Synchronization of Complex Dynamical Networks with Randomly Coupling via Nonfragile Control

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***Abstract*—This paper investigates the synchronization problem of complex dynamical networks (CDNs) with randomly occurring coupling and time-varying delay. A suitable Lyapunov-Krasovskii functional is formulated. Utilizing the well known extended Jensen's integral inequality and the properties of Bernoulli random variables, uncertainties in exchange of information within the networks is modeled. Sufficient delay dependent synchronization conditions are derived and nonfragile controllers proposed in the form of linear matrix inequalities (LMIs). Finally, an example is provided to indicate the effectiveness of our theoretical results.**

***Index Terms*—Complex dynamical networks (CDNs), extended Jensen's integral inequality, nonfragile controllers, synchronization.**

I. Introduction

In the last few years, the spearheading works of Watts and Strogatz in [1], which investigated complex networks have gained much attention as a results of its theoretical relevance and potential applications in most significant real world networks such as transportation networks, communication networks, social networks, biological networks, electric power grids and others [2]–[4]. A significant number of these networks display complexities in their entire topological and dynamic properties. One of the noteworthy and intriguing phenomena in complex dynamical systems is synchronization, which means the network nodes converge to a common state of behavior. The implementation of synchronization of chaotic system for communications are mainly dependent on robustness of the synchronization prevailing between the transmitters and receivers [5] . A great deal of useful and effective methodologies for CDNs synchronization have been proposed [6]–[10]. Control inputs are to ensure the robustness and stability with regards to network parameters hence, the numerous designed control strategies which require precise controllers. Accordingly, the designed control strategy requires precision of system parameters, which may not be generally conceivable in practical sense because of limited processing of information speed and round - off errors in numerical computation by computers. Based on these, it is important to ensure the outlined controllers do have the capacity to endure some fluctuations in their parameters for synchronization problems of CDNs. Controller fragility must be considered while implementing the designed controllers in practical applications, which inspired the nonfragile control problems been researched in [11], [12].

Furthermore, time delay is inevitable in numerous CDNs as a result of finite speed of information transmission which can eventually degrades the network's performance or destroys the perceived synchronization and stability [7], [13], [14].

Motivated by the above exposition, a nonfragile control scheme is designed for a delay dependent CDNs with randomly occurring coupling for synchronization problem in this paper. Also, an appropriate Lyapunov Krasovskii functional is constructed and applying the extended Jensen's integral inequality guaranteed less conservativeness from the derived delay dependent conditions. Finally, an example is given to illustrate the effectiveness of our proposed method.

Notations: The notations considered in this paper are standard. I and 0 represent identity and zero matrices with appropriate dimensions, $\mathbb{R}^n$ implies the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ represents the set of all $m \times n$ real matrices. , respectively. $P > 0$ means P is a real symmetric and positive definite matrix. The superscript "T" indicates matrix transposition. $*$ represent the symmetric elements beneath the main diagonal of a symmetric matrix and diag$\{....\}$ representing a block-diagonal matrix. When a matrix dimension is not stated, it is assumed to have a compatible dimension. A $\otimes$ B indicate the Kronecker product of matrices A and B.

II. Problem formulation

Consider the CDNs of N identical randomly coupled nodes with time varying delay as

$$\begin{aligned}\dot{x}_i(t) =&Ax_i(t) + B_1f(x_i(t)) + B_2f(x_i(t-\gamma(t))) + (1\\ &- \delta_1(t))\sum_{j=1}^{N}w_{ij}\Gamma x_j(t) + \delta_1(t)\sum_{j=1}^{N}w_{ij}\Gamma x_j(t-\gamma(t))\\ &+ \breve{u}_i(t), \qquad i = 1, 2, \ldots, N\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{in}(t))^T \in \mathbb{R}^n$denotes the state vector of the i^{th} node, $\check{u}_i(t) \in \mathbb{R}^n$. A, B_1, B_2 are constant matrices, $f : \mathbb{R}^n \to \mathbb{R}^n$ given as a nonlinear smooth function, $\gamma(t)$ represents time-varying delay satisfying the following: $0 \leq \gamma_1 \leq \gamma(t) \leq \gamma_2, \dot{\gamma}(t) \leq \mu$. $\Gamma \in \mathbb{R}^{n\times n}$ is an inner coupling matrix and $W = (w_{ij})_{N\times N}$ the outer coupling matrix configuration. When there exit a connection from node i to node j $(i \neq j)$, the coupling $w_{ij} \neq 0$; otherwise $w_{ij} = 0$. Furthermore, the diagonal matrix elements w_{ii} are denoted as $w_{ii} = -\sum_{j=1,j\neq i}^{N} w_{ij}$, $\delta_t \in \mathbb{R}$ shows a stochastic variable in the form of a Bernoulli distribution sequence given by

$$\delta_1(t) = \begin{cases} 1 & \text{information delayed exchange occur,} \\ 0 & \text{no information delayed exchange.} \end{cases}$$

Below indicates the probability occurrence of stochastic variable $\delta_1(t)$: $Pr\{\delta_1(t) = 1\} = \delta$, $Pr\{\delta_1(t) = 0\} = 1 - \delta$, with $\delta \in [0, 1]$ as a known constant.

