$$\mathrm{POS}_{A_2}(X) \subseteq \mathrm{POS}_{A_1}(X),\ \mathrm{NEG}_{A_2}(X) \subseteq \mathrm{NEG}_{A_1}(X),\ \mathrm{BND}_{A_2}(X) \supseteq \mathrm{BND}_{A_1}(X); \tag{4}$$

$$\mathrm{POS}_{A_2}(D) \subseteq \mathrm{POS}_{A_1}(D); \tag{5}$$

$$\gamma_{A_2}(D) \le \gamma_{A_1}(D). \tag{6}$$

These results reflect change monotonicity for the three-way regions, Cl-POS, and dependency measure. According to the integration definition (Formula (2)), Cl-POS change monotonicity originates from POS change monotonicity. Furthermore, Cl-POS preservation becomes a natural reduction criterion to define Pawlak-Reduct. Concretely, $B \subseteq C$ is Pawlak-Reduct of $C$ if it satisfies conditions (1) (2) or (1) (3):

(1) $\mathrm{POS}_B(D) = \mathrm{POS}_C(D)$;
(2) $\mathrm{POS}_{B'}(D) \neq \mathrm{POS}_C(D),\ \forall B' \subset B$.
(3) $\mathrm{POS}_{B-\{b\}}(D) \neq \mathrm{POS}_B(D),\ \forall b \in B$.

Pawlak-Reduct with regard to regional monotonicity could be generalized, and generalized reducts are discussed via monotonicity target preservation [66].

## 2.2. DTRS-Model

DTRS-Model comes from the two-category problem [52,57], where only two states and three actions exist. States $X$ and $\neg X$ indicate that an element is in $X$ and not in $X$, respectively, while actions $a_P$, $a_B$, and $a_N$ decide that an object is in sets POS($X$), BND($X$), and NEG($X$), respectively. When an object belongs to $X$, let $\lambda_{PP}$, $\lambda_{BP}$, and $\lambda_{NP}$ denote the costs of taking actions $a_P$, $a_B$, and $a_N$, respectively; in contrast, $\lambda_{PN}$, $\lambda_{BN}$, and $\lambda_{NN}$ denote the costs of taking the same three actions in the contrary condition. Thus, the loss functions are expressed as a $2 \times 3$ matrix:

| | $a_P$ | $a_B$ | $a_N$ |
|---|---|---|---|
| $X$ | $\lambda_{PP}$ | $\lambda_{BP}$ | $\lambda_{NP}$ |
| $\neg X$ | $\lambda_{PN}$ | $\lambda_{BN}$ | $\lambda_{NN}$ |

According to the Bayesian decision procedure, thresholds $\alpha$ and $\beta$ gain cost-based calculation formulas:

$$\begin{cases} \alpha = \dfrac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \\ \beta = \dfrac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \end{cases}$$

where $0 \le \beta < \alpha \le 1$. Three-way decisions (with regard to acceptance, rejection, and deferment) are established as follows:

$$\begin{cases} \text{If } Pr(X|[x]_A) \ge \alpha, \text{ then decide } [x]_A \subseteq \mathrm{POS}(X); \\ \text{If } Pr(X|[x]_A) \le \beta, \text{ then decide } [x]_A \subseteq \mathrm{NEG}(X); \\ \text{If } Pr(X|[x]_A) \in (\beta, \alpha), \text{ then decide } [x]_A \subseteq \mathrm{BND}(X), \end{cases}$$

where $Pr(X|[x]_A) = \frac{|X \cap [x]_A|}{|[x]_A|}$ denotes the conditional probability for $X$. Note that the three-way decisions are promoted to the three-way decision theory [5,7,13,18,26,61,63,74]. Furthermore, three-way regions (i.e., POS, NEG, and BND) are acquired as follows:

$$\begin{cases} \mathrm{POS}_A^{\alpha,\beta}(X) = \{x : Pr(X|[x]_A) \ge \alpha\}, \\ \mathrm{NEG}_A^{\alpha,\beta}(X) = \{x : Pr(X|[x]_A) \le \beta\}, \\ \mathrm{BND}_A^{\alpha,\beta}(X) = \{x : Pr(X|[x]_A) \in (\beta, \alpha)\}. \end{cases} \tag{7}$$

These regions exhibit a sort of quantitative description on the probability and threshold, and they determine DTRS-Model and its qualitative feature. In other words, the DTRS-Model becomes a quantitative model.

Probability $\frac{|X \cap [x]_A|}{|[x]_A|}$ serves as a core measure in RS-Theory [66]. It corresponds to several different terms, such as the rough membership [6], misclassification degree [75], and precision [67]. Generally, it is promoted to the inclusion degree to implement uncertainty measuring and approximation reasoning [36,63]. Herein, the probability could be utilized to provide a measure style for Pawlak-Model, i.e.,

$$\begin{cases} \mathrm{POS}_A(X) = \{x : Pr(X|[x]_A) = 1\}, \\ \mathrm{NEG}_A(X) = \{x : Pr(X|[x]_A) = 0\}, \\ \mathrm{BND}_A(X) = \{x : Pr(X|[x]_A) \in (0, 1)\}. \end{cases} \tag{8}$$

By comparing Formulas (7) and (8), DTRS-Model utilizes the probability measure to quantitatively expand the qualitative Pawlak-Model, which is only a special case with regard to $(\alpha, \beta) = (1, 0)$.

The above two-category DTRS model underlies multi-category generalization [72]. Next, Cl-POS is focused on in the multi-category case. At present, DTRS Cl-POS' construction exhibits two main approaches. One approach utilizes multiple two-category regions to make the composite construction [14,58], while the other resorts to performing an optimal decision in a high-dimensional space [10,71]. In view of regional hierarchical structures, the former is utilized in this paper's research. Thus, Cl-POS and the dependency degree, respectively, exhibit the following forms:

$$\mathrm{POS}_A^{\alpha,\beta}(D) = \bigcup_{X \in U/IND(D)} \mathrm{POS}_A^{\alpha,\beta}(X), \tag{9}$$

$$\gamma_A^{\alpha,\beta}(D) = \frac{|\mathrm{POS}_A^{\alpha,\beta}(D)|}{|U|}. \tag{10}$$

## 3. Knowledge-coarsening

RS-Theory originates from knowledge, while knowledge underlies granulation uncertainty and attribute reduction. Knowledge research, which concerns granular hierarchical structures, contributes to a thorough exploration for RS-Theory uncertainty essence. Thus, this section utilizes GrC to explore knowledge-coarsening, which accompanies attribute reduction. The relevant definition, types, and properties establish a macro GrC basis for extensive Rg-Change analyses.

### *3.1. Knowledge-coarsening's definition and types*

This section focuses on knowledge-coarsening's basic definition and two types.

**Lemma 3.1.1** (Kn-Monotonicity)**.** *Deleting $C'$ in $C$ implies $IND(C - C') \supseteq IND(C)$, i.e., $C \Longrightarrow C - C'$.*

$$\forall [x]_C \in U/IND(C),\ \exists [x]_{C-C'} \in U/IND(C - C'),\ s.t.,\ [x]_{C-C'} \supseteq [x]_C. \tag{11}$$

$$|U/IND(C - C')| \le |U/IND(C)|. \tag{12}$$

Attribute deletion, which is a basic process in attribute reduction, causes hierarchical transformation between knowledge structures. It has essential *knowledge monotonicity (Kn-Monotonicity)*. In fact, $C \Longrightarrow C - C'$ reflects a natural conclusion: knowledge never becomes finer for attribute deletion. Herein, two concrete granular results are explained as follows.

(1) Formula (11) uses granules to describe hierarchical structures between old and new knowledge. Concretely, each granule that regards old knowledge is necessarily included in a type of granule that regards new knowledge. This result implies the existence of a mapping (between knowledge structures), which will emerge in Proposition 3.1.5.
(2) Formula (12) exhibits the cardinality feature of knowledge-granules. Concretely, the knowledge-granular number never increases for attribute deletion.

With regard to attribute deletion, knowledge – a type of granular structure – is generally coarsened. This type of knowledge granulating acts as an essential source of external expressional uncertainty, including Rg-Change uncertainty, which is introduced later. Thus, Kn-Monotonicity is proposed to specifically describe the structural transformation process of attribute deletion.

**Definition 3.1.2** (Kn-Coarsening)**.** The granulating process in which old knowledge structure $U/IND(C)$ is generally coarsened to new knowledge structure $U/IND(C - C')$ is called *knowledge-coarsening* (*Kn-Coarsening*). This process is concretely denoted as follows:

$$C \overset{-C'}{\Longrightarrow} C - C' \ (\text{or}\, C \overset{-}{\Longrightarrow} C - C' \ \text{or}\, C \Longrightarrow C - C'), \tag{13}$$

where $C \overset{-c}{\Longrightarrow} C - \{c\}$ is directly used when $C' = \{c\}$. Acquired knowledge $U/IND(C - C')$ is called *coarsening knowledge* with regard to $C$; furthermore, *the coarsening knowledge set* with regard to $C$ becomes:

$$CKS_C = \{U/IND(C - C') : C' \subseteq C\}, \tag{14}$$

$$\text{where } U/IND(C - C) = U/IND(\emptyset) = \{U\},\ U/IND(C - \emptyset) = U/IND(C). \tag{15}$$

According to Kn-Monotonicity, Kn-Coarsening is formally defined from $C$. More generally, generalized Kn-Coarsening $A_1 \overset{-}{\Longrightarrow} A_2$ also exists in $C$. In essence, Kn-Coarsening refers to a type of knowledge granulating between knowledge hierarchies. Concretely, Kn-Coarsening corresponds to a coarsening transformation between two types of knowledge structures, where the term *coarsening* is in the generalized sense. Kn-Coarsening describes attribute deletion and provides its GrC essence. Thus, Kn-Coarsening plays an important role in attribute reduction and becomes a basic descriptive tool in later discussion.

As the opposite of Kn-Coarsening, knowledge-refining (*Kn-Refining*) can similarly be produced to describe attribute addition. Herein, both Kn-Coarsening and Kn-Refining concern horizontal attribute change with regard to a parallel structure. In fact, the terms *coarsening* and *refining* were previously concerned. In Refs. [2,3], the approximation maintenance for attribute value coarsening and refining are discussed; there, the coarsening and refining mainly involve longitudinal attribute change with regard to a hierarchical structure. In contrast, both Kn-Coarsening and Kn-Refining more closely adhere to attribute reduction.

**Definition 3.1.3.** For Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$,

(1) it is *knowledge preservation (Kn-Preservation)* if $U/IND(C - C') = U/IND(C)$;
(2) it is *knowledge non-preservation (Kn-Non-Preservation)* if $U/IND(C - C') \neq U/IND(C)$.

According to knowledge-structural relationships, Kn-Coarsening naturally exhibits two types, i.e., Kn-Preservation and Kn-Non-Preservation. Kn-Non-Preservation means strict knowledge coarsening in view of Kn-Monotonicity and, thus, becomes a main body. Both types could cause different granular results, as follows.

**Corollary 3.1.4.** *For Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$,*

(1) it is Kn-Preservation,

$$\text{iff } \forall [x]_C \in U/IND(C),\ \exists [x]_{C-C'} \in U/IND(C - C'),\ s.t.,\ [x]_{C-C'} = [x]_C, \text{ and iff } |U/IND(C - C')| = |U/IND(C)|;$$

(2) it is Kn-Non-Preservation,

$$\text{iff } \exists [x]_C \in U/IND(C),\ \exists [x]_{C-C'} \in U/IND(C - C'),\ s.t.,\ [x]_{C-C'} \supset [x]_C, \text{ and iff } |U/IND(C - C')| < |U/IND(C)|.$$

Based on Kn-Monotonicity, two knowledge-granular relationships are acquired for both Kn-Coarsening types. As a result, both types could be effectively identified by knowledge-granular cardinality-change. These different granular results will further cause distinctive regional manifestations (Section 4.1). In mathematics, they correspond to the following basic mappings.

**Proposition 3.1.5.** *In Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$, define mapping*

$$g : U/IND(C) \longrightarrow U/IND(C - C'),\ g([x]_C) = [x]_{C-C'} \supseteq [x]_C,\ \forall [x]_C \in U/IND(C). \tag{16}$$

*Then, g is a surjective mapping. $\forall [x]_{C-C'} \in U/IND(C - C')$, inverseimage set $g^{-1}(\{[x]_{C-C'}\})$ contains either an old granule $[x]_{C-C'}$ or multiple old granules in $[x]_{C-C'}$.*

(1) *g is a one-to-one mapping, iff $\forall [x]_{C-C'} \in U/IND(C - C')$ such that $g^{-1}(\{[x]_{C-C'}\}) = \{[x]_C\}$, and iff $C \overset{-C'}{\Longrightarrow} C - C'$ is Kn-Preservation.*
(2) *g is a non-injective mapping, iff $\exists [x]_{C-C'} \in U/IND(C - C')$ such that $g^{-1}(\{[x]_{C-C'}\}) \supset \{[x]_C\}$, and iff $C \overset{-C'}{\Longrightarrow} C - C'$ is Kn-Non-Preservation.*

Surjective mapping $g$ is the latent mapping, which is implied by Kn-Monotonicity (Formula (11)). Herein, the mapping $g$'s determination and surjection utilize the association $[x]_{C-C'} \supseteq [x]_C$. In contrast, $g^{-1}$ describes the inverseimage set and is not necessarily a mapping; $g$ and $g^{-1}$ describe knowledge hierarchies in two different directions. According to Proposition 3.1.5, Kn-Coarsening corresponds to a surjective mapping between knowledge-granular structures; furthermore, this mapping evolves into a one-to-one/non-injective mapping for Kn-Preservation/Kn-Non-Preservation.

### *3.2. Knowledge-coarsening's properties*

This section investigates knowledge-coarsening's properties, which arise mainly via the coarsening knowledge set.

For $C' \subseteq C$, all types of Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$ contain the same initial knowledge $C$ but different coarsening knowledge $C - C'$. Accordingly, the coarsening knowledge set $CKS_C$ carries all of the available knowledge, which can be realized by Kn-Coarsening from $C$. In other words, $CKS_C$ reflects all of the attribute deletion actions from $C$. Therefore, $CKS_C$'s description and measurement can thoroughly describe Kn-Coarsening or attribute deletion. Next, $CKS_C$'s structure and measure produce some basic mathematical results, which adhere to Kn-Coarsening.

**Theorem 3.2.1** ($CKS_C$ Complete Lattice)**.** *Knowledge deduction relation $\Longrightarrow$ is a partial order on $CKS_C$. Furthermore, $(CKS_C, \Longrightarrow)$ constitutes a complete lattice, where $U/IND(C)$ and $U/IND(\emptyset)$ serve as the least and greatest elements, respectively.*

$CKS_C$ becomes a complete lattice with regard to the deduction relation $\Longrightarrow$. This complete lattice of coarsening knowledge, which originates from hierarchical knowledge structures, especially describes a fundamental mathematical structure of Dc-Table Kn-Coarsening. As a result, it reflects the attribute reduction's mechanism and underlies rough reasoning.

**Corollary 3.2.2.** $\forall U/IND(C-C') \in CKS_C, U/IND(C) \Longrightarrow U/IND(C-C')$.

$U/IND(C)$, the least element of the complete lattice $(CKS_C, \Longrightarrow)$, can deduce any coarsening knowledge. This property accords with Kn-Monotonicity and further reflects Kn-Preservation's extremum. As a result, Kn-Preservation holds fundamental significance for attribute reduction. In fact, Kn-Preservation maintains the initial structures of knowledge $C$ to act as a basic state for deducibility/dependency. Furthermore, it becomes the most reliable action in attribute reduction; thus, the corresponding reduct is worthwhile to develop, especially in a complex quantitative environment. As is noted in [68,69], Kn-Preservation Reduct of Dc-Table $(U, C \cup D)$ becomes the strongest hierarchical Dc-Table reduct, while is equivalent to the classical reduct of the information table $(U, C)$.

**Theorem 3.2.3** (Complete Lattice Homomorphism)**.** *Define mapping*

$$f: 2^C \longrightarrow CKS_C,\ f(C') = U/IND(C-C'),\ \forall C' \in 2^C. \tag{17}$$

*Then, f is surjective to preserve the orders between two complete lattices:* $(2^C, \subseteq)$ *and* $(CKS_C, \Longrightarrow)$*. In other words, both complete lattices utilize f to exhibit a complete lattice homomorphism.*

**Proof.** Clearly, both $(2^C, \subseteq)$ and $(CKS_C, \Longrightarrow)$ are complete lattices, and $f$ is a surjective mapping according to its construction. Let $C'_1 \subseteq C'_2 \subseteq C$, then $C-C'_1, C-C'_2 \in CKS_C$ and $C-C'_1 \Longrightarrow C-C'_2$. Hence, $f$ preserves the relevant orders to establish a complete lattice homomorphism. □

In mathematics, a homomorphism can effectively describe the fundamental structure between systematic hierarchies. Herein, $f$ implements a complete lattice homomorphism from the classical set lattice $(2^C, \subseteq)$ to the coarsening knowledge lattice $(CKS_C, \Longrightarrow)$. This algebraic result deeply describes the structural relationship between the deleted condition attribute and final coarsening knowledge. As a result, the $\subseteq$ fundamentality is utilized to manifest the $\Longrightarrow$ scientificity. For the least and greatest elements, $\emptyset$ and $U$ correspond to $U/IND(C)$ and $U/IND(\emptyset)$, respectively; furthermore, the inverseimage set

$$f^{-1}(\{U/IND(C)\}) = \{C' : C' \subseteq C, f(C') = U/IND(C)\}$$

reflects the redundant attribute subsets with regard to Kn-Preservation. Note that the above complete lattice homomorphism cannot usually be promoted to a complete lattice isomorphism because $f$ is not always an injective mapping.

**Definition 3.2.4.** Mapping $m : CKS_C \longrightarrow [r_1, r_2] \subset \mathbf{R}^{0+}$ is called *a general measure* on the coarsening knowledge set, if it has the order-preservation feature, i.e.,

$$U/IND(C-C'_1) \Longrightarrow U/IND(C-C'_2),\ \text{then}\ m(U/IND(C-C'_1)) \le m(U/IND(C-C'_2)). \tag{18}$$

In mathematics, a measure is a mapping from the observed objects to $\mathbf{R}^{0+}$ and has some basic structures. For a measure on $CKS_C$, order-preservation becomes necessary because $CKS_C$ and $\mathbf{R}^{0+}$ have the partial orders $\Longrightarrow$ and $\le$, respectively. According to the order-preservation requirement, the general measure is proposed to satisfy natural monotonicity, where the order-preservation adopts a general direction. Herein, $[r_1, r_2]$ represents the limited range of measured values on $CKS_C$, and $([r_1, r_2], \le)$ also becomes a complete lattice even though $(\mathbf{R}^{0+}, \le)$ is not.

