Matrix

see Math Notation

notation

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Rank of a Matrix

the number of pivots in any REF of the Matrix

notation

$\boldsymbol{rank}\, A$, where

A is the Matrix to find the rank of

Multiplication by a Scalar

see Matrix Vector Space, Vector Space

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$

Matrix Addition

see Matrix Vector Space, Vector Space

$(A \cdot B)^{i,j} = A^{i,j} \cdot B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{m,n} B$

Matrix Multiplication

see Dot Product, Vector In Rn

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{N}n \vdash \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an $m'p$ Matrix)

$(AB)^{i,j} = A^{i,} \mid B^{,j} \dashv \mathbb{N}i \wedge \mathbb{N}j$, see Dot Product (the | here is a vector Dot Product, Think)

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

therefore,

$AA \ldots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

$AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$ — not commutative

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

$(AB)C = A(BC)$ — associative

$A(B \cdot C) = AB \cdot AC$ — distributive

$(B \cdot C)A = BA \cdot CA$ — distributive

$k(AB) = (kA)B = A(kB)$ — associative with scalars

examples

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Identity Matrix

definition

$I^{a,b} = 1 \land a = b \lor I^{a,b} = 0 \land a \neq b \dashv \mathbb{N}a \land \mathbb{N}b \land \mathbb{M}^{n,n}I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dots$

properties

$AI = A \land IA = A \dashv \mathbb{M}A$

Zero Matrix

see Matrix Vector Space, Vector Space

definition

$O^{a,b} = 0 \dashv \mathbb{N}a \land \mathbb{N}b \land \mathbb{M}^{n,m}O$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dots$

properties

$A \cdot O = A \wedge O \cdot A = A \dashv \mathbb{M} A$

$A_{m,n} O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p} O_{n,p} \wedge \mathbb{M}^{m,p} O_{m,p} \wedge \mathbb{M}^{m,n} A_{m,n}$

$O_{q,m} A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m} O_{q,m} \wedge \mathbb{M}^{q,n} O_{q,n} \wedge \mathbb{M}^{m,n} A_{m,n}$

Null Space (Nullspace, Kernel)

notation

$Ker\ A \equiv Null\ A$

definition

$Ker\ A = x \equiv Null\ A = x \equiv Ax = 0 \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a Matrix can be calculated using Row Reduction

properties

the Null Space of a Matrix is always a Vector Space

> **theorem**: the Spanning set of $Null\ A$ obtained from applying Row Reduction on the system $Ax = 0$ is a Basis for $Null\ A$
>
> therefore, as $\dim Null\ A = \text{number of free variables in } Ax = 0$, we deduce that $\dim Null\ A \cdot rank\ A = \text{number of columns in } A$

example

transforming a Vector Space into the null space of a certain Matrix

let $W = span(1,0,0,1),(1,1,1,0),(2,1,\circ 1,1)$

after solving the Linear System, we get $W(x,y,z,w) \equiv \circ x \cdot y \cdot w = 0$. therefore, W is the null space of $A = [\circ 1 \quad 1 \quad 0 \quad 1]$

Column Space, Row Space

the span of the set of columns or rows of a Matrix

see Vector In Rn Vector Space

notation

$\mathit{Col}\ A$

$\mathit{Row}\ A$

definition

$\mathit{Col}\ A = \mathit{span} A^{,n} \dashv \mathbb{N} n$

$\mathit{Row}\ A = \mathit{span} A^{n,} \dashv \mathbb{N} n$

properties

$\mathit{Col}\ A = \mathit{Row}\ A^{\intercal} \wedge \mathit{Row}\ A = \mathit{Col}\ A^{\intercal} \dashv \mathbb{M} A$, see transpose Matrix

> **theorem**: $\mathit{Row}\,A$ does not change when applying Elementary Operations on the rows of A (if A and B are Row Equivalent, $\mathit{Row}\ A = \mathit{Row}\ B$

