# Matrix

**see** math notation, eigen

**definition** *formally in my math notation* a matrix in $\mathbb{R}^{m,n}$ is a set theoryetical function with function > domain at least $\langle x, y \rangle \to \mathbb{N}x \land \mathbb{N}y \land 0 \leq x < m \land 0 \leq y < n$ that takes an ordered pair as an index and returns the element at that index

**notation**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

## Vector Space

**notation**

the vector space of $m$ by $n$ matrixes is denoted as follows:

*in my math notation*

$\mathbb{M}^{m,n}$

*in math notation*

$M_{m\,n}(\mathbb{R})$

## Multiplication by Scalar

**see** vector, vector space

**definition**

$kA$

**properties**

*commutativity with scalars* $kA = Ak$

## Addition

**see** vector, vector space

**definition**

$A : B$

## Multiplication

#todo mm

**see** dot product, vector in rn

### definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,p} B \wedge \mathbb{M}^{m,p} AB$ ($AB$ is defined if the number of columns in $A$ is equal to the number of rows in $B$. their product will be an $m$ by $p$ matrix)

$(AB)^{i,j} = \ : A^{i,} B^{,j}$ , see dot product

intuitively, matrix multiplication is the dot product of **every row** of the first matrix by **every column** of the second matrix

### notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

and therefore $AA \cdots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

### properties

*not commutative* $AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) ($AB$ being equal to $0$ does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

*associative* $(AB)C = A(BC)$

*distributive* $A(B : C) = AB : AC$

*distributive* $(B : C)A = BA : CA$

*associative with scalars* $k(AB) = (kA)B = A(kB)$

### applications

matrix > multiplication can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

matrix > multiplication can be used to represent any linear transformation

# Identity Matrix

#### definition

$(0 \cdots) \doteq (0 \cdots)$

#### examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#### properties

$AI = A \land IA = A \dashv \mathbb{M}A$

# Zero Matrix

**see** vector, vector space

#### definition

$O \doteq 0$

#### examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

#### properties

$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \land \mathbb{M}^{m,p}O_{m,p} \land \mathbb{M}^{m,n}A_{m,n}$

$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \land \mathbb{M}^{q,n}O_{q,n} \land \mathbb{M}^{m,n}A_{m,n}$

# Rank

*the number of pivots in any REF of the matrix*

#### notation

$rank\ A$, where

- $A$ is the matrix to find the matrix > rank of

# Element Count

**notation** $\# M$

**definition** the *element count* of a matrix is the total number of elements in the matrix

> **example** let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. then, $\mathbb{M}^{2,3} M \land \# M = 2 \mid 3 = 6$

# Null Space

# Column Space

# Row Space

**notations**

*kernel* $Ker\ A \equiv Null\ A$

*column space* $Col\ A$

*row space* $Row\ A$

**definitions**

*kernel* $(Ker\ A)\ x \equiv (Null\ A)\ x \equiv Ax = O \land \mathbb{M}^{m,n} A \land \mathbb{M}^{n,1} x$

*column space* $Col\ A = \text{span}\ A^{,n} \dashv \mathbb{N} n$

*row space* $Row\ A = \text{span}\ A^{n,} \dashv \mathbb{N} n$

**procedure** *computing the kernel of a matrix* use row reduction

**theorems**

the matrix > null space, matrix > row space and matrix > column space of a matrix are always vector spaces

$\text{number of free variables in } A : \text{number of pivots in } A = \text{number of columns in } A$

$\dim Null\ A = \text{number of free variables in } A$

$rank\ A = \text{number of pivots in } A$

the nonzero rows in any REF of a matrix $A$ forms a basis for $Row\ A$. therefore, $\dim Row\ A = rank\ A$, see matrix > rank

if $A$ and $B$ are row-equivalent, then $Row\ A = Row\ B$, see linear system the spanning set of $Null\ A$ obtained from applying row reduction on the system $Ax = O$ is a basis for $Null\ A$

$Row\ A$ does not change when applying linear system > elementary operations on the rows of $A$

**properties**

$Col\ A = Row\ A^\intercal \wedge Row\ A = Col\ A^\intercal \dashv \mathbb{M}A$, see matrix > transpose

**applications**

matrix > row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for a matrix > row space can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for a matrix > column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix

the same can be said for $Col\ A$

> **example** *transforming a vector space into the matrix > null space of a certain matrix*
>
> let $W = \operatorname{span} \langle\langle\ (1,0,0,1), (1,1,1,0), (2,1,\cdot 1,1)\ \rangle\rangle$
>
> after solving the linear system, we get $W\ (x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, $W$ is the matrix > null space of $A = [\cdot 1 \quad 1 \quad 0 \quad 1]$

# Diagonal

*the diagonal of a matrix*

**definition** the *diagonal* of a square matrix goes from its top left element to its bottom right element

# Transpose

*the transpose of a matrix*

*flips a matrix around its matrix > diagonal*

**definition**

$(A^\intercal)^{i,j} = A^{j,i}$

**properties**

$(A^\intercal)^\intercal = A$

$(AB)^\intercal = B^\intercal A^\intercal$ #todo mm

**representation**

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—
https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

# Conjugate Transpose

*the complex > conjugate of every entry of the matrix > transpose of a matrix*

> **aka** Hermitian transpose, adjoint matrix, transjugate

**definition**

$\operatorname{conj} A^\intercal$, where

- $A$ is the matrix to find the matrix > conjugate transpose of
- $\operatorname{conj}$ is the complex > conjugate function
- $^\intercal$ is the matrix > transpose operator