Assumption 1. *[4] Having a nonlinear function $f(\bullet) : \mathbb{R}^n \to \mathbb{R}^n$to be considered and satisfies $f(0) = 0$, the following sector bounded condition holds:*

$$[f(x)-f(y)-\ddot{U}_1(x-y)]^T[f(x)-f(y)-\ddot{U}_2(x-y)] \leq 0 \quad (2)$$

where $\ddot{U}_1$ and $\ddot{U}_2$ are known real constant matrices with appropriate dimensions.

From the well known kronecker properties,(1) becomes

$$\begin{aligned}\dot{x}(t) = &(I_N \otimes A)x(t) + (I_N \otimes B_1)g(x(t)) + (I_N \otimes B_2) \\ &\times g(x(t-\gamma(t)) + (1-\delta_1(t))(W\otimes\Gamma)x(t) + \delta_1(t) \\ &\times (W\otimes\Gamma)(t-\gamma(t)) + u(t)\end{aligned} \quad (3)$$

where,

$$\begin{aligned}x(t) &= [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T \\ g(x(t)) &= [f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t))]^T \\ u(t) &= [\check{u}_1^T(t), \check{u}_2^T(t), \ldots, \check{u}_N^T(t)]^T.\end{aligned}$$

Lemma II.1. *[15] For any given symmetric positive constant matrix $H = H^T > 0$ and a vector function $\alpha(\cdot) : [0 \quad \gamma] \to \mathbb{R}^n$ with a scalar $\gamma > 0$, such that the concerned integrations are properly defined, hence*
$\gamma \int_0^\gamma \alpha^T(s)H\alpha(s)ds \geq \left[\int_0^\gamma \alpha(s)ds\right]^T H \left[\int_0^\gamma \alpha(s)ds\right]$

Lemma II.2. *[16] For constant matrices $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{pmatrix}, L \in \mathbb{R}^{2n\times 2n}, \tilde{U} = \tilde{U}^T > 0, \tilde{U} \in \mathbb{R}^{n\times n}$, the continuous function satisfies $\hat{d}_1 \leq \hat{d}(t) \leq \hat{d}_2$, and a differentiable continuous function such that $x : [-\hat{d}_2, 0] \to \mathbb{R}^n$, the integration is well defined, which validates the following inequality:*

$\int_{t-\hat{d}_2}^{t-\hat{d}_1} \dot{x}^T(s)\tilde{U}\dot{x}(s)ds \geq \frac{1}{\hat{d}_{12}} v^T(t)\Psi v(t)$

where

$$\begin{aligned}
&v(t) = [\ v_1^T(t) \quad v_2^T(t) \quad v_3^T(t) \quad v_4^T(t)\ ]^T \\
&\hat{d}_{12} = \hat{d}_2 - \hat{d}_1,\ v_1(t) = x(t-\hat{d}_1) - x(t-\hat{d}(t)) \\
&v_2(t) = x(t-\hat{d}_1) + x(t-\hat{d}(t)) - \frac{2}{\hat{d}(t)-\hat{d}_1}\int_{t-\hat{d}(t)}^{t-\hat{d}_1} x(s)ds \\
&v_3(t) = x(t-\hat{d}(t)) - x(t-\hat{d}_2) \\
&v_4(t) = x(t-\hat{d}(t)) + x((t-\hat{d}_2) - \frac{2}{\hat{d}_2-\hat{d}(t)}\int_{t-\hat{d}_2}^{t-\hat{d}(t)} x(s)ds \\
&\Psi = \begin{bmatrix} \tilde{U} & 0 & L_{11} & L_{12} \\ * & 3\tilde{U} & L_{21} & L_{22} \\ * & * & \tilde{U} & 0 \\ * & * & * & 3\tilde{U}\end{bmatrix} \geq 0.
\end{aligned} \quad (4)$$

Remark 1. The use of Lemma (II.2), results in less conservativeness because it combines Jensen inequality and the reciprocal convex combination techniques. For the proof, we refer reader to [16].

Lemma II.3. *[17]: For a symmetric matrix $\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ * & \Lambda_{22}\end{pmatrix} < 0$, the following inequalities are valid:*

1) $\Lambda_{11} < 0, \Lambda_{22} - \Lambda_{12}^T\Lambda_{11}^{-1}\Lambda_{12} < 0$
2) $\Lambda_{22} < 0, \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{12}^T < 0$

Lemma II.4. *[4] Having some real matrices M,N,P of appropriate dimensions with the function $F(t)$ satisfying $F^T(t)F(t) < I$,then , $M + NF(t)P + P^TF^T(t)N^T < 0$ if and only if there is a scalar $\epsilon > 0$ which implies that,*

$$\begin{bmatrix} M & N & \epsilon P^T \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I\end{bmatrix} < 0.$$

Definition II.1. *The CDNs* (1) *is synchronized if the following condition holds: $\lim_{t\to\infty}[x_i(t) - \hat{s}(t)] = 0$, where $\hat{s}(t)$ is an isolated node and its dynamic system is $\dot{\hat{s}}(t) = A\hat{s}(t) + B_1f(\hat{s}((t))) + B_2f(\hat{s}(t-\gamma(t)))$. Let $\dot{e}_i(t) = x_i(t) - \hat{s}(t); (i = 1, 2, \ldots, N)$, then we have*

$$\begin{aligned}\dot{e}_i(t) = &Ae_i(t) + B_1g(e_i(t)) + B_2g(e_i(t-\gamma(t))) + (1-\delta_1(t)) \\ &\times \sum_{j=1}^{N} w_{ij}\Gamma e_j(t) + \delta_1(t)\sum_{j=1}^{N} w_{ij}\Gamma e_j(t-\gamma(t)) + u_i(t)\end{aligned} \quad (5)$$

Note: $g(e_i(t)) \equiv f(x_i(t)) - f(\hat{s}(t))$ and $g(e_i(t-\gamma(t))) \equiv f(x_i(t-\gamma(t))) - f(\hat{s}(t-\gamma(t)))$.