**Proposition 3.2.5.** *General measure m on $CKS_C$ is an order-preservation mapping from complete lattices $(CKS_C, \Longrightarrow)$ to $([r_1, r_2], \le)$. Moreover, $r_1 = m(U/IND(C))$, $r_2 = m(U/IND(\emptyset))$. Thus, measure m's boundedness is also expressed by:*

$$m(U/IND(C)) \le m(U/IND(C-C')) \le m(U/IND(\emptyset)),\ \forall U/IND(C-C') \in CKS_C. \tag{19}$$

The general measure establishes an order-preservation mapping between both complete lattices, which cannot usually evolve into a complete lattice homomorphism/isomorphism. Herein, the general measure $m$ has boundedness with regard to $[m(U/IND(C)), m(U/IND(\emptyset))]$, i.e., $m(U/IND(C))$ and $m(U/IND(\emptyset))$ determine the measure's infimum and supremum, respectively.

The general measure depends on order-preservation to consider the algebraic structures of two complete lattices; thus, it becomes scientific. It has two fundamental features with regard to boundedness and monotonicity. Furthermore, it can produce a composite description by combining the previous mapping $f$, which is another order-preservation mapping.

**Corollary 3.2.6.** *The composite mapping of f and m becomes:*

$$m \circ f : 2^C \longrightarrow [r_1, r_2] \subset \mathbf{R}^{0+},\ (m \circ f)(C') = m(f(C')) = m(U/IND(C-C')),\ \forall C' \in 2^C. \tag{20}$$

*$m \circ f$ constructs an order-preservation mapping from complete lattices $(2^C, \subseteq)$ to $([r_1, r_2], \le)$. Thus, $m \circ f$ has the monotonicity and boundedness, i.e.,*

(1) $(m \circ f)(C'_1) \le (m \circ f)(C'_2)$, if $C'_1 \subseteq C'_2 \subseteq C$;
(2) $(m \circ f)(C') \in [m(U/IND(C)), m(U/IND(\emptyset))]$, $\forall C' \subseteq C$.

Finally, the section that regards Kn-Coarsening is summarized. Kn-Coarsening describes attribute deletion by uncovering its GrC essence and, thus, underlies attribute reduction. For Kn-Coarsening, this section provides relevant knowledge granulating mechanisms. Note that Kn-Coarsening and its properties depend on only Dc-Table's formal structure, with its main condition attribute set $C$. Hence, they exhibit fundamentality and generalization for all concrete RS-Models.

## 4. Granule-merging and its region-distribution

For attribute deletion, its essential GrC origin – Kn-Coarsening – is described at the macro knowledge level (Section 3). Furthermore, its relevant GrC mechanism is worthwhile to explore at the micro granule level. This section begins studying internal granule-merging in Kn-Coarsening, mainly from a regional perspective. Concretely, this section discusses granule-merging and its region-distribution to deeply mine Rg-Change functions. The relevant results reveal the micro GrC mechanism for extensive Rg-Change analyses.

Rg-Change naturally forms in terms of RS-Model's three-way regions and closely adheres to GrC in attribute reduction. In this paper, Rg-Change is usually in the general/strict sense at the knowledge/granule level; moreover, the term *Rg-Changed* refers to strictness.

### 4.1. Granule-merging

This section focuses on granule-merging in Kn-Coarsening.

**Theorem 4.1.1.** *If Kn-Coarsening has strict Rg-Change, then it is necessarily Kn-Non-Preservation.*

Kn-Coarsening has two types. Kn-Preservation means invariableness not only for knowledge granules but also for model regions. In contrast, Kn-Non-Preservation implies strict knowledge coarsening and the necessary granule merging, and the latter causes possible Rg-Change between knowledge hierarchies. Therefore, at the knowledge level, only Kn-Non-Preservation is responsible for Rg-Change.

Kn-Coarsening usually consists of some groups of general granular merging. However, only strict granular merging plays a mechanism function for Rg-Change. Hence, it becomes the internal inducement of strict Rg-Change. In other words, at the granule level, strict granular merging is responsible for Rg-Change. Next, granule-merging (in the strict sense) and its regional function are studied with regard to the underlying Rg-Change analyses.

**Theorem 4.1.2** (Merging Principle)**.** *Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$ is Kn-Non-Preservation, iff*

$$\begin{aligned} &\exists [x]_C^1, \ldots, [x]_C^m \in U/IND(C)\ (m \geq 2),\ \exists [x]_{C-C'}^1 \in U/IND(C-C'), \\ &\quad s.t., [x]_C^1 \cup \cdots \cup [x]_C^m = [x]_{C-C'}^1, \\ &\quad i.e., g([x]_C^1) = \cdots = g([x]_C^m) = [x]_{C-C'}^1\ or\, g^{-1}(\{[x]_{C-C'}^1\}) = \{[x]_C^1, \ldots, [x]_C^m\}. \end{aligned} \tag{21}$$

For Kn-Coarsening, the surjective mapping $g$ between knowledge-granular sets describes the micro dynamic change (Proposition 3.1.5). Furthermore, for Kn-Non-Preservation, the Merging Principle conducts an in-depth GrC description. It reflects the objective existence of the granular merging action, which accords with the non-injective mapping $g$. In fact, granule merging implements a hierarchical structure transformation from old knowledge-granules to new ones, and this micro mechanism action underlies macro Kn-Non-Preservation. Herein, granule merging is formally proposed to describe Kn-Non-Preservation's internal process.

**Definition 4.1.3** (Gr-Merging)**.** *Granule-merging (Gr-Merging)* refers to the granular combination process in Kn-Non-Preservation/Kn-Coarsening. Concretely, the action that $m$ $(m \geq 2)$ old granules $[x]_C^1, \ldots, [x]_C^m$ are merged into one new granule $[x]_{C-C'}^1$ is as follows:

$$\text{Gr-Merging } [x]_C^1 \cup \cdots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1. \tag{22}$$

For Gr-Merging $[x]_C^1 \cup \cdots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1$, its granular superscript denotes the granular serial number; thus, $m \geq 2$ implies Gr-Merging's strictness. According to the Merging Principle, Gr-Merging becomes an essential characteristic of Kn-Non-Preservation. Because the unchanged granule has no effect on variability, Gr-Merging actually acts as the micro essential mechanism of Kn-Coarsening, especially from the GrC viewpoint. Thus, Kn-Coarsening's study mainly relies on Gr-Merging's analysis, and only a representative group $[x]_C^1 \cup \cdots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1$ is sufficient.

**Definition 4.1.4** (Gr-Preservation)**.** *Granule-preservation (Gr-Preservation)* refers to the granular unchanged process in Kn-Coarsening. Concretely, the action that one old granule $[x]_C^1$ is transformed into one new granule $[x]_{C-C'}^1$ is as follows:

$$\text{Gr-Preservation } [x]_C^1 \xrightarrow{=} [x]_{C-C'}^1. \tag{23}$$

Gr-Preservation describes the granular unchanged transformation in Kn-Coarsening and, thus, becomes Gr-Merging's opposite. Gr-Preservation and Gr-Merging involve invariance and a combination of partial granules, respectively. They fully describe Kn-Coarsening.

**Proposition 4.1.5.** *Kn-Coarsening is accompanied by two and only two types of granular transformation, i.e., Gr-Merging and Gr-Preservation.*

(1) Kn-Non-Preservation necessarily concerns Gr-Merging but also possibly concerns Gr-Preservation.
(2) Kn-Preservation necessarily consists of Gr-Preservation but never involves Gr-Merging.

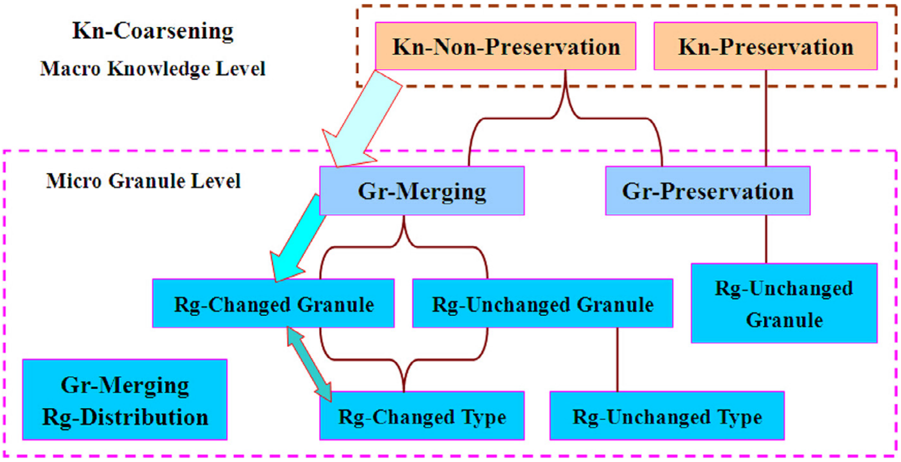


**Fig. 1.** Rg-Change's GrC hierarchy figure and cause probe route.

Herein, Gr-Merging and Gr-Preservation describe Kn-Coarsening and its two types by focusing on the granular mechanism. The concrete result is marked in Fig. 1 and actually reflects the relationships between the knowledge and granules. Thus, Kn-Coarsening usually consists of some groups of Gr-Preservation and Gr-Merging. Furthermore, Kn-Preservation demands purity of the Gr-Preservation group, while the Gr-Merging group exhibits a type of infectivity to cause Kn-Non-Preservation.

From a regional viewpoint, Gr-Merging underlies Rg-Change, especially when compared to Gr-Preservation. Next, we inspect Gr-Merging' position for Rg-Change analyses.

**Theorem 4.1.6** (Rg-Change Principle)**.** *In Kn-Non-Preservation/Kn-Coarsening, Rg-Change is determined by Gr-Merging (rather than by Gr-Preservation).*

**Proof.** This result is proved in macro and micro ways. (1) For both Kn-Coarsening types, only Kn-Non-Preservation has the Rg-Change possibility (Theorem 4.1.1). In fact, Gr-Merging acts as the main characteristic of Kn-Non-Preservation (Theorem 4.1.2). Thus, only it fully determines potential Rg-Change because Gr-Preservation has no influence on granules and further regions. (2) From the granular perspective, the granular micro action on regions radically determines macro Rg-Change because regions consist of granules. In fact, Kn-Coarsening includes two types of granular actions (Proposition 4.1.5), i.e., Gr-Preservation and Gr-Merging. The unchanged granules have no effects on Rg-Change, while all changed granules are necessarily concerned by Gr-Merging. Thus, Rg-Change is determined by only Gr-Merging. □

The Rg-Change Principle becomes fundamental for Rg-Change analyses. Its proof also provides some good explanations, such as Kn-Non-Preservation's necessity for strict Rg-Change and the changed granules' correspondence for Gr-Merging. For macro Rg-Change, Gr-Merging becomes the decisive factor from the granular transformation perspective. In fact, Gr-Merging also exhibits necessity for strict Rg-Change, and all of the Rg-Changed granules necessarily become changed granules in Gr-Merging. Thus, Rg-Change analyses rely on Gr-Merging analyses with regard to regional actions. In other words, mining Gr-Merging' regional function becomes a critical action that has decisive significance.

Strict Rg-Change at the macro region level completely corresponds to Rg-Changed granules at the micro granule level. In other words, Rg-Changed granule existence holds both sufficiency and necessity for strict Rg-Change. Thereby, Rg-Changed granules become a fundamental factor for Rg-Change analyses, especially from the granular mechanism. Next, Rg-Change granules are introduced for Gr-Merging to mine its regional function.

**Definition 4.1.7** (Rg-Changed/Rg-Unchanged Granule)**.** *Rg-Changed Granule* refers to the old granule that has strict Rg-Change, i.e., the old granule that belongs to two different regional types in Kn-Coarsening. The opposite of Rg-Changed Granule becomes *Rg-Unchanged Granule*.

**Proposition 4.1.8.**

(1) *Gr-Merging and Kn-Non-Preservation possibly concern Rg-Changed and Rg-Unchanged Granules.*
(2) *Gr-Preservation and Kn-Preservation involve only Rg-Unchanged Granule but no Rg-Changed Granule.*

Rg-Changed and Rg-Unchanged Granules are formally defined. They can effectively construct Gr-Merging/Gr-Preservation and Kn-Non-Preservation/Kn-Preservation. The relevant results are also reflected in Fig. 1. Herein, the Rg-Change Principle is utilized to mine Rg-Change's exact cause, i.e., the Rg-Changed Granule. Furthermore, exacting and analyzing Gr-Merging's Rg-Changed Granule could accurately provide Gr-Merging's regional function and further Kn-Coarsening's Rg-Change. However, this core process requires a concrete basis for Gr-Merging's regional description, which becomes the next focus.

### 4.2. Granule-merging region-distribution

By introducing three-way regions, this section establishes Gr-Merging's region-distribution to mine Gr-Merging's core function for Rg-Change. The obtained analysis algorithm and technology roadmap exhibit mechanism-based significance for Rg-Change analyses.

According to Section 4.1, Rg-Change analyses must analyze Gr-Merging's regional function based on Rg-Changed Granule. Thus, our primary task is to make an effective regional description for Gr-Merging. For this purpose, extensive POS, NEG, and BND are generally introduced regardless of RS-Model. Note that Gr-Merging is an action process and thus concerns two basic parts: some old granules and a new granule. Accordingly, the old and new granules' regional distributions radically determine and completely describe Gr-Merging's effect for Rg-Change. In other words, the two parts can construct a fundamental notion to extract Gr-Merging's regional function.

**Definition 4.2.1** (Gr-Merging Rg-Distribution)**.** *Gr-Merging Rg-Distribution* refers to the composite two-dimensional description of *Old Granules Rg-Distribution* and *New Granule Rg-Belonging* in Gr-Merging. Herein, the integrated description is constructed by the Cartesian product. Moreover, *Rg-Distribution/Rg-Belonging* denotes *region-distribution/region-belonging*, and *Rg-Distribution* applies to *Old Granules* and *Gr-Merging*.

**Example 1.** In Gr-Merging $[x]_C^1 \cup \cdots \cup [x]_C^m \stackrel{=}{\longrightarrow} [x]_{C-C'}^1$, if old $[x]_C^1, \ldots, [x]_C^m$ are in POS and NEG while new $[x]_{C-C'}^1$ belongs to BND, then *Gr-Merging Rg-Distribution* refers to ({POS, NEG}, {BND}). For simplicity, a vivid form "POS + NEG $\longrightarrow$ BND" and relevant tabular style could be adopted equivalently.

Gr-Merging is a dynamic action; thus, its regional description concerns two regional descriptions for the process beginning and ending. Continuing this thought, Gr-Merging Rg-Distribution utilizes the Cartesian product to combine the two basic regional factors, i.e., Old Granules Rg-Distribution and New Granule Rg-Belonging. Accordingly, Gr-Merging Rg-Distribution fully describes different region cases of the original and final granules, and Example 1 provides an explanation.

**Definition 4.2.2** (Rg-Changed/Rg-Unchanged Type)**.**

(1) If Gr-Merging concerns Rg-Changed Granule and thus holds the strict Rg-Changed function, then the relevant Gr-Merging Rg-Distribution is called *Rg-Changed Type*.
(2) If Gr-Merging concerns only Rg-Unchanged Granule and thus never holds the strict Rg-Changed function, then the relevant Gr-Merging Rg-Distribution is called *Rg-Unchanged Type*.

In Gr-Merging, all of the changed granules provide necessity for strict Rg-Change, while Rg-Changed Granule provides sufficiency and necessity. By virtue of Rg-Changed Granule, Gr-Merging Rg-Distributions reflect different Rg-Changed functions to produce two complementary qualitative types. Only Rg-Changed Type implies strict Rg-Change and thus provides relevant sufficiency and necessity. Both types and their relationships with Rg-Changed/Rg-Unchanged Granule are marked at the bottom of Fig. 1. Clearly, Rg-Changed Granule becomes a feature for discriminating the two types, and Rg-Changed Granule corresponds to Rg-Changed Type. Except for Rg-Unchanged Type, Rg-Changed Type can also involve Rg-Unchanged Granule.

Thus far, we can construct Rg-Change's GrC hierarchy figure, i.e., Fig. 1. There are four evolutionary hierarchies.

(1) The 1st hierarchy refers to Kn-Coarsening, which consists of Kn-Non-Preservation and Kn-Preservation.
(2) The 2nd hierarchy refers to granular transformation, which consists of Gr-Merging and Gr-Preservation.
(3) The 3rd hierarchy refers to the old granule, which consists of Rg-Changed and Rg-Unchanged Granules.
(4) The 4th hierarchy refers to Gr-Merging Rg-Distribution, which consists of Rg-Changed and Rg-Unchanged Types.

The 1st hierarchy and the latter three are located at the macro knowledge and micro-granule levels, respectively. The four hierarchies and their structural relationships are detailedly shown in Fig. 1.

Moreover, Fig. 1 marks the cause probe route via arrow labels. Kn-Non-Preservation and Gr-Merging reflect only necessity for strict Rg-Change. Furthermore, Rg-Changed Type and its corresponding Rg-Changed Granule provide relevant sufficiency and necessity. The former qualitatively carries Gr-Merging Rg-Distribution information, while the latter concretely gives Rg-Change feedback. In later discussions, Rg-Changed Type is emphatically focused on, while Rg-Changed Granule is mainly extracted to acquire specific Rg-Changed functions.

