

Matrix

see [math notation](#), [eigenvector](#) and [eigenvalues](#)

definition formally in my [math notation](#) a [matrix](#) in $\mathbb{R}^{m,n}$ is a [set theoretical function](#) with domain at least $\langle x, y \rangle \rightarrow \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$ that takes an [ordered pair](#) as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by a Scalar

see [matrix vector space](#), [vector space](#)

definition

$$(kA)^{i,j} = kA^{i,j} \vdash \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

properties

commutativity with [scalars](#) $kA = Ak$

Matrix Addition

see [matrix vector space](#), [vector space](#)

definition

$$(A : B)^{i,j} = A^{i,j} : B^{i,j} \vdash \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

Matrix Multiplication

see [dot product](#), [vector in rn](#)

definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B . their product will be an m by p [matrix](#))

$$(AB)^{i,j} = A^i \cdot B^j \vdash \mathbb{N}i \wedge \mathbb{N}j, \text{ see } \a href="#">dot product$$

intuitively, matrix multiplication is the [dot product](#) of **every row** of the first [matrix](#) by **every column** of the second [matrix](#)

notation

$$A^1 = A^2 = [A]^2 \vdash \mathbb{M}A$$

$$\text{and therefore } A^1 \dots A^1 = [A]^n \wedge \mathbb{N}n \vdash \mathbb{M}A$$

properties

$$\text{not commutative } AB = BA \not\vdash \mathbb{M}A \wedge \mathbb{M}B \text{ or } AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$$

$$AB = 0 \not\vdash A = 0 \vee B = 0 \text{ (it can happen that } AB = 0, \text{ but } A \neq 0 \text{ and } B \neq 0) \text{ (} AB \text{ being equal to 0 does not imply that } A = 0 \text{ or that } B = 0)$$

$$AC = BC \wedge C \neq 0 \not\vdash A = B \text{ (} AC = BC \text{ and } C \neq 0 \text{ does not imply that } A = B)$$

$$\text{associative } (AB)C = A(BC)$$

$$\text{distributive } A(B : C) = AB : AC$$

$$\text{distributive } (B : C)A = BA : CA$$

$$\text{associative with scalars } k(AB) = (kA)B = A(kB)$$

applications

matrix multiplication can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

matrix multiplication can be used to represent any linear transformation

Identity Matrix

definition

$$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \vdash \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

properties

$$AI = A \wedge IA = A \vdash \mathbb{M}A$$

Zero Matrix

see [matrix vector space](#), [vector space](#)

definition

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

properties

$$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$$

$$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$$

Rank of a Matrix

the number of pivots in any [REF](#) of the [matrix](#)

notation

rank A , where

- A is the [matrix](#) to find the rank of

Matrix Vector Spaces

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

kernel $Ker A \equiv Null A$

column space $Col A$

row space $Row A$

definitions

kernel $(Ker A) x \equiv (Null A) x \equiv Ax = O \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,1}x$

column space $Col A = \text{span } A'^n \dashv \mathbb{N}n$

row space $\text{Row } A = \text{span } A^n, \vdash \mathbb{N}n$

procedure computing the kernel of a matrix use row reduction

theorems

the Null Space, Row Space and Column Space of a matrix are always vector spaces

number of free variables in A : number of pivots in A = number of columns in A

$\dim \text{Null } A$ = number of free variables in A

$\text{rank } A$ = number of pivots in A

the nonzero rows in any REF of a matrix A forms a basis for $\text{Row } A$. therefore,
 $\dim \text{Row } A = \text{rank } A$ (see rank of a matrix)

if A and B are row-equivalent, then $\text{Row } A = \text{Row } B$, see linear system

the spanning set of $\text{Null } A$ obtained from applying row reduction on the system
 $Ax = 0$ is a basis for $\text{Null } A$

$\text{Row } A$ does not change when applying elementary operations on the rows of A , see
linear system

properties

$\text{Col } A = \text{Row } A^T \wedge \text{Row } A = \text{Col } A^T \vdash \mathbb{M}A$, see transpose matrix

applications

row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for the row space of a matrix can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for the column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix

the same can be said for $\text{Col } A$

example transforming a vector space into the null space of a certain matrix

let $W = \text{span} \langle (1, 0, 0, 1), (1, 1, 1, 0), (2, 1, -1, 1) \rangle$

after solving the linear system, we get $W(x, y, z, w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$

