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Detection Theory Notes

Max and Min of two IID Random Variables:

Let X_1 & X_2 are two iid random variables.
$X_1 \in A$ & $X_2 \in B$.

i. $\max\{X_1, X_2\} \leq x$ iff $X_1 \leq x$ & $X_2 \leq x$
ii. $\min\{X_1, X_2\} \geq x$ iff $X_1 \geq x$ & $X_2 \geq x$

Distribution of $\max\{X_1, X_2\}$

$$F_X(x) = P(X \leq x)$$
$$= P(\max\{X_1, X_2\} \leq x)$$
$$= P(X_1 \leq x \ \& \ X_2 \leq x)$$
$$F_X(x) = P(X_1 \leq x) \cdot P(X_2 \leq x)$$

As X_1 & X_2 are identically distributed.

So,

$$P(X_1 \leq x) = F_X(x)$$
$$P(X_1 \leq x) \cdot P(X_2 \leq x) = \{F_X(x)\}^2$$

Distribution function:

$$f_X(x) = \frac{d}{dx} F_X^2(x) = 2F(x)F'(x)$$

$$\boxed{f_X(x) = 2f(x)F(x)}$$

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Date

Page

Distribution of min$\{X_1, X_2\}$

$$F_X(x) = P(X \le x)$$
$$= 1 - P(X \ge x)$$
$$= 1 - P(\min\{X_1, X_2\} \ge x)$$
$$= 1 - P(X_1 \ge x \;\&\; X_2 \ge x)$$
$$= 1 - P(X_1 \ge x) \cdot P(X_2 \ge x)$$
$$F_X(x) = 1 - [1 - P(X_1 \le x)][1 - P(X_2 \le x)]$$

As X_1 & X_2 are identically distributed.

$$F_X(x) = 1 - [1 - P(X \le x)]^2$$

distribution function:

$$f_X(x) = \frac{d}{dx} F_X(x)$$
$$= \frac{d}{dx}\left\{1 - [1 - P(X \le x)]^2\right\}$$
$$= -\frac{d}{dx}\left\{[1 - F(x)]^2\right\}$$
$$= 2[1 - F(x)]\, F'(x)$$

$$\boxed{f_X(x) = 2 f(x)[1 - F(x)]}$$

Leibniz integral rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) =$$

$$= f(x, b(x))\,\frac{d}{dx}b(x) - f(x, a(x))\,\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

☆ Distribution of Product of Normal distribution:

☆ CDF of Product of Normal distribution:

$$Y \sim N^2(0,1) \quad ; \quad Y = X^2 \;\; \& \;\; X \sim N(0,1)$$

CDF of gaussian.

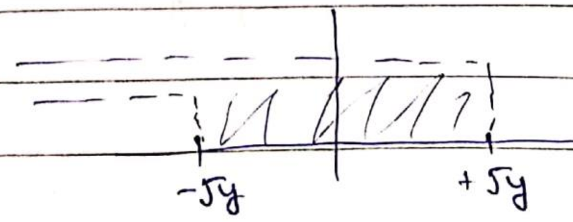
$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(|X| \le \sqrt{y})$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$P(-\sqrt{y} \le X \le \sqrt{y}) = \phi(\sqrt{y}) - \phi(-\sqrt{y})$$



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Date ____
Page ____

Now:

$$F_Y(y) = \phi(\sqrt{y}) - \phi(-\sqrt{y})$$

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

As:

$$\frac{d}{dy}\{\phi(\sqrt{y})\} = \frac{d}{dy}\left[\int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right]$$

from Leibniz integral rule:

$$\frac{d}{dy}\left[\int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right] = \frac{1}{2\sqrt{y}} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-y/2}\right) - 0$$

So.
$$\frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-y/2}\right) + \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}} e^{-y/2}\right)$$

So:

$$f_Y(y) = \frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}} e^{-y/2}\right)$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2}}$$

Bayesian Hypothesis Testing :

Let's assume there are two hypothesis H_0 & H_1 such that

H_0 : $Y \sim P_0$
(the transmitted information has probability distribution P_0)

H_1 : $Y \sim P_1$
(the transmitted information has probability distribution P_1)

Let the observation set is Γ

$$y \in \Gamma$$

observation set is partitioned into two sets Γ_0 & Γ_1 such that

$$\delta(y) = \begin{cases} 1 & \text{if } y \in \Gamma_1 \\ 0 & \text{if } y \in \Gamma_0 \end{cases}$$

$\delta(y)$ is decision rule.

{ If $y=0$ is transmitted and $y \in \Gamma_0$ observed then decision will be taken as '0' ~~of~~ if otherwise '1'.}

Now let's assume each decision comes with a different cost.

C_{ij} : cost of choosing hypothesis H_i when H_j was transmitted.

Example: C_{01} is we assume that Y has probability distribution 'P_0' but it's true distribution is 'P_1'

Now let's also assume that hypothesis H_0 and H_1 occurs with probabilities 'π_0' & 'π_1'.

Now let's define Bayes risk.
(Average cost by decision rule δ)

$$r(\delta) := \pi_0 R_0(\delta) + \pi_1 R_1(\delta)$$

$\pi_0 R_0(\delta)$: (Prob. that we choosed hypothesis H_0) * (cost of choosing hypothesis H_0 or H_1 when H_0 was the true hypothesis.

where :

$$R_0(\delta) = C_{10} P_0(\Gamma_1) + C_{00} P_0(\Gamma_0)$$

& $$R_1(\delta) = C_{11} P_1(\Gamma_1) + C_{01} P_1(\Gamma_0)$$

$R_0(\delta)$ = (cost of choosing H_1 when H_0 is true) * (Prob. that the observation set was Γ_1)
+
(cost of choosing H_0 when H_0 is true) * (Prob. that the observation set was Γ_0)

Now :

$$r(\delta) = \pi_0 \{ C_{10} P_0(\Gamma_1) + C_{00} P_0(\Gamma_0) \} + \pi_1 \{ C_{11} P_1(\Gamma_1) + C_{01} P_1(\Gamma_0) \}$$

$$P_j(\Gamma_1) = 1 - P_j(\Gamma_0)$$

{ When the hypothesis H_j was true then the prob that observation set was Γ_1 is complementary of the prob that the observation set was Γ_0. Given that the observation set was divided into only two sets Γ_0 & Γ_1 }

classmate

Date

Page

Now :

$$r(\delta) = \pi_0 \{ C_{10} (1 - P_0(\Gamma_0)) + C_{00} P_0(\Gamma_0) \} + \pi_1 \{ C_{11} (1 - P_1(\Gamma_0)) + C_{01} P_1(\Gamma_0) \}$$

or.

$$r(\delta) = \pi_0 \{ C_{10} P_0(\Gamma_1) + C_{00} (1 - P_0(\Gamma_1)) \} + \pi_1 \{ C_{11} P_1(\Gamma_1) + C_{01} (1 - P_1(\Gamma_1)) \}$$

$$r(\delta) = \pi_0 C_{00} + \pi_1 C_{01} + \pi_0 P_0(\Gamma_1)(C_{10} - C_{00}) + \pi_1 P_1(\Gamma_1)(C_{11} - C_{01})$$

Now to take optimum decision rule for H_0 vs H_1.