**Theorem 4.2.3.**

(1) *Kn-Non-Preservation/Kn-Coarsening has strict Rg-Change, iff its Gr-Merging has Rg-Changed Type Rg-Distribution.*
(2) *Kn-Non-Preservation/Kn-Coarsening preserves regions, iff its Gr-Merging has only Rg-Unchanged Type Rg-Distribution.*

**Proof.** Kn-Coarsening consists of Gr-Preservation and Gr-Merging, and Gr-Merging has two Rg-Distribution types. However, Gr-Preservation and Rg-Unchanged Type never have Rg-Changed functions, so strict Rg-Change relies on only Rg-Changed Type. In fact, Rg-Changed Type necessarily changes regions, while Rg-Unchanged Type never changes regions. □

Theorem 4.2.3 reflects relationships between Rg-Change/*Rg-Preservation (region-preservation)* and Rg-Changed/Rg-Unchanged Type. Concretely, strict Rg-Change and Rg-Preservation require Rg-Changed Type's existence and Rg-Unchanged Type's universality, respectively. Rg-Changed Type again reflects sufficiency and necessity for strict Rg-Change, thus radically

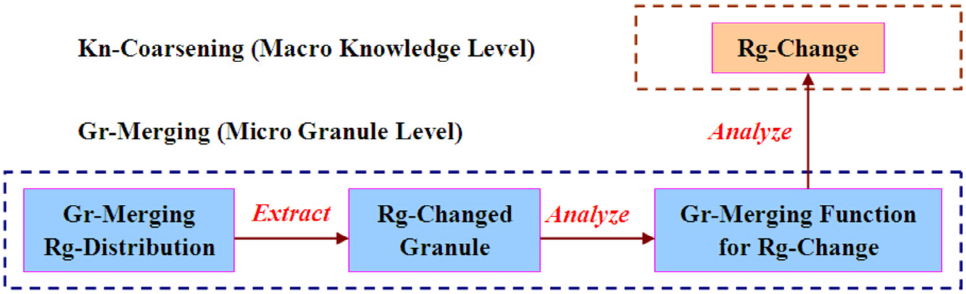


**Fig. 2.** Rg-Change analyses' technology roadmap based on Gr-Merging Rg-Distribution.

determining Rg-Change. Furthermore, regional analyses of all Rg-Changed Types can exactly exhibit Rg-Change results; thereby, extracting and analyzing Rg-Changed Granule in Gr-Merging become a basic step.

According to Theorem 4.2.3, we focus on Rg-Preservation, a natural reduction criterion that is weaker than Kn-Preservation [69]. Rg-Preservation requires whole purity of Rg-Unchanged Type, while Rg-Changed Type exhibits internal infectivity to cause strict Rg-Change. Therefore, pure Rg-Unchanged Type underlies Rg-Preservation's rule and reduct. This result sharply contrasts with that of Kn-Preservation (Proposition 4.1.5), where pure Gr-Preservation is basically required.

Based on the above studies, extracting Rg-Changed Granule can sufficiently exhibit Gr-Merging's regional influence for Rg-Change, and Gr-Merging Rg-Distribution establishes a basis. The Rg-Change analyses feasibility based on Gr-Merging Rg-Distribution is presented as follows.

**Proposition 4.2.4.** *According to Gr-Merging Rg-Distribution, Gr-Merging's Rg-Changed Granule can be extracted to effectively analyze Kn-Coarsening's Rg-Change.*

**Example 2.** Suppose $[x]_C^1 \cup \ldots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1$ has Gr-Merging Rg-Distribution "POS + NEG $\longrightarrow$ BND". Then, all old granules initially belong to POS or NEG but finally belong to BND, so they become Rg-Changed Granules to be successfully extracted. Herein, these granules $[x]_C^1 \ldots, [x]_C^m$ produce two classes.

(1) For the 1st class, all old granules enter new BND from old POS, so their effect for Rg-Change is to lessen POS while enlarge BND.
(2) For the 2nd class, all old granules enter new BND from old NEG, so their effect for Rg-Change is to lessen NEG while enlarge BND.

On the whole, Gr-Merging has the Rg-Change function to enlarge BND and to lessen POS/NEG.

Extracting Rg-Changed Granule and further analyzing Gr-Merging's regional function can gain Rg-Change in Kn-Coarsening. Example 2 provides a detailed illustration via a specific type of Gr-Merging Rg-Distribution. In fact, all of the types can be similarly analyzed, and by integrating all of the regional functions, Rg-Change is finally achieved. Thus, an algorithm process and a technology roadmap are naturally established for Rg-Change analyses.

To acquire Rg-Change, Algorithm 1 provides the evolutionary process via Gr-Merging Rg-Distribution. Step 1 provides the analyses basis: Gr-Merging Rg-Distribution. Step 2 extracts Rg-Changed Granule in Gr-Merging. Finally, Steps 3 and 4 analyze Rg-Change at the Gr-Merging and Kn-Coarsening levels, respectively. In particular, Step 4 reflects a comprehensive analysis process because Kn-Coarsening usually includes multiple groups of Gr-Merging.

**Algorithm 1** Rg-Change analyses based on Gr-Merging Rg-Distribution.

**Input:** Dc-Table $(C, D)$, Kn-Coarsening $C \overset{-C'}{\Longrightarrow} C - C'$, RS-Model;
**Output:** Rg-Change in Kn-Coarsening;
1: Establish Gr-Merging Rg-Distribution with regard to RS-Model;
2: Based on Gr-Merging Rg-Distribution, extract Rg-Changed Granule in Gr-Merging;
3: Based on Rg-Changed Granule, analyze Gr-Merging's function for Rg-Change;
4: Based on Gr-Merging's regional function, analyze Rg-Change by synthesizing all Gr-Merging;
5: **return** Rg-Change.

The relevant technology roadmap is provided in Fig. 2. Thereby, Proposition 4.2.4, Algorithm 1, and Fig. 2 constitute the whole Rg-Change analysis system with regard to Gr-Merging Rg-Distribution. The former one and the latter two provide theoretical feasibility and practical operability, respectively. The relevant operation steps correspond to the four structural hierarchies (from the bottom to top) in Fig. 1 and, thus, exhibit tightness. They will be illustrated by Example 5 (Section 6.1).

Aiming at Rg-Change, two opposite clues emerge, and they are explained by comparing Figs. 1 and 2.

**Table 1**
Complete Gr-Merging Rg-Distribution for extensive RS-Models.

| Old Granules Rg-Distribution | New Granule Rg-Belonging |
|---|---|
| (1) POS<br>(2) NEG<br>(3) BND<br>(4) POS, BND<br>(5) NEG, BND<br>(6) POS, NEG<br>(7) POS, NEG, BND | (1) POS<br>(2) NEG<br>(3) BND |

(1) In the theoretical analysis, we gradually construct the GrC hierarchy and deeply mine the internal hierarchical cause for Rg-Change. As is shown by the arrow labels in Fig. 1, the necessity for strict Rg-Change is searched from the macro knowledge to micro granule levels, and a diminishing route is achieved, i.e.,

*Kn-Coarsening/Kn-Non-Preservation* $\longrightarrow$ *Gr-Merging* $\longrightarrow$ *Rg-Changed Granule* $\longleftrightarrow$ *Rg-Changed Type.*

Finally, Rg-Changed Type and Rg-Changed Granule acquire sufficiency and necessity for strict Rg-Change.

(2) In a practical deduction, we successfully gain an effective technology process (Algorithm 1 and Fig. 2). For the concrete RS-Model, its Gr-Merging Rg-Distribution must first be established; then, further studies with regard to Rg-Changed Granule and Gr-Merging's Rg-Changed function accord with feature identification with regard to the Rg-Changed/Rg-Unchanged Type. In fact, these processes are located at the micro granule level. Furthermore, Rg-Change can be summarized at the macro knowledge level. Thus, Rg-Change analyses follow an evolutionary technology line:

*Rg-Changed Type* $\longleftrightarrow$ *Rg-Changed Granule* $\longrightarrow$ *Gr-Merging* $\longrightarrow$ *Kn-Non-Preservation/Kn-Coarsening,*

which becomes the opposite of the above theoretical route.

Based on the above algorithm and technology, Gr-Merging Rg-Distribution's construction and identification become the subsequent starting point for Rg-Change analyses. The next section first provides the broadest Gr-Merging Rg-Distribution regardless of the RS-Model.

### 4.3. Granule-merging region-distribution completeness

Aiming at the Gr-Merging Rg-Distribution, this section explores the largest completeness with regard to the general RS-Model, where extensive three-way regions are still utilized.

**Theorem 4.3.1** (Gr-Merging Rg-Distribution Completeness)**.** *In theory, there are at most 21 types of Gr-Merging Rg-Distribution. They are provided in* Table 1 *by fully combining the seven cases of Old Granules Rg-Distribution and the three cases of New Granule Rg-Belonging.*

**Proof.** With regard to POS, NEG, and BND, Old Granules Rg-Distribution and New Granule Rg-Belonging have seven and three cases, respectively. In terms of their full combination, Gr-Merging Rg-Distribution has at most 21 types in Table 1. □

**Proposition 4.3.2.** The 21 Rg-Distributional types accord with the complete many-to-many mathematical correspondence, which is from *the seven-element set of Old Granules Rg-Distribution* to *the three-element set of New Granule Rg-Belonging*.

**Example 3.** For Gr-Merging $[x]_C^1 \cup \cdots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1$, if old granules $[x]_C^1, \ldots, [x]_C^m$ come from POS and NEG, then new granule $[x]_{C-C'}^1$ possibly belongs to POS, NEG, and BND. Thus, three distributional types emerge, i.e.,

(1) "POS + NEG $\longrightarrow$ POS", (2) "POS + NEG $\longrightarrow$ NEG", and (3) "POS + NEG $\longrightarrow$ BND".

In contrast, if new granule $[x]_{C-C'}^1$ belongs to POS, then old granules $[x]_C^1, \ldots, [x]_C^m$ possibly have seven combined distributions with regard to POS, NEG, and BND. Thus, seven types emerge, i.e.,

(1) "POS $\longrightarrow$ POS", (2) "NEG $\longrightarrow$ POS", (3) "BND $\longrightarrow$ POS",
(4) "POS + BND $\longrightarrow$ POS", (5) "NEG + BND $\longrightarrow$ POS", (6) "POS + NEG $\longrightarrow$ POS",
and (7) "POS + NEG + BND $\longrightarrow$ POS".

For Gr-Merging Rg-Distribution, the 21 types in Table 1 actually provide the largest completeness, which applies to all RS-Models. In mathematics, all of the 21 types are related to a many-to-many correspondence (rather than a mapping) between two Rg-Distribution sets. The mathematical correspondence and distributional completeness are partly illustrated by Example 3.

The complete Gr-Merging Rg-Distribution holds theoretical significance for extensive RS-Models. However, some distributional types possibly never exist in reality, especially for the concrete RS-Model. In other words, we must practically inspect the distributional purity, which means real existence with regard to Kn-Coarsening or practical realizability with regard to Dc-Table. Furthermore, exact Gr-Merging Rg-Distribution can be finally established to underlie Rg-Change analyses, especially for a specific RS-Model. According to Pawlak-Model and DTRS-Model, Sections 5 and 6 will first identify Gr-Merging Rg-Distribution exactness, respectively, before in-depth Rg-Change analyses.

**Table 2**
Gr-Merging Rg-Distribution and Rg-Change analyses in Pawlak-Model.

| Type | Old Granules Rg-Distribution | New Granule Rg-Belonging | Rg-Changed Granule Source | Rg-Changed Function | Rg-Changed Type |
|---|---|---|---|---|---|
| (1) | POS | POS | – | – | No |
| (2) | NEG | NEG | – | – | No |
| (3) | BND | BND | – | – | No |
| (4) | POS, BND | BND | POS | ⊖POS, ⊕BND | Yes |
| (5) | NEG, BND | BND | NEG | ⊖NEG, ⊕BND | Yes |
| (6) | POS, NEG | BND | POS, NEG | ⊖POS, ⊖NEG, ⊕BND | Yes |
| (7) | POS, NEG, BND | BND | POS, NEG | ⊖POS, ⊖NEG, ⊕BND | Yes |

**Proposition 4.3.3.** *For Gr-Merging Rg-Distribution, only three types become Rg-Unchanged Type, i.e.,*

"POS $\longrightarrow$ POS", "NEG $\longrightarrow$ NEG", *and* "BND $\longrightarrow$ BND".

In contrast, all the surplus 18 types become Rg-Changed Type.

Note that Rg-Changed/Rg-Unchanged Type underlies Kn-Coarsening Rg-Change (Theorem 4.2.3). Thus, all of the 21 types are concretely divided into Rg-Changed and Rg-Unchanged Types. Furthermore, the core Rg-Changed Type can be identified and analyzed for the specific RS-Model, and the relevant studies will be conducted for Pawlak-Model and DTRS-Model (Sections 5 and 6).

Finally, this section that regards Gr-Merging is summarized by illustrating Gr-Merging's important significance to Rg-Change. Within the Kn-Coarsening framework, this section gradually investigates Gr-Merging, Gr-Merging Rg-Distribution, and Gr-Merging Rg-Distribution completeness. All of the mined results, including the Rg-Change's GrC hierarchy (Fig. 1), Rg-Change analysis method (Algorithm 1 and Fig. 2), and Rg-Distribution establishment technology (Table 1), establish an extensive GrC basis for Rg-Change analyses. In short, we thoroughly provide the GrC mechanism for Rg-Change analyses. Note that these studies mainly originate from GrC and never depend on RS-Model and, thus, they hold general significance. Next, the relevant GrC results will be utilized to implement concrete Rg-Change analyses for the qualitative Pawlak-Model and quantitative DTRS-Model.

## 5. Region-change certainty analyses in the qualitative Pawlak-Model

Thus far, Kn-Coarsening and Gr-Merging provide in-depth GrC mechanisms for Rg-Change analyses. Aiming at the qualitative Pawlak-Model, this section constructs Gr-Merging Rg-Distribution to conduct Rg-Change analyses, where the background Pawlak-Model is usually omitted and becomes an implicit condition. Concretely, we mainly mine *Rg-Change Certainty* and its relevant properties. The qualitative Rg-Change certainty analyses will establish a comparative basis for the later quantitative Rg-Change uncertainty analyses.

### *5.1. Qualitative granule-merging region-distribution*

As a basis for Rg-Change certainty analyses, this section presents Gr-Merging Rg-Distribution.

**Theorem 5.1.1.** *There are seven and only seven types of Gr-Merging Rg-Distribution, i.e., Types (1–7) in* Table 2.

**Proof.** For Gr-Merging $[x]_C^1 \cup \cdots \cup [x]_C^m \stackrel{=}{\longrightarrow} [x]_{C-C'}^1$, New Granule Rg-Belonging needs inspecting for the seven conditions of Old Granules Rg-Distribution. Let $i \in \{1, \ldots, m\}$.

(1) For three cases of Old Granules Rg-Distribution, "POS", "NEG", and "BND" mean that $[x]_C^i$ comes from old POS, NEG, and BND, respectively. Hence, $[x]_{C-C'}^1$ necessarily belongs to new POS, NEG, and BND, respectively. Thus,

"POS $\longrightarrow$ POS", "BND $\longrightarrow$ BND", and "BND $\longrightarrow$ BND"

exist and become Types (1), (2), and (3), respectively.

(2) Only four cases are surplus, i.e.,

"POS + BND", "NEG + BND", "POS + NEG", and "POS + NEG + BND".

For each case, new granule $[x]_{C-C'}^1$ satisfies the following condition:

$[x]_{C-C'}^1 \cap X \neq \emptyset$ and $[x]_{C-C'}^1 \cap (\neg X) \neq \emptyset$, i.e., $[x]_{C-C'}^1 \subseteq \mathrm{BND}_{C-C'}(X)$.

Hence, New Granule Rg-Belonging corresponds to only BND. Thus,

"POS + BND $\longrightarrow$ BND", "NEG + BND $\longrightarrow$ BND", "POS + NEG $\longrightarrow$ BND", and "POS + NEG + BND $\longrightarrow$ BND"

exist and become Types (4), (5), (6), and (7), respectively. □

**Proposition 5.1.2** (Rg-Distribution Mapping)**.** The seven Rg-Distributional types accord with a specific mathematical mapping, which is from *the seven-element set of Old Granules Rg-Distribution* to *the three-element set of New Granule Rg-Belonging*.

Gr-Merging Rg-Distribution is provided exactly in Theorem 5.1.1 and Table 2. The relevant proof utilizes the knowledge-granular certainty/uncertainty feature (or the Pawlak-Regional qualitative property). As a result, the seven types are related to a concrete mapping (rather than a correspondence) between two Rg-Distribution sets. For the 21 complete types (Theorem 4.3.1), we verify the purity of seven types and the impossibility of 14 types. Moreover, the relevant Rg-Distribution mapping with seven types becomes a sub-correspondence of the complete Rg-Distribution correspondence with 21 types (Proposition 4.3.2).

**Corollary 5.1.3.** *The seven types have exactness, which includes both completeness and purity. They constitute a type of partition from the viewpoint of being non-overlapping. Thus, arbitrary Gr-Merging corresponds to one and only one type.*

The seven exact types are provided in detail in Table 2. As an example, Type (4) "POS + BND $\longrightarrow$ BND" represents that old granules from POS and BND are merged into a new granule in BND. Based on Corollary 5.1.3, only one type is provided exactly for a Gr-Merging action. Next, the seven types are appropriately summarized.

**Proposition 5.1.4.**

(1) *Let $\sharp$ denote that all of Gr-Merging's old granules come from neither POS nor NEG. Then, Gr-Merging Rg-Distribution "$\sharp \longrightarrow$ BND" holds and represents the surplus types of* "POS $\longrightarrow$ POS" *and* "NEG $\longrightarrow$ NEG"*. From another viewpoint, $\sharp$ becomes BND's inverseimage with regard to Rg-Distribution Mapping.*
(2) *Let $\natural$ denote all of the subparts of* POS + NEG*, including* POS, NEG, POS + NEG*, and nothing. Then, Gr-Merging Rg-Distribution* "BND + $\natural \longrightarrow$ BND" *holds and represents the surplus types of* "POS $\longrightarrow$ POS", "NEG $\longrightarrow$ NEG", *and* "POS + NEG $\longrightarrow$ BND"*. In other words, New Granule Rg-Belonging is necessarily BND as long as Old Granules Rg-Distribution is related to BND.*

For the seven types, two inductive methods are given above and explained below. (1) From the inverseimage perspective, the seven types can be concluded to only three classes, i.e., "POS $\longrightarrow$ POS", "NEG $\longrightarrow$ NEG", and surplus "$\sharp \longrightarrow$ BND". According to the absorption function of BND's old granule, they can also be summarized by only four classes, i.e., "POS $\longrightarrow$ POS", "NEG $\longrightarrow$ NEG", "POS + NEG $\longrightarrow$ BND", and surplus "BND + $\natural \longrightarrow$ BND". (2) For three-way regions, only BND corresponds to uncertainty. Thus, "$\sharp \longrightarrow$ BND" and "BND + $\natural \longrightarrow$ BND" summarize several types that involve BND. They can be analyzed from the perspective of uncertainty. For "$\sharp \longrightarrow$ BND", the new granule necessarily belongs to BND as long as the Old Granules Rg-Distribution never involves pure POS or NEG. For "BND + $\natural \longrightarrow$ BND", the new granule necessarily belongs to BND as long as an old granule exists in old BND. Both distributional classes extensively transcend "BND $\longrightarrow$ BND" and, thus, manifest uncertainty reinforcement, which essentially originates from Kn-Coarsening/Kn-Non-Preservation. Furthermore, they also partially underlie later Rg-Change properties, especially Gr-Merging One-Way Convergence (Proposition 5.2.2).

