

Matrix

see [Math Notation](#)

definition

formally, in my [Math Notation](#), a [Matrix](#) in $\mathbb{R}^{m,n}$ is a [Set Theoretical Function](#) with domain $(x,y) \rightarrow \mathbb{N}x \wedge \mathbb{N}y \wedge 0 \leq x < m \wedge 0 \leq y < n$ that takes an ordered pair (a [Vector In Rn2](#)) as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by a Scalar

see [Matrix Vector Space](#), [Vector Space](#)

definition

$$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

properties

$$kA = Ak \text{ --- commutative with scalars}$$

Matrix Addition

see [Matrix Vector Space](#), [Vector Space](#)

$$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

Matrix Multiplication

see [Dot Product](#), [Vector In Rn](#)

definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B . their product will be an m by p Matrix)

$(AB)^{i,j} = A^i \cdot B^j \mapsto \mathbb{N}i \wedge \mathbb{N}j$, see Dot Product

intuitively, matrix multiplication is the Dot Product of **every row** of the first Matrix by **every column** of the second Matrix

notation

$$AA = A^2 = [A]^2 \mapsto \mathbb{M}A$$

therefore,

$$AA \dots A = [A]^n \wedge \mathbb{N}n \mapsto \mathbb{M}A$$

properties

$AB = BA \not\wedge \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$ — not commutative

$AB = 0 \not\Leftarrow A = 0 \vee B = 0$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$AC = BC \wedge C \neq 0 \not\Leftarrow A = B$ ($AC = BC$ and $C \neq 0$ does not imply that $A = B$)

$(AB)C = A(BC)$ — associative

$A(B : C) = AB : AC$ — distributive

$(B : C)A = BA : CA$ — distributive

$k(AB) = (kA)B = A(kB)$ — associative with scalars

applications

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be used to represent any Linear Transformation

Identity Matrix

definition

$$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...

properties

$$AI = A \wedge IA = A \dashv \mathbb{M}A$$

Zero Matrix

see [Matrix Vector Space](#), [Vector Space](#)

definition

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

...

properties

$$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$$

$$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$$

Rank of a Matrix

the number of pivots in any REF of the Matrix

notation

$\text{rank } A$, where

A is the Matrix to find the rank of

Null Space (Nullspace, Kernel), Column Space, Row Space

notation

$\text{Ker } A \equiv \text{Null } A$

$\text{Col } A$

$\text{Row } A$

definition

$(\text{Ker } A) x \equiv (\text{Null } A) x \equiv Ax = 0 \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a Matrix can be calculated using Row Reduction

$\text{Col } A = \text{span } A^{\cdot n} \vdash \mathbb{N}n$

$\text{Row } A = \text{span } A^{n, \cdot} \vdash \mathbb{N}n$

properties

theorem: the Null Space, Row Space and Column Space of a Matrix are always Vector Spaces

theorem:

number of free variables in A : number of pivots in A = number of columns in A

theorem: $\dim \text{Null } A$ = number of free variables in A

theorem: $\text{rank } A$ = number of pivots in A

theorem: the nonzero rows in any REF of a Matrix A forms a Basis for $Row A$.
therefore, $\dim Row A = rank A$ (see rank of a Matrix)

theorem: if A and B are Row Equivalent, $Row A = Row B$

theorem: the Spanning Set of $Null A$ obtained from applying Row Reduction on the system $Ax = 0$ is a Basis for $Null A$

theorem: $Row A$ does not change when applying Elementary Operations on the rows of A

$Col A = Row A^T \wedge Row A = Col A^T \dashv \mathbb{M}A$, see transpose Matrix

applications

row spaces can be used to find a Basis for a Spanning Set of vectors through Row Reduction

the basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for $Col A$

example

transforming a Vector Space into the null space of a certain Matrix

let $W = \text{span} (1,0,0,1), (1,1,1,0), (2,1,1,1)$

after solving the Linear System, we get $W(x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = \begin{bmatrix} \cdot & 1 & 0 & 1 \end{bmatrix}$