Consider the designed nonfragile feedback control to ensure synchronization in the following form:

$$u_i(t) = (k_i + \sigma(t)\triangle k_i(t))e(t) + k_{\tau i}e(t-\gamma(t)),\ i = 1, 2, \ldots, N. \quad (6)$$

$k_i, k_{\tau i} \in \mathbb{R}^{n\times n}$ are the control feedback gain matrices to be estimated with $\triangle k_i$ been the controller gain fluctuation. The

$\triangle k_i(t)$ term is considered as follows:

$$\triangle k_i(t) \equiv H_i \tilde{\Upsilon}_i(t) W_i \tag{7}$$

where $\tilde{\Upsilon}_i(t) \in \mathbb{R}^{k\times l}, \quad i = 1,2,\dots,N$ as an unknown time-varying matrix satisfying the following: $\tilde{\Upsilon}_i(t)^T \tilde{\Upsilon}_i(t) \leq I, \quad H_i$ and W_i are known constant matrices. $\sigma(t)$ is a stochastic variable describing the randomly occurring controller gain fluctuations. This satisfies the Bernoulli distribution and is defined as:

$$\sigma(t) = \begin{cases} 1 & \text{Occurrence of control gain fluctuations,} \\ 0 & \text{No control gain fluctuations} \end{cases} \tag{8}$$

with stochastic probability variable of $\sigma(t)$ being given as: $Pr\{\sigma(t) = 1\} = \sigma, \quad Pr\{\sigma(t) = 0\} = 1 - \sigma; \quad \sigma \in [0,1]$
The control input is substituted into the error dynamics (5) which results as follows:

$$\begin{aligned}\dot{e}(t) = {} & (\bar{A} + \bar{k} + \sigma \bar{H}\bar{\Upsilon}\bar{W} + (W\otimes\Gamma) + \bar{k}_\tau)e(t) + (\sigma(t) - \sigma) \\ & \times \bar{H}\bar{\Upsilon}\bar{W}e(t) + \bar{B}_1 G(e(t)) + \bar{B}_2 G(e(t-\gamma(t))) \\ & - (\bar{k}_\tau + \delta(W\otimes\Gamma)) \int_{t-\gamma(t)}^{t} \dot{e}(s)\,\mathrm{d}s - (\delta_1(t) - \delta) \\ & \times (W\otimes\Gamma) \int_{t-\gamma(t)}^{t} \dot{e}(s)\,\mathrm{d}s\end{aligned} \tag{9}$$

where, $\bar{A} \equiv I_N \otimes A,\ \bar{B}_1 \equiv I_N \otimes B_1,\ \bar{B}_2 \equiv I_N \otimes B_2$
$\bar{k} \equiv diag(k_1, k_2, \dots, k_N),\ \bar{k}_\tau \equiv diag(k_{\tau 1}, k_{\tau 2}, \dots, k_{\tau N})$,
$\bar{H} \equiv diag(H_1, H_2, \dots, H_N),\ \bar{\Upsilon} \equiv diag(\tilde{\Upsilon}_1, \tilde{\Upsilon}_2, \dots, \tilde{\Upsilon}_N)$
$\bar{W} \equiv diag(W_1, W_2, \dots, W_N)$, $e(t) \equiv [e_1^T(t), e_2^T(t), \dots, e_N^T(t)]^T$, $G(e(t)) \equiv [g^T(e_1(t)), g^T(e_2(t)), \dots, g^T(e_N(t))]^T$
$G(e(t-\gamma(t))) \equiv [g^T(e_1(t-\gamma(t))), g^T(e_2(t-\gamma(t))), \dots, g^T(e_N(t-\gamma(t)))]^T$

III. Main Results

In this portion, we will establish sufficient conditions which ensures synchronization of the CDNs.