Next, we introduce the Rg-Change analysis process from Proposition 4.2.4, Algorithm 1, and Fig. 2 (Section 4.2). Gr-Merging Rg-Distribution is first provided in Theorem 5.1.1 and Table 2. At the micro Gr-Merging level, the Rg-Changed Granule must then be extracted to further analyze Gr-Merging's Rg-Changed function. At the macro Kn-Coarsening level, Rg-Change is finally analyzed by systematically inspecting all of the Gr-Merging.

Along the above strategy, Table 2 also endows each type with Rg-Changed Granule Source, Rg-Changed Function, and Rg-Changed Type Identification. There, the symbol "–" means that the result is nothing for the considered item; the symbols $\ominus$ and $\oplus$ express the lessening and enlargement functions for their object regions, respectively; and the type identification symbols Yes and No correspond to Rg-Changed and Rg-Unchanged Types, respectively. As an example, Type (6) "POS + NEG $\longrightarrow$ BND" is explained. There, the Rg-Changed Granule comes from POS and BND, and this type of key granule leads to the Gr-Merging's Rg-Changed function, which is to lessen POS and NEG while is to enlarge BND; thus, Type (6) becomes the Rg-Changed Type. In fact, Table 2 provides the complete information that is required by the Rg-Change analysis process; there, the Rg-Changed Types and their Rg-Changed Functions become especially important. Table 2 and its Rg-Changed Types will be utilized to mine the properties at both the Gr-Merging and Kn-Coarsening levels (Section 5.2).

**Proposition 5.1.5.** *The seven distributional types produce four classes according to their reginal roles.*

(1) The 1st class includes Types (1–3), where Gr-Merging never changes any region.
(2) The 2rd class includes Type (4), where Gr-Merging lessens POS and enlarges BND.
(3) The 3rd class includes Type (5), where Gr-Merging lessens NEG and enlarges BND.
(4) The 4th class includes Types (6) and (7), where Gr-Merging lessens POS and BND while enlarges BND.

The first class and the latter three correspond to the Rg-Unchanged and Rg-Changed Types, respectively.

Herein, Gr-Merging Rg-Distributions are summarized into four classes. Thus, the concrete Rg-Change functions of Gr-Merging are acquired to underlie the next Rg-Change analyses.

### *5.2. Qualitative region-change certainty and its relevant properties*

Based on Gr-Merging Rg-Distribution, especially the complete Table 2, this section begins Rg-Change analyses. Concretely, Rg-Change Certainty and its relevant properties are mined from the Gr-Merging to Kn-Coarsening levels.

**Theorem 5.2.1** (Gr-Merging Rg-Certainty)**.** *Gr-Merging holds certainty for the Pawlak-Regions, i.e., Old Granules Rg-Distribution completely determines New Granule Rg-Belonging in Gr-Merging.*

**Proof.** This result is easily concluded by the seven distributional types (Table 2). $\square$

Gr-Merging Rg-Certainty accords with Rg-Distribution Mapping from Old Granules Rg-Distribution to New Granule Rg-Belonging (Proposition 5.1.2). Thus, New Granule Rg-Belonging becomes clear if Old Granules Rg-Distribution is clear. In other words, the Gr-Merging action holds certainty from the Pawlak-Region perspective.

**Theorem 5.2.2** (Gr-Merging One-Way Convergence)**.** *In Gr-Merging, all of the new granules gather together in new BND as much as possible. In other words, all of the old granules converge to new BND to the greatest extent.*

**Proof.** This result is proven by summarizing the seven distributional types (Table 2). For Types (1) and (2), the new granule never has a convergence possibility with regard to new BND; in fact, the new granule necessarily belongs to new POS or NEG. For surplus Types (3–7), i.e., "$\sharp \longrightarrow$ BND" (Proposition 5.1.4), new granules necessarily belong to new BND regardless of the Old Granules Rg-Distribution. The new granular aggregation with regard to BND implies the one-way convergence of old granules. Moreover, the old granule convergence can utilize Rg-Changed and Rg-Unchanged Types (Table 2). Rg-Unchanged Types (i.e., Types (1–3)) never have an Rg-Changed Granule, while all of the Rg-Changed Granules of the Rg-Changed Types (i.e., Types (4–7)) converge to new BND. In other words, if possible, all of the old granules converge to new BND. □

One-Way Convergence comes from the synthesis of all Rg-Distribution types; thus, it becomes a granular statistical law with regard to the Pawlak-Region. In fact, this BND-directional conclusion highly accords with the inductive result "$\sharp \longrightarrow$ BND" (Proposition 5.1.4). Thus, One-Way Convergence acts as a concrete embodiment of uncertainty enhancement. In essence, it originates from Kn-Coarsening/Kn-Non-Preservation.

Gr-Merging Rg-Certainty and One-Way Convergence serve as two fundamental granular properties at the micro Gr-Merging level. The former underlies the latter. Based on these granular mechanisms, the corresponding regional manifestations can further be mined at the macro Kn-Coarsening level.

**Theorem 5.2.3** (Rg-Change Certainty)**.** *Rg-Change holds certainty in Kn-Coarsening, i.e., all three-way regions have their own stable change directions with regard to regional enlargement or lessening.*

**Proof.** Rg-Preservation and Rg-Unchanged Type have regional invariance, which is a sort of regional change certainty. In contrast, strict Rg-Change is determined by Rg-Changed Type (Theorem 4.2.3). Thus, only Rg-Changed Types (i.e., Types (4–7)) need analyzing. According to Table 2, each Rg-Changed Type reflects the specific Rg-Changed direction with regard to BND enlargement or POS/NEG lessening. Furthermore, by integrating all Rg-Changed functions (Table 2 or Proposition 5.1.5), three-way regions exhibit definite change directions: POS and NEG are generally lessened while BND is generally enlarged. □

Rg-Change Certainty is summarized by Gr-Merging Rg-Distribution (Table 2), mainly its Rg-Changed Types. From the granular mechanism, Rg-Change Certainty can also be caused by Gr-Merging Rg-Certainty. In fact, Rg-Change Certainty mainly focuses on the monotonic direction of regional enlargement or lessening, thus underlying the next monotonic change.

**Theorem 5.2.4** (Rg-Change Monotonicity)**.** *Rg-Change holds monotonicity in Kn-Coarsening. Concretely, BND is not lessened, while POS/NEG is not enlarged.*

**Proof.** This conclusion naturally holds based on Theorem 5.2.3's proof. □

Rg-Change Monotonicity, which is a fundamental feature for qualitative Pawlak-Model, arises directly from Rg-Change Certainty (or its inductive proof). Moreover, it is caused by One-Way Convergence from the granular mechanism.

Both Rg-Change Certainty and Monotonicity reflect the Rg-Change law. In fact, monotonicity describes a type of certainty with regard to the change direction; hence, Rg-Change Monotonicity becomes a concrete embodiment of Rg-Change Certainty. In contrast, certainty refers to a type of monotonicity stability; hence, Rg-Change Certainty can be explained by Rg-Change Monotonicity. Furthermore, a combination of both fully describes the Rg-Change essence.

**Theorem 5.2.5** (Qualitative Rg-Change Essence)**.** Qualitative Rg-Change has the essence of *certain monotonicity*, i.e., three-way Pawlak-Regions have a certain monotonic change in Kn-Coarsening.

At the Kn-Coarsening level, Rg-Change Certainty underlies Rg-Change Monotonicity, while the latter manifests Rg-Change Certainty. Thus, Theorem 5.2.5 uncovers the qualitative Rg-Change essence: certain monotonicity. Furthermore, both fundamental properties, especially Rg-Change Monotonicity, naturally deduce the next Cl-POS Change Monotonicity.

**Corollary 5.2.6** (Cl-POS Change Monotonicity)**.** *Cl-POS change holds monotonicity in Kn-Coarsening. Concretely, Cl-POS is not enlarged.*

**Proof.** In fact, Cl-POS is POS's combination of multiple concepts (Formula (2)). Thus, this result comes from POS Change Monotonicity (Theorem 5.2.4) with regard to each concept. □

Thus far, two and three properties are mined at the Gr-Merging and Kn-Coarsening levels, respectively. Before commenting on their logical relationships and basic roots, the relevant figure, Fig. 3, is provided.

Kn-Coarsening and Gr-Merging act as two fundamental GrC notions. They are located at the macro knowledge and micro granule levels, respectively. Furthermore, Gr-Merging concerns two main parts, i.e., its Rg-Distribution and its properties.

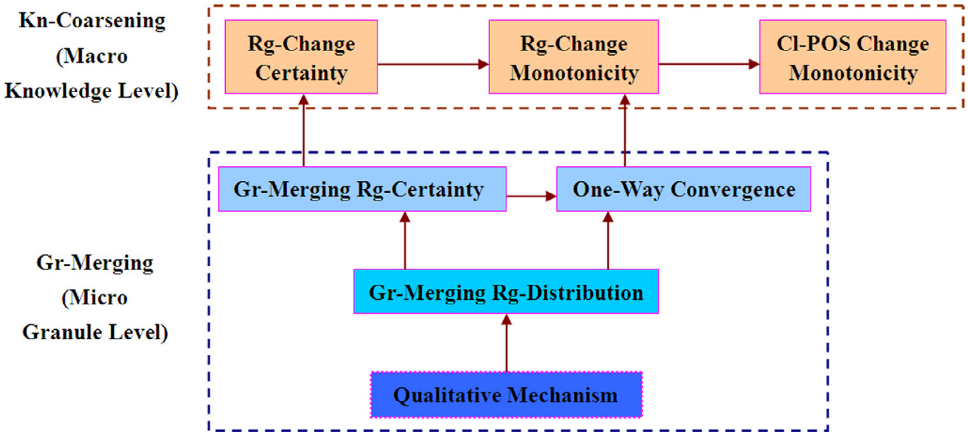


**Fig. 3.** Qualitative Rg-Change certainty's relevant properties and their relationships.

(1) According to the Rg-Change Principle (Theorem 4.1.6), Gr-Merging becomes the sole action cause at the granule level to alter the granular regional belonging. Hence, the internal regional features of Gr-Merging determine the external Rg-Change of Kn-Coarsening. Thus, Gr-Merging Rg-Distribution (Theorem 5.1.1), especially Rg-Changed Type, establishes a solid foundation for Rg-Change. Table 2 and its Rg-Changed Function (with Rg-Changed Granule) are utilized for relevant Rg-Change analyses and certainty mining.
(2) Based on Gr-Merging Rg-Distribution (Table 2), two granular statistical properties are concluded at the micro Gr-Merging level, i.e., Gr-Merging Rg-Certainty and One-Way Convergence. The former underlies the latter.
(3) Three regional integral properties are further manifested at the macro Kn-Coarsening level, i.e., Rg-Change Certainty, Rg-Change Monotonicity, and Cl-POS Change Monotonicity. The three exhibit a decisive relationship from the former to the latter.

The above three items exhibit Gr-Merging Rg-Distribution, Gr-Merging's regional properties, Kn-Coarsening's regional properties, and their internal relationships. All of these results are hierarchically described in Fig. 3, where the internal relationships are horizontally exhibited.

Herein, hierarchical relationships are analyzed from the causality perspective.

(1) According to the purity and completeness detection (Theorem 5.1.1), the qualitative mechanism (with POS/NEG certainty and BND uncertainty) provides Gr-Merging Rg-Distribution exactness; thus, it becomes a basic mechanism root. In contrast, Kn-Coarsening acts as the essential action cause.
(2) Gr-Merging Rg-Distribution directly underlies Gr-Merging Rg-Certainty and One-Way Convergence, which are two statistical properties at the micro Gr-Merging level.
(3) Gr-Merging Rg-Certainty and Rg-Change Certainty become the regional certainty at the micro granule and macro knowledge levels, respectively; both hold an inference relationship. One-Way Convergence and Rg-Change Monotonicity become regional embodiments of uncertainty enhancement at the two relevant levels, respectively; both have a mechanism explanation relationship.

The above three items provide a causality development line, i.e.,

*Qualitative Mechanism*$\longrightarrow$*Gr-Merging Rg-Distribution*$\longrightarrow$
*Gr-Merging's regional properties*$\longrightarrow$*Kn-Coarsening's regional properties.*

All of these results are vertically exhibited (from the bottom to the top) in Fig. 3.

Finally, we summarize this section with regard to qualitative Rg-Change certainty analyses. First, Gr-Merging Rg-Distribution is established via the qualitative mechanism; then, some fundamental regional properties are mined at both the micro Gr-Merging and macro Kn-Coarsening levels. In fact, we mainly depend on the micro Gr-Merging level to make systematic Rg-Change analyses; furthermore, the relevant results provide systematic Rg-Change mechanisms at the macro Kn-Coarsening level. As a result, we clearly interpret the well-known phenomena of attribute reduction, which include Rg-Change Monotonicity and Cl-POS Change Monotonicity, and in addition, we thoroughly reveal a steady background, i.e., Rg-Change Certainty. Note that *certain monotonicity* becomes the essence of the qualitative Rg-Change.

## 6. Region-change uncertainty analyses in the quantitative DTRS-Model

Aiming at the quantitative DTRS-Model, this section establishes Gr-Merging Rg-Distribution to make Rg-Change analyses, where the background DTRS-Model is usually omitted to become a latent restriction. Concretely, we mainly probe the hidden Rg-Change uncertainty and relevant properties. Herein, the quantitative Rg-Change uncertainty analyses refers, in contrast, to the qualitative Rg-Change certainty analyses (Section 5).

**Table 3**
Gr-Merging Rg-Distribution and Rg-Change analyses in DTRS-Model.

| Type/Class | Old Granules Rg-Distribution | New Granule Rg-Belonging | Rg-Changed Granule Source | Rg-Changed Function | Rg-Changed Type |
|---|---|---|---|---|---|
| (1) | POS | POS | – | – | No |
| (2) | NEG | NEG | – | – | No |
| (3) | BND | BND | – | – | No |
| (4) | POS, BND | POS or BND | | | |
| (4a) | POS, BND | POS | BND | $\ominus$BND, $\oplus$POS | Yes |
| (4b) | POS, BND | BND | POS | $\ominus$POS, $\oplus$BND | Yes |
| (5) | NEG, BND | NEG or BND | | | |
| (5a) | NEG, BND | NEG | BND | $\ominus$BND, $\oplus$NEG | Yes |
| (5b) | NEG, BND | BND | NEG | $\ominus$NEG, $\oplus$BND | Yes |
| (6) | POS, NEG | POS or NEG or BND | | | |
| (6a) | POS, NEG | POS | NEG | $\ominus$NEG, $\oplus$POS | Yes |
| (6b) | POS, NEG | NEG | POS | $\ominus$POS, $\oplus$NEG | Yes |
| (6c) | POS, NEG | BND | POS, NEG | $\ominus$POS, $\ominus$NEG, $\oplus$BND | Yes |
| (7) | POS, NEG, BND | POS or NEG or BND | | | |
| (7a) | POS, NEG, BND | POS | NEG, BND | $\ominus$NEG, $\ominus$BND, $\oplus$POS | Yes |
| (7b) | POS, NEG, BND | NEG | POS, BND | $\ominus$POS, $\ominus$BND, $\oplus$NEG | Yes |
| (7c) | POS, NEG, BND | BND | POS, NEG | $\ominus$POS, $\ominus$NEG, $\oplus$BND | Yes |

**Table 4**
Dc-Table of Example 4.

| $U$ | $a$ | $b$ | $d$ |
|---|---|---|---|
| $x_1$ | 1 | 1 | 1 |
| $x_2$ | 0 | 1 | 1 |
| $x_3$ | 0 | 1 | 2 |
| $x_4$ | 0 | 0 | 1 |
| $x_5$ | 0 | 0 | 1 |
| $x_6$ | 0 | 0 | 1 |
| $x_7$ | 0 | 2 | 0 |

### 6.1. Quantitative granule-merging region-distribution

As a basis for Rg-Change uncertainty analyses, this section reveals Gr-Merging Rg-Distribution, mainly by taking two approaches with regard to completeness and purity.

**Lemma 6.1.1.** *In Gr-Merging* $[x]_C^1 \cup \cdots \cup [x]_C^m \xrightarrow{=} [x]_{C-C'}^1$,

$$Pr([x]_{C-C'}^1) \in [\min_{i\in\{1,\ldots,m\}} Pr(X|[x]_C^i), \max_{i\in\{1,\ldots,m\}} Pr(X|[x]_C^i)]. \quad (24)$$

Gr-Merging's old granules have a limited probability distribution. Under the merging action, the new granule naturally has a probability boundedness with regard to the minimum and maximum. Thus, a type of Gr-Merging probabilistic mechanism is concretely mined by Lemma 6.1.1.

**Theorem 6.1.2.** *There are at most 13 types of Gr-Merging Rg-Distribution, i.e., Types (1), (2), (3), (4a), (4b), (5a), (5b), (6a), (6b), (6c), (7a), (7b), and (7c) in* Table 3.

Theorem 6.1.2 embodies Gr-Merging Rg-Distribution Completeness. Its proof is in Appendix A and mainly utilizes the Gr-Merging probability boundedness (Lemma 6.1.1). Theorem 6.1.2 acquires a more accurate completeness (with only 13 types), which narrows down the largest completeness (with 21 types) in Theorem 4.3.1. Can the 13 types be further narrowed down? In other words, are there some types that never exist in practice? Clearly, both questions concern considerable purity, i.e., the real possibility in theory or concrete realizability in reality.

**Corollary 6.1.3.** *Types (1–3) are pure, i.e., they can emerge in practice.*

Corollary 6.1.3 is deduced by the proof of Theorem 6.1.2 (Appendix A) and verifies Type (1–3)'s purity. Then, the purity of the surplus ten types must be identified. Next, the purity conclusion for all 13 types is given first, and its correctness is illustrated by the relevant Dc-Table construction.

