

# Matrix

see [Math Notation](#)

## notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

## Multiplication by a Scalar

see [Matrix Vector Space](#), [Vector Space](#)

## definition

$$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

## properties

$$kA = Ak \text{ --- commutative with scalars}$$

## Matrix Addition

see [Matrix Vector Space](#), [Vector Space](#)

$$(A \cdot B)^{i,j} = A^{i,j} \cdot B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

## Matrix Multiplication

see [Dot Product](#), [Vector In Rn](#)

## definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{N}n \vdash \mathbb{M}^{m,p}AB$  ( $AB$  is defined if the number of columns in  $A$  is equal to the number of rows in  $B$ . their product will be an  $m \times p$  [Matrix](#))

$$(AB)^{i,j} = A^{i, \mid \circ B^j \dashv \mathbb{N}i \wedge \mathbb{N}j}, \text{ see } \a href="#">Dot Product$$

## notation

$$AA = A^2 = [A]^2 \dashv \mathbb{M}A$$

therefore,

$$AA \dots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$$

## properties

$$AB = BA \not\vdash \mathbb{M}A \wedge \mathbb{M}B \text{ or } AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B \text{ — not commutative}$$

$$AB = 0 \not\vdash A = 0 \vee B = 0 \text{ (it can happen that } AB = 0, \text{ but } A \neq 0 \text{ and } B \neq 0) \text{ (} AB \text{ being equal to } 0 \text{ does not imply that } A = 0 \text{ or that } B = 0)$$

$$AC = BC \wedge C \neq 0 \not\vdash A = B \text{ (} AC = BC \text{ and } C \neq 0 \text{ does not imply that } A = B)$$

$$(AB)C = A(BC) \text{ — associative}$$

$$A(B \cdot C) = AB \cdot AC \text{ — distributive}$$

$$(B \cdot C)A = BA \cdot CA \text{ — distributive}$$

$$k(AB) = (kA)B = A(kB) \text{ — associative with scalars}$$

## applications

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be used to represent any Linear Transformation

## Identity Matrix

### definition

$$I^{a,b} = 1 \wedge a = b \vee I^{a,b} = 0 \wedge a \neq b \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

### examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...

properties

$$AI = A \wedge IA = A \dashv \mathbb{M}A$$

Zero Matrix

see Matrix Vector Space, Vector Space

definition

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

...

properties

$$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$$

$$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$$

Rank of a Matrix

*the number of pivots in any REF of the Matrix*

notation

*rank*  $A$ , where

$A$  is the Matrix to find the rank of

# Null Space (Nullspace, Kernel), Column Space, Row Space

## notation

$$\text{Ker } A \equiv \text{Null } A$$

$$\text{Col } A$$

$$\text{Row } A$$

## definition

$$(\text{Ker } A) x \equiv (\text{Null } A) x \equiv Ax = O \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$$

the Kernel of a Matrix can be calculated using Row Reduction

$$\text{Col } A = \text{span } A^n \vdash \mathbb{N}n$$

$$\text{Row } A = \text{span } A^{n,} \vdash \mathbb{N}n$$

## properties

**theorem:** the Null Space, Row Space and Column Space of a Matrix are always Vector Spaces

**theorem:**

number of free variables in  $A$  · number of pivots in  $A$  = number of columns in  $A$

**theorem:**  $\dim \text{Null } A$  = number of free variables in  $A$

**theorem:**  $\text{rank } A$  = number of pivots in  $A$

**theorem:** the nonzero rows in any REF of a Matrix  $A$  forms a Basis for  $\text{Row } A$ .  
therefore,  $\dim \text{Row } A = \text{rank } A$  (see rank of a Matrix)

**theorem:** if  $A$  and  $B$  are Row Equivalent,  $\text{Row } A = \text{Row } B$

**theorem:** the Spanning Set of  $\text{Null } A$  obtained from applying Row Reduction on the system  $Ax = O$  is a Basis for  $\text{Null } A$

**theorem:**  $\text{Row } A$  does not change when applying Elementary Operations on the rows of  $A$

$Col A = Row A^T \wedge Row A = Col A^T \dashv \mathbb{M}A$ , see transpose Matrix

## applications

row spaces can be used to find a Basis for a Spanning Set of vectors through Row Reduction

the basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for  $Col A$

## example

*transforming a Vector Space into the null space of a certain Matrix*

let  $W = \text{span } (1,0,0,1), (1,1,1,0), (2,1,0,1)$

after solving the Linear System, we get  $W(x,y,z,w) \equiv 0x \cdot y \cdot w = 0$ . therefore,  $W$  is the null space of  $A = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$

