Matrix

see math notation

definition

formally, in my math notation, a matrix in $\mathbb{R}^{m,n}$ is a set theoryetical function with domain at least $\langle x, y\rangle \to \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$ that takes an ordered pair as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by a Scalar

see matrix vector space, vector space

definition

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$

properties

$kA = Ak$ — commutative with scalars

Matrix Addition

see matrix vector space, vector space

$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$

Matrix Multiplication

see dot product, vector in rn

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an m by p matrix)

$(AB)^{i,j} = A^{i,} \dot{|} B^{,j} \dashv \mathbb{N}i \wedge \mathbb{N}j$, see dot product

intuitively, matrix multiplication is the dot product of **every row** of the first matrix by **every column** of the second matrix

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

therefore,

$AA \ldots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

$AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$ — not commutative

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

$(AB)C = A(BC)$ — associative

$A(B : C) = AB : AC$ — distributive

$(B : C)A = BA : CA$ — distributive

$k(AB) = (kA)B = A(kB)$ — associative with scalars

applications

can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be used to represent any linear transformation

Identity Matrix

definition

$(I^{a,b} = 1 \land a = b) \lor (I^{a,b} = 0 \land a \neq b) \dashv \mathbb{N}a \land \mathbb{N}b \land \mathbb{M}^{n,n} I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\ldots$

properties

$AI = A \land IA = A \dashv \mathbb{M}A$

Zero Matrix

see matrix vector space, vector space

definition

$O^{a,b} = 0 \dashv \mathbb{N}a \land \mathbb{N}b \land \mathbb{M}^{n,m} O$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ldots$

properties

$A_{m,n} O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p} O_{n,p} \land \mathbb{M}^{m,p} O_{m,p} \land \mathbb{M}^{m,n} A_{m,n}$

$O_{q,m} A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m} O_{q,m} \land \mathbb{M}^{q,n} O_{q,n} \land \mathbb{M}^{m,n} A_{m,n}$

Rank of a Matrix

the number of pivots in any REF of the matrix

notation

$\boldsymbol{rank}\ A$, where

A is the matrix to find the rank of

Matrix Vector Spaces

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

$Ker\ A \equiv Null\ A$

$Col\ A$

$Row\ A$

definitions

$(Ker\ A)\ x \equiv (Null\ A)\ x \equiv Ax = O \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a matrix can be calculated using row reduction

$Col\ A = \text{span}\ A^{,n} \dashv \mathbb{N} n$

$Row\ A = \text{span}\ A^{n,} \dashv \mathbb{N} n$

properties

> **theorem**: the Null Space, Row Space and Column Space of a matrix are always vector spaces

> **theorem**:
> number of free variables in A : number of pivots in A = number of columns in A

> **theorem**: $\dim Null\ A$ = number of free variables in A

> **theorem**: $rank\ A$ = number of pivots in A

> **theorem**: the nonzero rows in any REF of a matrix A forms a basis for $\mathit{Row}\ A$. therefore, $\dim \mathit{Row}\ A = \mathit{rank}\ A$ (see rank of a matrix)

> **theorem**: if A and B are row-equivalent, then $\mathit{Row}\ A = \mathit{Row}\ B$, see linear system

> **theorem**: the spanning set of $\mathit{Null}\ A$ obtained from applying row reduction on the system $Ax = O$ is a basis for $\mathit{Null}\ A$

> **theorem**: $\mathit{Row}\ A$ does not change when applying elementary operations on the rows of A, see linear system

$\mathit{Col}\ A = \mathit{Row}\ A^\intercal \land \mathit{Row}\ A = \mathit{Col}\ A^\intercal \dashv \mathbb{M}A$, see transpose matrix

applications

row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for the row space of a matrix can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for the column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix

the same can be said for $\mathit{Col}\ A$

example

transforming a vector space into the null space of a certain matrix

let $W = \mathbf{span}\,\langle\langle\ (1,0,0,1),(1,1,1,0),(2,1,\cdot 1,1)\ \rangle\rangle$

after solving the linear system, we get $W\ (x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = [\cdot 1 \quad 1 \quad 0 \quad 1]$

Transpose Matrix

the transpose of a matrix

definition

flips a matrix around its diagonal

> **note**: the *diagonal* of a square <u>matrix</u> goes from its top left element to its bottom right element (triplicate)

$(A^{\intercal})^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$

properties

$(A^{\intercal})^{\intercal} = A \dashv \mathbb{M}A$

$(AB)^{\intercal} = B^{\intercal}A^{\intercal} \dashv \mathbb{M}A \wedge \mathbb{M}B$

representation

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Matrix Inverse

the inverse of a <u>matrix</u>

definition

$AA^{-} = A^{-}A = I \dashv \mathbb{M}A$, where

A is a square <u>matrix</u>

A^{-} is the *inverse matrix* of A

Invertability

> **definition**: an *invertible matrix* has a corresponding inverse <u>matrix</u>

see theorems below for invertability criteria

properties

let A and C be invertible matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$. then,