**Theorem 6.1.4.** *The 13 types are all pure.*

**Table 5**
Gr-Merging Rg-Distribution and Kn-Coarsening Rg-Change of Examples 4 and 5.

| Case | Threshold Distribution | $\{x_1\}\cup\{x_2,x_3\} \stackrel{=}{\longrightarrow} \{x_1,x_2,x_3\}$ | Type | Rg-Changed Granule | Rg-Change |
|---|---|---|---|---|---|
| (1) | $\beta<\alpha<0.5<2/3$ | POS + POS $\longrightarrow$ POS | (1) | – | – |
| (2) | $\beta<0.5<\alpha<2/3$ | POS + BND $\longrightarrow$ POS | (4a) | $\{x_2,x_3\}$ | BND$\ominus$, POS$\oplus$ |
| (3) | $\beta<0.5<2/3<\alpha$ | POS + BND $\longrightarrow$ BND | (4b) | $\{x_1\}$ | POS$\ominus$, BND$\oplus$ |
| (4) | $0.5<\beta<\alpha<2/3$ | POS + NEG $\longrightarrow$ POS | (6a) | $\{x_2,x_3\}$ | NEG$\ominus$, POS$\oplus$ |
| (5) | $0.5<2/3<\beta<\alpha$ | POS + NEG $\longrightarrow$ NEG | (6b) | $\{x_1\}$ | POS$\ominus$, NEG$\oplus$ |
| (6) | $0.5<\beta<2/3<\alpha$ | POS + NEG $\longrightarrow$ BND | (6c) | $\{x_1\}, \{x_2,x_3\}$ | POS$\ominus$, NEG$\ominus$, BND$\oplus$ |

**Example 4.** A three-category Dc-Table $(U, C\cup D)$ is provided in Table 4. Herein, $U=\{x_1,\ldots,x_7\}$, $C=\{a,b\}$, $D=\{d\}$, and $X=\{x_1,x_2,x_4,x_5,x_6\}$.

$$\begin{cases} U/IND(\{a,b\})=\{\{x_1\},\{x_2,x_3\},\{x_4,x_5,x_6\},\{x_7\}\},\\ U/IND(\{b\})=\{\{x_1,x_2,x_3\},\{x_4,x_5,x_6\},\{x_7\}\},\\ U/IND(\{d\})=\{\{x_1,x_2,x_4,x_5,x_6\},\{x_3\},\{x_7\}\}.\end{cases}$$

In Kn-Coarsening $\{a,b\} \stackrel{-a}{\Longrightarrow} \{b\}$, $\{a,b\}$ and $\{b\}$ become the old and new knowledge, respectively.

$$|U/IND(\{b\})|=3<4=|U/IND(\{a,b\})|,$$

so knowledge is strictly coarsened and Kn-Coarsening becomes Kn-Non-Preservation. Furthermore, there are three groups of granular transformation, i.e.,

$$\begin{cases} \text{Gr-Merging } \{x_1\}\cup\{x_2,x_3\} \stackrel{=}{\longrightarrow} \{x_1,x_2,x_3\},\\ \text{Gr-Preservation } \{x_4,x_5,x_6\} \stackrel{=}{\longrightarrow} \{x_4,x_5,x_6\},\\ \text{Gr-Preservation } \{x_7\} \stackrel{=}{\longrightarrow} \{x_7\}.\end{cases}$$

For the sole Gr-Merging $\{x_1\}\cup\{x_2,x_3\} \stackrel{=}{\longrightarrow} \{x_1,x_2,x_3\}$,

$$Pr(X|\{x_1\})=1,\ Pr(X|\{x_2,x_3\})=0.5,\ Pr(X|\{x_1,x_2,x_3\})=2/3.$$

Thus, thresholds $\alpha$ and $\beta$ exhibit only six distributional cases with regard to probability constants 0.5, 2/3, and 1. According to each case, its Gr-Merging Rg-Distribution and its type label are identified in Table 5. For Types (1), (4a), (4b), (6a), (6b), and (6c), each type concretely emerges in specific Kn-Coarsening. Hence, the purity of the six types is verified.

Example 4 proves the purity of six types. This construction strategy is based on Dc-Table and can be generalized for purity verification. In practice, we similarly prove the purity of the additional seven types. Thus, Theorem 6.1.4 holds to provide the Gr-Merging Rg-Distribution Purity.

**Theorem 6.1.5.** *There are 13 and only 13 types of Gr-Merging Rg-Distribution, which are presented in Table 3.*

Gr-Merging Rg-Distribution Completeness and Purity are provided by Theorems 6.1.2 and 6.1.4, respectively. Furthermore, Gr-Merging Rg-Distribution Exactness is naturally reached in Theorem 6.1.5. Thus, Table 3, which contains 13 types, exhibits exact Gr-Merging Rg-Distribution. As an example, Type (7b) "POS + NEG + BND $\longrightarrow$ NEG" means that old granules from POS, NEG, and BND are merged into a new granule in NEG. The 13 types produce seven classes, i.e., Classes (1–7) in Table 3. For example, three types, (7a), (7b), and (7c), constitute Class (7). In contrast, Classes (1), (2), and (3) include one type, and Classes (4) and (5) include two types, while Classes (6) and (7) include three types.

**Proposition 6.1.6** (Rg-Distribution Correspondence)**.** The 13 distributional types accord with a specific mathematical correspondence, which is from *the seven-element set of Old Granules Rg-Distribution* to *the three-element set of New Granule Rg-Belonging*.

For the entire set of 21 types (Theorem 4.3.1), we confirm both the impossibility of 8 types and the purity of 13 types. In mathematics, the 13 exact types construct a concrete correspondence (rather than a mapping) between two Rg-Distribution sets. Moreover, the correspondence becomes a subpart of the complete Rg-Distribution correspondence (Proposition 4.3.2).

For Rg-Change analyses, Gr-Merging Rg-Distribution can produce some basic preparations by considering the technology algorithm/roadmap process (Section 4.2). As a result, Table 3 also endows the 13 types with Rg-Changed Granule Source, Rg-Changed Function, and Rg-Changed Type Identification. Therefore, Table 3 contains complete information for the Rg-Change analyses, where the Rg-Changed Types and their Rg-Changed Functions become especially important. Next, in Section 6.2, Table 3 and its Rg-Changed Types will be utilized to mine extensive properties at both the Gr-Merging and Kn-Coarsening levels.

Except for Gr-Merging Rg-Distribution, Example 4 also illustrates Kn-Coarsening and Gr-Merging. In the subsequent example, the Rg-Change analysis road (Algorithm 1 and Fig. 2) is illustrated by Rg-Changed Granule, Rg-Changed Function, and Rg-Change.

**Example 5.** Herein, Table 4 is used again. Sole Gr-Merging $\{x_1\}\cup\{x_2,x_3\} \stackrel{=}{\longrightarrow} \{x_1,x_2,x_3\}$ completely determines Rg-Change in Kn-Coarsening $\{a,b\} \stackrel{-a}{\Longrightarrow} \{b\}$. As a result, Gr-Merging could be promoted to completely describe Rg-Change. In fact, Rg-Changed

Granule consistently exists at the Gr-Merging and Kn-Coarsening levels. By virtue of Gr-Merging Rg-Distribution, Rg-Changed Granule is extracted to achieve Rg-Change in $\{a, b\} \stackrel{-a}{\Longrightarrow} \{b\}$. Relevant results are appended to the last two columns of Table 5.

As an example, Case (2) is explained by using the processes of Algorithm 1 and Fig. 2.

(1) When $\beta < 0.5 < \alpha < 2/3$,
$\{x_1\} \subseteq POS^{\alpha,\beta}_{\{a,b\}}(X)$ and $\{x_2, x_3\} \subseteq BND^{\alpha,\beta}_{\{a,b\}}(X)$.
Under Gr-Merging's action,
$\{x_1, x_2, x_3\} \subseteq POS^{\alpha,\beta}_{\{b\}}(X)$.
Hence, $\{x_1\} \cup \{x_2, x_3\} \stackrel{=}{\longrightarrow} \{x_1, x_2, x_3\}$ corresponds to Type (4a) "POS + BND $\longrightarrow$ POS".
(2) $\{x_2, x_3\}$ from old BND becomes unique Rg-Changed Granule in Gr-Merging. In fact, it is adjusted into new POS from old BND.
(3) Gr-Merging mainly functions by old granule $\{x_2, x_3\}$ from the regional perspective. Thus, it has the individual influence on lessening BND and enlarging POS.
(4) The Rg-Changed function of Gr-Merging directly leads to the following Kn-Coarsening Rg-Change (which is denoted as BND$\ominus$ and POS$\oplus$), i.e., BND is lessened, POS is enlarged, while NEG is preserved. Herein, the consistent regional facts are also provided, i.e., old POS, NEG, and BND are $\{x_1, x_4, x_5, x_6\}$, $\{x_7\}$, and $\{x_2, x_3\}$, respectively, while new POS, NEG, and BND become $\{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\{x_7\}$, and $\emptyset$, respectively.

Other cases can be similarly analyzed, and all Rg-Change results concern only monotonicity. Non-monotonicity will be exhibited in Section 6.2, e.g., Example 7.

### 6.2. Quantitative region-change uncertainty and its relevant properties

Based on Gr-Merging Rg-Distribution, especially Table 3, this section makes Rg-Change analyses. Concretely, Rg-Change Uncertainty and its relevant properties are mined from the Gr-Merging to Kn-Coarsening levels. Herein, the quantitative mining comparatively refers to the qualitative mining (Section 5.2).

**Theorem 6.2.1** (Gr-Merging Rg-Uncertainty). *Gr-Merging holds uncertainty for DTRS-Regions, i.e., Old Granules Rg-Distribution cannot completely determine New Granule Rg-Belonging in Gr-Merging.*

**Proof.** This result is concluded by the 13 distributional types (Table 3). For example, if Old Granules Rg-Distribution is "POS + BND", then New Granule Rg-Belonging has two possible results with regard to POS and BND, i.e., Types (4a) and (4b) are concerned. □

Gr-Merging Rg-Uncertainty accords with Rg-Distribution Correspondence from Old Granules Rg-Distribution to New Granule Rg-Belonging (Proposition 6.1.6). Although Old Granules Rg-Distribution is initially clear, the new granule has regional belonging selectivity to some extent. As an example, Class (7) is explained by its Types, (7a), (7b), and (7c). There, old granules come from POS, NEG, and BND; however, the regional distribution information cannot determine the new granular belonging region, and all of the new three-way regions have the selected possibility; therefore, the Gr-Merging action holds uncertainty from the DTRS-Region perspective. Clearly, Gr-Merging Rg-Uncertainty radically refers to a type of selection possibility of New Granule Rg-Belonging, and this possibility feature is independent of Old Granules Rg-Distribution.

**Corollary 6.2.2.** *Gr-Merging Rg-Uncertainty mainly depends on Rg-Changed Type.*

According to Table 3, Gr-Merging Rg-Uncertainty completely focuses on Classes (4–7). These classes are thereby called uncertainty classes, and they correspond to Rg-Changed Type. As a result, Corollary 6.2.2 holds, and for uncertainty analyses, Rg-Changed Type plays a core role, while Rg-Unchanged Type has no effect.

**Theorem 6.2.3** (Gr-Merging Multi-Way Convergence). *In Gr-Merging, all of the new granules gather together in their possible belonging region. In other words, all of the old granules converge to their relevant new region, which can involve multiple regions with regard to POS, NEG, and BND.*

**Proof.** Herein, the Classes (1–7) of Table 3 are utilized. For Classes (1–3) with the Rg-Unchanged Type, the new granule belongs to its certain region because it has no selectivity. For Classes (4–7) with the Rg-Changed Type, the new granule can belong to at least two regions because of its selectivity possibility. Hence, both the new and old granules' convergence can exhibit a multi-way feature with regard to POS, NEG, and BND. □

**Corollary 6.2.4.** *In Gr-Merging, Multi-Way Convergence is mainly presented by the Rg-Changed Type.*

Based on the proof of Theorem 6.2.3, Multi-Way Convergence mainly comes from Rg-Unchanged Type. For example, for uncertainty Class (6) "POS + NEG $\longrightarrow$ POS or NEG or BND", the new granule can belong to POS or NEG or BND, i.e., Rg-Changed Types (6a), (6b), and (6c) can be involved. In contrast, Gr-Merging with Rg-Unchanged Type exhibits one-way convergence with regard to the sole region that is involved.

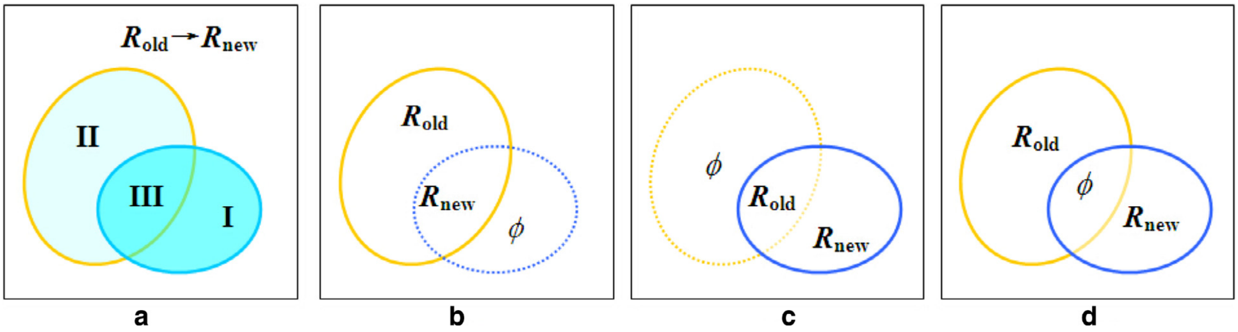


**Fig. 4.** Rg-Change and Drifting's schematic diagram.

Gr-Merging Rg-Uncertainty and Multi-Way Convergence become two fundamental granular properties at the micro Gr-Merging level. The former underlies the latter. Furthermore, the corresponding regional manifestations can be caught at the macro Kn-Coarsening level.

**Theorem 6.2.5** (Rg-Change Uncertainty)**.** *Rg-Change holds uncertainty in Kn-Coarsening, i.e., all of the three-way regions have their own unstable change directions with regard to the regional increase and decrease.*

**Proof.** Because Rg-Unchanged Type never has Rg-Changed functions, only Rg-Changed Type must be considered for Rg-Change analyses (Theorem 4.2.3). According to Table 3, each Rg-Changed Type corresponds to multiple Rg-Changed directions with regard to regional increases and decreases. Furthermore, by integrating all of the Rg-Changed functions, three-way regions exhibit indefinite change directions, i.e., POS, NEG, and BND can be increased or decreased. Hence, three-way regions can hold both the increased and decreased parts. □

Rg-Change Uncertainty is summarized by Gr-Merging Rg-Distribution (Table 3), mainly the Rg-Changed Type. From the granular mechanism, Rg-Change Uncertainty is caused by Gr-Merging Rg-Uncertainty. In contrast, which Rg-Change feature is caused by Multi-Way Convergence? In view of Rg-Change Uncertainty, Rg-Change Non-Monotonicity will become a natural and objective conclusion. Thus, for the Rg-Change description, monotonicity no longer becomes suitable, but non-monotonicity appears to be too ambiguous. A more accurate notion is worthwhile to propose. Note that Rg-Change Uncertainty has shifted the Rg-Change focus from the monotonic enlargement/lessening to the non-monotonic increase/decrease. Thus, drifting with an increase/decrease is introduced to exactly exhibit Rg-Change and finely describe Rg-Change Uncertainty.

**Definition 6.2.6** (Rg-Change and Drifting)**.** Let two regions $R_{old}, R_{new} \subseteq U$ have the same type region with regard to POS, NEG, and BND. Both constitute Rg-Change $R_{old} \longrightarrow R_{new}$. Thus, *drifting* is defined to vividly describe Rg-Change $R_{old} \longrightarrow R_{new}$, as follows:

(1) $R_{old}$ drifts to $R_{new}$ in Rg-Change $R_{old} \longrightarrow R_{new}$;
(2) the type region exhibits drifting in Rg-Change $R_{old} \longrightarrow R_{new}$;
(3) Rg-Change $R_{old} \longrightarrow R_{new}$ maintains drifting.

Rg-Change and Drifting are described in Definition 6.2.6. Rg-Change $R_{old} \longrightarrow R_{new}$ summarizes all Kn-Coarsening Rg-Change, i.e., it has three types with regard to POS, NEG, and BND. In mathematics, Rg-Change corresponds to the usual set transformation; thus, Drifting becomes a general dynamic characteristic with regard to the sets. Herein, three basic parts of Rg-Change must be emphasized, i.e.,

$$\begin{cases} \text{(I) increased part } R_{new} - R_{old}, \\ \text{(II) decreased part } R_{old} - R_{new}, \\ \text{(III) unchanged part } R_{new} \cap R_{old}. \end{cases}$$

Rg-Change and Drifting are schematically exhibited in Fig. 4. There, sub-figure (a) exhibits the usual case with non-empty parts (I–III), while sub-figures (b), (c), and (d) describe the special case that has the empty parts (I), (II), and (III), respectively; moreover, sub-figure (a) has inclusiveness to generalize the cases (b), (c), and (d). In general, Rg-Change and Drifting concern the regional scale change because $|R_{new}| = |R_{old}|$ usually does not hold.