Theorem III.1. *Let Assumption 1 hold for some given scalars $\delta, \sigma \in [0,1], \gamma_1, \gamma_2$, and $\mu < 1$, the CDNs synchronization is achieved with given controller gains $k_i, k_{\tau i}, i = 1,2,\dots,N$, when there exist positive definite matrices $\bar{P}, \hat{S}, W_1, W_2, Z \in \mathbb{R}^{nN\times nN}$, any matrix $\begin{bmatrix} L_{11} & L_{12} \\ * & L_{22}\end{bmatrix}$ such that*

$$\Psi = \begin{bmatrix} \hat{S} & 0 & L_{11} & L_{12} \\ * & 3\hat{S} & L_{21} & L_{22} \\ * & * & \hat{S} & 0 \\ * & * & * & 3\hat{S}\end{bmatrix} \geq 0,$$

Then

$$\begin{bmatrix} \Phi_1 & N & \varepsilon W_*^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I\end{bmatrix} < 0, \tag{10}$$

holds where

$$\Phi_1 = \begin{bmatrix} \Omega_1 & M_{1*}^T\bar{P} & \sqrt{\sigma(1-\sigma)}M_2^T\bar{P} & 0 \\ * & \Theta - 2\bar{P} & 0 & 0 \\ * & * & \Theta - 2\bar{P} & 0 \\ * & * & * & \Theta - 2\bar{P}\end{bmatrix}$$

$$N^T = \begin{bmatrix} \sigma(\bar{P}\bar{H})^T & \underbrace{0\cdots 0}_{9\,elements} & \sigma(\bar{P}\bar{H})^T & 0 & \hat{\varpi}\end{bmatrix}$$

$$W_* = \begin{bmatrix} W & \underbrace{0\cdots 0}_{12\,elements}\end{bmatrix}$$

$$M_1 = \begin{bmatrix} \bar{Z} & \bar{B}_1 & \bar{B}_2 & -(k_\tau + \delta(W\otimes\Gamma)) & \underbrace{0\cdots 0}_{6\,elements}\end{bmatrix}$$

$\bar{Z} = \bar{A} + \bar{k} + \sigma\bar{H}\bar{\Upsilon}\bar{W} + W\otimes\Gamma + \bar{k}_\tau$
$M_{1*} = M_1 - \begin{bmatrix} \sigma\bar{H}\bar{\Upsilon}\bar{W} & \underbrace{0\cdots 0}_{9\,elements}\end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 0 & 0 & W\otimes\Gamma & \underbrace{0\cdots 0}_{6\,elements}\end{bmatrix}$, $\Omega_1 \equiv \begin{bmatrix}\Omega_1^1 & \Omega_1^2 \\ * & \Omega_1^3\end{bmatrix} - \frac{1}{\gamma_{12}}\Pi^T\Psi\Pi$, $\Omega_1^1 \equiv \begin{bmatrix} ß_1 + ß_2 & \bar{P}\bar{B}_1 + F_2 \\ * & -I\end{bmatrix}$,
$ß_1 = \bar{P}\bar{A} + \bar{A}^T\bar{P} + \bar{P}\bar{k} + \bar{k}^T\bar{P} + \bar{P}(W\otimes\Gamma) + (W\otimes\Gamma)^T\bar{P}$
$ß_2 = \bar{P}\bar{k}_\tau + \bar{k}_\tau^T\bar{P} + W_1 + \gamma_1 W_2 - F_1$

$$\Omega_1^2 \equiv \begin{bmatrix} \bar{P}\bar{B}_2 & -\delta\bar{P}(W\otimes\Gamma) - \bar{P}\bar{k}_\tau & \underbrace{0\cdots 0}_{6\,elements} \\ 0 & \underbrace{0\cdots 0}_{6\,elements} & \end{bmatrix},$$

$$\Omega_1^3 \equiv \begin{bmatrix} -I & 0 & 0 & F_2^T & 0 & 0 & 0 & 0 \\ 0 & -\frac{1-\mu}{\gamma_2}Z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\Delta}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\gamma_1}W_2\end{bmatrix}$$

Where:
$\Theta = \gamma_{12}\hat{S} + \gamma_2 Z,\ \tilde{\Delta}_1 = -(1-\mu)W_1 - F_1$,
$\hat{\varpi} = \sqrt{\sigma(1-\sigma)}(\bar{P}\tilde{H})^T$, $F_1 = I_N \otimes \frac{\ddot{U}_1^T\ddot{U}_2 + \ddot{U}_2^T\ddot{U}_1}{2}$, $F_2 = I_N \otimes \frac{\ddot{U}_1^T + \ddot{U}_2^T}{2}$

Proof. Choose the following as the Lyapunov-Krasovskii function

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{11}$$

Where

$$\begin{aligned} V_1(t) &= e^T(t)\bar{P}e(t) \\ V_2(t) &= \int_{t-\gamma(t)}^{t} e^T(s)W_1 e(s)\mathrm{d}s + \int_{-\gamma_1}^{0}\int_{t+\phi}^{t} e^T(s)W_2 e(s) \\ &\quad \times \mathrm{d}s\mathrm{d}\phi \\ V_3(t) &= \int_{-\gamma_2}^{-\gamma_1}\int_{t+\delta}^{t} \dot{e}^T(s)\hat{S}\dot{e}(s)\,\mathrm{d}s\mathrm{d}\delta + \int_{-\gamma(t)}^{0}\int_{t+\delta}^{t} \dot{e}^T(s) \\ &\quad \times Ze(s)\mathrm{d}s\mathrm{d}\delta \end{aligned}$$

Taking the infinitesimal operator Ł of $V(t)$ as follows:
$ŁV(t) = \lim_{\triangle\to 0^+} \frac{1}{\triangle}\{E\{V(t+\triangle)\} - V(t)\}$
It should be noted that:
$E\{\delta_1(t) - \delta\} = 0, \quad E\{(\delta_1(t) - \delta)^2\} = \delta(1-\delta)$,
$E\{\sigma(t) - \sigma\} = 0, \quad E\{(\sigma(t) - \sigma)^2\} = \sigma(1-\sigma)$.

