

Matrix

see [math notation](#), [eigen](#)

definition formally in my [math notation](#) a [matrix](#) in $\mathbb{R}^{m,n}$ is a [set theoretical function](#) with [function > domain](#) at least $\langle x, y \rangle \rightarrow \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$ that takes an [ordered pair](#) as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by Scalar

see [matrix vector space](#), [vector space](#)

definition

$$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

properties

commutativity with [scalars](#) $kA = Ak$

Addition

see [matrix vector space](#), [vector space](#)

definition

$$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

Multiplication

see [dot product](#), [vector in rn](#)

definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B . their product will be an m by p [matrix](#))

$$(AB)^{i,j} = A^i, \mid B^j \dashv \mathbb{N}i \wedge \mathbb{N}j, \text{ see } \a href="#">dot product$$

intuitively, matrix multiplication is the [dot product](#) of **every row** of the first [matrix](#) by **every column** of the second [matrix](#)

notation

$$AA = A^2 = [A]^2 \vdash \mathbb{M}A$$

$$\text{and therefore } AA \cdots A = [A]^n \wedge \mathbb{N}n \vdash \mathbb{M}A$$

properties

$$\text{not commutative } AB = BA \not\vdash \mathbb{M}A \wedge \mathbb{M}B \text{ or } AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$$

$$AB = 0 \not\vdash A = 0 \vee B = 0 \text{ (it can happen that } AB = 0, \text{ but } A \neq 0 \text{ and } B \neq 0) \text{ (} AB \text{ being equal to 0 does not imply that } A = 0 \text{ or that } B = 0)$$

$$AC = BC \wedge C \neq 0 \not\vdash A = B \text{ (} AC = BC \text{ and } C \neq 0 \text{ does not imply that } A = B)$$

$$\text{associative } (AB)C = A(BC)$$

$$\text{distributive } A(B : C) = AB : AC$$

$$\text{distributive } (B : C)A = BA : CA$$

$$\text{associative with scalars } k(AB) = (kA)B = A(kB)$$

applications

matrix > multiplication can be used to represent a linear system of linear equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

matrix > multiplication can be used to represent any linear transformation

Identity Matrix

definition

$$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \vdash \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

properties

$$AI = A \wedge IA = A \vdash \mathbb{M}A$$

Zero Matrix

see [matrix vector space](#), [vector space](#)

definition

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

properties

$$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$$

$$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$$

Rank

the number of pivots in any [REF](#) of the [matrix](#)

notation

rank A , where

- A is the [matrix](#) to find the [matrix > rank](#) of

Element Count

notation $\# M$

definition the *element count* of a [matrix](#) is the total number of elements in the [matrix](#)

$$\left| \begin{array}{l} \text{example let } M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}. \text{ then, } \mathbb{M}^{2,3}M \wedge \# M = 2 \mid 3 = 6 \end{array} \right.$$

Vector Spaces

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

kernel $Ker A \equiv Null A$

column space $Col A$

row space $Row A$

definitions

kernel ($Ker A$) $x \equiv (Null A) x \equiv Ax = O \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

column space $Col A = \text{span } A^n \rightarrow \mathbb{N}n$

row space $Row A = \text{span } A^n \rightarrow \mathbb{N}n$

procedure computing the kernel of a matrix use row reduction

theorems

the Null Space, Row Space and Column Space of a matrix are always vector spaces

number of free variables in A : number of pivots in A = number of columns in A

$\dim Null A$ = number of free variables in A

$rank A$ = number of pivots in A

the nonzero rows in any REF of a matrix A forms a basis for $Row A$. therefore,
 $\dim Row A = rank A$, see matrix > rank

if A and B are row-equivalent, then $Row A = Row B$, see linear system

the spanning set of $Null A$ obtained from applying row reduction on the system
 $Ax = O$ is a basis for $Null A$

$Row A$ does not change when applying linear system > elementary operations on the rows of A

properties

$Col A = Row A^T \wedge Row A = Col A^T \rightarrow \mathbb{M}A$, see matrix > transpose

applications

row spaces can be used to find a basis for a spanning set of vectors through row reduction

the basis for the row space of a matrix can be found by applying row reduction and spanning the **row-reduced columns** in the REF form of the matrix

the basis for the column space of a matrix can be found by applying row reduction and spanning the **original columns** that became pivots in the REF form of the matrix

the same can be said for *Col A*

example transforming a vector space into the null space of a certain matrix

let $W = \text{span} \langle (1, 0, 0, 1), (1, 1, 1, 0), (2, 1, 1, 1) \rangle$

after solving the linear system, we get $W(x, y, z, w) \equiv x : y : w = 0$. therefore, W is the null space of $A = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$

Diagonal

the diagonal of a matrix

definition the *diagonal* of a square matrix goes from its top left element to its bottom right element

