Matrix

see math notation

definition *formally in my math notation* a matrix in $\mathbb{R}^{m,n}$ is a set theoryetical function with domain at least $\langle x, y\rangle \to \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$ that takes an ordered pair as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by a Scalar

see matrix vector space, vector space

definition

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$

properties

commutativity with scalars $kA = Ak$

Matrix Addition

see matrix vector space, vector space

definition

$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$

Matrix Multiplication

see dot product, vector in rn

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an m by p matrix)

$(AB)^{i,j} = A^{i,} \overset{\cdot}{|} B^{,j} \dashv \mathbb{N}i \wedge \mathbb{N}j$, see dot product

intuitively, matrix multiplication is the dot product of **every row** of the first matrix by **every column** of the second matrix

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

and therefore $AA \ldots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

not commutative $AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

associative $(AB)C = A(BC)$

distributive $A(B : C) = AB : AC$

distributive $(B : C)A = BA : CA$

associative with scalars $k(AB) = (kA)B = A(kB)$

applications

matrix multiplication can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

matrix multiplication can be used to represent any linear transformation

Identity Matrix

definition

$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

properties

$AI = A \wedge IA = A \dashv \mathbb{M}A$

Zero Matrix

see matrix vector space, vector space

definition

$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

properties

$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$

$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$

Rank of a Matrix

the number of pivots in any REF of the matrix

notation

rank A, where

- A is the matrix to find the rank of

Matrix Vector Spaces

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

kernel $\mathit{Ker}\ A \equiv \mathit{Null}\ A$

column space $\mathit{Col}\ A$

row space $\mathit{Row}\ A$

definitions

kernel $(\mathit{Ker}\ A)\ x \equiv (\mathit{Null}\ A)\ x \equiv Ax = O \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,1}x$

column space $\mathit{Col}\ A = \operatorname{span} A^{,n} \dashv \mathbb{N}n$

row space $\mathit{Row}\ A = \operatorname{span} A^{n,} \dashv \mathbb{N}n$

procedure *computing the kernel of a matrix* use row reduction

theorems

the Null Space, Row Space and Column Space of a matrix are always vector spaces

number of free variables in A : number of pivots in A = number of columns in A

$\dim \mathit{Null}\ A =$ number of free variables in A

$\mathit{rank}\ A =$ number of pivots in A

the nonzero rows in any REF of a matrix A forms a basis for $\mathit{Row}\,A$. therefore, $\dim \mathit{Row}\ A = \mathit{rank}\ A$ (see rank of a matrix)

if A and B are row-equivalent, then $\mathit{Row}\ A = \mathit{Row}\ B$, see linear system

the spanning set of $\mathit{Null}\ A$ obtained from applying row reduction on the system $Ax = O$ is a basis for $\mathit{Null}\ A$

$\mathit{Row}\ A$ does not change when applying elementary operations on the rows of A, see linear system

properties

$\mathit{Col}\ A = \mathit{Row}\ A^{\intercal} \wedge \mathit{Row}\ A = \mathit{Col}\ A^{\intercal} \dashv \mathbb{M}A$, see transpose matrix

applications

row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for the row space of a matrix can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for the column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix

the same can be said for $Col\ A$

> **example** *transforming a vector space into the null space of a certain matrix*
>
> let $W = \text{span}\,\langle\langle\ (1,0,0,1),(1,1,1,0),(2,1,\cdot 1,1)\ \rangle\rangle$
>
> after solving the linear system, we get $W\,(x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = \begin{bmatrix}\cdot 1 & 1 & 0 & 1\end{bmatrix}$

Transpose Matrix

the transpose of a matrix

flips a matrix around its diagonal

> **note** the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

definition

$(A^\intercal)^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$

properties

$(A^\intercal)^\intercal = A \dashv \mathbb{M}A$

$(AB)^\intercal = B^\intercal A^\intercal \dashv \mathbb{M}A \wedge \mathbb{M}B$

representation

$$A$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Matrix Inverse

the inverse of a matrix

definition

$AA^- = A^-A = I \dashv \mathbb{M}A$, where

- A is a square matrix
- A^- is the *inverse matrix* of A

Invertability

definition an *invertible matrix* has a corresponding inverse matrix

see theorems below for invertability criteria

properties

let A and C be invertible matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$. then,