Calculating the derivative of $V(t)$ along the trajectory of error system (9)

$$\begin{aligned}\mathbb{E}\{LV_1(t)\} = \mathbb{E}\{2e^T(t)\bar{P}[(\bar{A}+\bar{k}+\sigma\bar{H}\bar{\Upsilon}\bar{W}+(W\otimes\Gamma)\\ +\bar{k_\tau})e(t)+\bar{B}_1G(e(t))+\bar{B}_2G(e(t-\gamma(t)))\\ -\delta(W\otimes\Gamma)\int_{t-\gamma(t)}^{t}\dot{e}(s)\,\mathrm{d}s-\bar{k_\tau}\int_{t-\gamma(t)}^{t}\dot{e}(s)\,\mathrm{d}s]\}\end{aligned} \tag{12}$$

$$\begin{aligned}\mathbb{E}\{LV_2(t)\} \le \mathbb{E}\{e^T(t)(W_1+\gamma_1W_2)e(t)-(1-\mu)\\ \times e^T(t-\gamma(t))W_1 \quad \times e(t-\gamma(t))-\frac{1}{\gamma_1}\\ \times\left(\int_{t-\gamma_1}^{t}e(\theta)\,\mathrm{d}\theta\right)^T W_2\left(\int_{t-\gamma_1}^{t}e(\theta)\,\mathrm{d}\theta\right)\}\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbb{E}\{LV_3(t)\} \le \mathbb{E}\{(\gamma_2-\gamma_1)\dot{e}^T(t)\hat{S}\dot{e}(t)+\gamma_2\dot{e}^T(t)Z\dot{e}(t)\\ -\int_{t-\gamma_2}^{t-\gamma_1}\dot{e}^T(\delta)\hat{S}\dot{e}(\delta)\,\mathrm{d}\delta-(1-\mu)\int_{t-\gamma(t)}^{t}\\ \times\dot{e}^T(s)Z\dot{e}(s)\,\mathrm{d}s\}\end{aligned} \tag{14}$$

From lemma (II.2), the integral part of (14) satisfies the inequality stated below:

$$-\int_{t-\gamma_2}^{t-\gamma_1}\dot{e}^T(\delta)\hat{S}\dot{e}(\delta)\,\mathrm{d}\delta \le -\zeta^T(t)\frac{1}{\gamma_{12}}\Pi^T\Psi\Pi\zeta(t) \tag{15}$$

Where,

$$\gamma_{12}=\gamma_2-\gamma_1$$

$$\begin{aligned}v(t) &= \begin{pmatrix} \underbrace{0\cdots0}_{4\,elements} & I & -I & 0 & 0 & 0 & 0\\ \underbrace{0\cdots0}_{4\,elements} & I & I & 0 & -2I & 0 & 0\\ \underbrace{0\cdots0}_{4\,elements} & 0 & I & -I & 0 & 0 & 0\\ \underbrace{0\cdots0}_{4\,elements} & 0 & I & I & 0 & -2I & 0\end{pmatrix}\zeta(t)\\ &= \Pi\zeta(t)\end{aligned} \tag{16}$$

Let,

$$\begin{aligned}\zeta(t) = [e^T(t), G^T(e(t)), G^T(e(t-\gamma(t))), \int_{t-\gamma(t)}^{t}\dot{e}^T(s)\,\mathrm{d}s,\\ e^T(t-\gamma_1), e^T(t-\gamma(t)), e^T(t-\gamma_2), \frac{1}{\gamma(t)-\gamma_1}\int_{t-\gamma(t)}^{t-\gamma_1}\times\\ e^T(\omega)\,\mathrm{d}\omega, \frac{1}{\gamma_2-\gamma(t)}\int_{t-\gamma_2}^{t-\gamma(t)}e^T(\omega)\,\mathrm{d}\omega, \int_{t-\gamma_1}^{t}e^T(\theta)\,\mathrm{d}\theta]^T\end{aligned}$$

$$\begin{aligned}M_1 = [\bar{A}+\bar{k}+\sigma\bar{H}\bar{\Upsilon}\bar{W}+W\otimes\Gamma+\bar{k}_\tau, \bar{B}_1, \bar{B}_2, -(k_\tau\\ +\delta(W\otimes\Gamma)), \underbrace{0\cdots0}_{6\,elements}\ ]\end{aligned}$$

$$\begin{aligned}\dot{e}(t) = M_1\zeta(t)+(\sigma(t)-\sigma)\bar{H}\bar{\Upsilon}\bar{W}e(t)+(\delta-\delta_1(t))(W\otimes\Gamma)\\ \times\int_{t-\gamma(t)}^{t}\dot{e}(s)\,\mathrm{d}s\end{aligned}$$

From (14), we represent $M_2 = \begin{bmatrix} 0 & 0 & 0 & (W\otimes\Gamma) & \underbrace{0\cdots0}_{6\,elements}\end{bmatrix}$ $M_3 = \begin{bmatrix}\bar{H}\bar{\Upsilon}\bar{W} & \underbrace{0\cdots0}_{9\,elements}\end{bmatrix}$

