Matrix

see math notation, eigen

definition *formally in my math notation* a matrix in $\mathbb{R}^{m,n}$ is a set theoryetical function with function > domain at least $\langle x, y \rangle \to \mathbb{N}x \land \mathbb{N}y \land 0 \leq x < m \land 0 \leq y < n$ that takes an ordered pair as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by Scalar

see matrix vector space, vector space

definition

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \land \mathbb{N}j \land \mathbb{R}k \land \mathbb{M}A$

properties

*commutativity with scalar*s $kA = Ak$

Addition

see matrix vector space, vector space

definition

$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \land \mathbb{N}j \land \mathbb{M}^{m,n}A \land \mathbb{M}^{m,n}B$

Multiplication

see dot product, vector in rn

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \land \mathbb{M}^{n,p}B \land \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an m by p matrix)

$(AB)^{i,j} = A^{i,} \overset{\cdot}{|} B^{,j} \dashv \mathbb{N}i \land \mathbb{N}j$, see dot product

intuitively, matrix multiplication is the dot product of **every row** of the first matrix by **every column** of the second matrix

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

and therefore $AA \cdots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

not commutative $AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

associative $(AB)C = A(BC)$

distributive $A(B : C) = AB : AC$

distributive $(B : C)A = BA : CA$

associative with scalars $k(AB) = (kA)B = A(kB)$

applications

matrix > multiplication can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

matrix > multiplication can be used to represent any linear transformation

Identity Matrix

definition

$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

properties

$AI = A \wedge IA = A \dashv \mathbb{M}A$

Zero Matrix

see matrix vector space, vector space

definition

$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$

examples

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

properties

$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$

$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$

Rank

the number of pivots in any REF of the matrix

notation

$rank\ A$, where

- A is the matrix to find the matrix > rank of

Element Count

notation $\#\ M$

definition the *element count* of a matrix is the total number of elements in the matrix

> **example** let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. then, $\mathbb{M}^{2,3}M \wedge \#\ M = 2 \mid 3 = 6$

Vector Spaces

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

kernel $Ker\ A \equiv Null\ A$

column space $Col\ A$

row space $Row\ A$

definitions

kernel $(Ker\ A)\ x \equiv (Null\ A)\ x \equiv Ax = O \land \mathbb{M}^{m,n} A \land \mathbb{M}^{n,1} x$

column space $Col\ A = \operatorname{span} A^{,n} \dashv \mathbb{N} n$

row space $Row\ A = \operatorname{span} A^{n,} \dashv \mathbb{N} n$

procedure *computing the kernel of a matrix* use row reduction

theorems

the Null Space, Row Space and Column Space of a matrix are always vector spaces

$\text{number of free variables in } A : \text{number of pivots in } A = \text{number of columns in } A$

$\dim Null\ A = \text{number of free variables in } A$

$rank\ A = \text{number of pivots in } A$

the nonzero rows in any REF of a matrix A forms a basis for $Row\ A$. therefore, $\dim Row\ A = rank\ A$, see matrix > rank

if A and B are row-equivalent, then $Row\ A = Row\ B$, see linear system

the spanning set of $Null\ A$ obtained from applying row reduction on the system $Ax = O$ is a basis for $Null\ A$

$Row\ A$ does not change when applying linear system > elementary operations on the rows of A

properties

$Col\ A = Row\ A^\intercal \land Row\ A = Col\ A^\intercal \dashv \mathbb{M} A$, see matrix > transpose

applications

row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for the row space of a matrix can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for the column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix the same can be said for $Col\ A$

> **example** *transforming a vector space into the null space of a certain matrix*
>
> let $W = \text{span}\,\langle\langle\ (1,0,0,1),(1,1,1,0),(2,1,\cdot 1,1)\ \rangle\rangle$
>
> after solving the linear system, we get $W\ (x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = \begin{bmatrix} \cdot 1 & 1 & 0 & 1 \end{bmatrix}$

Diagonal

the diagonal of a matrix

definition the *diagonal* of a square matrix goes from its top left element to its bottom right element

Transpose

the transpose of a matrix

flips a matrix around its matrix > diagonal

definition

$(A^\intercal)^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \land \mathbb{N}j \land \mathbb{M}A$

properties

$(A^\intercal)^\intercal = A \dashv \mathbb{M}A$

$(AB)^\intercal = B^\intercal A^\intercal \dashv \mathbb{M}A \land \mathbb{M}B$

$$\mathbf{A}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

representation — https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Inverse

the inverse of a matrix

definition

$AA^{-} = A^{-}A = I \dashv \mathbb{M}A$, where

- A is a square matrix
- A^{-} is the *inverse matrix* of A

Invertability

definition an *invertible matrix* has a corresponding matrix > inverse

see theorems below for invertability criteria

properties

let A and C be invertible matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$. then,

