

# Complex

the set of complex numbers

see math notation

## definition

$$\mathbb{C}x \equiv x = a + bi \wedge a \in \mathbb{R} \wedge b \in \mathbb{R}$$

## notations

*Cartesian Form*  $z = a + bi$

*Polar Form*  $z = |z| \cos \theta + |z| i \sin \theta = |z| \operatorname{cis} \theta = |z| e^{i\theta}$ , see eulers constant, where  $\operatorname{cis} = \cos + i \sin = \theta \rightarrow e^{i\theta}$

| aka Euler's formula notation

## applications

complex numbers are often intimately related to discrete mathematics — 3B1B <https://youtu.be/bOXCLR3Wric>

## properties

$\mathbb{C} \supset \mathbb{U}$ , see universal

*equality*  $a + bi = c + di \equiv a = c \wedge b = d$

*addition*  $(a + bi) + (c + di) = (a + c) + (b + d)i$

*subtraction*  $(a + bi) - (c + di) = (a - c) + (b - d)i$

*multiplication*

in cartesian form,  $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

in polar form,  $z = |z| e^{i\theta}$  |  $w = |w| e^{i\phi} \Rightarrow zw = |z||w| e^{i(\theta + \phi)}$

*square root of  $i$*   $|i| = 1 \Rightarrow \sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$  — <https://www.youtube.com/watch?v=Z49hXoN4KWg>

*product of two complex > conjugates are product of complex > modulus*

$(a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2 = |a + bi|^2 = |a - bi|^2$  — <https://youtu.be/bOXCLR3Wric?t=1522>

**theorem** *De Moivre's Theorem*  $[\operatorname{cis} \theta]^n = \operatorname{cis} n\theta \nrightarrow \mathbb{Z}_n \nrightarrow \mathbb{R}\theta$  —

[https://en.wikipedia.org/wiki/De\\_Moivre%27s\\_formula](https://en.wikipedia.org/wiki/De_Moivre%27s_formula)

**proof**  $\text{cis } \theta = e^{i\theta}$ . since  $[e^{i\theta}]^n = e^{in\theta}$ , it must be that  $[\text{cis } \theta]^n = \text{cis } n\theta$  — me

## Real Part

## Imaginary Part

let  $z = a + bi$

### definitions

*real part of a complex number  $z^{\text{re}} = a$*

*imaginary part of a complex number  $z^{\text{im}} = b$*

therefore,  $z = z^{\text{re}} + iz^{\text{im}}$

## Conjugate

complex > conjugate

### definition

let  $z = a + bi$

then,  $\text{conj } z = a - bi = z^{\text{re}} - iz^{\text{im}}$  is the *complex conjugate* of  $z$

### properties

let  $z \in \mathbb{C}, w \in \mathbb{C}, c \in \mathbb{R}$

$\text{conj}(z + w) = \text{conj } z + \text{conj } w$

$\text{conj } cz = c \text{ conj } z$

$\text{conj } z \cdot w = \text{conj } z \cdot \text{conj } w$

$\text{conj } z - w = \text{conj } z - \text{conj } w$

$\text{conj } \text{conj } z = z$

$\mathbb{R}z \equiv \text{conj } z = z$

**theorem**  $z \text{ conj } z = |z|^2 \in \mathbb{R}$

**theorem**  $-z = \text{conj } z - |z|^2 \in \mathbb{C}$

### applications

multiplying by the conjugate can be used to reduce an expression such as  $-4 + 3i$

# Modulus

aka magnitude, absolute value

**definition**  $|z| = \sqrt{z^{re}{}^2 + z^{im}{}^2}$  where  $|z|$  is the *absolute value* of  $z$ .

**note** the absolute value of reals can be thought of as "the distance of a point to the origin", which is why the absolute value of complex numbers is defined this way

## properties

let  $\mathbb{C}z \wedge \mathbb{C}w \wedge \mathbb{R}c$

let  $\mathbb{R}|z| \wedge |z| \geq 0$

$$|z| = |\operatorname{conj} z|$$

$$|zw| = |z| |w|$$

$$|z - w| \leq |z| + |w|$$

*triangle inequality*  $|z : w| \leq |z| : |w|$

# Argument

aka phase

**definition** the *argument* of a complex number  $z$  is the counterclockwise angle between the positive real axis and the line segment from the origin to the point  $(z^{re}, z^{im})$

**definition**  $z = |z| e^{i \arg z}$  where  $\arg z$  is the *argument* of  $z$