Hence,

$$\begin{aligned}\mathbb{E}\{LV_3(t)\} \le \mathbb{E}\{\zeta^T(t)(M_1^T(\gamma_{12}\hat{S}+\gamma_2Z)M_1+\delta(1-\delta)M_2\\ \times(\gamma_{12}\hat{S}+\gamma_2Z)M_2^T+\sigma(1-\sigma)M_3^T(\gamma_{12}\hat{S}\\ +\gamma_2Z)M_3) \quad -\frac{1}{\gamma_{12}}\Pi^T\Psi\Pi)\zeta(t)\}\\ -\frac{1-\mu}{\gamma_2}(\int_{t-\gamma(t)}^{t}\dot{e}(s)\mathrm{d}s)^TZ(\int_{t-\gamma(t)}^{t}\dot{e}(s)\,\mathrm{d}s)\}\end{aligned} \tag{17}$$

By some computations, it can be obtained by Assumption (1) that:

$$-\begin{bmatrix}e(t)\\ G(e(t))\end{bmatrix}^T\begin{bmatrix}F_1 & -F_2\\ * & I\end{bmatrix}\begin{bmatrix}e(t)\\ G(e(t))\end{bmatrix}\ge 0, \tag{18}$$

$$F_1 = I_N\otimes\frac{\ddot{U}_1^T\ddot{U}_2+\ddot{U}_2^T\ddot{U}_1}{2},\ F_2 = I_N\otimes\frac{\ddot{U}_1^T+\ddot{U}_2^T}{2}$$

$$-\begin{bmatrix}e(t-\gamma(t))\\ G(e(t-\gamma(t)))\end{bmatrix}^T\begin{bmatrix}F_1 & -F_2\\ * & I\end{bmatrix}\begin{bmatrix}e(t-\gamma(t))\\ G(e(t-\gamma(t)))\end{bmatrix}\ge 0, \tag{19}$$

combining equations (12)-(13), (17)-(19), hence the following:

$$\mathbb{E}\{LV(t)\} \le \mathbb{E}\{\zeta^T(t)\Phi\zeta(t)\} \tag{20}$$

We can conclude that, $\mathbb{E}\{LV(t)\} < 0$, if $\mathbb{E}\{\zeta^T(t)\Phi\zeta(t)\} < 0$, holds when $\Phi < 0$:
$\Phi = \Omega_1+M_1^T\Theta M_1+\delta(1-\delta)M_2^T\Theta M_2+\sigma(1-\sigma)M_3^T\Theta M_3+\Omega_2+\Omega_2^T$
From theorem (III.1), we represent

$$\Omega_2 = \begin{bmatrix}\sigma P\bar{H}\bar{\Upsilon}\bar{W}\\ 0_{9n\times n}\end{bmatrix}\begin{bmatrix} I & \underbrace{0\cdots0}_{9\,elements}\end{bmatrix}, \Theta = \gamma_{12}\hat{S}+\gamma_2Z$$

From the Schur's complements formula,

$$\tilde{\Phi} = \begin{bmatrix}\Omega_z & M_1^T\bar{P} & \Omega_{zz} & \sqrt{\sigma(1-\sigma)}M_3^T\bar{P}\\ * & -\Theta^{-1} & 0 & 0\\ * & * & -\Theta^{-1} & 0\\ * & * & * & -\Theta^{-1}\end{bmatrix} < 0 \tag{21}$$

where, $\Omega_z = \Omega_1+\Omega_2+\Omega_2^T$, $\Omega_{zz} = \sqrt{\delta(1-\delta)}M_2^T\bar{P}$. By using the method of congruence transformation with $diag\{\underbrace{I, I, \ldots, I}_{10}, \bar{P}, \bar{P}, \bar{P}\}$ on (21) and utilizing the inequality $\bar{P}\Theta^{-1}\bar{P} \ge 2\bar{P}-\Theta$, then it can be concluded that $\tilde{\Phi} < 0$ if the following inequality is true:

By some computations and with Lemma (II.4), let $\Phi_1 = \begin{bmatrix} \Omega_1 & M_{1*}^T\bar{P} & \Omega_{zz} & 0 \\ * & \Theta-2\bar{P} & 0 & 0 \\ * & * & \Theta-2\bar{P} & 0 \\ * & * & * & \Theta-2\bar{P} \end{bmatrix}$,
$M_{1*} = M_1 - \begin{bmatrix} \sigma\bar{H}\Upsilon\bar{W} & \underbrace{0\cdots0}_{9\,elements} \end{bmatrix}$
$N^T = \begin{bmatrix} \sigma(\bar{P}\bar{H})^T & 0_{n,9n} & \sigma(\bar{P}\bar{H})^T & 0 & \sqrt{\sigma(1-\sigma)}(\bar{P}\bar{H})^T \end{bmatrix}$
$W_* = \begin{bmatrix} W & 0_{n,12n} \end{bmatrix}$. Then we have

$$\begin{bmatrix} \Phi_1 & N & \varepsilon W_*^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0 \tag{22}$$

which concludes based on Lemma (II.4) that $\tilde{\Phi} < 0$ holds then, $\mathbb{E}\{LV(t)\} < 0$. Thus the synchronization error is asymptotically stable, and hence synchronization is achieved. This concludes the proof. □

The nonfragile controller is design based on the derived error synchronization results, hence the following theorem.