**Proposition 6.2.7** (Monotonicity/Non-Monotonicity)**.**

(1) *Rg-Change $R_{old} \longrightarrow R_{new}$ exhibits a monotonic decrease (i.e., the type region is lessened), iff $R_{new} - R_{old} = \emptyset$ (i.e., part (I) is empty). The relevant Rg-Change and Drifting correspond to sub-figure (b).*
(2) *Rg-Change $R_{old} \longrightarrow R_{new}$ exhibits a monotonic increase (i.e., the type region is enlarged), iff $R_{old} - R_{new} = \emptyset$ (i.e., part (II) is empty). The relevant Rg-Change and Drifting correspond to sub-figure (c).*

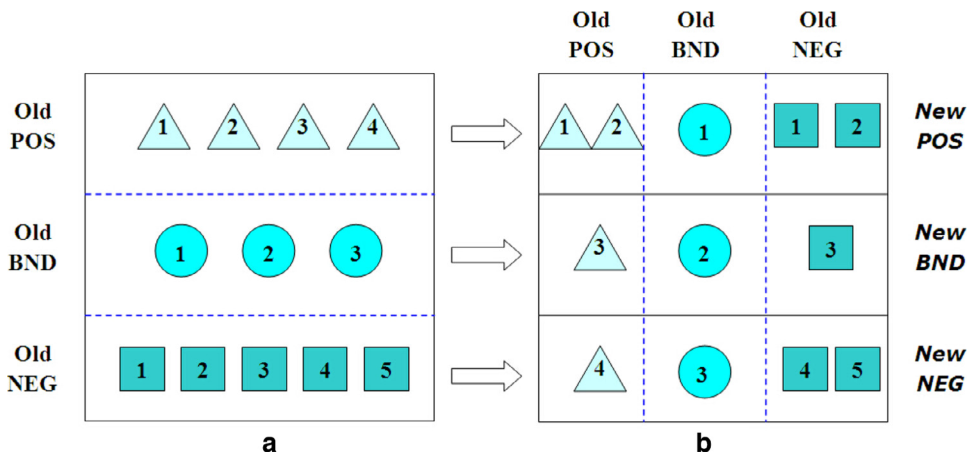


**Fig. 5.** Rg-Change and Drifting of Example 6.

(3) *Rg-Change $R_{\text{old}} \longrightarrow R_{\text{new}}$ is not monotonic (i.e., the type region is neither lessened nor enlarged), iff both $R_{\text{new}} - R_{\text{old}} \neq \emptyset$, and $R_{\text{old}} - R_{\text{new}} \neq \emptyset$ (i.e., neither part (I) nor part (II) is empty). The relevant Rg-Change and Drifting correspond to sub-figures (a) and (d), where the surplus part (III) is non-empty and empty, respectively.*

Herein, the drifting includes both the usual monotonicity and the new non-monotonicity. Both of the features could be effectively identified by the increased and decreased parts, i.e., parts (I) and (II). In Fig. 4, monotonic and non-monotonic drifting are exhibited by sub-figures (b), (c) and (a), (d), respectively. Thus, drifting transcends monotonicity and non-monotonicity to become a generalized notion; it can both vividly describe Rg-Change and accurately manifest Rg-Change uncertainty.

**Example 6.** Under the action of Kn-Coarsening,

(1) let old POS, BND, and NEG originally consist of “four triangles”, “three circles”, and “five squares”, respectively;
(2) let new POS, BND, and NEG finally consist of “two triangles, one circle, two squares”, “one triangle, one circle, one square”, and “one triangle, one circle, two squares”, respectively.

The old and new three-way regions are provided in sub-figures (a) and (b) of Fig. 5, respectively. They are also exhibited by the columns and rows of only sub-figure (b). Both of the sub-figures can be used to vividly observe the relevant Rg-Change and Drifting. Herein, the three-way region drifting exhibits non-monotonicity, which is a usual case (reflected by sub-figure (a) in Fig. 4).

From the granular mechanism, the above Drifting could also be explained by internal component blocks. For example, NEG Drifting concerns removing three squares (1–3) and adding one triangle (4), one circle (3). In fact, both synthetic moves and systematic reorganization of granular blocks realize macro regional Drifting (or Rg-Change Drifting). Herein, Drifting is accompanied by a regional scale change because the internal block number has changed.

**Theorem 6.2.8** (Rg-Change Drifting)**.** *In Kn-Coarsening, Rg-Change holds Drifting, i.e., three-way regions exhibit Drifting in Rg-Change.*

**Proof.** According to the proof of Theorem 6.2.5, each Rg-Changed Type has the following Rg-Changed functions: increasing one region and decreasing one region or two regions. By summarizing all of the Rg-Change functions, POS, NEG, and BND can possess their own increased and decreased parts. Hence, they exhibit Drifting in Kn-Coarsening, i.e., Rg-Change holds Drifting in Kn-Coarsening. □

Herein, we basically comment on Rg-Change Uncertainty, Rg-Change Drifting, and their relationships.

(1) Rg-Change Uncertainty is an in-depth statistical result of Gr-Merging Rg-Distribution; additionally, it is a macro combination conclusion of granular regional selection-possibility. In other words, for Rg-Change Uncertainty, Gr-Merging Rg-Distribution (especially Rg-Changed Type) acts as its inductive basis, while Gr-Merging Rg-Uncertainty serves as its granular formation mechanism.
(2) Rg-Change Drifting objectively reflects the regional drifting characteristic of Kn-Coarsening. From the granular mechanism, it is caused by Multi-Way Convergence. In fact, Multi-Way Convergence serves as a granular statistical property at the micro Gr-Merging level; furthermore, Rg-Change Drifting correspondingly becomes a regional comprehensive manifestation at the macro Kn-Coarsening level.
(3) Both Rg-Change Uncertainty and Rg-Change Drifting reflect the Rg-Change law. In fact, Drifting describes a type of uncertainty with regard to the change direction. Thus, Rg-Change Drifting becomes a concrete embodiment of Rg-Change Uncertainty. In contrast, the uncertainty refers to a type of drifting possibility. Thus, Rg-Change Uncertainty could be explained by Rg-Change Drifting. Furthermore, a combination of both exactly describes the essence of Rg-Change.

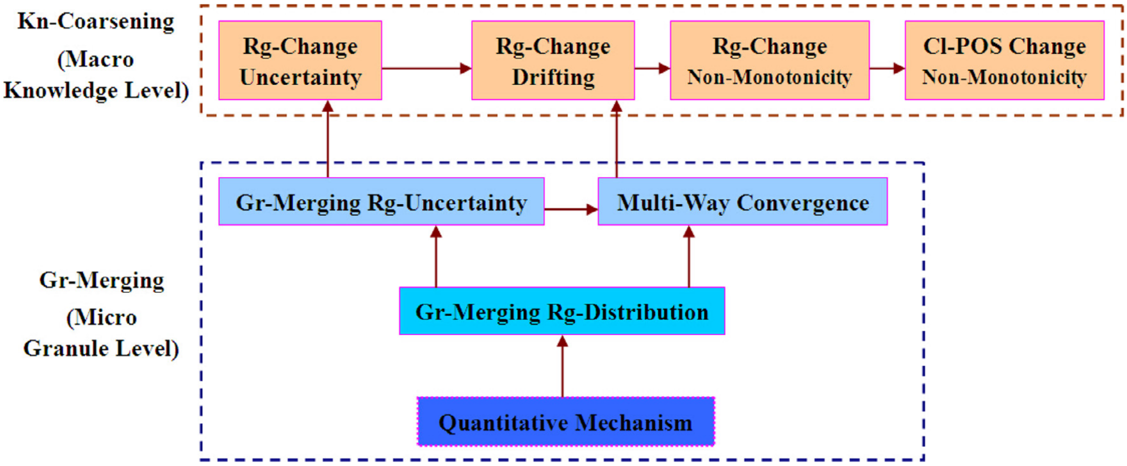


**Fig. 6.** Quantitative Rg-Change uncertainty's relevant properties and their relationships.

**Theorem 6.2.9** (Quantitative Rg-Change Essence)**.** *Quantitative Rg-Change has the essence of uncertain drifting, i.e., three-way DTRS-Regions have a drifting possibility in Kn-Coarsening.*

At the Kn-Coarsening level, Rg-Change Uncertainty underlies Rg-Change Drifting, while Rg-Change Drifting manifests Rg-Change Uncertainty. Furthermore, the two fundamental properties, especially Rg-Change Drifting, could naturally infer two basic conclusions.

**Corollary 6.2.10** (Rg-Change Non-Monotonicity)**.** *In Kn-Coarsening, Rg-Change exhibits non-monotonicity.*

**Corollary 6.2.11** (Cl-POS Change Non-Monotonicity)**.** *In Kn-Coarsening, Cl-POS Change exhibits non-monotonicity.*

For Rg-Change, Drifting includes not only monotonicity but also non-monotonicity (Proposition 6.2.7). Thus, Rg-Change Drifting naturally deduces Rg-Change Non-Monotonicity. Furthermore, Cl-POS Change Non-Monotonicity holds as well because of Cl-POS's definition based on POS (Formula (9)). Generally, Cl-POS Change Non-Monotonicity usually holds, as long as Cl-POS is constructed by three-way regions; except for non-monotonicity, Cl-POS change also concerns drifting and uncertainty. On the whole, these properties are located at the Kn-Coarsening level, including Rg-Change Non-Monotonicity, Cl-POS Change Non-Monotonicity, and their roots: Rg-Change Uncertainty, Rg-Change Drifting. All of the properties concern not only transcendental non-monotonicity but also confused uncertainty.

Thus far, two and four properties are mined at the Gr-Merging and Kn-Coarsening levels, respectively. Before analyzing their logical relationships and basic roots, Fig. 6, which is relevant, is first provided.

Kn-Coarsening and Gr-Merging are two fundamental GrC notions that are at the macro knowledge and micro granule levels, respectively. Gr-Merging further involves its Rg-Distribution and its properties.

(1) Internal Gr-Merging regional features determine external Kn-Coarsening Rg-Change. Thus, Gr-Merging Rg-Distribution, especially Rg-Changed Type, establishes a solid foundation for Rg-Change. Furthermore, Table 3 is utilized for relevant Rg-Change analyses and uncertainty mining.
(2) Based on Gr-Merging Rg-Distribution, two granular statistical properties are summarized at the micro Gr-Merging level, i.e., Gr-Merging Rg-Uncertainty and Multi-Way Convergence. The former underlies the latter.
(3) Four regional integral properties are manifested at the macro Kn-Coarsening level, i.e., Rg-Change Uncertainty, Rg-Change Drifting, Rg-Change Non-Monotonicity, and Cl-POS Change Non-Monotonicity. These four exhibit a decisive relationship from the former to the latter.

The above three items exhibit Gr-Merging Rg-Distribution, Gr-Merging's regional properties, Kn-Coarsening's regional properties, and their internal relationships. All of these results are hierarchically reflected in Fig. 6, where the internal relationships are horizontally exhibited.

Next, the hierarchical relationships are analyzed from the perspective of causality.

(1) According to the completeness verification and purity construction, the quantitative mechanism (with the probability measure) provides Gr-Merging Rg-Distribution Exactness; thus, it becomes a basic mechanism root. Meanwhile, Kn-Coarsening acts as the essential action cause.
(2) Gr-Merging Rg-Distribution (mainly Rg-Changed Type) directly underlies Gr-Merging Rg-Uncertainty and Multi-Way Convergence, two statistical properties that are at the micro Gr-Merging level.
(3) Gr-Merging Rg-Uncertainty and Rg-Change Uncertainty become regional uncertainty at the micro granule and macro knowledge levels, respectively; both have an inference relationship. Multi-Way Convergence and Rg-Change Drifting be-

**Table 6**
Dc-Table of Example 7.

| $U$ | $a$ | $b$ | $d$ |
|---|---|---|---|
| $x_1$ | 0 | 1 | 1 |
| $x_2$ | 0 | 1 | 1 |
| $x_3$ | 0 | 1 | 1 |
| $x_4$ | 1 | 1 | 1 |
| $x_5$ | 1 | 1 | 0 |
| $x_6$ | 1 | 1 | 0 |
| $x_7$ | 2 | 2 | 1 |
| $x_8$ | 2 | 2 | 1 |
| $x_9$ | 2 | 2 | 1 |
| $x_{10}$ | 3 | 2 | 1 |
| $x_{11}$ | 3 | 2 | 1 |
| $x_{12}$ | 3 | 2 | 0 |

come regional embodiments of quantitative uncertainty at the two relevant levels, respectively; both have a mechanism explanation relationship.

The above three items provide a causality development line, i.e.,

*Quantitative Mechanism⟶Gr-Merging Rg-Distribution⟶*
*Gr-Merging's regional properties⟶Kn-Coarsening's regional properties.*

All of these results are vertically exhibited (from the bottom to the top) in Fig. 6. In particular, Rg-Change Drifting plays an important role in Rg-Change analyses. By virtue of Multi-Way Convergence, Rg-Change Drifting represents Rg-Change Uncertainty, while it guides Rg-Change Non-Monotonicity and Cl-POS Change Non-Monotonicity.

Finally, we summarize Sections 6.1 and 5.2 with regard to quantitative Rg-Change uncertainty analyses. First, Gr-Merging Rg-Distribution is established via the quantitative mechanism; then, some fundamental regional properties are mined at both the Gr-Merging and Kn-Coarsening levels. Micro Gr-Merging is dependent on making systematic Rg-Change analyses, and then, the relevant results naturally cause Rg-Change properties in macro Kn-Coarsening. In summary, we not only clearly interpret the transcendental phenomena of attribute reduction, which are Rg-Change Non-Monotonicity and Cl-POS Change Non-Monotonicity, but also thoroughly uncover the extensive basis, which is Rg-Change Uncertainty and Rg-Change Drifting. In particular, *uncertain drifting* becomes the truth that is hidden behind non-monotonicity and finally reflects the essence of quantitative Rg-Change.

### 6.3. An example of quantitative region-change analyses

In this section, a Dc-Table example [69] is utilized to make quantitative Rg-Change analyses.

**Example 7.** The concrete Dc-Table $(U, C \cup D)$ is provided in Table 6. Herein,

$$U = \{x_1, x_2, \ldots, x_{12}\},\ C = \{a, b\},\ D = \{d\};$$
$$U/IND(D) = \{X, \neg X\},\ X = \{x_1, x_2, x_3, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}.$$

By calculation, we can obtain:

$$U/IND(\{a, b\}) = U/IND(\{a\}) = \{Y_1, Y_2, Y_3, Y_4\},\ U/IND(\{b\}) = \{Z_1, Z_2\};$$
$$Y_1 = \{x_1, x_2, x_3\},\ Y_2 = \{x_4, x_5, x_6\},\ Y_3 = \{x_7, x_8, x_9\},\ Y_4 = \{x_{10}, x_{11}, x_{12}\},$$
$$Z_1 = \{x_1, \ldots, x_6\},\ Z_2 = \{x_7 \ldots, x_{12}\}.$$

As a result, Kn-Coarsening $\{a, b\} \overset{-a}{\Longrightarrow} \{b\}$ becomes Kn-Non-Preservation and involves two groups of granular transformation: Gr-Merging $Y_1 \cup Y_2 \overset{=}{\longrightarrow} Z_1$ and Gr-Merging $Y_3 \cup Y_4 \overset{=}{\longrightarrow} Z_2$.

$$\begin{cases} Pr(X|Y_1) = 1,\ Pr(X|Y_2) = \frac{1}{3},\ Pr(X|Z_1) = \frac{2}{3}, \\ Pr(X|Y_3) = 1,\ Pr(X|Y_4) = \frac{2}{3},\ Pr(X|Z_2) = \frac{5}{6}; \end{cases}$$

according to these probabilities, thresholds $\alpha$ and $\beta$ are set in range:

$$0 < \beta < \frac{1}{3} < \frac{2}{3} < \alpha < \frac{5}{6} < 1.$$

For macro Rg-Change, old three-way regions are initially:

$$\mathrm{POS}^{\alpha,\beta}_{\{a,b\}}(X) = Y_1 \cup Y_3,\ \mathrm{BND}^{\alpha,\beta}_{\{a,b\}}(X) = Y_2 \cup Y_4,\ \mathrm{NEG}^{\alpha,\beta}_{\{a,b\}}(X) = \emptyset. \tag{25}$$

By Kn-Coarsening $\{a, b\} \overset{-a}{\Longrightarrow} \{b\}$, new three-way regions finally become:

$$\mathrm{POS}^{\alpha,\beta}_{\{b\}}(X) = Z_2 = Y_3 \cup Y_4,\ \mathrm{BND}^{\alpha,\beta}_{\{b\}}(X) = Z_1 = Y_1 \cup Y_2,\ \mathrm{NEG}^{\alpha,\beta}_{\{b\}}(X) = \emptyset. \tag{26}$$

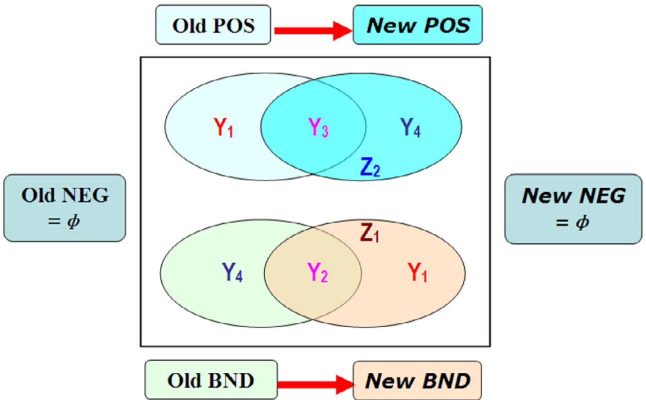


**Fig. 7.** Rg-Change Drifting and Non-Monotonicity of Example 7.

Thus, both POS and NEG exhibit usual Drifting (rather than monotonicity) in Kn-Coarsening $\{a, b\} \overset{-a}{\Longrightarrow} \{b\}$; in contrast, BND has no change because it is always empty. Relevant Rg-Change Drifting and Non-Monotonicity are exhibited in Fig. 7.

For micro Gr-Merging, $Y_1 \cup Y_2 \overset{=}{\longrightarrow} Z_1$ and $Y_3 \cup Y_4 \overset{=}{\longrightarrow} Z_2$ correspond to Type (4b) "POS + BND ⟶ BND" and Type (4a) "POS + BND ⟶ POS", respectively. By inspecting Rg-Changed Granule, $Y_1$ changes from old POS to new BND, while $Y_4$ changes from old BND to new POS. POS and BND have both the increased and decreased parts to achieve Drifting and Non-Monotonicity. Accordingly, the above macro Rg-Change emerges, and the relevant GrC mechanism is clarified.

$POS^{\alpha,\beta}_{\{a,b\}}(\neg X) = \emptyset = POS^{\alpha,\beta}_{\{b\}}(\neg X)$, so old and new Cl-POS become:

$$POS^{\alpha,\beta}_{\{a,b\}}(D) = POS^{\alpha,\beta}_{\{a,b\}}(X),\ POS^{\alpha,\beta}_{\{b\}}(D) = POS^{\alpha,\beta}_{\{b\}}(X). \tag{27}$$

Cl-POS change has the same result as that of POS change, i.e., Cl-POS also exhibits Drifting but Non-Monotonicity from $Y_1 \cup Y_3$ to $Y_3 \cup Y_4$.

In this example, both Rg-Change and Cl-POS change naturally exhibit monotonicity for Pawlak-Model. In contrast, all change of POS, BND, and Cl-POS exhibit non-monotonic drifting for DTRS-Model. For Rg-Change, the exactness and vividness of Drifting are illustrated by this example, where monotonicity no longer holds while non-monotonicity becomes too ambiguous.

