Matrix

see Math Notation

definition

formally, in my Math Notation, a Matrix in $\mathbb{R}^{m,n}$ is a Set Theoryetical Function with domain at least $\langle x, y\rangle \to \mathbb{N}x \land \mathbb{N}y \land 0 \leq x < m \land 0 \leq y < n$ that takes an Ordered Pair as an index and returns the element at that index

notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiplication by a Scalar

see Matrix Vector Space, Vector Space

definition

$(kA)^{i,j} = kA^{i,j} \dashv \mathbb{N}i \land \mathbb{N}j \land \mathbb{R}k \land \mathbb{M}A$

properties

$kA = Ak$ — commutative with scalars

Matrix Addition

see Matrix Vector Space, Vector Space

$(A : B)^{i,j} = A^{i,j} : B^{i,j} \dashv \mathbb{N}i \land \mathbb{N}j \land \mathbb{M}^{m,n}A \land \mathbb{M}^{m,n}B$

Matrix Multiplication

see Dot Product, Vector In Rn

definition

$AB \neq \varnothing \equiv \mathbb{M}^{m,n}A \wedge \mathbb{M}^{n,p}B \wedge \mathbb{M}^{m,p}AB$ (AB is defined if the number of columns in A is equal to the number of rows in B. their product will be an m by p Matrix)

$(AB)^{i,j} = A^{i,} \mid B^{,j} \dashv \mathbb{N}i \wedge \mathbb{N}j$, see Dot Product

intuitively, matrix multiplication is the Dot Product of **every row** of the first Matrix by **every column** of the second Matrix

notation

$AA = A2 = [A]2 \dashv \mathbb{M}A$

therefore,

$AA \ldots A = [A]n \wedge \mathbb{N}n \dashv \mathbb{M}A$

properties

$AB = BA \not\dashv \mathbb{M}A \wedge \mathbb{M}B$ or $AB \neq BA \wedge \mathbb{M}A \wedge \mathbb{M}B$ — not commutative

$AB = 0 \nvdash A = 0 \vee B = 0$ (it can happen that $AB = \mathbf{0}$, but $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$) (AB being equal to $\mathbf{0}$ does not imply that $A = \mathbf{0}$ or that $B = \mathbf{0}$)

$AC = BC \wedge C \neq 0 \nvdash A = B$ ($AC = BC$ and $C \neq \mathbf{0}$ does not imply that $A = B$)

$(AB)C = A(BC)$ — associative

$A(B : C) = AB : AC$ — distributive

$(B : C)A = BA : CA$ — distributive

$k(AB) = (kA)B = A(kB)$ — associative with scalars

applications

can be used to represent a Linear System of Linear Equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be used to represent any Linear Transformation

Identity Matrix

definition

$(I^{a,b} = 1 \wedge a = b) \vee (I^{a,b} = 0 \wedge a \neq b) \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,n}I$

examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\ldots$

properties

$AI = A \wedge IA = A \dashv \mathbb{M}A$

Zero Matrix

see Matrix Vector Space, Vector Space

definition

$O^{a,b} = 0 \dashv \mathbb{N}a \wedge \mathbb{N}b \wedge \mathbb{M}^{n,m}O$

examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ldots$

properties

$A_{m,n}O_{n,p} = O_{m,p} \dashv \mathbb{M}^{n,p}O_{n,p} \wedge \mathbb{M}^{m,p}O_{m,p} \wedge \mathbb{M}^{m,n}A_{m,n}$

$O_{q,m}A_{m,n} = O_{q,n} \dashv \mathbb{M}^{q,m}O_{q,m} \wedge \mathbb{M}^{q,n}O_{q,n} \wedge \mathbb{M}^{m,n}A_{m,n}$

Rank of a Matrix

the number of pivots in any REF of the Matrix

notation

$rank\ A$, where

A is the Matrix to find the rank of

Null Space (Nullspace, Kernel), Column Space, Row Space

notations

$Ker\ A \equiv Null\ A$

$Col\ A$

$Row\ A$

definitions

$(Ker\ A)\ x \equiv (Null\ A)\ x \equiv Ax = O \wedge \mathbb{M}^{m,n} A \wedge \mathbb{M}^{n,1} x$

the Kernel of a Matrix can be calculated using Row Reduction

$Col\ A = \operatorname{span} A^{,n} \dashv \mathbb{N} n$

$Row\ A = \operatorname{span} A^{n,} \dashv \mathbb{N} n$

properties

> **theorem**: the Null Space, Row Space and Column Space of a Matrix are always Vector Spaces