$AA^- = A^-A = I$

$(A^-)^- = A$

$(A^p)^- = (A^-)^p$

$(kA)^- = -k \mid A^-$ (restriction might not be necessary, see improved expression evaluation)

$(AC)^{-} = C^{-}A^{-}$

> **note** in the equation above, the order of the matrixes has changed. this is significant as matrix multiplication is not commutative

if AC is invertible, then A is invertible and C is invertible

> **procedure** *computing the inverse of a matrix*
>
> let $\mathbb{M}^{n,n}A$
>
> solve the system $AA^{-} = I$ by extending the matrix with the identity matrix and solve the linear system up to RREF using row reduction. $[A \quad | \quad I] \sim \ldots [I \quad | \quad A^{-}]$

> **procedure** *computing the inverse of a* 2 *by* 2 *matrix*
>
> see determinant
>
> let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
>
> A is invertible if and only if $|A| \neq 0$
>
> $A^{-} = -|A| \;\; | \;\; \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$

applications *using a matrix inverse to solve a linear system*

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{-}$

this can be used to solve a linear system such as:

$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$

Triangular Matrix

definition a matrix is said to be *triangular* if every entry below its diagonal **or** above its diagonal is 0

> **note** the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

Diagonal Matrix

definition a matrix is said to be *diagonal* if every entry below its diagonal **and** above its diagonal is 0

> **note** the *diagonal* of a square matrix goes from its top left element to its bottom right element (triplicate)

let D be a diagonal matrix

applications

$[D]x$ can be calculated by raising every entry of D to the power x

Diagonalizable Matrix

see eigenvector

definition an n by n matrix A is said to be *diagonalizable over the reals* if there exists a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A

a matrix is *diagonalizable* if and only if the geometic multiplicity of an eigenvalue is equal to the algebraic multiplicity of said eigenvalue, for every eigenvalue of the matrix (see eigenvector and eigenvalue)

> **note** a matrix may also be diagonalizable over other number fields such as the set of complex numbers $\mathbb{C}$

> **note** some matrixes do not have "enough" real eigenvalues or "enough" eigenvectors to be diagonalizable

> **example** the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle\langle\ (1,1),(1,\text{-}1)\ \rangle\rangle$ is a basis of $\mathbb{R}^2$ consisting entirely of eigenvectors of A

> **example** the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigenvalues

> **example** the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigenvalue, and therefore only one set of linearly dependent eigenvectors (see eigenvector and eigenvalue)

> **example** the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single eigenvalue $\lambda = 1$, its eigenspace spans $\mathbb{R}^2$. this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$
>
> > **proof** let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see eigenvector and eigenvalue

> **example** let $\mathbb{M}^{n,n}A \wedge \mathbb{N}n$ and suppose A has n distinct eigenvalues. deduce that A is diagonalizable over the reals
>
> > **proof** A has at most n eigenvalues $\rightarrow$ the algebraic multiplicity of every eigenvalue of A is 1 as they are all distinct and must be greater than 1 $\rightarrow$ the geometric multiplicity of every eigenvalue of A is 1 as it must be greater than 1 and less than its algebraic multiplicity $\rightarrow$ all algebraic multiplicityes and geometric multiplicityes are equal $\rightarrow$ A is diagonalizable. see eigenvector and eigenvalue

eigenvector and eigenvalues

theorems

see linear system

theorem

let $\mathbb{M}^{m,n}A$ (see matrix). the following logic statements are equivalent:

- every variable is a leading variable
- there is a leading variable in every column of the RREF of A
- the system $Ax = O$ has a unique solution

- the columns of A are linearly independent
- $Ker\ A = \langle\langle 0\rangle\rangle$
- $\dim Ker\ A = 0$
- $rank\ A = n$

see linear system theorem proof

theorem

let $\mathbb{M}^{n,n} A$ (see matrix). the following logic statements are equivalent:

> **note** all logic statements below are valid for both A and $A^\intercal$, see transpose matrix

- $rank\ A = n$
- every linear system of the form $Ax = b$ has a unique solution
- the RREF of A is the identity matrix
- $Ker\ A = \langle\langle 0\rangle\rangle$
- $Col\ A = \mathbb{R}^n$
- $Row\ A = \mathbb{R}^n$
- the columns of A are linearly independent
- the rows of A are linearly independent
- the columns of A form a basis for $\mathbb{R}^n$
- the rows of A form a basis for $\mathbb{R}^n$
- A is an invertible matrix
- $\det A \neq 0$