For Drifting, the region scale usually exhibits variation. However, this example concerns only the same region scale in Kn-Coarsening. This special result mainly comes from the same scale of two Rg-Changed Granules (which implement exchange), i.e., $|Y_1| = |Y_4|$. According to Formula (10),

$$\gamma^{\alpha,\beta}_{\{a,b\}}(D) = 0.5 = \gamma^{\alpha,\beta}_{\{b\}}(D).$$

In other words, the dependency degree $\gamma^{\alpha,\beta}_{\{a,b\}}(D)$ never changes in Kn-Coarsening. However, Cl-POS and regions have had the essential transformation with regard to non-monotonic Drifting. These facts trigger our in-depth thinking, as follows:

(1) Regional migration (i.e., regional Drifting with the same scale) never occurs in the Pawlak-Model. This special phenomenon transcends the usual rules based on the qualitative mechanism. Hence, it depends on the quantitative mechanism to become a specific characteristic for the quantitative model.
(2) Within the non-monotonic Drifting framework, the attribute dependency degree cannot adequately measure Rg-Change. How to describe and measure regional Drifting becomes a new and valuable problem.

## 7. Comparative region-change analyses and further experiment verification

Thus far, Kn-Coarsening and Gr-Merging provide GrC mechanisms for Rg-Change (Sections 3 and 4). On this basis, relevant Rg-Change certainty and uncertainty analyses are completed in Pawlak-Model and DTRS-Model (Sections 5 and 6), respectively, where the comparative strategy is adopted. As a result, the qualitative and quantitative principles/properties that are acquired above exhibit a sharp contrast, and the latter become more complex and in-depth. This section first summarizes the comparative Rg-Change analyses and then provides further experiment verification.

### *7.1. Comparative region-change analyses*

This section summarizes the comparative Rg-Change analyses. To highlight the quantitative studies and results, quantitative Rg-Change analyses are chosen as a descriptive subject, while qualitative Rg-Change analyses act as a comparative basis. Note that both have some common GrC mechanisms with regard to Kn-Coarsening and Gr-Merging.

Gr-Merging Rg-Distribution establishes the basis for Rg-Change analyses; thus, it is focused on first. Herein, the relevant processes of qualitative and quantitative mining are compared. In Section 5.1, Gr-Merging Rg-Distribution is directly established for Pawlak-Model. Because Pawlak-Model is a qualitative model that has a simple qualitative mechanism, Rg-Distributional completeness and purity are easily identified via POS/NEG certainty and BND uncertainty. In contrast, DTRS-Model becomes a quantitative model with expansion and transcendence to hold a complex quantitative mechanism. In particular, Gr-Merging's probability boundedness (Lemma 6.1.1) becomes a specific quantitative property with regard to the probability measure to establish Rg-Distributional completeness. For purity inspection, the probability-based technology becomes essential but difficult; thus, a simple but effective approach of Dc-Table construction is chosen. In Section 6.1, exact Gr-Merging Rg-Distribution is finally established for DTRS-Model.

After the above mining, qualitative and quantitative Gr-Merging Rg-Distributions are presented exactly in Tables 2 and 3, respectively. On this basis, the Gr-Merging Rg-Distribution types are compared, as follows:

(1) Pawlak-Model and DTRS-Model have 7 and 13 types, respectively. The latter expands the former, and the additional 6 quantitative types exhibit transcendence. In fact, the common 7 types have a similar function for Rg-Change. In contrast, the transcendental 6 types mainly cause specific uncertainty and manifestations for Rg-Change; hence, they become the main GrC roots for quantitative Rg-Change uncertainty analyses.
(2) 7 qualitative types and 13 quantitative types constitute a mathematical mapping and correspondence, respectively. The former mapping is included in the latter correspondence, and both become subparts of the complete Rg-Distribution correspondence. In fact, they basically provide mathematical mechanisms for the relevant certainty and uncertainty analyses, respectively.

Now, concrete Rg-Change analyses are focused on. Qualitative and quantitative parts, which are fixed in Sections 5.2 and 5.2, respectively, are mainly presented by Figs. 3 and 6, respectively. For both figures, their contrast can effectively clarify the comparative region-change analyses. In fact, they exhibit a substantial amount of structural similarity, except for the latter Rg-Change Drifting. Thus, qualitative and quantitative Rg-Change analyses have almost the same property points and deduction relationships. In general, *uncertainty*, *non-monotonicity*, and *multi-way* in the quantitative properties replace *certainty*, *monotonicity*, and *one-way* in the qualitative properties, respectively. Moreover, Rg-Change Drifting is added to vividly describe the quantitative Rg-Change with complexity.

Next, we show some detailed contrast for the several property points from the micro Gr-Merging to macro Kn-Coarsening levels.

Gr-Merging Rg-Uncertainty in DTRS-Model contrasts with Gr-Merging Rg-Certainty in Pawlak-Model, thus manifesting quantitative transcendence. By comparing Tables 2 and 3, Types (1–3) exhibit Gr-Merging Rg-Certainty in both cases, while the uncertainty classes (4–7) in Table 3 become the micro source for quantitative Gr-Merging Rg-Uncertainty.

With regard to Gr-Merging, Multi-Way Convergence in DTRS-Model contrasts with One-Way Convergence in Pawlak-Model, thus also reflecting quantitative transcendence. In fact, Multi-Way Convergence is a granular statistical property with regard to DTRS-Region. As a result, knowledge granules gather as quickly as possible not only in new BND but also in new POS or NEG. This transcendental phenomenon can be explained by a quantitative mechanism. Herein, both convergence types are mainly explained from the viewpoint of uncertainty.

(1) For qualitative Pawlak-Regions, only BND corresponds to uncertainty. Kn-Coarsening/Kn-Non-Preservation leads to uncertainty enhancement. As a result, knowledge granules necessarily exhibit convergence with regard to only BND. Furthermore, Rg-Change Monotonicity naturally emerges at the macro Kn-Coarsening level.
(2) For quantitative DTRS-Regions, POS, NEG, and BND are all related to uncertainty in view of quantitative expansion. Although Gr-Merging still implies strict Kn-Coarsening (i.e., Kn-Non-Preservation), the corresponding uncertainty enhancement cannot already ensure sole convergence with regard to only BND. In fact, directional convergence with regard to POS/NEG emerges as well. Therefore, Gr-Merging utilizes the quantitative uncertainty mechanism to exhibit new multi-way convergence with regard to three-way DTRS-Regions. Furthermore, Rg-Change in macro Kn-Coarsening transcends the basic monotonicity framework.

Gr-Merging thoroughly gains Rg-Uncertainty/Rg-Certainty and Convergence. These micro properties depend on granular mechanisms to naturally cause macro manifestations in Kn-Coarsening. Thus, Rg-Change properties require only direct contrasts at the Kn-Coarsening level. Rg-Change Uncertainty, Rg-Change Non-Monotonicity, and Cl-POS Change Non-Monotonicity in DTRS-Model sharply contrast with Rg-Change Certainty, Rg-Change Monotonicity, and Cl-POS Change Monotonicity in Pawlak-Model, respectively. The former three, which never exist in the qualitative model, also manifest quantitative inherence and transcendence. In particular, Rg-Change Drifting is produced as a novel notion for the quantitative environment with complexity, and it is emphatically analyzed for quantitative Rg-Change. Rg-Change Drifting holds within the Rg-Change Uncertainty framework, and uncertain drifting becomes the quantitative Rg-Change essence (Theorem 6.2.9); furthermore, Rg-Change Non-Monotonicity and Cl-POS Change Non-Monotonicity become two natural results. In contrast, Rg-Change Monotonicity holds within the Rg-Change Certainty framework, and certain monotonicity becomes the qualitative Rg-Change essence (Theorem 5.2.5); furthermore, Rg-Change Monotonicity and Cl-POS Change Monotonicity become two clear properties.

### 7.2. Further experiment verification

To verify the comparative Rg-Change analyses, this section performs some further experiment studies. Concretely, five databases from the UCI Machine Learning Repository [76] are adopted, where part cores and reducts were previously acquired

**Table 7**
Description of five UCI databases.

| ID | Databases | $\|U\|$ | $\|C\|$ | $\|U/IND(D)\|$ |
|---|---|---|---|---|
| (1) | Voting | 435 | 16 | 2 |
| (2) | SPECT Heart | 267 | 22 | 2 |
| (3) | Tic-Tac-Toe | 958 | 9 | 2 |
| (4) | Monks-3 | 432 | 6 | 2 |
| (5) | Balance | 625 | 4 | 3 |

[69]. Relevant Dc-Table $(U, C \cup D)$ information is summarized in Table 7, where both two-category and multi-category cases are concerned.

Knowledge-based regions become a radical basis for Rg-Change analyses. This regional emphasis inspires the following experimental design.

(1) First, Kn-Coarsening is mainly determined by coarsening knowledge, a subset of $C$. Preliminary knowledge selectively arises from some important notions, such as the core. Subsequent knowledge is internally determined by extracting a hierarchical chain. This monotonicity strategy fully manifests Kn-Coarsening; thus, relevant deducibility/dependency can well motivate general Rg-Change. Moreover, knowledge requires amending by the relevant region feedback, to gain effectiveness.
(2) Based on the above theoretical research, Gr-Merging provides a granular mechanism for Rg-Change in Kn-Coarsening. In the practical database, Gr-Merging usually has massive groups. Thus, we neglect the relevant mechanism and function of micro Gr-Merging and directly concern macro regions and their properties in Kn-Coarsening.
(3) Regions are directly computed. Only POS, NEG, and Cl-POS are concerned in both Pawlak-Model and DTRS-Model, where surplus BND becomes clear based on complementarity. Simple regional symbols are used in contrast, and decision concept $X$ usually corresponds to the first element. For the two-category DTRS-Model, the symmetrical threshold $\beta = 1 - \alpha$ leads to the simplified calculation of Cl-POS with regard to $X$, i.e.,

$$\text{Cl-POS}^{\alpha,\beta} = \text{POS}^{\alpha,\beta}(X) \cup \text{POS}^{\alpha,\beta}(\neg X) = \text{POS}^{\alpha,\beta}(X) \cup \text{NEG}^{\alpha,\beta}(X),$$

which is similar to that in two-category Pawlak-Model. Thus, we mainly adopt the symmetrical and representative threshold $(\alpha, \beta) = (0.8, 0.2)$, which is sufficiently effective.
(4) The regions are accurately calculated. However, they cannot be completely presented because the object numbers are large. In fact, our purpose is only to verify Rg-Change. Thus, the whole region results become complex and unnecessary; in contrast, portion-based typicality becomes effective and sufficient. Based on concrete experimental results, representative elements will be selectively extracted. They not only represent the whole regions but also include Rg-Changed granules. Thus, they are finally utilized to perform the effective verification for Rg-Change analyses.

Based on the above preparations, regional experiment results are provided in detail in Table 8. The processes and results in Table 8 are explained, as follows. There are eight columns in total. The 1st column shows the illustrated cases, which are linked to the selected knowledge and portion-based regions. Database $*$ has multiple Cases $(*.1)$ $(*.2)$..., and Case $(*.1)$ corresponds to the initial knowledge $C$ and the regions. The 2nd reflects the knowledge with the usual deducibility/dependency hierarchy. The 3rd, 4th, 5th columns provide qualitative POS, NEG, and Cl-POS, respectively. The latter column is the union of the former two for two-category Databases (1–4). In contrast, the 6rd, 7th, 8th columns, respectively, provide quantitative POS, NEG, and Cl-POS with the threshold (0.8, 0.2). Additionally, the latter column is the union of the former two for Databases (1–4). Next, the choices of knowledge, concepts, and objects are explained in each database, respectively.

(1) Database (1) produces five cases. Preliminary knowledge $c_1c_2c_3c_9c_{11}c_{13}c_{16}$ and its internal knowledge $c_1c_2c_3c_9c_{13}c_{16}$ correspond to cores $C57513$ and $C57481$, respectively [69]; the surplus $c_1c_2c_3c_{13}c_{16}$ and $c_1c_2c_3c_{16}$ gradually become coarser. $X$ reflects the republican class. Representative elements refer to the first ten objects, which have the mixed republican and democrat results.
(2) Database (2) produces five cases. In [69], the binary code represents knowledge and $C2945871$ becomes a core. In fact, $C2945871$ is related to binary code 1011001111001101001111 and, thus, represents preliminary knowledge $c_1c_3c_4c_7 \ldots c_{10}c_{13}c_{14}c_{16}c_{19} \ldots c_{22}$. Furthermore, $C2945856$, $C62272$, and $C61504$ mean the surplus hierarchical knowledge. For example, $C61504$ corresponds to 22-bit code 0000001111000001000000 to represent knowledge $c_7 \ldots c_{10}c_{16}$. Herein, the 187/80 test/train objects are arranged first/second, and $X$ refers to OVERALL DIAGNOSIS with 1. Representative elements choose the first 20 objects, which are in $X$.
(3) Database (3) provides six cases. In fact, eight arbitrary conditional attributes constitute several types of reducts [69]. Thus, preliminary knowledge is used to choose seven attribute-based $c_1 \ldots c_7$, and the attribute number of the surplus knowledge is gradually decreased to three. Herein, $X$ reflects the positive class. Representative elements choose the final four objects, $x_{954}$ ... $x_{958}$, which are in negative class $\neg X$.
(4) Database (4) provides four cases. Preliminary knowledge is $c_2c_4c_5$, which is a core and reduct [69]. According to the 432 test elements, $X$ reflects the class with the value 1. Representative elements choose the first ten objects, which have mixed class attributes.

**Table 8**
Reginal experiment results of five UCI databases.

| Case | Knowledge | POS | NEG | CI-POS | $POS^{0.8,\,0.2}$ | $NEG^{0.8,\,0.2}$ | $CI\text{-}POS^{0.8,\,0.2}$ |
|---|---|---|---|---|---|---|---|
| (1.1) | $C$ | $x_1x_2x_8x_9\ldots$ | $x_3\ldots x_7x_{10}\ldots$ | $x_1\ldots x_{10}\ldots$ | $x_1x_2x_8x_9\ldots$ | $x_3\ldots x_7x_{10}\ldots$ | $x_1\ldots x_{10}\ldots$ |
| (1.2) | $c_1c_2c_3c_9c_{11}c_{13}c_{16}$ | $x_1x_2\ldots$ | $x_3x_4x_5x_7x_{10}\ldots$ | $x_1\ldots x_5x_7x_{10}\ldots$ | $x_1x_2x_8x_9\ldots$ | $x_3x_4x_5x_7x_{10}\ldots$ | $x_1\ldots x_5x_7\ldots x_{10}\ldots$ |
| (1.3) | $c_1c_2c_3c_9c_{13}c_{16}$ | $x_2\ldots$ | $x_3x_7x_{10}\ldots$ | $x_2x_3x_7x_{10}\ldots$ | $x_1x_2x_8x_9\ldots$ | $x_3x_4x_6x_7x_{10}\ldots$ | $x_1\ldots x_4x_6\ldots x_{10}\ldots$ |
| (1.4) | $c_1c_2c_3c_{13}c_{16}$ | $\ldots$ | $x_3x_7x_{10}\ldots$ | $x_3x_7x_{10}\ldots$ | $x_1x_2x_8x_9\ldots$ | $x_3\ldots x_7x_{10}\ldots$ | $x_1\ldots x_{10}\ldots$ |
| (1.5) | $c_1c_2c_3c_{16}$ | $\ldots$ | $x_3x_{10}\ldots$ | $x_3x_{10}\ldots$ | $x_1x_2x_7x_8x_9\ldots$ | $x_3\ldots x_6x_{10}\ldots$ | $x_1\ldots x_{10}\ldots$ |
| (2.1) | $C$ | $x_1\ldots x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{20}\ldots$ | $x_1\ldots x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{20}\ldots$ |
| (2.2) | $C$2945871 | $x_1\ldots x_{11}x_{13}x_{14}x_{15}x_{17}x_{19}x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{11}x_{13}x_{14}x_{15}x_{17}x_{19}x_{20}\ldots$ | $x_1\ldots x_{15}x_{17}x_{19}x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{15}x_{17}x_{19}x_{20}\ldots$ |
| (2.3) | $C$2945856 | $x_1\ldots x_{11}x_{13}x_{15}x_{17}x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{11}x_{13}x_{15}x_{17}x_{20}\ldots$ | $x_1\ldots x_{11}x_{13}x_{14}x_{15}x_{17}x_{20}\ldots$ | $\ldots$ | $x_1\ldots x_{11}x_{13}x_{14}x_{15}x_{17}x_{20}\ldots$ |
| (2.4) | $C$62272 | $x_1x_2x_4\ldots x_7x_9x_{10}x_{11}$ $x_{13}x_{15}x_{17}x_{20}\ldots$ | $\ldots$ | $x_1x_2x_4\ldots x_7x_9x_{10}x_{11}$ $x_{13}x_{15}x_{17}x_{20}\ldots$ | $x_1x_2x_4\ldots x_{11}x_{13}\ldots x_{17}x_{20}\ldots$ | $\ldots$ | $x_1x_2x_4\ldots x_{11}x_{13}\ldots x_{17}x_{20}\ldots$ |
| (2.5) | $C$61504 | $x_1x_4x_6x_{10}x_{15}x_{17}x_{20}\ldots$ | $\emptyset$ | $x_1x_4x_6x_{10}x_{15}x_{17}x_{20}\ldots$ | $x_1x_2x_4\ldots x_{18}x_{20}\ldots$ | $\emptyset$ | $x_1x_2x_4\ldots x_{18}x_{20}\ldots$ |
| (3.1) | $C$ | $\ldots$ | $\ldots x_{954}\ldots x_{958}$ | $\ldots x_{954}\ldots x_{958}$ | $\ldots$ | $\ldots x_{954}\ldots x_{958}$ | $\ldots x_{954}\ldots x_{958}$ |
| (3.2) | $c_1\ldots c_7$ | $\ldots$ | $\ldots x_{954}x_{955}x_{956}x_{958}$ | $\ldots x_{954}x_{955}x_{956}x_{958}$ | $\ldots$ | $\ldots x_{954}x_{955}x_{956}x_{958}$ | $\ldots x_{954}x_{955}x_{956}x_{958}$ |
| (3.3) | $c_1\ldots c_6$ | $\ldots$ | $\ldots x_{954}x_{956}$ | $\ldots x_{954}x_{956}$ | $\ldots$ | $\ldots x_{954}x_{956}$ | $\ldots x_{954}x_{956}$ |
| (3.4) | $c_1\ldots c_5$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots x_{955}x_{958}$ | $\ldots x_{954}x_{956}$ | $\ldots x_{954}x_{955}x_{956}x_{958}$ |
| (3.5) | $c_1\ldots c_4$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots x_{954}$ | $\ldots x_{954}$ |
| (3.6) | $c_1\ldots c_3$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots x_{958}$ | $\ldots$ | $\ldots x_{958}$ |
| (4.1) | $C$ | $x_1\ldots x_6x_9x_{10}\ldots$ | $x_7x_8\ldots$ | $x_1\ldots x_{10}\ldots$ | $x_1\ldots x_6x_9x_{10}\ldots$ | $x_7x_8\ldots$ | $x_1\ldots x_{10}\ldots$ |
| (4.2) | $c_2c_4c_5$ | $x_1\ldots x_6x_9x_{10}\ldots$ | $x_7x_8\ldots$ | $x_1\ldots x_{10}\ldots$ | $x_1\ldots x_6x_9x_{10}\ldots$ | $x_7x_8\ldots$ | $x_1\ldots x_{10}\ldots$ |
| (4.3) | $c_4c_5$ | $x_5x_6\ldots$ | $x_7x_8\ldots$ | $x_5\ldots x_8\ldots$ | $x_5x_6\ldots$ | $x_7x_8\ldots$ | $x_5\ldots x_8\ldots$ |
| (4.4) | $c_5$ | $\emptyset$ | $x_7x_8\ldots$ | $x_7x_8\ldots$ | $\emptyset$ | $x_7x_8\ldots$ | $x_7x_8\ldots$ |
| (5.1) | $C$ | $x_{26}x_{51}x_{52}x_{56}\ldots$ | $x_1\ldots x_{25}x_{27}\ldots x_{50}x_{53}x_{54}x_{55}\ldots$ | $x_1\ldots x_{56}\ldots$ | $x_{26}x_{51}x_{52}x_{56}\ldots$ | $x_1\ldots x_{25}x_{27}\ldots x_{50}x_{53}x_{54}x_{55}\ldots$ | $x_1\ldots x_{56}\ldots$ |
| (5.2) | $c_2c_3c_4$ | $x_{26}x_{51}x_{52}x_{56}\ldots$ | $x_5x_8x_9x_{10}x_{12}x_{13}x_{14}$ $x_{15}x_{17}\ldots x_{25}x_{35}x_{39}x_{40}x_{43}x_{44}$ $x_{45}x_{47}x_{48}x_{49}x_{50}\ldots$ | $x_8x_9x_{10}x_{12}\ldots x_{15}$ $x_{17}\ldots x_{20}x_{22}\ldots x_{26}x_{39}x_{40}$ $x_{43}x_{44}x_{45}x_{48}\ldots x_{52}x_{56}\ldots$ | $x_1x_{26}x_{27}x_{28}x_{31}x_{36}x_{51}\ldots x_{56}\ldots$ | $x_4x_5x_7\ldots x_{10}x_{12}\ldots x_{25}$ $x_{34}x_{35}x_{38}x_{39}x_{40}$ $x_{42}\ldots x_{45}x_{47}\ldots x_{50}\ldots$ | $x_1x_5x_8x_9x_{10}x_{12}\ldots x_{15}$ $x_{17}\ldots x_{28}x_{31}x_{35}x_{36}x_{38}x_{39}x_{40}$ $x_{43}x_{44}x_{45}x_{47}\ldots x_{56}\ldots$ |
| (5.3) | $c_2c_3$ | $\emptyset$ | $x_{21}\ldots x_{25}\ldots$ | $\emptyset$ | $x_{51}\ldots x_{55}\ldots$ | $x_6\ldots x_{25}x_{41}\ldots x_{50}\ldots$ | $x_{11}\ldots x_{25}x_{46}\ldots x_{55}\ldots$ |