We need to minimize $r(\delta)$. {Average cost}

For a observation set 'Γ_1'

Let $P_j(\Gamma_1)$ has prob. density $p_j(y)$

$$\pi_0 P_0(\Gamma_1)(C_{10} - C_{00}) + \pi_1 P_1(\Gamma_0)(C_{11} - C_{01}) \leq 0$$

$$\pi_0 P_0(y)(C_{10} - C_{00}) - \pi_1 P_1(y)(C_{01} - C_{11}) \leq 0$$

$$\Rightarrow \quad \pi_1 P_1(y)(C_{11} - C_{01}) \leq \pi_0 P_0(y)(C_{00} - C_{10})$$

Now assume : $p_1(y) \geq \tau p_0(y)$

So.

$$\tau(y) = \frac{P_1(y)}{P_0(y)} \qquad y \in T$$

$$\tau \triangleq \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

$L(y)$ is known as the likelihood - ratio

So. if $L(y) \geq \tau$ then decision will be '1'.

That means.

If remember:

$$\delta(y) = \begin{cases} 1 & \text{if } y \in T_1 \\ 0 & \text{if } y \in T_0 \end{cases}$$

And for observation set T_1 the cost will be minimum if

$$\pi_1 P_1(y)(C_{11} - C_{01}) \leq \pi_0 P_0(y)(C_{00} - C_{10})$$

if

$$\Rightarrow \quad L(y) \geq \tau.$$

So,

$$\delta(y) = \begin{cases} 1 & \text{if } L(y) \geq \tau \\ 0 & \text{if } L(y) < \tau \end{cases}$$

Classmate

Date
Page

Bayesian Hypothesis testing with Gaussian Error:

Let's assume the observation 'X' and following two Hypotheses

$$H_0 : X \sim P_0 \quad (X \sim N(\mu_0, \sigma^2))$$

&

$$H_1 : X \sim P_1 \quad (X \sim N(\mu_1, \sigma^2))$$

X has Gaussian distribution.

So the likelihood ratio will be

$$L(x) = \frac{P_1(x)}{P_0(x)}$$

$$L(x) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu_1)^2/2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu_0)^2/2\sigma^2}}$$

$$L(x) = e^{-\left\{\frac{(x-\mu_1)^2 - (x-\mu_0)^2}{2\sigma^2}\right\}}$$

$$L(x) = e^{+\left\{\left(x - \frac{\mu_0+\mu_1}{2}\right) / \left(\frac{\sigma^2}{\mu_1-\mu_0}\right)\right\}}$$

The Bayes test

$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \qquad \left\{ \tau = \frac{\pi_0}{\pi_1} \right\}$$

(for uniform cost)

$$\delta(x) = \begin{cases} 1 & \text{if } L(x) \geq \tau \\ 0 & \text{if } L(x) \leq \tau \end{cases}$$

Let's assume both the hypothesis are equally likely. $\pi_0 = \pi_1 = 1/2$

so. $\tau = 1$

Therefore

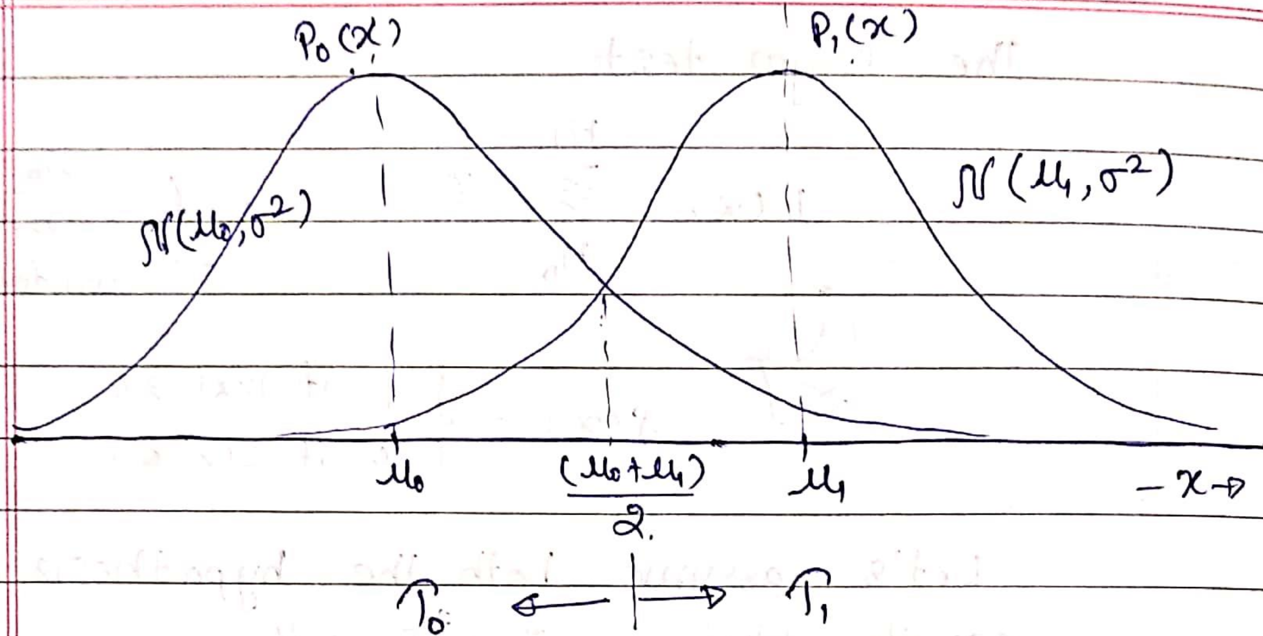
$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow \quad e^{\left(\frac{x - (\mu_0 + \mu_1)/2}{\sigma^2/(\mu_1 - \mu_0)} \right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

take log.

$$\frac{x - (\mu_0 + \mu_1)/2}{\sigma^2/(\mu_1 - \mu_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\boxed{x \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\mu_0 + \mu_1}{2} \right)}$$



so, the expression, $x \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\mu_0+\mu_1}{2}\right)$

says that if $x \geq \left(\frac{\mu_0+\mu_1}{2}\right)$ then

we choose hypothesis H_1 and our observation x will belong to set Γ_1. In this case it will have prob. distribution $P_1(x)$. otherwise we choose hypothesis H_0 and transmitted information belongs to observation set Γ_0.

Example:

Let H_0 : 0 volt was sent
H_1 : A volt was sent.

$\longrightarrow$ AWGN $\longrightarrow$

So.

H_0 : $Y = 0 + N \sim \mathcal{N}(0, \sigma^2)$

H_1 : $Y = A + N \sim \mathcal{N}(A, \sigma^2)$

therefore

$\mu_1 = A$ & $\mu_0 = 0$

So,

$$\boxed{x \underset{H_0}{\overset{H_1}{\gtrless}} A/2}$$

This says if $x \geq A/2$ then 'A' was sent (as we choose hypothesis H_1) and if $x < A/2$ then '0' volt was sent).

Now Let:

H_0 : $Y \sim P_0$ $(P_0(x) \sim \mathcal{N}(0, \sigma_0^2))$

H_1 : $Y \sim P_1$ $(P_1(x) \sim \mathcal{N}(0, \sigma_1^2))$

Let's also assume that hypothesis are not equally likely $\pi_0 \neq \pi_1$

So, likelihood ratio.

$$L(x) = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-x^2/2\sigma_1^2}}{\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-x^2/2\sigma_0^2}}$$

$$L(x) = \frac{\sigma_0}{\sigma_1} e^{-\frac{x^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}$$

$$L(x) = \frac{\sigma_0}{\sigma_1} e^{-\frac{x^2}{2}\left(\frac{\sigma_0^2 - \sigma_1^2}{\sigma_0^2 \sigma_1^2}\right)}$$

The Bayes test:

$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

$$\frac{\sigma_0}{\sigma_1} e^{-x^2\left(\frac{\sigma_0^2-\sigma_1^2}{2\sigma_0^2\sigma_1^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