> **theorem**:
> number of free variables in A : number of pivots in A = number of columns in A

> **theorem**: $\dim Null\ A$ = number of free variables in A

> **theorem**: $rank\ A$ = number of pivots in A

> **theorem**: the nonzero rows in any REF of a Matrix A forms a Basis for $\textit{Row } A$. therefore, $\dim \textit{Row } A = \textit{rank } A$ (see rank of a Matrix)

> **theorem**: if A and B are Row Equivalent, $\textit{Row } A = \textit{Row } B$

> **theorem**: the Spanning Set of $\textit{Null } A$ obtained from applying Row Reduction on the system $Ax = O$ is a Basis for $\textit{Null } A$

> **theorem**: $\textit{Row } A$ does not change when applying Elementary Operations on the rows of A

$\textit{Col } A = \textit{Row } A^\intercal \land \textit{Row } A = \textit{Col } A^\intercal \dashv \mathbb{M}A$, see transpose Matrix

applications

row spaces can be used to find a Basis for a Spanning Set of vectors through Row Reduction

the Basis for the row space of a Matrix can be found by applying Row Reduction and Spanning the **row-reduced columns** in the REF form of the Matrix

the Basis for the column space of a Matrix can be found by applying Row Reduction and Spanning the **original columns** that became pivots in the REF form of the Matrix

the same can be said for $\textit{Col } A$

example

transforming a Vector Space into the null space of a certain Matrix

let $W = \mathrm{span}\{(1,0,0,1),(1,1,1,0),(2,1,\cdot 1,1)\}$

after solving the Linear System, we get $W(x,y,z,w) \equiv \cdot x : y : w = 0$. therefore, W is the null space of $A = [\cdot 1 \quad 1 \quad 0 \quad 1]$

Transpose Matrix

the transpose of a Matrix

definition

flips a Matrix around its diagonal

> **note**: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

$(A^{\intercal})^{i,j} = (A)^{j,i} \dashv \mathbb{N}i \wedge \mathbb{N}j \wedge \mathbb{M}A$

properties

$(A^{\intercal})^{\intercal} = A \dashv \mathbb{M}A$

$(AB)^{\intercal} = B^{\intercal}A^{\intercal} \dashv \mathbb{M}A \wedge \mathbb{M}B$

example

$$A$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

—

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Matrix_transpose.gif/200px-Matrix_transpose.gif

Matrix Inverse

the Inverse of a Matrix

definition

$AA^{-} = A^{-}A = I \dashv \mathbb{M}A$, where

A is a (square) Matrix

A^{-} is the *inverse matrix* of A

Invertability

*an **invertible** Matrix has a corresponding inverse Matrix* see theorems below for invertability criteria

properties

let A and C be invertible Matrixes, let $\mathbb{Z}p$ and let $\mathbb{R}k \wedge k \neq 0$

$AA^{-} = A^{-}A = I$

$(A^{-})^{-} = A$

$(A^{p})^{-} = (A^{-})^{p}$

$(kA)^{-} =$ 1-k | A^{-} (restriction might not be necessary, see Improved Expression Evaluation)

$(AC)^{-} = C^{-}A^{-}$

> **note**: in the equation above, the order of the matrices has changed. this is significant as Matrix multiplication is not commutative)

if AC is invertible, then A is invertible and C is invertible

finding a matrix inverse

let $\mathbb{M}^{n,n}A$

solve the system $AA^{-} = I$ by extending the Matrix with the identity Matrix and solve the Linear System up to RREF using Row Reduction. $[A \quad | \quad I] \sim \ldots [I \quad | \quad A^{-}]$

shortcut with Matrixes in $\mathbb{M}^{2,2}$

see Determinant

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A is invertible if and only if $|A| \neq 0$

$A^{-} = -|A| \ \mid \ \begin{bmatrix} d & \cdot b \\ \cdot c & a \end{bmatrix}$

application example

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

then, calculate B such that $B \equiv A^{-}$

this can be used to solve a system such as:

$$Ax = \begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$BAx = B\begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

$$Ix = x = B\begin{bmatrix} \cdot 1 \\ 1 \end{bmatrix}$$

Triangular Matrix

a Matrix is *triangular* if every entry below its diagonal **or** above its diagonal is 0

