

# Matrix

see [Math Notation](#)

## definition

formally, in my [Math Notation](#), a [Matrix](#) in  $\mathbb{R}^{m,n}$  is a [Set Theoretical Function](#) with domain at least  $\langle x,y \rangle \rightarrow \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$  that takes an [Ordered Pair](#) as an index and returns the element at that index

## notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

# Multiplication by a Scalar

see [Matrix Vector Space](#), [Vector Space](#)

## definition

$$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

## properties

$$kA = Ak \text{ --- commutative with scalars}$$

# Matrix Addition

see [Matrix Vector Space](#), [Vector Space](#)

$$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

# Matrix Multiplication

see [Dot Product](#), [Vector In Rn](#)

## definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$  ( $AB$  is defined if the number of columns in  $A$  is equal to the number of rows in  $B$ . their product will be an  $m$  by  $p$  Matrix)

$(AB)^{i,j} = A^{i,\cdot} \cdot B^{\cdot,j} \vdash \mathbb{N}i \wedge \mathbb{N}j$ , see Dot Product

intuitively, matrix multiplication is the Dot Product of **every row** of the first Matrix by **every column** of the second Matrix

## notation

$$AA = A^2 = [A]^2 \vdash \mathbb{M}A$$

therefore,

$$AA \dots A = [A]^n \wedge \mathbb{N}n \vdash \mathbb{M}A$$

## properties

$AB = BA \not\vdash \mathbb{M}A \wedge \mathbb{M}B$  or  $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$  — not commutative

$AB = 0 \not\vdash A = 0 \vee B = 0$  (it can happen that  $AB = 0$ , but  $A \neq 0$  and  $B \neq 0$ ) ( $AB$  being equal to 0 does not imply that  $A = 0$  or that  $B = 0$ )

$AC = BC \wedge C \neq 0 \not\vdash A = B$  ( $AC = BC$  and  $C \neq 0$  does not imply that  $A = B$ )

$(AB)C = A(BC)$  — associative

$A(B : C) = AB : AC$  — distributive

$(B : C)A = BA : CA$  — distributive

$k(AB) = (kA)B = A(kB)$  — associative with scalars

## applications

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be used to represent any Linear Transformation

## Identity Matrix

**definition**

$$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

**examples**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...

**properties**

$$AI = A \wedge IA = A \dashv \mathbb{M}A$$

**Zero Matrix**

see [Matrix Vector Space](#), [Vector Space](#)

**definition**

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

**examples**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

...

**properties**

$$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$$

$$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$$

# Rank of a Matrix

the number of pivots in any REF of the Matrix

## notation

$\text{rank } A$ , where

$A$  is the Matrix to find the rank of

# Null Space (Nullspace, Kernel), Column Space, Row Space

## notations

$\text{Ker } A \equiv \text{Null } A$

$\text{Col } A$

$\text{Row } A$

## definitions

$(\text{Ker } A) x \equiv (\text{Null } A) x \equiv Ax = O \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a Matrix can be calculated using Row Reduction

$\text{Col } A = \text{span } A^{\cdot n} \dashv \mathbb{N}n$

$\text{Row } A = \text{span } A^n \dashv \mathbb{N}n$

## properties

**theorem:** the Null Space, Row Space and Column Space of a Matrix are always Vector Spaces

**theorem:**

number of free variables in  $A$  : number of pivots in  $A$  = number of columns in  $A$

**theorem:**  $\dim \text{Null } A$  = number of free variables in  $A$

**theorem:**  $\text{rank } A$  = number of pivots in  $A$

**theorem:** the nonzero rows in any REF of a Matrix  $A$  forms a Basis for  $Row A$ .  
therefore,  $\dim Row A = rank A$  (see rank of a Matrix)

**theorem:** if  $A$  and  $B$  are Row Equivalent,  $Row A = Row B$

**theorem:** the Spanning Set of  $Null A$  obtained from applying Row Reduction on the system  $Ax = 0$  is a Basis for  $Null A$

**theorem:**  $Row A$  does not change when applying Elementary Operations on the rows of  $A$

$Col A = Row A^T \wedge Row A = Col A^T \dashv \mathbb{M}A$ , see transpose Matrix

## applications

row spaces can be used to find a Basis for a Spanning Set of vectors through Row Reduction

the Basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the Basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for  $Col A$

## example

*transforming a Vector Space into the null space of a certain Matrix*

let  $W = \text{span}\{(1, 0, 0, 1), (1, 1, 1, 0), (2, 1, -1, 1)\}$

after solving the Linear System, we get  $W(x, y, z, w) \equiv \cdot x : y : z : w = 0$ . therefore,  $W$  is the null space of  $A = \begin{bmatrix} \cdot & 1 & 0 & 1 \end{bmatrix}$

## Transpose Matrix

*the transpose of a Matrix*

## definition

*flips a Matrix around its diagonal*

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

$$(A^{\top})^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

## properties

$$(A^{\top})^{\top} = A \dashv \mathbb{M}A$$

$$(AB)^{\top} = B^{\top}A^{\top} \dashv \mathbb{M}A \wedge \mathbb{M}B$$

## representation

**A**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—  
[https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix\\_transpose.gif/200px-Matrix\\_transpose.gif](https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif)

