

Matrix

see [Math Notation](#)

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Rank of a Matrix

the number of pivots in any [REF](#) of the [Matrix](#)

notation

rank A , where

A is the [Matrix](#) to find the rank of

Multiplication by a Scalar

see [Matrix Vector Space](#), [Vector Space](#)

$$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{R}k \wedge \mathbb{M}A$$

Matrix Addition

see [Matrix Vector Space](#), [Vector Space](#)

$$(A \cdot B)^{i,j} = A^{i,j} \cdot B^{i,j} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}^{m,n}A \wedge \mathbb{M}^{m,n}B$$

Matrix Multiplication

see [Dot Product](#), [Vector In \$\mathbb{R}^n\$](#)

definition

$AB \neq \emptyset \equiv \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,p} B \wedge \mathbb{N} n \vdash \mathbb{M}^{m,p} AB$ (AB is defined if the number of columns in A is equal to the number of rows in B . their product will be an $m'p$ Matrix)

$(AB)^{i,j} = A^i \cdot B^j \vdash \mathbb{N} i \wedge \mathbb{N} j$, see Dot Product (the \cdot here is a vector Dot Product, Think)

notation

$$AA = A2 = [A]2 \vdash \mathbb{M} A$$

therefore,

$$AA \dots A = [A]n \wedge \mathbb{N} n \vdash \mathbb{M} A$$

properties

$$AB = BA \vdash \mathbb{M} A \wedge \mathbb{M} B \equiv \perp \text{ or } AB \neq BA \wedge \mathbb{M} A \wedge \mathbb{M} B \text{ — not commutative}$$

$AB = 0 \vdash A = 0 \vee B = 0 \equiv \perp$ (it can happen that $AB = 0$, but $A \neq 0$ and $B \neq 0$) (AB being equal to 0 does not imply that $A = 0$ or that $B = 0$)

$$AC = BC \wedge C \neq 0 \vdash A = B \equiv \perp \text{ (} AC = BC \text{ and } C \neq 0 \text{ does not imply that } A = B\text{)}$$

$$(AB)C = A(BC) \text{ — associative}$$

$$A(B \cdot C) = AB \cdot AC \text{ — distributive}$$

$$(B \cdot C)A = BA \cdot CA \text{ — distributive}$$

$$k(AB) = (kA)B = A(kB) \text{ — associative with scalars}$$

examples

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Identity Matrix

definition

$$I^{a,b} = 1 \wedge a = b \vee I^{a,b} = 0 \wedge a \neq b \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...

properties

$$AI = A \wedge IA = A \dashv \mathbb{M}A$$

Zero Matrix

see [Matrix Vector Space](#), [Vector Space](#)

definition

$$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

...

properties

$$A \cdot O = A \wedge O \cdot A = A \vdash \mathbb{M}A$$

$$A_{m,n} O_{n,p} = O_{m,p} \vdash \mathbb{M}^{n,p} O_{n,p} \wedge \mathbb{M}^{m,p} O_{m,p} \wedge \mathbb{M}^{m,n} A_{m,n}$$

$$O_{q,m} A_{m,n} = O_{q,n} \vdash \mathbb{M}^{q,m} O_{q,m} \wedge \mathbb{M}^{q,n} O_{q,n} \wedge \mathbb{M}^{m,n} A_{m,n}$$

Null Space (Nullspace, Kernel)

notation

$$Ker A \equiv Null A$$

definition

$$Ker A = x \equiv Null A = x \equiv Ax = 0 \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$$

the Kernel of a Matrix can be calculated using Row Reduction

properties

the Null Space of a Matrix is always a Vector Space

theorem: the Spanning set of $Null A$ obtained from applying Row Reduction on the system $Ax = 0$ is a Basis for $Null A$

therefore, as $\dim Null A = \text{number of free variables in } Ax = 0$, we deduce that $\dim Null A \cdot rank A = \text{number of columns in } A$

example

transforming a Vector Space into the null space of a certain Matrix

$$\text{let } W = span(1,0,0,1), (1,1,1,0), (2,1,0,1)$$

after solving the Linear System, we get $W(x,y,z,w) \equiv 0x \cdot y \cdot w = 0$. therefore, W is the null space of $A = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$

Column Space, Row Space

see Vector In \mathbb{R}^n Vector Space

notation

$Col A$

$Row A$

definition

$$Col A = span A \cdot^n \rightarrow \mathbb{R}^n$$

$$Row A = span A^n \rightarrow \mathbb{R}^n$$

properties

$$Col A = Row A^T \wedge Row A = Col A^T \rightarrow \mathbb{M}A, \text{ see transpose } \underline{\text{Matrix}}$$

theorem: $Row A$ does not change when applying Elementary Operations on the rows of A (if A and B are Row Equivalent, $Row A = Row B$)

theorem: the nonzero rows in any REF of a Matrix A forms a Basis for $Row A$. therefore, $\dim Row A = rank A$ (see rank of a Matrix)

row spaces can be used to find a Basis for a Spanning set of vectors through Row Reduction

the basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for $Col A$

Transpose Matrix

the Transpose of a Matrix

definition

flips a Matrix around its diagonal

$$(A^T)^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$$

properties

$$A^{TT} = A \dashv \mathbb{M}A$$

$$(AB)^T = B^T A^T \dashv \mathbb{M}A \wedge \mathbb{M}B$$

example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix Inverse

the Inverse of a Matrix

definition

$$AA^{-1} = I, \text{ where}$$

A is a (square) Matrix

A^{-1} is the *inverse matrix* of A

invertability

an *invertible* Matrix has an inverse

see Linear System for invertability criteria

properties

let A and C be invertible Matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$

$$AA^{\circ 1} = A^{\circ 1}A = I$$

$$(A^{\circ 1})^{\circ 1} = A$$

$$(A^p)^{\circ 1} = (A^{\circ 1})^p$$

$$(kA)^{\circ 1} = \frac{1}{k}A^{\circ 1}$$

$(AC)^{\circ 1} = C^{\circ 1}A^{\circ 1}$ (note that the order has changed as Matrix multiplication is not commutative)

if AC is invertible, then A is invertible and C is invertible

finding a matrix inverse

let $\mathbb{M}^{n,n} A$

solve the system $AA^{-1} = I$ by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction.
 $[A \mid I] \sim \dots [I \mid A^{-1}]$

shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

example usage

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

then, calculate B such that $B \equiv A^{-1}$

this can be used to solve a system such as:

$$Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$BAx = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Triangular Matrix

a Matrix is *triangular* if every entry below its diagonal or above its diagonal is 0

the *diagonal* of a square Matrix goes from its top left element to its bottom right element