Transpose

the transpose of a matrix

flips a matrix around its matrix > diagonal

definition

$$(A^T)^{i,j} = (A)^{j,i} \vdash \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

properties

$$(A^T)^T = A \vdash \mathbb{M}A$$

$$(AB)^T = B^T A^T \vdash \mathbb{M}A \wedge \mathbb{M}B$$

representation

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Inverse

the inverse of a [matrix](#)

definition

$AA^{-1} = A^{-1}A = I \in \mathbb{M}A$, where

- A is a square [matrix](#)
- A^{-1} is the *inverse matrix* of A

Invertability

definition an *invertible matrix* has a corresponding [matrix > inverse](#)

see theorems below for invertability criteria

properties

let A and C be invertible [matrixes](#), let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$. then,

$$AA^{-1} = A^{-1}A = I$$

$$(A^{-1})^{-1} = A$$

$$(A^p)^{-1} = (A^{-1})^p$$

$$(kA)^{-1} = -k \mid A^{-1} \text{ (see [improved expression evaluation](#))}$$

$$(AC)^{-1} = C^{-1}A^{-1}$$

note in the equation above, the order of the [matrixes](#) has changed. this is significant as [matrix > multiplication](#) is not commutative

if AC is invertible, then A is invertible and C is invertible

procedure computing the [matrix > inverse](#) of a [matrix](#)

let $\mathbb{M}^{n,n}A$

solve the system $AA^{-1} = I$ by extending the [matrix](#) with the [matrix > identity matrix](#) and solve the [linear system](#) up to [RREF](#) using [row reduction](#).

$$[A \mid I] \sim \cdots [I \mid A^{-1}]$$

procedure computing the [matrix > inverse](#) of a 2 by 2 [matrix](#)

see [determinant](#)

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

applications using a matrix > inverse to solve a linear system

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate B such that $B \equiv A^{-1}$

this can be used to solve a linear system such as:

$$Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Ix = x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Triangular Matrix

definition a matrix is said to be *triangular* if every entry below its matrix > diagonal or above its matrix > diagonal is 0

Diagonal Matrix

definition a matrix is said to be *diagonal* if every entry below its matrix > diagonal and above its matrix > diagonal is 0

applications

$[D]^x$ can be calculated by raising every entry of D to the power x

Diagonalizable Matrix

see eigen > vector

definition an n by n matrix A is said to be *diagonalizable over the reals* if there exists a basis of \mathbb{R}^n consisting entirely of eigen > vectors of A

a matrix is *diagonalizable* if and only if the geometric eigen > multiplicity of an eigen > value is equal to the algebraic eigen > multiplicity of said eigen > value, for every eigen > value of the matrix

note a matrix may also be diagonalizable over other number fields such as the set of complex numbers \mathbb{C}

note some matrixes do not have "enough" real eigen > values or "enough" eigen > vectors to be diagonalizable

example the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\langle (1, 1), (1, -1) \rangle$ is a basis of \mathbb{R}^2 consisting entirely of eigen > vectors of A

example the matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real eigen > values

example the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one eigen > value, and therefore only one set of linearly dependent eigen > vectors

example the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single eigen > value $\lambda = 1$, its eigen > space spans \mathbb{R}^2 . this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$

proof let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see eigen

example let $\mathbb{M}^{n,n} A \wedge \mathbb{N}n$ and suppose A has n distinct eigen > values. deduce that A is diagonalizable over the reals

proof A has at most n eigen > values \rightarrow the algebraic eigen > multiplicity of every eigen > value of A is 1 as they are all distinct and must be greater than 1 \rightarrow the geometric eigen > multiplicity of every eigen > value of A is 1 as it must be greater than 1 and less than its algebraic eigen > multiplicity \rightarrow all algebraic eigen > multiplicities and geometric eigen > multiplicities are equal $\rightarrow A$ is diagonalizable. see eigen

theorems

see linear system

theorem

let $\mathbb{M}^{m,n} A$ (see matrix). the following logic statements are equivalent:

- every variable is a leading variable
- there is a leading variable in every column of the RREF of A
- the system $Ax = O$ has a unique solution

- the columns of A are linearly independent
- $\text{Ker } A = \langle \langle 0 \rangle \rangle$
- $\dim \text{Ker } A = 0$
- $\text{rank } A = n$

see linear system theorem proof

theorem

let $\mathbb{M}^{n,n} A$ (see matrix). the following logic statements are equivalent:

note all logic statements below are valid for both A and A^T , see matrix > transpose

- $\text{rank } A = n$
- every linear system of the form $Ax = b$ has a unique solution
- the RREF of A is the matrix > identity matrix
- $\text{Ker } A = \langle \langle 0 \rangle \rangle$
- $\text{Col } A = \mathbb{R}^n$
- $\text{Row } A = \mathbb{R}^n$
- the columns of A are linearly independent
- the rows of A are linearly independent
- the columns of A form a basis for \mathbb{R}^n
- the rows of A form a basis for \mathbb{R}^n
- A is an invertible matrix
- $\det A \neq 0$