**properties**

let a matrix of reals $A$. then, $\operatorname{conj} A^\intercal \equiv A^\intercal$

# Inverse

#todo mm

*the inverse of a matrix*

**definition**

$AA^- = A^-A = I \dashv \mathbb{M}A$, where

- $A$ is a square matrix
- $A^-$ is the *inverse matrix* of $A$

# Invertability

**definition** an *invertible matrix* has a corresponding matrix > inverse

**see** theorems below for invertability criteria

**properties**

let $A$ and $C$ be invertible matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$. then,

$AA^- = A^-A = I$

$(A^-)^- = A$

$(A^p)^- = (A^-)^p$

$(kA)^- = \text{-}k \mid A^-$ (see improved expression evaluation)

$(AC)^- = C^-A^-$

> **note** in the equation above, the order of the matrixes has changed. this is significant as matrix > multiplication is not commutative

if $AC$ is invertible, then $A$ is invertible and $C$ is invertible

> **procedure** *computing the matrix > inverse of a matrix*
>
> let $\mathbb{M}^{n,n}A$
>
> solve the system $AA^- = I$ by extending the matrix with the matrix > identity matrix and solve the linear system up to RREF using row reduction.
> $[A \quad | \quad I] \sim \cdots [I \quad | \quad A^-]$

> **procedure** *computing the matrix > inverse of a $2$ by $2$ matrix*
>
> **see** determinant
>
> let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
>
> $A$ is invertible if and only if $\det A \neq 0$
>
> $A^- = -\det A \mid \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$

**applications** *using a matrix > inverse to solve a linear system*

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate $B$ such that $B \equiv A^{-}$

this can be used to solve a linear system such as:

$$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

# Triangular Matrix

**definition** a matrix is said to be *triangular* if every entry below its matrix > diagonal **or** above its matrix > diagonal is $0$

# Diagonal Matrix

**definition** a matrix is said to be *diagonal* if every entry below its matrix > diagonal **and** above its matrix > diagonal is $0$

**applications**

$[D]x$ can be calculated by raising every entry of $D$ to the power $x$ #todo mm

# Diagonalizable Matrix

**see** eigen > vector

**definition** an $n$ by $n$ matrix $A$ is said to be *diagonalizable over the reals* if there exists a basis of $\mathbb{R}^n$ consisting entirely of eigen > vectors of $A$

a matrix is *diagonalizable* if and only if the geometic eigen > multiplicity of an eigen > value is equal to the algebraic eigen > multiplicity of said eigen > value, for every eigen > value of the matrix

> **note** a matrix may also be diagonalizable over other number fields such as the set of complex numbers $\mathbb{C}$

> **note** some matrixes do not have "enough" real eigen > values or "enough" eigen > vectors to be diagonalizable

> **example** the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle\langle\ (1, 1), (1, \cdot 1)\ \rangle\rangle$ is a basis of $\mathbb{R}^2$ consisting entirely of eigen > vectors of $A$

> **example** the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigen > values

> **example** the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigen > value, and therefore only one set of linearly dependent eigen > vectors

> **example** the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though $A$ has a single eigen > value $\lambda = 1$, its eigen > space spans $\mathbb{R}^2$. this is the case for both $A = I \land \lambda = 1$ and $A = O \land \lambda = 0$
>
> > **proof** let $A = I \land \lambda = 1 \land E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$ . therefore, $E_1 \equiv \mathbb{R}^2$. see eigen

> **example** let $\mathbb{M}^{n,n} A \land \mathbb{N} n$ and suppose $A$ has $n$ distinct eigen > values. deduce that $A$ is diagonalizable over the reals
>
> > **proof** $A$ has at most $n$ eigen > values $\rightarrow$ the algebraic eigen > multiplicity of every eigen > value of $A$ is $1$ as they are all distinct and must be greater than $1$ $\rightarrow$ the geometric eigen > multiplicity of every eigen > value of $A$ is $1$ as it must be greater than $1$ and less than its algebraic eigen > multiplicity $\rightarrow$ all algebraic eigen > multiplicityes and geometric eigen > multiplicityes are equal $\rightarrow$ $A$ is diagonalizable. see eigen

# theorems

**see** linear system

### theorem

let $\mathbb{M}^{m,n} A$ (see matrix). the following logic statements are equivalent:

- every variable is a leading variable
- there is a leading variable in every column of the RREF of $A$
- the system $Ax = O$ has a unique solution
- the columns of $A$ are linearly independent
- $Ker\ A = \langle\langle 0 \rangle\rangle$
- $\dim Ker\ A = 0$
- $rank\ A = n$

**see** linear system theorem proof

**theorem**

let $\mathbb{M}^{n,n} A$ (see matrix). the following logic statements are equivalent:

> **note** all logic statements below are valid for both $A$ and $A^\intercal$, see matrix > transpose

- $rank\ A = n$
- every linear system of the form $Ax = b$ has a unique solution
- the RREF of $A$ is the matrix > identity matrix
- $Ker\ A = \langle\langle 0 \rangle\rangle$
- $Col\ A = \mathbb{R}^n$
- $Row\ A = \mathbb{R}^n$
- the columns of $A$ are linearly independent
- the rows of $A$ are linearly independent
- the columns of $A$ form a basis for $\mathbb{R}^n$
- the rows of $A$ form a basis for $\mathbb{R}^n$
- $A$ is an invertible matrix
- $\det A \neq 0$