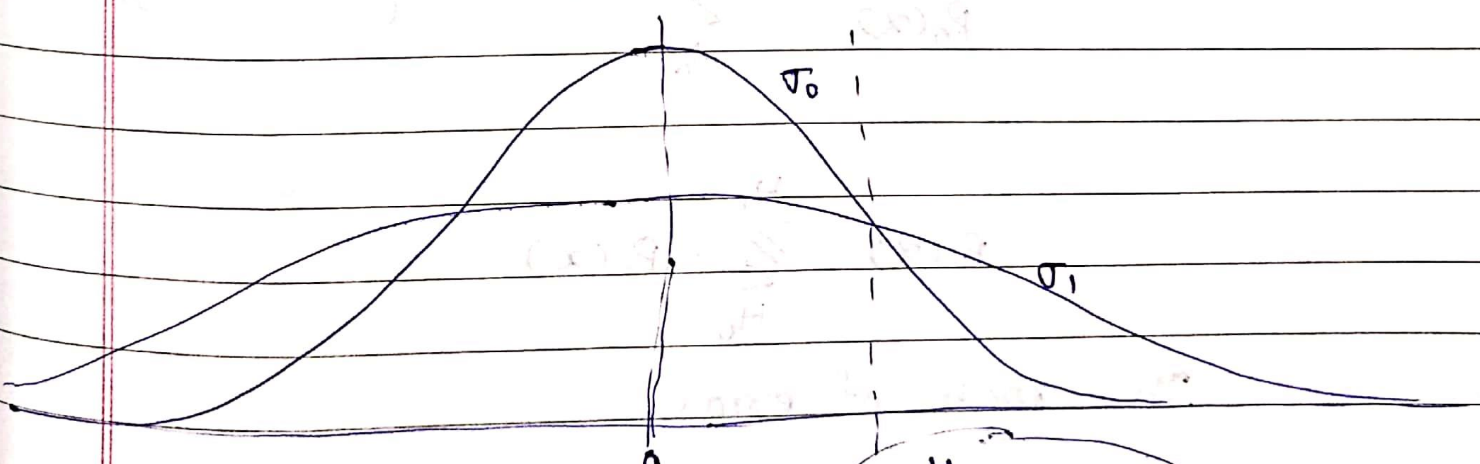
$$\Rightarrow \log\left(\frac{\sigma_0}{\sigma_1}\right) + x^2\left(\frac{\sigma_1^2-\sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log\left(\frac{\pi_0}{\pi_1}\right)$$

$$\Rightarrow x^2\left(\frac{\sigma_1^2-\sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log\left(\frac{\pi_0}{\pi_1}\right) - \log\left(\frac{\sigma_0}{\sigma_1}\right)$$

Let's assume $\sigma_1 > \sigma_0$

so. $$x^2 \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{2(\sigma_0\sigma_1)^2}{\sigma_1^2-\sigma_0^2}\right)\left(\log\left(\frac{\pi_0}{\pi_1}\right)+\log\left(\frac{\sigma_1}{\sigma_0}\right)\right)$$

for $\sigma_1 = 4$; $\sigma_0 = 2$; $\pi_0 = \pi_1 = \pi$



$$x \underset{H_0}{\overset{H_1}{\gtrless}} 2.7191.$$

Probability of errors Minimax Hypothesis

A decision rule which minimizes the conditional risk is known as minimax rule.

Let $\pi_0 = \pi_1 = 1/2$; Hypothesis are equally likely.

The likelihood ratio

$$L(x) = \frac{P_1(x)}{P_0(x)}$$

so,

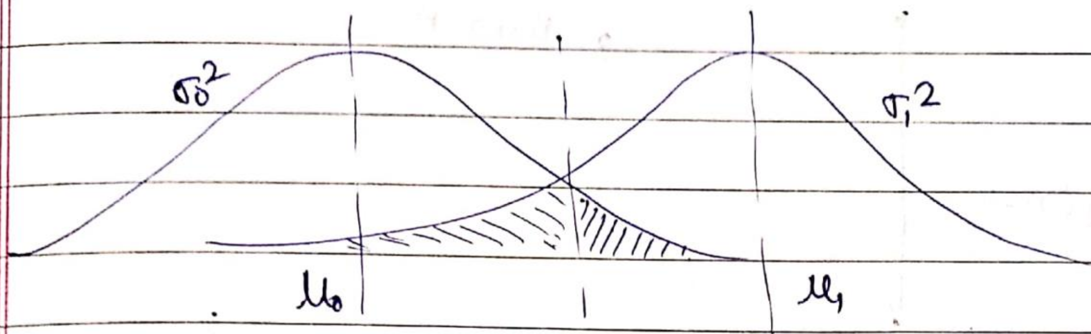
$$\frac{P_1(x)}{P_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \quad \left\{ \text{as } \tau = \frac{\pi_0}{\pi_1} = 1 \right\}$$

$$P_1(x) \underset{H_0}{\overset{H_1}{\gtrless}} P_0(x)$$

The prob. of error:

$$P_e = \frac{1}{2} \text{Prob}\{\text{Decide } H_1 / H_0\} + \frac{1}{2} \text{Prob}\{\text{Decide } H_0 / H_1\}$$

{ ∵ 1/2 as hypothesis are equally likely and equal to 1/2 }



$$P_e = \frac{1}{2}\int_R \min\{P_1(x), P_0(x)\}\,dx$$

when $\pi_0 \neq \pi_1$

then

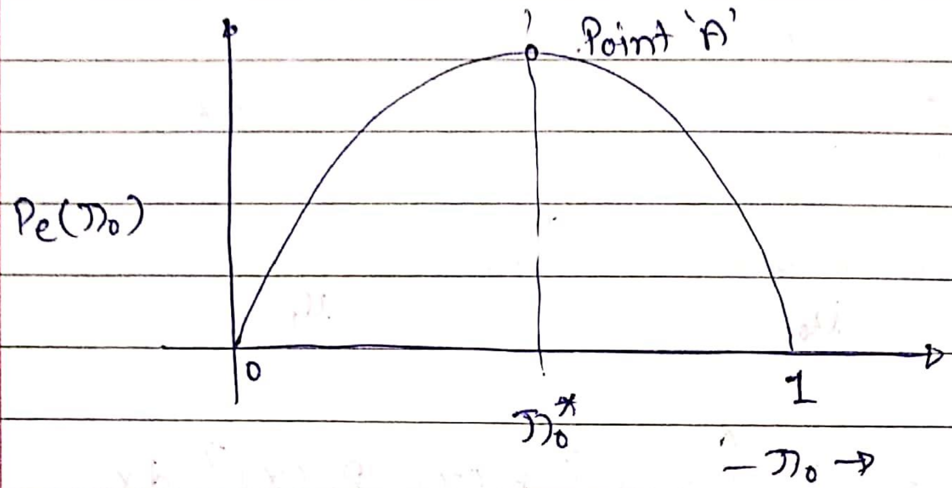
$$\pi_1 P_1(x) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 P_0(x)$$

$$P_e = \int_\Gamma \min\{\pi_0 P_0(x), \pi_1 P_1(x)\}\,dx$$

$$\Gamma = \{x : \pi_1 P_1(x) \geq \pi_0 P_0(x)\}$$

as $\pi_1 = 1 - \pi_0$

$$P_e(\pi_0) = \int_\Gamma \min\{\pi_0 P_0(x), (1-\pi_0) P_1(x)\}\,dx$$



If we design the detector for point A, which has the worst prob. of error then for other points the prob. of error will be less.

[minimax detector]

Example:

Let Gaussian error and non-uniform cost.