Theorem III.2. *For given scalars $\delta, \sigma \in [0,1], \gamma_1, \gamma_2$, and $\mu < 1$, the CDNs synchronization can be realized, with the existence of symmetric positive definite matrices $P = diag\{\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N\}, \hat{S} = diag\{\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_N\}$ and matrices $Y_1 = diag\{Y_1^1, Y_2^1, \ldots, Y_N^1\} \in \mathbb{R}^{nN\times nN}$ and $Y_2 = diag\{Y_1^2, Y_2^2, \ldots, Y_N^2\} \in \mathbb{R}^{nN\times nN}$ with a scalar $\varepsilon > 0$, hence $\bar{\Omega} < 0$, where*

$$\bar{\Omega} \equiv \begin{bmatrix} \bar{\Omega}_1 & \bar{\Omega}_2 \\ * & \bar{\Omega}_3 \end{bmatrix} < 0, \tag{23}$$

$$\bar{\Omega}_1 \equiv \begin{bmatrix} \bar{\Omega}_1^1 & \bar{\Omega}_1^2 \\ * & \bar{\Omega}_1^3 \end{bmatrix},$$

$$\bar{\Omega}_1^1 \equiv \begin{bmatrix} \bar{\Omega}*_1^1 & P\bar{B}_1+F_2 & PB_2 & -\delta P(W\otimes\Gamma)-Y_2 \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\frac{1-\mu}{\gamma_2}Z \end{bmatrix},$$

$$\bar{\Omega}_1^2 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Omega}*_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_1^TP & 0 \\ 0 & F_2^T & 0 & 0 & 0 & 0 & B_2^TP & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta(W\otimes\Gamma)^T & \wp \end{bmatrix},$$

$$\bar{\Omega}_1^3 \equiv \begin{bmatrix} \bar{\Omega}_1^{31} & \bar{\Omega}_1^{32} \end{bmatrix},\ \bar{\Omega}_2 \equiv \begin{bmatrix} 0 & \sigma P\bar{H} & \varepsilon\bar{W}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{k}_\tau^T & 0 & 0 \\ 0_{6n\times n} & 0_{6n\times n} & 0_{6n\times n} \\ 0 & \sigma P\bar{H} & 0 \\ 0 & \tilde{\varpi} & 0 \end{bmatrix},$$

$$\bar{\Omega}_3 \equiv \begin{bmatrix} -\Theta^{-1} & 0 & 0 \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix}$$

$$\bar{\Omega}_1^{32} \equiv \begin{bmatrix} 0_{5n\times n} & 0_{5n\times n} & 0_{5n\times n} \\ -\frac{1}{\gamma_1}W_2 & 0 & 0 \\ * & \varpi_3 & 0 \\ * & * & -\Theta^{-1} \end{bmatrix},$$

$$\bar{\Omega}_1^{31} \equiv \begin{bmatrix} -4\hat{S} & \Delta & \Delta_1 & 6\hat{S} & \varpi_1 \\ * & \Delta_2 & \tilde{\Delta}_1 & \tilde{\Delta}_2 & \tilde{\Delta} \\ * & * & -4\hat{S} & \varpi_2 & 6\hat{S} \\ * & * & * & -12\hat{S} & -4L_{22} \\ * & * & * & * & -12\hat{S} \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

Where:
$\bar{\Omega}*_1^2 = \bar{A}^TP + Y_1^T + (W\otimes\Gamma)^TP + Y_2^T$
$\bar{\Omega}*_1^1 = P\bar{A} + \bar{A}^TP + Y_1 + Y_1^T + P(W\otimes\Gamma) + (W\otimes\Gamma)^TP + Y_2 + Y_2^T + W_1 + \gamma_1 W_2 - F_1$, $\tilde{\Delta} = 6\hat{S} - 2L_{12} + 2L_{22}$
$\wp = \sqrt{\sigma(1-\sigma)}(W\otimes\Gamma)^T$, $\tilde{\Delta}_1 = -2\hat{S} + L_{21} - L_{22} - L_{11} + L_{12}$
$\varpi_1 = 2L_{22} + 2L_{12}$, $\varpi_2 = 2L_{22}^T - L_{12}^T$, $\varpi_3 = \Theta - 2P$
$\tilde{\Delta}_2 = 2L_{12}^T + 6\hat{S} + 2L_{22}^T$, $\tilde{\varpi} = \sqrt{\sigma(1-\sigma)}\hat{S}\bar{H}$
$\Delta = -2\hat{S} - L_{11} - L_{12} - L_{21} - L_{22}$; $\Delta_1 = L_{11} - L_{12} + L_{21} - L_{22}$; $\Delta_2 = -(1-\mu)W_1 - F_1 - 8\hat{S} + L_{11} - L_{11}^T + L_{12} - L_{12}^T + L_{21}^T - L_{22}^T - L_{21} - L_{22}$;
Let $Y_1 = P\bar{k}$ and $Y_2 = P\bar{k}_\tau$, then the controlled gains become $\bar{k} = P^{-1}Y_1$ and $\bar{k}_\tau = P^{-1}Y_2$, the proof can be emulated from Theorem (III.1).