$AA^{-} = A^{-}A = I$

$(A^{-})^{-} = A$

$(A^{p})^{-} = (A^{-})^{p}$

$(kA)^{-} = \text{-}k \mid A^{-}$ (see improved expression evaluation)

$(AC)^{-} = C^{-}A^{-}$

> **note** in the equation above, the order of the matrixes has changed. this is significant as matrix > multiplication is not commutative

if AC is invertible, then A is invertible and C is invertible

> **procedure** *computing the matrix > inverse of a matrix*
>
> let $\mathbb{M}^{n,n}A$
>
> solve the system $AA^{-} = I$ by extending the matrix with the matrix > identity matrix and solve the linear system up to RREF using row reduction.
> $[A \quad | \quad I] \sim \cdots [I \quad | \quad A^{-}]$

> **procedure** *computing the matrix > inverse of a 2 by 2 matrix*
>
> **see** determinant
>
> let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
>
> A is invertible if and only if $|A| \neq 0$

> $A^{-} = -|A| \mid \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$

applications *using a matrix > inverse to solve a linear system*

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{-}$

this can be used to solve a linear system such as:

$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

Triangular Matrix

definition a matrix is said to be *triangular* if every entry below its matrix > diagonal **or** above its matrix > diagonal is 0

Diagonal Matrix

definition a matrix is said to be *diagonal* if every entry below its matrix > diagonal **and** above its matrix > diagonal is 0

applications

$[D]x$ can be calculated by raising every entry of D to the power x

Diagonalizable Matrix

see eigen > vector

definition an n by n matrix A is said to be *diagonalizable over the reals* if there exists a basis of $\mathbb{R}^n$ consisting entirely of eigen > vectors of A

a matrix is *diagonalizable* if and only if the geometic eigen > multiplicity of an eigen > value is equal to the algebraic eigen > multiplicity of said eigen > value, for every eigen > value of the matrix

> **note** a matrix may also be diagonalizable over other number fields such as the set of complex numbers $\mathbb{C}$

> **note** some matrixes do not have "enough" real eigen > values or "enough" eigen > vectors to be diagonalizable

> **example** the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle\langle\, (1,1),(1,\cdot 1)\,\rangle\rangle$ is a basis of $\mathbb{R}^2$ consisting entirely of eigen > vectors of A

> **example** the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigen > values

> **example** the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigen > value, and therefore only one set of linearly dependent eigen > vectors

> **example** the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single eigen > value $\lambda = 1$, its eigen > space spans $\mathbb{R}^2$. this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$
>
> > **proof** let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see eigen

> **example** let $\mathbb{M}^{n,n} A \wedge \mathbb{N} n$ and suppose A has n distinct eigen > values. deduce that A is diagonalizable over the reals
>
> > **proof** A has at most n eigen > valuess $\rightarrow$ the algebraic eigen > multiplicity of every eigen > value of A is 1 as they are all distinct and must be greater than 1 $\rightarrow$ the geometric eigen > multiplicity of every eigen > value of A is 1 as it must be greater than 1 and less than its algebraic eigen > multiplicity $\rightarrow$ all algebraic eigen > multiplicityes and geometric eigen > multiplicityes are equal $\rightarrow$ A is diagonalizable. see eigen

theorems

see linear system

theorem

let $\mathbb{M}^{m,n} A$ (see matrix). the following logic statements are equivalent:

- every variable is a leading variable
- there is a leading variable in every column of the RREF of A
- the system $Ax = O$ has a unique solution
- the columns of A are linearly independent
- $Ker\ A = \langle\langle 0 \rangle\rangle$
- $\dim Ker\ A = 0$

- $rank\ A = n$

see linear system theorem proof

theorem

let $\mathbb{M}^{n,n} A$ (see matrix). the following logic statements are equivalent:

> **note** all logic statements below are valid for both A and $A^\intercal$, see matrix > transpose

- $rank\ A = n$
- every linear system of the form $Ax = b$ has a unique solution
- the RREF of A is the matrix > identity matrix
- $Ker\ A = \langle\langle 0 \rangle\rangle$
- $Col\ A = \mathbb{R}^n$
- $Row\ A = \mathbb{R}^n$
- the columns of A are linearly independent
- the rows of A are linearly independent
- the columns of A form a basis for $\mathbb{R}^n$
- the rows of A form a basis for $\mathbb{R}^n$
- A is an invertible matrix
- $\det A \neq 0$