H_0 : $Y \sim P_0$ $(\mathcal{N}(\mu_0, \sigma_0^2))$, π_0

H_1 : $Y \sim P_1$ $(\mathcal{N}(\mu_1, \sigma_1^2))$, π_1

Let $\sigma_0 = \sigma_1 = \sigma$

so the likelihood ratio

$$L(x) = \frac{e^{-(x-\mu_1)^2/2\sigma^2}}{e^{-(x-\mu_0)^2/2\sigma^2}}$$

So.
$$e^{-\left\{\frac{(x-\mu_1)^2-(x-\mu_0)^2}{2\sigma^2}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

$$\Rightarrow \quad e^{\left\{\frac{x-(\mu_0+\mu_1)/2}{\sigma^2/(\mu_1-\mu_0)}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

$$\Rightarrow \quad \frac{x-(\mu_0+\mu_1)/2}{\sigma^2/(\mu_1-\mu_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \log\left(\frac{\pi_0}{\pi_1}\right)$$

$$\Rightarrow \quad x \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\mu_0+\mu_1}{2}\right) + \left(\frac{\sigma^2}{\mu_1-\mu_0}\right)\log\left(\frac{\pi_0}{\pi_1}\right)$$

As $\pi_1 = 1-\pi_0$

So.
$$x \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\left(\frac{\mu_0+\mu_1}{2}\right) + \left(\frac{\sigma^2}{\mu_1-\mu_0}\right)\log\left(\frac{\pi_0}{1-\pi_0}\right)}_{\text{the } \tau \text{ is function of } \pi_0}$$

$\tau(\pi_0)$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \tau(\pi_0)$$

$$\text{Prob.}(\text{decide } H_1 | H_0) = \int_{\tau(\pi_0)}^{\infty} P_0(x)\, dx = \phi_0(\tau(\pi_0))$$

$$\text{Prob}(\text{decide } H_0 | H_1) = \int_{-\infty}^{\tau(\pi_0)} P_1(x)\, dx$$

$$= 1 - \int_{\tau(\pi_0)}^{\infty} P_1(x)\, dx = 1 - \phi_1(\tau(\pi_0))$$

$$\boxed{\int_{\tau(\pi_0)}^{\infty} \left(P_0(x) + P_1(x) \right) dx = 1}$$

from solving the above equation & from symmetry we get

$$\tau(\pi_0) = \frac{\mu_0 + \mu_1}{2}.$$

Which concludes that the least favarable $\pi_0^* = 1/2$.

Let's assume both the hypothesis has equal probability.

$$\pi_0 = \pi_1$$

so $\tau(\pi_0)$ will be $= \left(\frac{\mu_0 + \mu_1}{2}\right)$

so.

$$\text{Prob}(\text{decide } H_1 | H_0) = \phi\left(\frac{\tau(\pi_0) - \mu_0}{\sigma}\right)$$

$$= \phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right)$$

$$\text{Prob}(\text{decide } H_0 / H_1) = 1 - \phi\left(\frac{\tau(\pi_1) - \mu_1}{\sigma}\right)$$

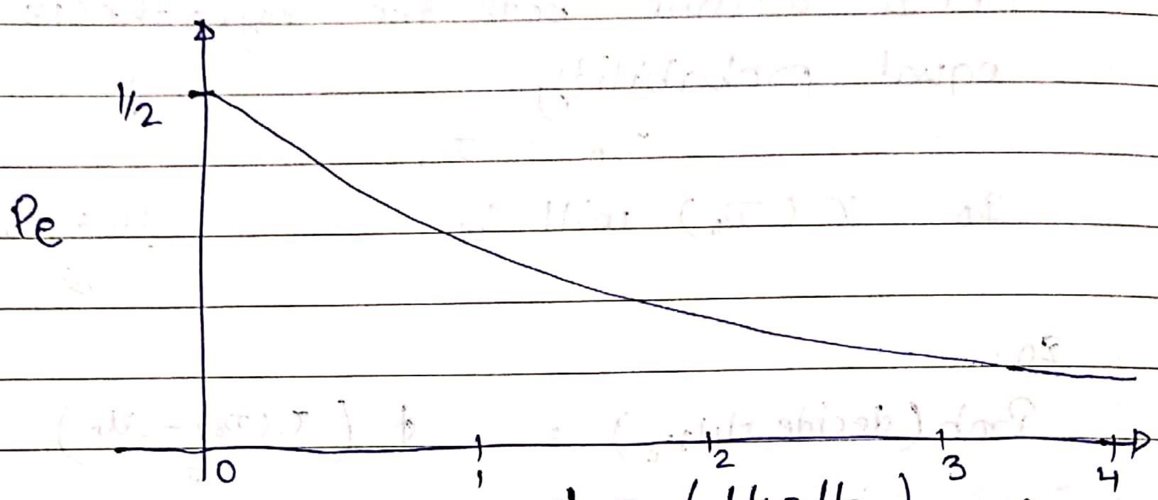
$$= 1 - \phi\left(\frac{\mu_0 - \mu_1}{2\sigma}\right) = \phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right)$$

Now Prob of error: $\{\mu_1 > \mu_0\}$

$$P_e = \min\left\{\phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right), \phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right)\right\}$$

$$P_e = \phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right) = \phi(d/2)$$

where $d = \frac{\mu_1 - \mu_0}{\sigma}$ {separation between both the means}



$d = \left(\frac{\mu_1 - \mu_0}{\sigma}\right) \rightarrow$

The above plot explains 'prob of error' or 'Bayes risk' with respect to the separation between both the means.

As the separation increases, the prob. of error decreases.

We can consider 'd' as a simple version of signal-to-noise ratio.

Example:

Let Gaussian error with non-uniform cost:

$$H_0 : \quad X \sim P_0 = \mathcal{N}(0, \sigma_0^2) \quad , \pi_0$$
$$H_1 : \quad X \sim P_1 = \mathcal{N}(0, \sigma_1^2) \quad , \pi_1$$

where $\sigma_1 > \sigma_0$

So the likelihood ratio

$$L(x) = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-x^2/2\sigma_1^2}}{\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-x^2/2\sigma_0^2}}$$

$$L(x) = \frac{\sigma_0}{\sigma_1} e^{+\frac{x^2}{2}\left(\frac{\sigma_1^2 - \sigma_0^2}{(\sigma_0\sigma_1)^2}\right)}$$

our hypothesis:

$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

$$L(x) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{1-\pi_0}$$

$$\frac{\sigma_0}{\sigma_1}\, e^{\frac{x^2}{2}\left(\frac{\sigma_1^2-\sigma_0^2}{(\sigma_0\sigma_1)^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{1-\pi_0}$$

$$\log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{x^2}{2}\cancel{\log}\left(\frac{(\sigma_1^2-\sigma_0^2)}{(\sigma_0\sigma_1)^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log\left(\frac{\pi_0}{1-\pi_0}\right)$$

$$\frac{x^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{(\sigma_0\sigma_1)^2}{\sigma_1^2-\sigma_0^2}\right)\left[\log\left(\frac{\pi_0}{1-\pi_0}\right)-\log\left(\frac{\sigma_0}{\sigma_1}\right)\right]$$

$$|x| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\left(\frac{2(\sigma_0\sigma_1)^2}{\sigma_1^2-\sigma_0^2}\right)\log\left(\frac{\pi_0\sigma_1}{(1-\pi_0)\sigma_0}\right)}$$

Let $\tau(\pi_0) = \sqrt{\left(\frac{2(\sigma_0\sigma_1)^2}{\sigma_1^2-\sigma_0^2}\right)\log\left(\frac{\pi_0\sigma_1}{(1-\pi_0)\sigma_0}\right)}$