Transpose Matrix

the transpose of a matrix

flips a matrix around its diagonal

note the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

definition

$$(A^T)^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

properties

$$(A^T)^T = A \dashv \mathbb{M}A$$

$$(AB)^T = B^T A^T \dashv \mathbb{M}A \wedge \mathbb{M}B$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

representation

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Matrix Inverse

the inverse of a matrix

definition

$$AA^{-1} = A^{-1}A = I \dashv \mathbb{M}A, \text{ where}$$

- A is a square matrix
- A^{-1} is the *inverse matrix* of A

Invertability

definition an *invertible matrix* has a corresponding inverse matrix

see theorems below for invertability criteria

properties

let A and C be invertible matrixes, let \mathbb{Z}_p and let $\mathbb{R}k \wedge k \neq 0$. then,

$$AA^{-1} = A^{-1}A = I$$

$$(A^{-1})^{-1} = A$$

$$(A^p)^{-1} = (A^{-1})^p$$

$$(kA)^{-1} = -k \mid A^{-1} \text{ (see improved expression evaluation)}$$

$$(AC)^{-1} = C^{-1}A^{-1}$$

note in the equation above, the order of the matrixes has changed. this is significant as matrix multiplication is not commutative

if AC is invertible, then A is invertible and C is invertible

procedure *computing the inverse of a matrix*

let $\mathbb{M}^{n,n} A$

solve the system $AA^{-1} = I$ by extending the matrix with the identity matrix and solve the linear system up to RREF using row reduction. $[A \mid I] \sim \dots [I \mid A^{-1}]$

procedure *computing the inverse of a 2 by 2 matrix*

see determinant

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $|A| \neq 0$

$$A^{-1} = -|A| \mid \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

applications *using a matrix inverse to solve a linear system*

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate B such that $B \equiv A^{-1}$

this can be used to solve a linear system such as:

$$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

Triangular Matrix

definition a matrix is said to be *triangular* if every entry below its diagonal **or** above its diagonal is 0

note the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

Diagonal Matrix

definition a matrix is said to be *diagonal* if every entry below its diagonal **and** above its diagonal is 0

note the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

let D be a diagonal matrix

applications

$[D]^x$ can be calculated by raising every entry of D to the power x

Diagonalizable Matrix

see eigenvector

definition an n by n matrix A is said to be *diagonalizable over the reals* if there exists a basis of \mathbb{R}^n consisting entirely of eigenvectors of A

a matrix is *diagonalizable* if and only if the geometric multiplicity of an eigenvalue is equal to the algebraic multiplicity of said eigenvalue, for every eigenvalue of the matrix (see eigenvector and eigenvalue)

note a matrix may also be diagonalizable over other number fields such as the set of complex numbers \mathbb{C}

note some matrixes do not have "enough" real eigenvalues or "enough" eigenvectors to be diagonalizable

example the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle\langle (1, 1), (1, -1) \rangle\rangle$ is a basis of \mathbb{R}^2 consisting entirely of eigenvectors of A

example the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigenvalues

example the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigenvalue, and therefore only one set of linearly dependent eigenvectors (see eigenvector and eigenvalue)

example the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single eigenvalue $\lambda = 1$, its eigenspace spans \mathbb{R}^2 . this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$

proof let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see eigenvector and eigenvalue

example let $\mathbb{M}^{n,n} A \wedge \mathbb{N}n$ and suppose A has n distinct eigenvalues. deduce that A is diagonalizable over the reals

proof A has at most n eigenvalues \rightarrow the algebraic multiplicity of every eigenvalue of A is 1 as they are all distinct and must be greater than 1 \rightarrow the geometric multiplicity of every eigenvalue of A is 1 as it must be greater than 1 and less than its algebraic multiplicity \rightarrow all algebraic multiplicities and geometric multiplicities are equal $\rightarrow A$ is diagonalizable. see eigenvector and eigenvalue

theorems

see linear system

theorem

let $\mathbb{M}^{m,n} A$ (see matrix). the following logic statements are equivalent:

- every variable is a leading variable
- there is a leading variable in every column of the RREF of A
- the system $Ax = O$ has a unique solution
- the columns of A are linearly independent
- $\text{Ker } A = \langle \langle 0 \rangle \rangle$
- $\dim \text{Ker } A = 0$
- $\text{rank } A = n$

see linear system theorem proof

theorem

let $\mathbb{M}^{n,n} A$ (see matrix). the following logic statements are equivalent:

note all logic statements below are valid for both A and A^T , see transpose matrix

- $\text{rank } A = n$
- every linear system of the form $Ax = b$ has a unique solution
- the RREF of A is the identity matrix
- $\text{Ker } A = \langle \langle 0 \rangle \rangle$
- $\text{Col } A = \mathbb{R}^n$
- $\text{Row } A = \mathbb{R}^n$
- the columns of A are linearly independent
- the rows of A are linearly independent
- the columns of A form a basis for \mathbb{R}^n
- the rows of A form a basis for \mathbb{R}^n
- A is an invertible matrix
- $\det A \neq 0$