## Matrix Inverse

the *Inverse* of a Matrix

### definition

$$AA^{-} = A^{-}A = I \dashv \mathbb{M}A, \text{ where}$$

$A$  is a (square) Matrix

$A^{-}$  is the *inverse matrix* of  $A$

## Invertability

**definition:** an *invertible matrix* has a corresponding inverse Matrix

see theorems below for invertability criteria

## properties

let  $A$  and  $C$  be invertible Matrixes, let  $\mathbb{Z}p$  and let  $\mathbb{R}k \wedge k \neq 0$

$$AA^{-} = A^{-}A = I$$

$$(A^{-})^{-} = A$$

$$(A^p)^{-} = (A^{-})^p$$

$$(kA)^{-} = -k \mid A^{-} \text{ (restriction might not be necessary, see Improved Expression Evaluation)}$$

$$(AC)^{-} = C^{-}A^{-}$$

**note:** in the equation above, the order of the matrices has changed. this is significant as Matrix multiplication is not commutative

if  $AC$  is invertible, then  $A$  is invertible and  $C$  is invertible

## procedure

let  $\mathbb{M}^{n,n}A$

solve the system  $AA^{-} = I$  by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction.  $[A \mid I] \sim \dots [I \mid A^{-}]$

## shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$A$  is invertible if and only if  $|A| \neq 0$

$$A^{-} = -|A| \mid \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$$

## application example

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate  $B$  such that  $B \equiv A^{-}$

this can be used to solve a Linear System such as:

$$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

## Triangular Matrix

**definition:** a Matrix is said to be *triangular* if every entry below its diagonal **or** above its diagonal is 0

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

## Diagonal Matrix

**definition:** a Matrix is said to be *diagonal* if every entry below its diagonal **and** above its diagonal is 0

**note:** the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

## properties

let  $D$  be a diagonal Matrix

$[D]x$  can be calculated by raising every entry of  $D$  to the power  $x$

## Diagonalizable Matrix

see Eigenvector

### definition

an  $n$  by  $n$  Matrix  $A$  is said to be *diagonalizable over the reals* if there exists a Basis of  $\mathbb{R}^n$  consisting entirely of Eigenvectors of  $A$



a Matrix is *diagonalizable* if and only if the geometric Multiplicity of an Eigenvalue is equal to the algebraic Multiplicity of said Eigenvalue, for every Eigenvalue of the Matrix (see Eigenvector And Eigenvalue)

**note:** a Matrix may also be diagonalizable over other Number Fields such as the Set of Complex numbers  $\mathbb{C}$

**note:** some Matrixes do not have "enough" real Eigenvalues or "enough" Eigenvectors to be diagonalizable

## examples and counterexamples

#example

the Matrix  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  is diagonalizable over the reals as  $\{(1,1), (1,-1)\}$  is a Basis of  $\mathbb{R}^2$  consisting entirely of Eigenvectors of  $A$

the Matrix  $A = \begin{bmatrix} 1 & 1 \\ .1 & 1 \end{bmatrix}$  is not diagonalizable over the reals as it does not have any real Eigenvalues

the Matrix  $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  is not diagonalizable over the reals as it only has one Eigenvalue, and therefore only one set of Linearly Dependent Eigenvectors (see Eigenvector And Eigenvalue)

the Matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is diagonalizable over the reals as, even though  $A$  has a single Eigenvalue  $\lambda = 1$ , its Eigenspace Spans  $\mathbb{R}^2$ . this is the case for both  $A = I \wedge \lambda = 1$  and  $A = O \wedge \lambda = 0$

**proof:** let  $A = I \wedge \lambda = 1 \wedge E_1 = x$ . we then have  $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$ . therefore,  $E_1 \equiv \mathbb{R}^2$ . see Eigenvector And Eigenvalue

let  $\mathbb{M}^{n,n}A \wedge \mathbb{N}n$  and suppose  $A$  has  $n$  distinct Eigenvalues. deduce that  $A$  is diagonalizable over the reals

**proof:**  $A$  has at most  $n$  Eigenvalues  $\rightarrow$  the algebraic Multiplicity of every Eigenvalue of  $A$  is 1 as they are all distinct and must be greater than 1  $\rightarrow$  the geometric Multiplicity of every Eigenvalue of  $A$  is 1 as it must be greater than 1 and less than its algebraic Multiplicity  $\rightarrow$  all algebraic Multiplicities and geometric Multiplicities are equal  $\rightarrow A$  is diagonalizable. see Eigenvector And Eigenvalue

# Eigenvector And Eigenvalues

## theorems

see Linear System

**theorem:** let  $M^{m,n}A$  (see Matrix). the following Logic Statements are equivalent:

1. every variable is a leading variable
2. there is a leading variable in every column of the RREF of  $A$
3. the system  $Ax = O$  has a unique solution
4. the columns of  $A$  are Linearly Independent
5.  $\text{Ker } A = \{0\}$
6.  $\dim \text{Ker } A = 0$
7.  $\text{rank } A = n$

see Linear System Theorem Proof

**theorem:** let  $M^{n,n}A$  (see Matrix). the following Logic Statements are equivalent:

**note:** all Logic Statements below are valid for both  $A$  and  $A^T$ , see transpose Matrix

1.  $\text{rank } A = n$
2. every linear system of the form  $Ax = b$  has a unique solution
3. the RREF of  $A$  is the identity Matrix
4.  $\text{Ker } A = \{0\}$
5.  $\text{Col } A = \mathbb{R}^n$
6.  $\text{Row } A = \mathbb{R}^n$
7. the columns of  $A$  are Linearly Independent
8. the rows of  $A$  are Linearly Independent
9. the columns of  $A$  form a Basis for  $\mathbb{R}^n$
10. the rows of  $A$  form a Basis for  $\mathbb{R}^n$
11.  $A$  is an invertible Matrix
12.  $\det A \neq 0$