So,

$$|x| \underset{H_0}{\overset{H_1}{\gtrless}} \tau(\pi_0)$$

if $\quad -\tau(\pi_0) < x < \tau(\pi_0) \rightarrow H_0$

and,

$$x \le -\tau(\pi_0) \text{ or } x \geq \tau(\pi_0) \rightarrow H_1$$

$$Pr.\{\text{decide } H_0 | H_1\} \Rightarrow \int_{-\tau(\eta_0)}^{\tau(\eta_0)} P_0(x)\, dx$$

$$= 2\int_{0}^{\tau(\eta_0)} P_0(x)\, dx = 2\left(\frac{1}{2} - \phi\left(\frac{\tau_{\eta_0}}{\sigma_0}\right)\right)$$

$$\text{Prob}\{\text{decide } H_1 | H_0\} = \int_{-\infty}^{-\tau(\eta_0)} P_0(x)\, dx + \int_{\tau(\eta_0)}^{\infty} P_0(x)\, dx$$

$$= 2\int_{\tau(\eta_0)}^{\infty} P_1(x)\, dx, \quad = 2\,\phi\left(\frac{\tau_{\eta_0}}{\sigma_1}\right)$$

for minimax detector / for worst prob. of error.

$$2\phi\left(\frac{\tau}{\sigma_1}\right) = 2\left(\frac{1}{2} - \phi\left(\frac{\tau}{\sigma_0}\right)\right)$$

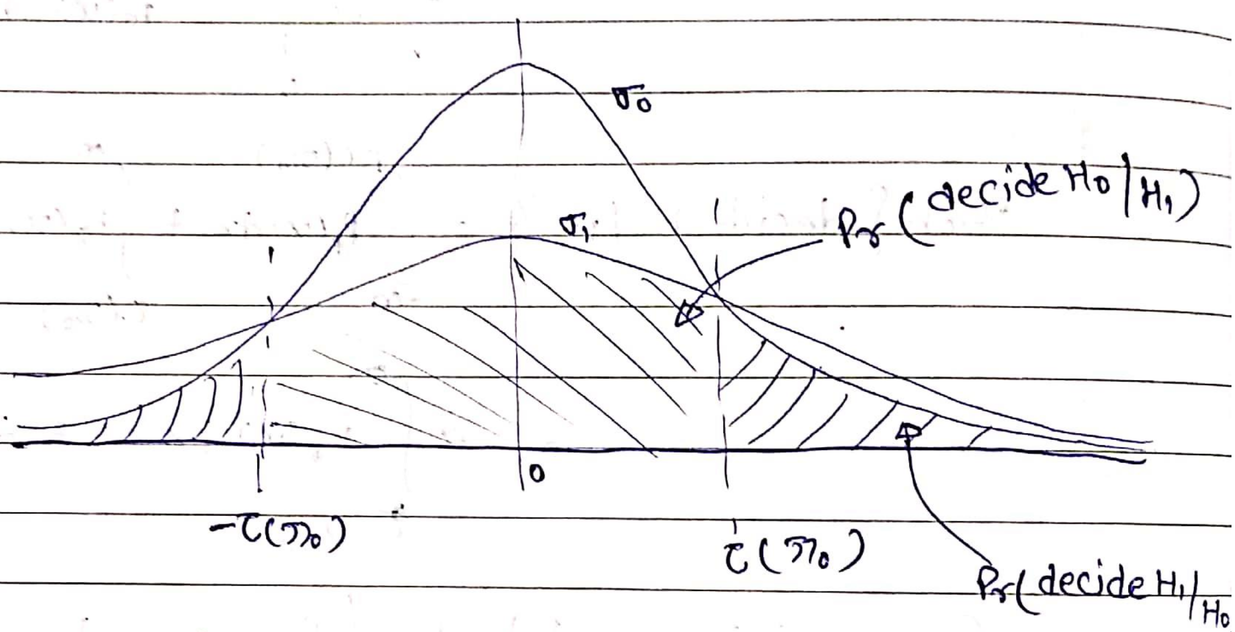
$$\boxed{\phi\left(\frac{\tau}{\sigma_0}\right) + \phi\left(\frac{\tau}{\sigma_1}\right) = 1/2}$$

Intution:

let $d = \sigma_1/\sigma_0$ and let the hypothesis has equal probability.

So. $$\tau = \sqrt{\frac{2\, d^2}{(d^2-1)} \log d}$$

So as the d increases (the difference between σ_1 & σ_0 increases) τ also increases but very slowly.



As the difference between σ_0 & σ_1 increases, $\tau(\eta_0)$ moves far from mean (0) and the prob. of error decreases (Area under the curve decreases).

Neyman Pearson Hypothesis Testing.

⇒ Bayesion :

Minimize the over all expected cost (average risk) where we assume that the hypothesis probabilities are prior available.

$$\frac{P_1(x)}{P_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

⇒ Minimax :

Minimize the max. of the conditional expected cost assuming that the hypothesis probabilities are prior not available.

$$P_e = \min\{\pi_0 P_0(x), (1-\pi_0) P_1(x)\}$$

⇒ Neyman - Pearson :

The specific cost structure is not desired. We place a bound on the false alarm probability and then minimize the miss probability within the bound to maximize the detection probability.

$$\max P_D(\delta) \text{ subject to } P_F(\delta) \leq \alpha$$

Neyman - Pearson Hypothesis:

False alarm: (P_F)

when H_0 or H_1 can be falsely rejected.

(Prob of miss $\Rightarrow$ Prob. (detect H_0 / H_1) or

(Prob of false alarm $\Rightarrow$ Prob. (detect H_1 / H_0)

Detection Probability : (P_D)

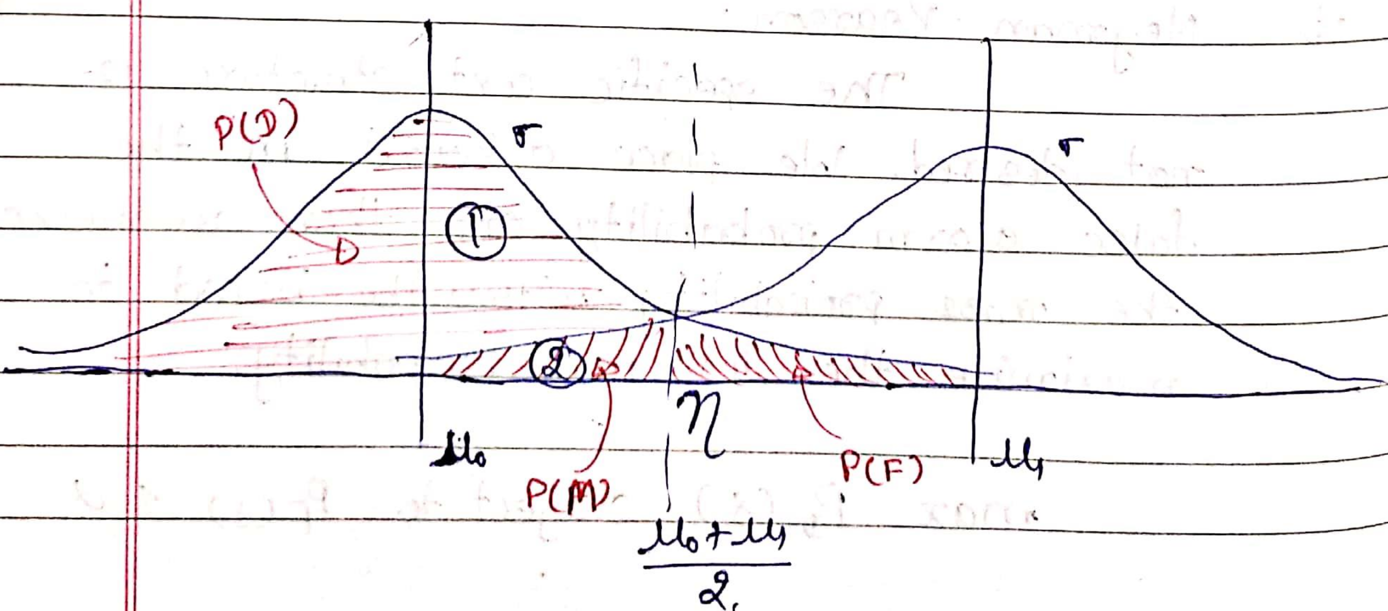
Prob. (detect H_0 / H_0) or

Prob. (detect H_1 / H_1)

Example: Gaussian Error:

H_0 : $X \sim N(\mu_0, \sigma)$, D_0

H_1 : $X \sim N(\mu_1, \sigma)$, D_1



Let's assume 'H_0' was the true hypothesis.