> **note**: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

Diagonal Matrix

a Matrix is *diagonal* if every entry below its diagonal **and** above its diagonal is 0

> **note**: the *diagonal* of a square Matrix goes from its top left element to its bottom right element (triplicate)

properties

let D be a diagonal Matrix

$[D]x$ can be calculated by multiplying every entry of D by x

Diagonalizable Matrix

see Eigenvector

definition

an n by n Matrix A is said to be *diagonalizable over the reals* if there exists a Basis of $\mathbb{R}^n$ consisting entirely of Eigenvectors of A

a Matrix is *diagonalizable* if and only if the geometic Multiplicity of an Eigenvalue is equal to the algebraic Multiplicity of said Eigenvalue, for every Eigenvalue of the Matrix (see Eigenvector And Eigenvalue)

> **note**: a Matrix may also be diagonalizable over other Number Fields such as the Set of Complex numbers $\mathbb{C}$

> **note**: some Matrixes do not have "enough" real Eigenvalues or "enough" Eigenvectors to be diagonalizable

examples and counterexamples

#example

the Matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is diagonalizable over the reals as $\{(1,1),(1,\cdot 1)\}$ is a Basis of $\mathbb{R}^2$ consisting entirely of Eigenvectors of A

the Matrix $A = \begin{bmatrix} 1 & 1 \\ \cdot 1 & 1 \end{bmatrix}$ is not diagonalizable over the reals as it does not have any real Eigenvalues

the Matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable over the reals as it only has one Eigenvalue, and therefore only one set of Linearly Dependent Eigenvectors (see Eigenvector And Eigenvalue)

the Matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable over the reals as, even though A has a single Eigenvalue $\lambda = 1$, its Eigenspace Spans $\mathbb{R}^2$. this is the case for both $A = I \wedge \lambda = 1$ and $A = O \wedge \lambda = 0$

> **proof**: let $A = I \wedge \lambda = 1 \wedge E_1 = x$. we then have $O = A \cdot \lambda I \mid x = I \cdot 1I \mid E_1 = O \mid E_1$. therefore, $E_1 \equiv \mathbb{R}^2$. see Eigenvector And Eigenvalue

let $\mathbb{M}^{n,n}A \wedge \mathbb{N}n$ and suppose A has n distinct Eigenvalues. deduce that A is diagonalizable over the reals

> **proof**: A has at most n Eigenvalues $\to$ the algebraic Multiplicity of every Eigenvalue of A is 1 as they are all distinct and must be greater than 1 $\to$ the geometric Multiplicity of every Eigenvalue of A is 1 as it must be greater than 1 and less than its algebraic Multiplicity $\to$ all algebraic Multiplicityes and geometric Multiplicityes are equal $\to$ A is diagonalizable. see Eigenvector And Eigenvalue

Eigenvector And Eigenvalues

theorems

see Linear System

> **theorem**: let $\mathbb{M}^{m,n} A$ (see Matrix). the following Logic Statements are equivalent:
>
> 1. every variable is a leading variable
> 2. there is a leading variable in every column of the RREF of A
> 3. the system $Ax = O$ has a unique solution
> 4. the columns of A are Linearly Independent
> 5. $Ker\ A = \{0\}$
> 6. $\dim Ker\ A = 0$
> 7. $rank\ A = n$

see Linear System Theorem Proof

> **theorem**: let $\mathbb{M}^{n,n} A$ (see Matrix). the following Logic Statements are equivalent:
>
> **note**: all Logic Statements below are valid for both A and $A^{\intercal}$, see transpose Matrix
>
> 1. $rank\ A = n$
> 2. every linear system of the form $Ax = b$ has a unique solution
> 3. the RREF of A is the identity Matrix
> 4. $Ker\ A = \{0\}$
> 5. $Col\ A = \mathbb{R}^n$
> 6. $Row\ A = \mathbb{R}^n$
> 7. the columns of A are Linearly Independent
> 8. the rows of A are Linearly Independent
> 9. the columns of A form a Basis for $\mathbb{R}^n$
> 10. the rows of A form a Basis for $\mathbb{R}^n$
> 11. A is Invertible
> 12. $\det A \neq 0$