$AA^{-} = A^{-}A = I$

$(A^{-})^{-} = A$

$(A^{p})^{-} = (A^{-})^{p}$

$(kA)^{-} = \text{-}k \mid A^{-}$ (restriction might not be necessary, see improved expression evaluation)

$(AC)^{-} = C^{-}A^{-}$

> **note**: in the equation above, the order of the matrixes has changed. this is significant as matrix multiplication is not commutative

if AC is invertible, then A is invertible and C is invertible

procedure

let $\mathbb{M}^{n,n}A$

solve the system $AA^{-} = I$ by extending the matrix with the identity matrix and solve the linear system up to RREF using row reduction. $[A \quad | \quad I] \sim \ldots [I \quad | \quad A^{-}]$

shortcut with matrixes in $\mathbb{M}^{2,2}$

see determinant

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if and only if $|A| \neq 0$

$A^{-} = -|A| \mid \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$

application example

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{-}$

this can be used to solve a linear system such as:

$$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

Triangular Matrix

> **definition**: a matrix is said to be *triangular* if every entry below its diagonal **or** above its diagonal is $\mathbf{0}$

> **note**: the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

Diagonal Matrix

> **definition**: a matrix is said to be *diagonal* if every entry below its diagonal **and** above its diagonal is $\mathbf{0}$

> **note**: the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

properties

let D be a diagonal matrix

$[D]x$ can be calculated by raising every entry of D to the power x

Diagonalizable Matrix

see eigenvector

definition

an n by n matrix A is said to be *diagonalizable over the reals* if there exists a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A

a matrix is *diagonalizable* if and only if the geometic multiplicity of an eigenvalue is equal to the algebraic multiplicity of said eigenvalue, for every eigenvalue of the matrix (see eigenvector and eigenvalue)

> **note**: a matrix may also be diagonalizable over other number fields such as the set of complex numbers $\mathbb{C}$

> **note**: some matrixes do not have "enough" real eigenvalues or "enough" eigenvectors to be diagonalizable

examples and counterexamples

#example

the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle\langle\ (1,1),(1,\cdot 1)\ \rangle\rangle$ is a basis of $\mathbb{R}^2$ consisting entirely of eigenvectors of A

the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigenvalues

the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigenvalue, and therefore only one set of linearly dependent eigenvectors (see eigenvector and eigenvalue)

the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single eigenvalue $\lambda = 1$, its eigenspace spans $\mathbb{R}^2$. this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$

> **proof**: let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see eigenvector and eigenvalue

let $\mathbb{M}^{n,n} A \wedge \mathbb{N} n$ and suppose A has n distinct eigenvalues. deduce that A is diagonalizable over the reals

> **proof**: A has at most n eigenvalues $\to$ the algebraic multiplicity of every eigenvalue of A is 1 as they are all distinct and must be greater than 1 $\to$ the geometric multiplicity of every eigenvalue of A is 1 as it must be greater than 1 and less than its algebraic multiplicity $\to$ all algebraic multiplicityes and geometric multiplicityes are equal $\to$ A is diagonalizable. see eigenvector and eigenvalue

eigenvector and eigenvalues

theorems

see linear system

> **theorem**: let $\mathbb{M}^{m,n}A$ (see matrix). the following logic statements are equivalent:
>
> 1. every variable is a leading variable
> 2. there is a leading variable in every column of the RREF of A
> 3. the system $Ax = O$ has a unique solution
> 4. the columns of A are linearly independent
> 5. $Ker\ A = \langle\langle 0\rangle\rangle$
> 6. $\dim Ker\ A = 0$
> 7. $rank\ A = n$

see linear system theorem proof

> **theorem**: let $\mathbb{M}^{n,n}A$ (see matrix). the following logic statements are equivalent:
>
> **note**: all logic statements below are valid for both A and $A^\intercal$, see transpose matrix
>
> 1. $rank\ A = n$
> 2. every linear system of the form $Ax = b$ has a unique solution
> 3. the RREF of A is the identity matrix
> 4. $Ker\ A = \langle\langle 0\rangle\rangle$
> 5. $Col\ A = \mathbb{R}^n$
> 6. $Row\ A = \mathbb{R}^n$
> 7. the columns of A are linearly independent
> 8. the rows of A are linearly independent
> 9. the columns of A form a basis for $\mathbb{R}^n$
> 10. the rows of A form a basis for $\mathbb{R}^n$
> 11. A is an invertible matrix
> 12. $\det A \neq 0$