The prob. of detection:
(area under curve ①)

$$P_D = 1 - \phi\left(\frac{\eta - \mu_0}{\sigma}\right)$$

where $\eta = \left(\frac{\mu_0 + \mu_1}{2}\right) + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log\left(\frac{\pi_0}{\pi_1}\right)$

let $\pi_0 = \pi_1$ so,

$$\eta = \left(\frac{\mu_0 + \mu_1}{2}\right)$$

The prob. of false alarm.:
(area of curve 2)

$$P_F = 1 - \phi\left(\frac{\eta - \mu_1}{\sigma}\right)$$

Let's choose an 'α' such that
$1 \geq \alpha \geq 0$
α reprents max. false alarm prob.
for given 'α'.

$$\alpha = 1 - \phi\left(\frac{\eta_0 - \mu_1}{\sigma}\right).$$

$$\eta_0 = \sigma \phi^{-1}(1-\alpha) + \mu_1$$

Now the prob. of detection for given 'α'.

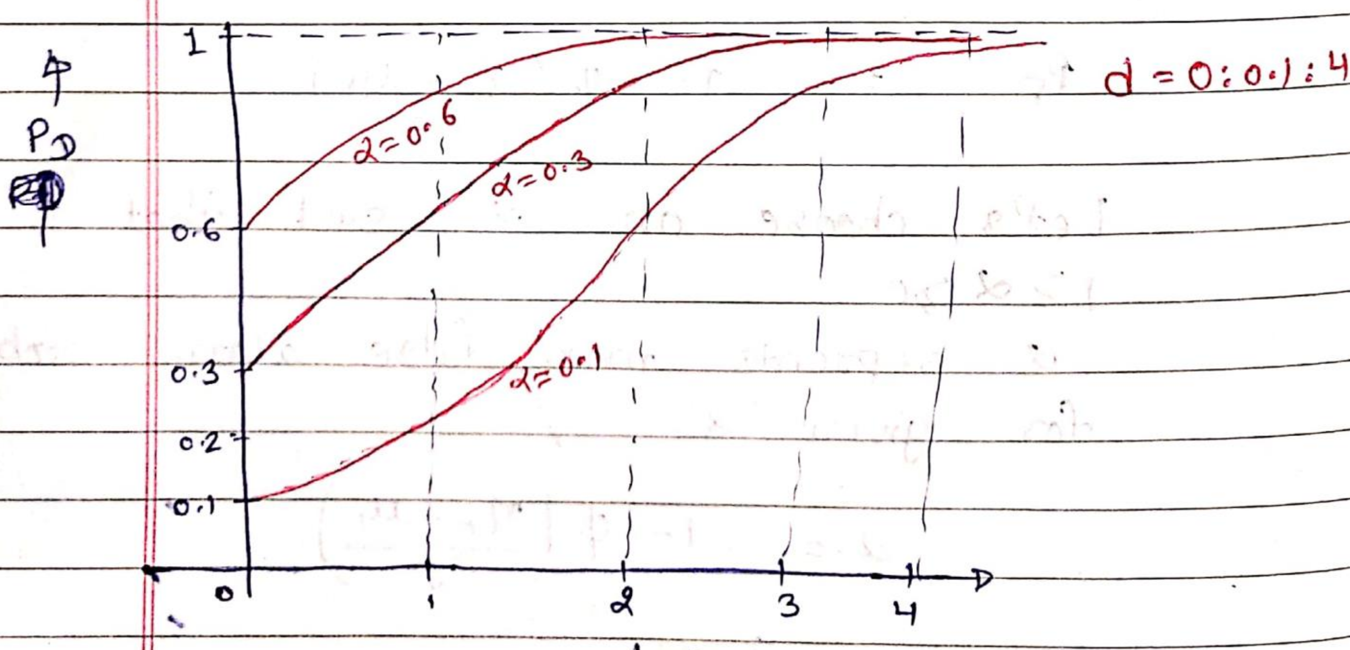
$$P_D = 1 - \phi\left(\frac{\sigma \phi^{-1}(1-\alpha) + \mu_1 - \mu_0}{\sigma}\right)$$

$$P_D = 1 - \phi\left(\phi^{-1}(1-\alpha) + \frac{\mu_1 - \mu_0}{\sigma}\right)$$

Let's assume : $d = \frac{\mu_1 - \mu_0}{\sigma}$

So, $$\boxed{P_D = 1 - \phi\left(\phi^{-1}(1-\alpha) + d\right)}$$

Intution:



From the plot:

For $\alpha = 0.1$;

Prob. that you should get false alarm is very low, so the prob. of detection (P_D) will also be less, and to get good prob. of detection you need more separation in μ_0 & μ_1 (large 'd').

For $\alpha = 0.6$:

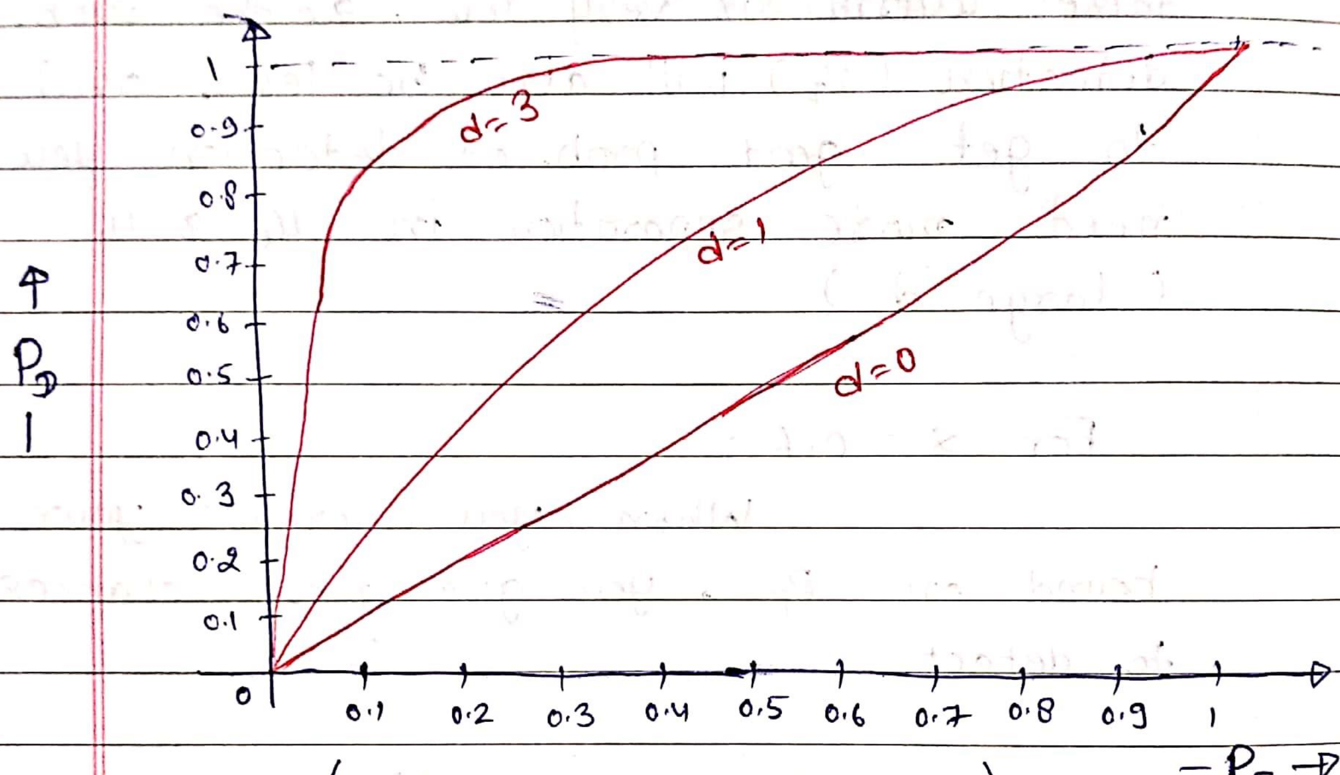
When you increase your bound on P_F, you give more chances to detect.