## Transpose Matrix

*the **transpose** of a Matrix*

### definition

*flips a Matrix around its diagonal*

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element

$$(A^T)^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

### properties

$$(A^T)^T = A \dashv \mathbb{M}A$$

$$(AB)^T = B^T A^T \dashv \mathbb{M}A \wedge \mathbb{M}B$$

### example

**A**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

## Matrix Inverse

*the Inverse of a Matrix*

### definition

$AA^{-} = A^{-}A = I \dashv \mathbb{M}A$ , where

$A$  is a (square) Matrix

$A^{-}$  is the *inverse matrix* of  $A$

### Invertability

an ***invertible** Matrix* has a corresponding inverse Matrix

see theorems below for invertability criteria

### properties

let  $A$  and  $C$  be invertible Matrixes, let  $\mathbb{Z}p$  and let  $\mathbb{R}k \wedge k \neq 0$

$$AA^{-} = A^{-}A = I$$

$$(A^{-})^{-} = A$$

$$(A^p)^{-} = (A^{-})^p$$

$$(kA)^{-} = 1\text{-}k \mid A^{-} \text{ (restriction might not be necessary, see Improved Expression Evaluation)}$$

$$(AC)^{-} = C^{-}A^{-}$$

**note:** in the equation above, the order of the matrices has changed. this is significant as Matrix multiplication is not commutative)

if  $AC$  is invertible, then  $A$  is invertible and  $C$  is invertible

## finding a matrix inverse

let  $\mathbb{M}^{n,n}A$

solve the system  $AA^{-1} = I$  by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction.  $[A \mid I] \sim \dots [I \mid A^{-1}]$

## shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$A$  is invertible if and only if  $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## application example

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate  $B$  such that  $B \equiv A^{-1}$

this can be used to solve a system such as:

$$Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

## Triangular Matrix

a Matrix is *triangular* if every entry below its diagonal **or** above its diagonal is 0

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element

## Diagonal Matrix

a Matrix is *diagonal* if every entry below its diagonal **and** above its diagonal is 0

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element

### properties

let  $D$  be a diagonal Matrix

$[D]x$  can be calculated by multiplying every entry of  $D$  by  $x$

## Diagonalizable Matrix

see Eigenvector

### definition

an  $n$  by  $n$  Matrix  $A$  is said to be *diagonalizable over the reals* if there exists a Basis of  $\mathbb{R}^n$  consisting entirely of Eigenvectors of  $A$

a Matrix is *diagonalizable* if and only if the geometric Multiplicity of an Eigenvalue is equal to the algebraic Multiplicity of said Eigenvalue, for every Eigenvalue of the Matrix (see Eigenvector And Eigenvalue)

**note:** a Matrix may also be diagonalizable over other Number Fields such as the Set of Complex numbers  $\mathbb{C}$

**note:** some Matrixes do not have "enough" real Eigenvalues or "enough" Eigenvectors to be diagonalizable

### examples and counterexamples

the Matrix  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  is diagonalizable over the reals as  $\{(1,1), (1,-1)\}$  is a Basis of  $\mathbb{R}^2$  consisting entirely of Eigenvectors of  $A$



the Matrix  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  is not diagonalizable over the reals as it does not have any real Eigenvalues

the Matrix  $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  is not diagonalizable over the reals as it only has one Eigenvalue, and therefore only one set of Linearly Dependent Eigenvectors (see Eigenvector And Eigenvalue)

the Matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is diagonalizable over the reals as, even though  $A$  has a single Eigenvalue  $\lambda = 1$ , its Eigenspace Spans  $\mathbb{R}^2$ . this is the case for both  $A = I \wedge \lambda = 1$  and  $A = O \wedge \lambda = 0$

**proof:** let  $A = I \wedge \lambda = 1 \wedge E_1 = x$ . we then have  $O = A \circ \lambda I \mid x = I \circ 1I \mid E_1 = O \mid E_1$ .  
therefore,  $E_1 \equiv \mathbb{R}^2$ . see Eigenvector And Eigenvalue

let  $\mathbb{M}^{n,n} A \wedge \mathbb{N} n$  and suppose  $A$  has  $n$  distinct Eigenvalues. deduce that  $A$  is diagonalizable over the reals

**proof:**  $A$  has at most  $n$  Eigenvalues  $\rightarrow$  the algebraic Multiplicity of every Eigenvalue of  $A$  is 1 as they are all distinct and must be greater than 1  $\rightarrow$  the geometric Multiplicity of every Eigenvalue of  $A$  is 1 as it must be greater than 1 and less than its algebraic Multiplicity  $\rightarrow$  all algebraic Multiplicities and geometric Multiplicities are equal  $\rightarrow A$  is diagonalizable. see Eigenvector And Eigenvalue

## Eigenvector And Eigenvalues

### theorems

see Linear System

**theorem:** let  $\mathbb{M}^{m,n} A$  (see Matrix). the following statements are equivalent:

1. every variable is a leading variable
2. there is a leading variable in every column of the RREF of  $A$
3. the system  $Ax = O$  has a unique solution
4. the columns of  $A$  are Linearly Independent
5.  $\text{Ker } A = \{0\}$
6.  $\dim \text{Ker } A = 0$
7.  $\text{rank } A = n$

see Linear System Theorem Proof

**theorem:** let  $M^{n,n} A$  (see Matrix). the following statements are equivalent:

**note:** all statements below are valid for both  $A$  and  $A^T$ , see transpose Matrix

1.  $\text{rank } A = n$
2. every linear system of the form  $Ax = b$  has a unique solution
3. the RREF of  $A$  is the identity Matrix
4.  $\text{Ker } A = \{0\}$
5.  $\text{Col } A = \mathbb{R}^n$
6.  $\text{Row } A = \mathbb{R}^n$
7. the columns of  $A$  are Linearly Independent
8. the rows of  $A$  are Linearly Independent
9. the columns of  $A$  form a Basis for  $\mathbb{R}^n$
10. the rows of  $A$  form a Basis for  $\mathbb{R}^n$
11.  $A$  is Invertible
12.  $\det A \neq 0$