> **theorem**: the nonzero rows in any REF of a Matrix A forms a Basis for $\mathit{Row}\ A$. therefore, $\dim \mathit{Row}\ A = \mathit{rank}\ A$ (see rank of a Matrix)

row spaces can be used to find a Basis for a Spanning set of vectors through Row Reduction

the basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for $\mathit{Col}\ A$

Transpose Matrix

the Transpose of a Matrix

definition

flips a Matrix around its diagonal

> **note**: the *diagonal* of a square Matrix goes from its top left element to its bottom right element

$(A^\intercal)^{i,j} = (A)^{j,i} \dashv \mathbb{N}\boldsymbol{i} \land \mathbb{N}\boldsymbol{j} \land \mathbb{M}A$

properties

$A^{\intercal^\intercal} = A \dashv \mathbb{M}A$

$(AB)^\intercal = B^\intercal A^\intercal \dashv \mathbb{M}A \land \mathbb{M}B$

example

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix Inverse

the Inverse of a Matrix

definition

$AA^{\circ 1} = I$, where

A is a (square) Matrix

$A^{\circ 1}$ is the *inverse matrix* of A

invertability

*an **invertible** Matrix has a corresponding inverse Matrix*

see theorems below for invertability criteria

properties

let A and C be invertible Matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$

$AA^{\circ 1} = A^{\circ 1}A = I$

$(A^{\circ 1})^{\circ 1} = A$

$(A^{p})^{\circ 1} = (A^{\circ 1})^{p}$

$(kA)^{\circ 1} = 1\text{-}kA^{\circ 1}$

$(AC)^{\circ 1} = C^{\circ 1}A^{\circ 1}$

> **note**: in the equation above, the order of the matrices multiplied together has changed as Matrix multiplication is not commutative)

if AC is invertible, then A is invertible and C is invertible

finding a matrix inverse

let $\mathbb{M}^{n,n}A$

solve the system $AA^{\circ 1} = I$ by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction. $[A \quad | \quad I] \sim \ldots [I \quad | \quad A^{\circ 1}]$

shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if and only if $|A| \neq 0$

$A^{\circ 1} = 1\text{-}|A|\ | \begin{bmatrix} d & \circ b \\ \circ c & a \end{bmatrix}$

example usage

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{\circ 1}$

this can be used to solve a system such as:

$Ax = \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

$BAx = B \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

$Ix = x = B \begin{bmatrix} \circ 1 \\ 1 \end{bmatrix}$

Triangular Matrix

a Matrix is *triangular* if every entry below its diagonal or above its diagonal is 0

> **note**: the *diagonal* of a square Matrix goes from its top left element to its bottom right element

Theorems

see Linear System

> **theorem**: let $\mathbb{M}^{m,n} A$ (see Matrix). the following statements are equivalent:
>
> 1. every variable is a leading variable

2. there is a leading variable in every column of the RREF of A
3. the system $Ax = 0$ has a unique solution
4. the columns of A are Linearly Independent
5. $Ker\ A = 0$
6. $\dim Ker\ A = 0$
7. $rank\ A = n$

see Linear System Theorem Proof

> **theorem**: let $\mathbb{M}^{n,n} A$ (see Matrix). the following statements are equivalent:
>
> **note**: all statements below are valid for both A and $A^\intercal$, see transpose Matrix
>
> 1. $rank\ A = n$
> 2. every linear system of the form $Ax = b$ has a unique solution
> 3. the RREF of A is the identity Matrix
> 4. $Ker\ A = 0$
> 5. $Col\ A = \mathbb{R}^n$
> 6. $Row\ A = \mathbb{R}^n$
> 7. the columns of A are Linearly Independent
> 8. the rows of A are Linearly Independent
> 9. the columns of A form a Basis for $\mathbb{R}^n$
> 10. the rows of A form a Basis for $\mathbb{R}^n$
> 11. A is Invertible
> 12. $\det A \neq 0$