When $\alpha = 1$: prob of false alarm becomes '1'. ($P_F \leq 1$) which will always be true, so the prob of detection will also be '1'.

So, there is a trade-off. If you want to get good prob. of detection then you will have to loose the bound on prob. of false alarm.

Receiver Operating characteristics (ROC)

ROC is the plot between P_D & P_F.



(where $d = \frac{\mu_1 - \mu_0}{\sigma}$)

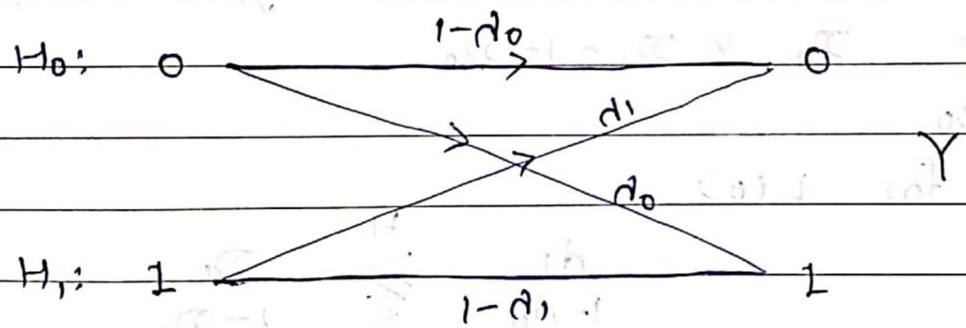
- For $d=0$, both the pdf ($P_0(x)$ & $P_1(x)$) collapse to single pdf ($\mu_0 = \mu_1$ & $\sigma_0 = \sigma_1$). In that case $P_D = P_F$.
- When you increase bound on false alarm (α) prob. of detection increases.
- When you increase sepration between both the pdf, prob. of detection increases.

Example: Binary Channel

Let:

$$H_0 \; : \; Y \sim P_0$$
$$H_1 \; : \; Y \sim P_1$$

Hypothesis 'H_0' that received signal Y has distribution 'P_0' & hypothesis 'H_1' that received signal 'Y' has distribution 'P_1'.



It says that transmitted signal '0' received as '1' with probability 'd_0' & received as '0' with probability '$1-d_0$'.

Let's see likelihood ratio:

When the received signal is $y=0$

$$L(0) = \frac{\text{Prob.}(0/H_1)}{\text{Prob.}(0/H_0)}$$

$$L(0) = \frac{\alpha_1}{1-\alpha_0}$$

Similarly likelihood ratio when received signal is '1'.

$$L(1) = \frac{1-\alpha_1}{\alpha_0}$$

For Bayesian Decision rule:

Let's assume the hypothesis probabilities are π_0 & $\pi_1 = 1-\pi_0$.

so,

for $L(0)$:

$$\frac{\alpha_1}{1-\alpha_0} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{1-\pi_0}$$

$$\Rightarrow \quad \alpha_1 - \alpha_1 \pi_0 \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 - \pi_0 \alpha_0$$

$$\Rightarrow \quad \pi_0 (\alpha_0 - \alpha_1 - 1) \gtrless -\alpha_1$$

$$\Rightarrow \quad \pi_0 (\alpha_1 + 1 - \alpha_0) \underset{H_0}{\overset{H_1}{\lessgtr}} \alpha_1$$

$$\Rightarrow \quad \pi_0 \underset{H_0}{\overset{H_1}{\lessgtr}} \left(\frac{\alpha_1}{\alpha_1 + 1 - \alpha_0} \right)$$

For equally likely hypothesis.

$$\pi_0 = \pi_1 = 1/2$$

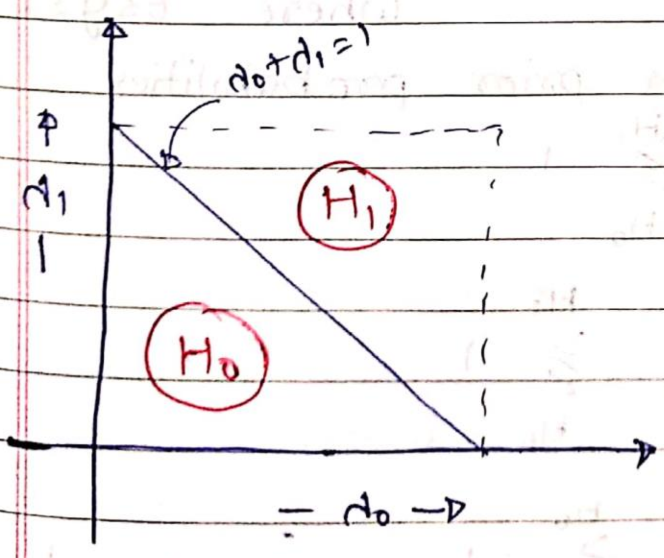
Bayesion Decision rule :

for $L(0)$: $\dfrac{\alpha_1}{1-\alpha_0} \underset{H_0}{\overset{H_1}{\gtrless}} 1$

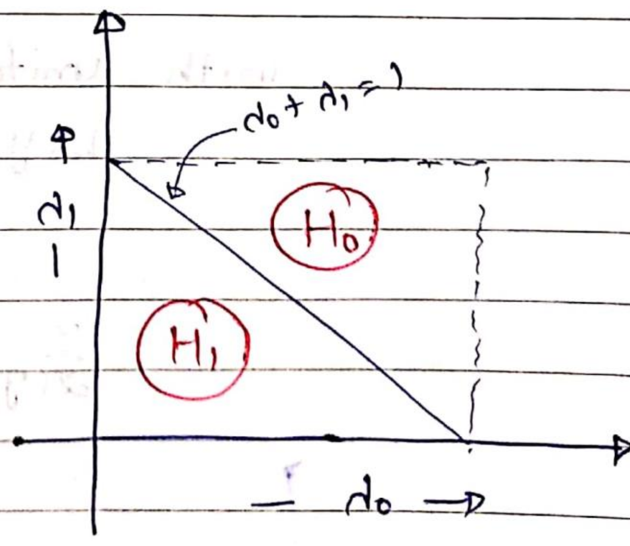
$$\boxed{\alpha_1 \underset{H_0}{\overset{H_1}{\gtrless}} 1-\alpha_0}$$

for $L(1)$: $\dfrac{1-\alpha_1}{\alpha_0} \underset{H_0}{\overset{H_1}{\gtrless}} 1$

$$\boxed{1-\alpha_1 \underset{H_0}{\overset{H_1}{\gtrless}} \alpha_0}$$



for $y=0$



for $y=1$

Example for Bayes rule, minimax rule and Neyman-Pearson rule :

Let Y is a random variable and following are two hypothesis :

H_0 : $y \sim P_0(y) = \frac{2}{3}(y+1)$

H_1 : $y \sim P_1(y) = 1$

where $0 \leq y \leq 1$

&. $\pi_0 = \pi_1 = 1/2$.

