Matrix

see [[math-notation]]

notation

[[vector-space]] of $m \times n$ matrices:

$\mathbb{M}^{m,n}$ in my [[math-notation]]

$M_{m\,n}(\mathbb{R})$ in [[classical-math-notation]]

Rank of a Matrix

the number of pivots in any [[REF]] of the matrix

notation

$rank\ A$, where

A is the matrix to find the rank of

determining the type of the general solution

see [[linear-system]]

let $[A \quad | \quad b]$ be an augmented matrix.

- the system has no solutions if $rank(A) < rank([A \quad | \quad b])$
- the system has a unique solution if and only if $rank(A) = rank([A \quad | \quad b]) = \text{number of columns in A}$
- the system infinite solutions if and only if $rank(A) = rank([A \quad | \quad b]) < \text{number of columns in A}$

Multiplication by a Scalar

definition

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$

Matrix Addition

definition

$(A \cdot B)^{i,j} = A^{i,j} \cdot B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B \quad \text{(matrix addition)}$

Matrix Multiplication

see [[dot-product]], [[vector-in-rn]]

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{N}n \vdash \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an $m'p$ matrix)

$(AB)^{i,j} = A^{i,} \mid B^{,j} \dashv \mathbb{N}i \wedge \mathbb{N}j$, see [[dot-product]] (the | here is a vector [[dot-product]], [[think]])

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

therefore,

$AA \ldots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

$AB = BA \dashv \mathbb{M}A \wedge \mathbb{M}B \equiv \perp$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$ — not commutative

$AB = 0 \vdash A = 0 \vee B = 0 \equiv \perp$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \vdash A = B \equiv \perp$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

$(AB)C = A(BC)$ — associative

$A(B \cdot C) = AB \cdot AC$ — distributive

$(B \cdot C)A = BA \cdot CA$ — distributive

$k(AB) = (kA)B = A(kB)$ — associative with scalars

examples

can be used to represent a [[linear-system]] of [[linear-equation]]s:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Identity Matrix

$I^{a,b} = 1 \wedge a = b \vee I^{a,b} = 0 \wedge a \neq b \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n} I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\ldots$

properties

$AI = A \wedge IA = A \dashv \mathbb{M}A$

Zero Matrix

$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m} O$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dots$

properties

$A \cdot O = A \wedge O \cdot A = A \dashv \mathbb{M} A$

$A_{m,n} O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p} O_{n,p} \wedge \mathbb{M}^{m,p} O_{m,p} \wedge \mathbb{M}^{m,n} A_{m,n}$

$O_{q,m} A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m} O_{q,m} \wedge \mathbb{M}^{q,n} O_{q,n} \wedge \mathbb{M}^{m,n} A_{m,n}$

Nullspace (Kernel)

notation

$Ker\ A \equiv Null\ A$

definition

$Ker\ A = x \equiv Null\ A = x \equiv Ax = 0 \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a matrix can be calculated using [[row-reduction]]

properties

the Null Space of a [[matrix]] is always a [[vector-space]]

> **theorem**: the [[span]]ning set of $Null\ A$ obtained from applying [[row-reduction]] on the system $Ax = 0$ is a [[basis]] for $Null\ A$
>
> therefore, as $\dim Null\ A = \text{number of free variables in } Ax = 0$, we deduce that $\dim Null\ A \cdot rank\ A = \text{number of columns in } A$

example

transforming a vector space into the null space of a certain matrix

let $W = span(1,0,0,1),(1,1,1,0),(2,1,\circ 1,1)$

after solving the [[linear-system]], we get $W(x,y,z,w) \equiv \circ x \cdot y \cdot w = 0$. therefore, W is the nullspace of $A = [\circ 1 \quad 1 \quad 0 \quad 1]$

Column Space, Row Space

see [[vector-in-rn-vector-space]]

notation

$Col\ A$

$Row\ A$

definition

$Col\ A = span A^{,n} \dashv \mathbb{N}n$

$Row\ A = span A^{n,} \dashv \mathbb{N}n$

properties

$Col\ A = Row\ A^\intercal \wedge Row\ A = Col\ A^\intercal$, see transpose [[matrix]]

> **theorem**: $Row\ A$ does not change when doing [[linear-system|elementary-operations]] on the rows of A (if A and B are [[linear-system|row-equivalent]], $Row\ A = Row\ B$
>
> **theorem**: the nonzero rows in any [[REF]] of a [[matrix]] A forms a [[basis]] for $Row\ A$. therefore, $\dim Row\ A = rank\ A$ (see rank of a [[matrix]])

row spaces can be used to find a [[basis]] for a [[span]]ning set of vectors through [[row-reduction]] the basis for the row space of a [[matrix]] can be found by applying [[row-reduction]] and [[span]]ning the **row-reduced columns** in the [[REF]] form of the [[matrix]]

the basis for the column space of a [[matrix]] can be found by applying [[row-reduction]] and [[span]]ning the **original columns** that became pivots in the [[REF]] form of the [[matrix]]

the same can be said for ***Col A***

Transpose Matrix

the Transpose of a Matrix

definition

flips a matrix around its diagonal

$(A^{\intercal})^{i,j} = (A)^{j,i} \dashv \mathbb{N}\boldsymbol{i} \wedge \mathbb{N}\boldsymbol{j} \wedge \mathbb{M}A$

properties

$A^{\intercal^{\intercal}} = A \dashv \mathbb{M}A$

$(AB)^{\intercal} = B^{\intercal}A^{\intercal} \dashv \mathbb{M}A \wedge \mathbb{M}B$

example

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix Inverse

the Inverse of a Matrix

definition

$AA^{-1} = I$, where

A is a (square) [[matrix]]

A^{-1} is the *inverse matrix* of A

invertability

*an **invertible** [[matrix]] has an inverse*

see [[linear-system]] for invertability criteria

properties

let A and C be invertible [[matrix]]es, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$

$AA^{\circ 1} = A^{\circ 1}A = I$

$(A^{\circ 1})^{\circ 1} = A$

$(A^{p})^{\circ 1} = (A^{\circ 1})^{p}$

$(kA)^{\circ 1} = 1\text{-}kA^{\circ 1}$

$(AC)^{\circ 1} = C^{\circ 1}A^{\circ 1}$ (note the order has changed as [[matrix]] multiplication is not commutative)

if AC is invertible, then A is invertible and C is invertible

finding a matrix inverse

let $\mathbb{M}^{n,n}A$

solve the system $AA^{-1} = I$ by extending the [[matrix]] with the identity [[matrix]] and solve the [[linear-system]] up to [[RREF]] using [[row-reduction]].

$[A \quad | \quad I] \sim \ldots [I \quad | \quad A^{-1}]$

shortcut with [[matrix]]es in $\mathbb{M}^{2,2}$

see [[determinant]]

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if and only if $|A| \neq 0$

$A^{-1} = 1\text{-}|A| \begin{bmatrix} d & \circ b \\ \circ c & a \end{bmatrix}$

example usage

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{-1}$

this can be used to solve a system such as:

$Ax = \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

$BAx = B \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

$Ix = x = B \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

Triangular Matrix

a [[matrix]] is *triangular* if every entry below its diagonal or above its diagonal is 0

the *diagonal* of a square [[matrix]] goes from its top left element to its bottom right element