(5) In Database (5), $C$ has four attributes, so only knowledge $c_2c_3c_4$ and $c_2c_3$ are concerned. $X$ selects the class with result L. Representative elements refer to the first 56 ones, whose main class label is R. Note that there are three classes so Cl-POS becomes union of three kinds of POS (rather than union of POS and BND).

Now, Table 8 is utilized to make comparative Rg-Change analyses. First, qualitative Rg-Change is simple for Pawlak-Model. For the databases, POS, NEG, and Cl-POS, all exhibit clear monotonic change for the given Kn-Coarsening. Thus, all of the qualitative Rg-Change properties are verified in practice, including Rg-Change Certainty/Monotonicity and Cl-POS Change Monotonicity (Theorems 5.2.3, 5.2.4, Corollary 5.2.6, or Fig. 3). In contrast, quantitative Rg-Change becomes complex for DTRS-Model. For an effective illustration, partial Kn-Coarsening and objects are selected in each database, and non-monotonic drifting (with both increasing and decreasing parts) becomes a focus of concern.

(1) In Kn-Coarsening $C \overset{-}{\Longrightarrow} c_1c_2c_3c_{16}$ from Cases (1.1) and (1.5),

$$x_7 \in \mathrm{POS}^{0.8,0.2}_{\mathrm{new}} - \mathrm{POS}^{0.8,0.2}_{\mathrm{old}}.$$

In generalized Kn-Coarsening $c_1c_2c_3c_9c_{11}c_{13}c_{16} \overset{-}{\Longrightarrow} c_1c_2c_3c_9c_{13}c_{16}$ from Cases (1.2) and (1.3),

$$x_6 \in \mathrm{NEG}^{0.8,0.2}_{\mathrm{new}} - \mathrm{NEG}^{0.8,0.2}_{\mathrm{old}},\ x_6 \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}};$$
$$x_5 \in \mathrm{NEG}^{0.8,0.2}_{\mathrm{old}} - \mathrm{NEG}^{0.8,0.2}_{\mathrm{new}},\ x_5 \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}}.$$

Herein, quantitative POS is never lessened, while quantitative NEG and Cl-POS change exhibit drifting, uncertainty, and non-monotonicity.

(2) In generalized Kn-Coarsening $C2945856 \overset{-}{\Longrightarrow} C62272$ from Cases (2.3) and (2.4),

$$x_{16} \in \mathrm{POS}^{0.8,0.2}_{\mathrm{new}} - \mathrm{POS}^{0.8,0.2}_{\mathrm{old}},\ x_3 \in \mathrm{POS}^{0.8,0.2}_{\mathrm{old}} - \mathrm{POS}^{0.8,0.2}_{\mathrm{new}}.$$

Thus, the quantitative POS change exhibits drifting, uncertainty, and non-monotonicity, as does quantitative Cl-POS change, because Cl-POS is equal to POS with regard to the 20 representative elements.

(3) In generalized Kn-Coarsening $c_1c_2c_3c_4 \overset{-}{\Longrightarrow} c_1c_2c_3$ from Cases (3.5) and (3.6),

$$x_{958} \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}},\ x_{954} \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}}.$$

Thus, quantitative Cl-POS change exhibits drifting, uncertainty, and non-monotonicity.

(4) With regard to the knowledge and object selected in Database (4), quantitative Rg-Change exhibits only monotonicity, which is similar to that of qualitative Rg-Change.

(5) In Kn-Coarsening $C \overset{-}{\Longrightarrow} c_2c_3$ from Cases (5.1) and (5.3),

$$x_{53}, x_{54}, x_{55} \in \mathrm{POS}^{0.8,0.2}_{\mathrm{new}} - \mathrm{POS}^{0.8,0.2}_{\mathrm{old}},\ x_{26}, x_{56} \in \mathrm{POS}^{0.8,0.2}_{\mathrm{old}} - \mathrm{POS}^{0.8,0.2}_{\mathrm{new}}.$$

In generalized Kn-Coarsening $c_2c_3c_4 \overset{-}{\Longrightarrow} c_2c_4$ from Cases (5.2) and (5.3),

$$x_6, x_{11}, x_{41}, x_{46} \in \mathrm{NEG}^{0.8,0.2}_{\mathrm{new}} - \mathrm{NEG}^{0.8,0.2}_{\mathrm{old}};\ x_4, x_5, x_{34}, x_{35}, x_{38}, x_{39}, x_{40} \in \mathrm{NEG}^{0.8,0.2}_{\mathrm{old}} - \mathrm{NEG}^{0.8,0.2}_{\mathrm{new}};$$
$$x_{11}, x_{16}, x_{46} \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}};\ x_1, x_5, x_8, x_9, x_{10} \in \text{Cl-POS}^{0.8,0.2}_{\mathrm{old}} - \text{Cl-POS}^{0.8,0.2}_{\mathrm{new}}.$$

Thus, all of quantitative POS, NEG, and Cl-POS change exhibit drifting, uncertainty, and non-monotonicity.

Based on the five analysis items above, all of the quantitative Rg-Change properties gain effective verification, including Rg-Change Uncertainty/Drifting/Non-Monotonicity and Cl-POS Change Non-Monotonicity (Theorems 6.2.5, 6.2.8, Corollaries 6.2.10, 6.2.11, or Fig. 6).

Thus far, the comparative Rg-Change analyses are effectively verified via the above database experiments. Herein, we mainly adopt the two-category/three-category database, hierarchical knowledge with fewer attributes, and the symmetrical threshold (0.8, 0.2). More generally, effective verification could also utilize the multi-category database, chaotic knowledge with more attributes, and unsymmetrical threshold.

## 8. Conclusions

The Rg-Change law is fundamental for attribute reduction. However, it only rests on non-monotonicity/monotonicity in the quantitative/qualitative model. In other words, it lacks an in-depth exploration for the Rg-Change essence, mechanism, and property. From the uncertainty/certainty viewpoint, this paper utilizes DTRS-Model and Pawlak-Model to perform comparative Rg-Change analyses based on GrC. As a result, quantitative/qualitative Rg-Change uncertainty/certainty is mined and analyzed to underlie attribute reduction.

(1) Kn-Coarsening and Gr-Merging, two GrC levels, are first investigated. The GrC hierarchy (Fig. 1) and technological algorithm/roadmap (Algorithm 1 and Fig. 2) deeply underlie Rg-Change analyses. As a result, Gr-Merging Rg-Distribution and Rg-Changed Type become especially important. For Rg-Change analyses, these studies clarify GrC mechanisms and hold fundamental significance for extensive RS-Models.

(2) Aimed at DTRS-Model and Pawlak-Model, we establish Gr-Merging Rg-Distribution and inspect Rg-Changed Type to make comparative Rg-Change analyses. Quantitative/qualitative Rg-Change Uncertainty/Certainty and relevant properties are fully acquired. Herein, we adopt both a research direction from the micro Gr-Merging mechanism to macro Kn-Coarsening manifestation and a comparative research strategy between quantitative Rg-Change uncertainty analyses and qualitative Rg-Change certainty analyses. As a result, Rg-Change laws and relevant uncertainty/certainty mechanisms are uncovered for both basic RS-Models.

Next, the main DTRS Rg-Change analyses are especially summarized. First, Gr-Merging's Rg-Uncertainty and Multi-Way Convergence are mined at the micro granule level; then, Rg-Change's Uncertainty and Drifting are naturally gained at the macro knowledge level; furthermore, the latter two deduce Rg-Change Non-Monotonicity and Cl-POS Change Non-Monotonicity. Thus, Rg-Change holds uncertainty, drifting, and non-monotonicity, while Cl-POS Change exhibits non-monotonicity. In particular, we discover Rg-Change Uncertainty and Rg-Change Drifting and provide their mechanisms and rules. Both fundamental features are utilized to mine the quantitative Rg-Change essence: uncertain drifting, which deepens and underlies non-monotonicity. In other words, essential uncertain drifting becomes the truth that is hidden behind the non-monotonicity phenomenon, and the former radically provides the cause and explanation for the latter. Note that all of the obtained properties exhibit quantitative transcendence because they never emerge in qualitative Pawlak-Model. In fact, these transcendental characteristics originate from probability-based quantitative expansion and embody generality for probabilistic/quantitative RS-Models. Thereby, the relevant technologies of Rg-Change uncertainty analyses are worthwhile to generalize for quantitative RS-Models (such as PRS-Models). Moreover, two points are retained to be treated in a more in-depth fashion.

(1) The probabilistic/quantitative mechanism determines Gr-Merging Rg-Distribution, e.g., Gr-Merging's probability boundedness (Formula (24)) establishes Rg-Distributional Completeness. Thus, the probabilistic/quantitative mechanism is worthwhile to explore thoroughly. In particular, it is expected to probe Rg-Distributional Purity, whose proof in this paper adopts only a construction approach.
(2) The exact Gr-Merging Rg-Distribution (Table 3) underlies Rg-Change analyses. There, Rg-Changed Type is worthwhile to utilize to mine in-depth Rg-Change properties. For example, Types (4a), (6a), (7a) and (5a), (6b), (7b) have an increasing function for POS and NEG, respectively; moreover, Type (6c) can construct disjoint drifting.

In summary, the relevant possibility, distribution, mechanism, explanation, and rule are provided exactly for both quantitative Rg-Change Uncertainty/Drifting and qualitative Rg-Change Certainty/Monotonicity. Therefore, this paper provides some new insights into quantitative expansion and transcendence. The acquired results could largely eliminate current reduction confusions about Rg-Change; thus, they underlie rational construction for further attribute reduction, especially with regard to quantitative models. In fact, they also inspire a new attribute reduction challenge. Within an objective framework of Rg-Change Uncertainty/Drifting, how to scientifically define quantitative reducts becomes a considerable problem. In our opinion, Rg-Preservation is natural for qualitative Rg-Change Monotonicity, while it appears to be more necessary in the quantitative expansion with not only Rg-Change Non-Monotonicity but also Rg-Change Uncertainty/Drifting. In other words, Rg-Preservation is still a rational choice for quantitative reduction construction. Following this simple idea, Rg-Preservation was previously utilized to construct DTRS-Reduction [68,69]. More generally, quantitative attribute reduction is worthwhile to develop for extensive probabilistic/quantitative RS-Models.

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## Appendix A. Proof of Theorem 6.1.2

**Proof.** For Gr-Merging Rg-Distribution, the largest completeness with 21 types is provided in Theorem 4.3.1. In contrast, this theorem aims to establish more accurate completeness. In other words, we need to verify the nonexistence of some types. Old Granules Rg-Distribution has seven cases. By comparing Tables 1 and 3, all types with regard to "POS + NEG" and "POS + NEG + BND" have the common existence. Thus, only surplus five cases of Old Granules Rg-Distribution need considering, and they are:

"POS", "NEG", "BND", "POS + BND", and "NEG + BND".

In fact, we only need to identify their eight impossible types for New Granule Rg-Belonging (or Gr-Merging Rg-Distribution).

Give Gr-Merging $[x]_C^1 \cup \cdots \cup [x]_C^m \overset{=}{\longrightarrow} [x]_{C-C'}^1$, and let

$$\begin{cases} \min\limits_{i\in\{1,\ldots,m\}} Pr(X|[x]_C^i) = min, \\ \max\limits_{i\in\{1,\ldots,m\}} Pr(X|[x]_C^i) = max. \end{cases} \tag{A.1}$$

Then, according to Lemma 6.1.1,

$$min \le Pr([x]^1_{C-C'}) \le max. \tag{A.2}$$

(1) For Old Granules Rg-Distribution case “POS”, $[x]^i_C$ comes from POS so $Pr([x]^i_C) \ge \alpha$. According to Formula (A.2),

$$Pr([x]^1_{C-C'}) \ge min \ge \alpha,$$

so $[x]^1_{C-C'}$ belongs to POS. Thus, “POS $\longrightarrow$ BND” and “POS $\longrightarrow$ NEG” never emerge, while “POS $\longrightarrow$ POS” necessarily exists and becomes Type (1) (Table 3).

(2) For case “NEG”, $[x]^i_C$ comes from NEG so $Pr([x]^i_C) \le \beta$. According to Formula (A.2),

$$Pr([x]^1_{C-C'}) \le max \le \beta,$$

so $[x]^1_{C-C'}$ belongs to NEG. Thus, “NEG $\longrightarrow$ POS” and “NEG $\longrightarrow$ BND” never emerge, while “NEG $\longrightarrow$ NEG” necessarily exists and becomes Type (2).

(3) For case “BND”, $[x]^i_C$ comes from BND so $Pr([x]^i_C) \in (\beta, \alpha)$. According to Formula (A.2),

$$\beta < min \le Pr([x]^1_{C-C'}) \le max < \alpha,$$

so $[x]^1_{C-C'}$ belongs to BND. Thus, “BND $\longrightarrow$ POS” and “BND $\longrightarrow$ NEG” never emerge, while “BND $\longrightarrow$ BND” necessarily exists and becomes Type (3).

(4) For case “POS + BND”, $[x]^i_C$ comes from POS or BND so $Pr([x]^i_C) \ge \alpha$ or $Pr([x]^i_C) \in (\beta, \alpha)$. Then, $min \in (\beta, \alpha)$ and $max \ge \alpha$, i.e., $\beta < min < \alpha \le max$. According to Formula (A.2),

$$Pr([x]^1_{C-C'}) \in [min, max],\ Pr([x]^1_{C-C'}) \nleq \beta,$$
$$\text{so } Pr([x]^1_{C-C'}) \ge \alpha \text{ or } Pr([x]^1_{C-C'}) \in (\beta, \alpha).$$

Accordingly, “POS + BND $\longrightarrow$ NEG” never emerges; meanwhile, there are at most two possible types, i.e., Type (4a) “POS + BND $\longrightarrow$ POS” and Type (4b) “POS + BND $\longrightarrow$ BND”.

(5) For case “NEG + BND”, $[x]^i_C$ comes from NEG or BND so $Pr([x]^i_C) \le \beta$ or $Pr([x]^i_C) \in (\beta, \alpha)$. Then, $min \le \beta$ and $max \in (\beta, \alpha)$, i.e., $min \le \beta < max < \alpha$. According to Formula (A.2),

$$Pr([x]^1_{C-C'}) \in [min, max],\ Pr([x]^1_{C-C'}) \ngeq \alpha,$$
$$\text{so } Pr([x]^1_{C-C'}) \le \beta \text{ or } Pr([x]^1_{C-C'}) \in (\beta, \alpha).$$

Accordingly, “BND + NEG $\longrightarrow$ POS” never emerges; meanwhile, there are at most two possible types, i.e., Type (5a) “BND + NEG $\longrightarrow$ NEG” and Type (5b) “BND + NEG $\longrightarrow$ BND”. □

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