Transpose Matrix

*the **transpose** of a Matrix*

definition

flips a Matrix around its diagonal

note: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

$$(A^{\top})^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

properties

$$(A^{\top})^{\top} = A \dashv \mathbb{M}A$$

$$(AB)^{\top} = B^{\top}A^{\top} \dashv \mathbb{M}A \wedge \mathbb{M}B$$

example

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Matrix Inverse

the *Inverse* of a Matrix

definition

$AA^{-} = A^{-}A = I \dashv \mathbb{M}A$, where

A is a (square) Matrix

A^{-} is the *inverse matrix* of A

Invertability

an **invertible** Matrix has a corresponding inverse Matrix

see theorems below for invertability criteria

properties

let A and C be invertible Matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$

$$AA^{-} = A^{-}A = I$$

$$(A^{-})^{-} = A$$

$$(A^p)^{-} = (A^{-})^p$$

$$(kA)^{-} = 1/k \mid A^{-} \text{ (restriction might not be necessary, see Improved Expression Evaluation)}$$

$$(AC)^{-} = C^{-}A^{-}$$

note: in the equation above, the order of the matrices has changed. this is significant as Matrix multiplication is not commutative)

if AC is invertible, then A is invertible and C is invertible

finding a matrix inverse

let $\mathbb{M}^{n,n}A$

solve the system $AA^{-} = I$ by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction. $[A \mid I] \sim \dots [I \mid A^{-}]$

shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $|A| \neq 0$

$$A^{-} = -|A| \mid \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$$

application example

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate B such that $B \equiv A^{-}$

this can be used to solve a system such as:

$$Ax = \begin{bmatrix} .1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} .1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} .1 \\ 1 \end{bmatrix}$$

Triangular Matrix

a Matrix is *triangular* if every entry below its diagonal **or** above its diagonal is 0

note: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

Diagonal Matrix

a Matrix is *diagonal* if every entry below its diagonal **and** above its diagonal is 0

note: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

properties

let D be a diagonal Matrix

$[D]x$ can be calculated by multiplying every entry of D by x

Diagonalizable Matrix

see Eigenvector

definition

an n by n Matrix A is said to be *diagonalizable over the reals* if there exists a Basis of \mathbb{R}^n consisting entirely of Eigenvectors of A

a Matrix is *diagonalizable* if and only if the geometric Multiplicity of an Eigenvalue is equal to the algebraic Multiplicity of said Eigenvalue, for every Eigenvalue of the Matrix (see

Eigenvector And Eigenvalue)

note: a Matrix may also be diagonalizable over other Number Fields such as the Set of Complex numbers \mathbb{C}

note: some Matrixes do not have "enough" real Eigenvalues or "enough" Eigenvectors to be diagonalizable

examples and counterexamples

the Matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\{(1,1), (1,-1)\}$ is a Basis of \mathbb{R}^2 consisting entirely of Eigenvectors of A

the Matrix $A = \begin{bmatrix} 1 & 1 \\ .1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real Eigenvalues

the Matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one Eigenvalue, and therefore only one set of Linearly Dependent Eigenvectors (see Eigenvector And Eigenvalue)

the Matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single Eigenvalue $\lambda = 1$, its Eigenspace Spans \mathbb{R}^2 . this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$

proof: let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$.
therefore, $E_1 \equiv \mathbb{R}^2$. see Eigenvector And Eigenvalue

let $\mathbb{M}^{n,n} A \wedge \mathbb{N} n$ and suppose A has n distinct Eigenvalues. deduce that A is diagonalizable over the reals

proof: A has at most n Eigenvalues \rightarrow the algebraic Multiplicity of every Eigenvalue of A is 1 as they are all distinct and must be greater than 1 \rightarrow the geometric Multiplicity of every Eigenvalue of A is 1 as it must be greater than 1 and less than its algebraic Multiplicity \rightarrow all algebraic Multiplicities and geometric Multiplicities are equal $\rightarrow A$ is diagonalizable. see Eigenvector And Eigenvalue

Eigenvector And Eigenvalues

theorems

see [Linear System](#)

theorem: let $M^{m,n} A$ (see [Matrix](#)). the following [Logic Statements](#) are equivalent:

1. every variable is a leading variable
2. there is a leading variable in every column of the [RREF](#) of A
3. the system $Ax = O$ has a unique solution
4. the columns of A are [Linearly Independent](#)
5. $\text{Ker } A = \{0\}$
6. $\dim \text{Ker } A = 0$
7. $\text{rank } A = n$

see [Linear System Theorem Proof](#)

theorem: let $M^{n,n} A$ (see [Matrix](#)). the following [Logic Statements](#) are equivalent:

note: all [Logic Statements](#) below are valid for both A and A^T , see transpose [Matrix](#)

1. $\text{rank } A = n$
2. every linear system of the form $Ax = b$ has a unique solution
3. the [RREF](#) of A is the identity [Matrix](#)
4. $\text{Ker } A = \{0\}$
5. $\text{Col } A = \mathbb{R}^n$
6. $\text{Row } A = \mathbb{R}^n$
7. the columns of A are [Linearly Independent](#)
8. the rows of A are [Linearly Independent](#)
9. the columns of A form a [Basis](#) for \mathbb{R}^n
10. the rows of A form a [Basis](#) for \mathbb{R}^n
11. A is [Invertible](#)
12. $\det A \neq 0$