IV. Numerical Illustration

We consider an isolated standard Chua's circuit as an unforced node of CDNs been described by the following state equations as a special case of (1) when, A= $B_2 = 0, B_1 = I_3$ and δ= 0.5. The standard Chua's circuit is normalized and transformed to:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - m_1x_1 + \bar{\varphi}) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\tilde{b}x_2 \end{aligned} \tag{24}$$

Let a =10, $\bar{b}$ =14.87 and $\bar{\varphi}(x_1) = \tilde{\sigma}_1x_1 + 0.5(\tilde{\sigma}_2 - \tilde{\sigma}_1)\bar{\psi}(x_1)$ where $\tilde{\sigma}_1$= -0.68, $\tilde{\sigma}_2$= -1.27, and $\bar{\psi}(x_1)$ = $(|x_1+1| - |x_1-1|)$. Denote $x = [x_1, x_2, x_3]^T$, $\bar{\phi} = -0.5(\tilde{\sigma}_2 - \tilde{\sigma}_1)$,
$f(s) = \begin{bmatrix} -a-\bar{b}\tilde{\sigma}_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -\bar{b} & 0 \end{bmatrix} + \begin{bmatrix} \bar{\phi}\bar{\psi}(x_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
f(s) satisfies Assumption (1) with
$\ddot{U}_1 = \begin{bmatrix} 2.7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}, \ddot{U}_2 = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}$
The inner coupling $\Gamma$and the network topology Wmatrices are given as
$\Gamma_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, W = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$,
The selected time-varying delay and other variables are: $\gamma(t) = 0.4 + 0.01\sin(10t), \mu = 0.1, \gamma_1 = 0.01, \gamma_2 = 0.41, \sigma = 0$. Using MATLAB LMI toolbox for Theorem (III.1), the guaranteed feasible solution is obtained for (9).The controller gain matrices obtained are
$k_1 = \begin{bmatrix} -1.6882 & 0.4418 & 0.2129 \\ 0.5491 & -1.6881 & 0.4721 \\ 0.0420 & 0.27952 & -0.9744 \end{bmatrix}$

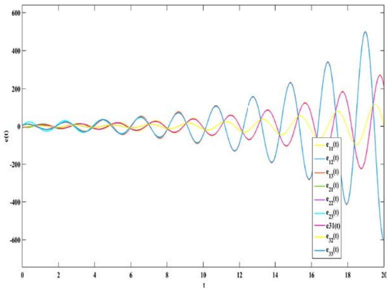

Fig. 1: State error trajectories without control inputs.

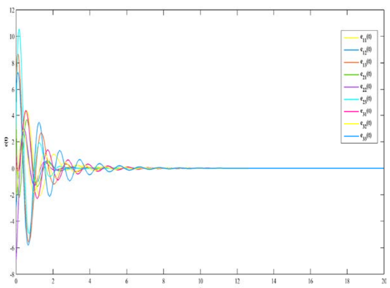

Fig. 2: The state error trajectories with control inputs.

$$k_2 = \begin{bmatrix} -2.2666 & 0.3964 & 0.2390 \\ 0.4522 & -1.1723 & 0.3876 \\ 0.0098 & 0.2049 & -1.4268 \end{bmatrix}$$
$$k_3 = \begin{bmatrix} -1.2863 & 0.3905 & 0.1771 \\ 0.5728 & -0.9870 & -0.1493 \\ 0.0788 & -0.0162 & -0.9256 \end{bmatrix}$$
$$k_{\tau 1} = \begin{bmatrix} -0.2932 & -0.0510 & -0.0273 \\ -0.0724 & 0.1297 & 0.9611 \\ -0.0144 & 0.4599 & -0.3422 \end{bmatrix}$$
$$k_{\tau 2} = \begin{bmatrix} -0.3277 & 0.0798 & -0.0081 \\ -0.0841 & -1.3480 & 1.3627 \\ -0.0062 & 0.5451 & -0.3051 \end{bmatrix}$$
$$k_{\tau 3} = \begin{bmatrix} -0.1915 & 0.1252 & 0.0186 \\ -0.1170 & -0.4970 & 1.9512 \\ -0.0395 & 0.9863 & 0.1134 \end{bmatrix}$$

The coupled Chua's circuits is used modeled the case of transmitter and receivers employed in secure communication [5]. The initial conditions of CDNs are chosen as $x_1(0) = [1, -2, 8]^T$, $x_2(0) = [4, -6, 4]^T$, $x_3(0) = [1, -1, 7]^T$,and $s(0) = [0, -1, 1]^T$.The state error trajectories without control input is depicted in Fig.(1). Applying the controller gain matrices, the state error trajectories of (9) converges to zero which indicate the synchronization of the transmitter and receivers when considering secure communication for signal transmission as shown in Fig. (2).

V. Conclusion

In this paper, the nonfragile synchronization control problem for a CDNs with randomly occurring coupling and time-varying delay is investigated. Nonfragile controllers have been designed and introduced to the CDNs which ensured the desired synchronization. The extended Jensen's inequality is used fully in deriving the LMIs which guaranteed the feasibility of our results. The synchronized CDNs is used to model synchronization of Chua's circuits used secure communication as a special case. Finally, the numerical simulations presented show the effectiveness of our proposed designed control.

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