Bayes rule :

likelihood ratio will be

$$L(y) = \frac{P_1(y)}{P_0(y)} = \frac{3}{2(y+1)}$$

where $0 \leq y \leq 1$

with uniform prior probabilities :

$$L(y) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\frac{3}{2(y+1)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\boxed{y \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{2}}$$ Bayes rule

- Minimum Bayes risk:

$$r(y) = \frac{1}{2}\int_0^{1/2} \frac{2}{3}(y+1)\,dy + \frac{1}{2}\int_{1/2}^{1} 1\cdot dy$$

$$\boxed{r(y) = \frac{11}{24}}$$ minimum Bayes risk.

- Minimax rule:

We need to choose 'τ' such that

$\pi_0 P_0(x) = \pi_1 P_1(x)$

$$P_0(\tau) = \int_0^{\tau} \frac{2}{3}(y+1)\,dy = \frac{2}{3}\left(\frac{\tau^2}{2}+\tau\right)$$

$$P_1(\tau) = \int_{\tau}^{1} 1\,dy = 1-\tau$$

Now:

$$\frac{1}{2}P_0(\tau) = \frac{1}{2}P_1(\tau) \quad \{\text{as } \pi_0 = \pi_1 = 1/2\}$$

$$\Rightarrow \frac{2}{3}\tau\left(\frac{\tau}{2}+1\right) = 1-\tau$$

$$\Rightarrow \tau = \frac{-5+\sqrt{37}}{2}$$

$$\boxed{y \underset{H_1}{\overset{H_0}{\gtrless}} \frac{-5+\sqrt{37}}{2}}$$ Minimax Rule

- Minimax Risk:

$$r(y) = \frac{1}{2}\int_0^t \frac{2}{3}(y+1)\,dy + \frac{1}{2}\int_t^1 1.\,dy$$

where $t = \frac{-5+\sqrt{37}}{2}$

as. $\int_0^t \frac{2}{3}(y+1)\,dy = \int_t^1 1.\,dy = 1-t$

so, $r(y) = \frac{1}{2}(1-t) + \frac{1}{2}(1-t)$

$$\boxed{r(y) = \frac{7-\sqrt{37}}{2}}$$ Minimax Risk.

- Neyman - Pearson rule:

Probability of false alarm

$= \text{Prob}\{\text{detect } H_1 / H_0\}$

$$P_{FA} = \int_0^{\tau} \frac{2}{3}(y+1)\, dy$$

Let's assume threshold is α.

so,

$$\int_0^{\tau} \frac{2}{3}(y+1) = \alpha$$

$$\frac{2}{3}\tau\left(\frac{\tau}{2}+1\right) = \alpha$$

$$\tau = \sqrt{1+3\alpha} - 1$$

so,

$$\boxed{r(y) \underset{H_1}{\overset{H_0}{\gtrless}} \sqrt{1+3\alpha} - 1}$$

Neyman-pearson rule.

Now probability of detection:

$$P_D = \int_0^{\tau} 1 \cdot dy = \tau$$

$$P_D = \sqrt{1+3\alpha} - 1 \quad \text{for} \quad 0 < \alpha < 1$$

Composite Hypothesis testing:

The hypothesis testing problems in which each of the two hypothesis corresponds to only a single distribution for a observation are known as Simple hypothesis testing.

The hypothesis which deals with multiple distributions under each hypothesis are known as Composite hypothesis testing.

Example:

A radar signal has many unknown parameters such as exact time of arrival, Doppler shift etc.

Hypothesis Model:

Let:

Λ : A parameter set

θ : a parameter which takes values in Λ

Θ : a random parameter which takes values from Λ

P_θ : Prob. distribution of the observation given that θ is the true parameter.

Example:

Let's assume our hypothesis are

$$H_0 : \begin{cases} Y_1 = \epsilon_1 \\ Y_2 = \epsilon_2 \end{cases}$$

versus

$$H_1 : \begin{cases} Y_1 = A\cos\psi + \epsilon_1 \\ Y_2 = A\sin\psi + \epsilon_2 \end{cases}$$

where A is positive constant,
$\psi \in [0, 2\pi]$ uniform distribution,
ϵ_1 & $\epsilon_2 \sim \mathcal{N}(0, \sigma^2)$ independent random variables.

Two parameters can be taken as:

$$\theta = (\theta_1, \theta_2)$$

$\theta_1 \in \{0, A\}$ &

$\theta_2 \in [0, 2\pi]$

So the parameter set

$$\Lambda \in \{0, A\} \times [0, 2\pi]$$

The density of Y at $\theta = \theta$ is joint density of two Independent random variables

$$p_\theta(y_1, y_2) = \frac{1}{(\sqrt{2\pi\sigma^2})^2} e^{-(y_1^2+y_2^2)/2\sigma^2}$$

$$p_\theta(y_1, y_2) = \frac{1}{(\sqrt{2\pi\sigma^2})^2} e^{-\frac{\{(y_1 - A\cos\psi)^2 + (y_2 - A\sin\psi)^2\}}{2\sigma^2}}$$

So, for hypothesis H_0:

$$p_0(y_1, y_2) = \frac{1}{2\pi\sigma^2} e^{-(y_1^2+y_2^2)/2\sigma^2}$$

{ can also be represented as $p(y_1, y_2 / \theta \in \Lambda_0)$ }

For hypothesis H_1:

$$p_{1/\psi}(y_1, y_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{\{(y_1 - A\cos\psi)^2 + (y_2 - A\sin\psi)^2\}}{2\sigma^2}}$$

So Now the likelihood ratio will be:

$$L(y_1, y_2) = \frac{p_1(y_1, y_2)}{p_0(y_1, y_2)}$$

As ψ is uniformly distributed over $[0, 2\pi]$

so,

$p_1(y_1, y_2)$ = average of $p_{1/\psi}(y_1, y_2)$ for all ψ.

So,

$$p_1(y_1, y_2) = E\left[p_{1|\psi}(y_1, y_2) \right]$$

$$p_1(y_1, y_2) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi\sigma^2}\right) e^{-\frac{\{(y_1 - A\cos\psi)^2 + (y_2 - A\sin\psi)^2\}}{2\sigma^2}} d\psi$$

So now:

$$L(y_1, y_2) = \frac{\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi\sigma^2}\right) e^{-\frac{\{(y_1 - A\cos\psi)^2 + (y_2 - A\sin\psi)^2\}}{2\sigma^2}} d\psi}{\left(\frac{1}{2\pi\sigma^2}\right) e^{-(y_1^2 + y_2^2)/2\sigma^2}}$$

$$\Rightarrow = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{\{A^2\cos^2\psi - 2y_1 A\cos\psi + A^2\sin^2\psi - 2y_2 A\sin\psi\}}{2\sigma^2}} d\psi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{+\frac{A}{\sigma^2}\left(\frac{A}{2} + y_1\cos\psi + y_2\sin\psi\right)} d\psi$$

$$= \frac{1}{2\pi} e^{-A^2/2\sigma^2} \int_0^{2\pi} e^{\left\{\frac{A}{\sigma^2}(y_1\cos\psi + y_2\sin\psi)\right\}} d\psi$$

Let's assume:

$$y_1 = r\cos\phi$$
$$y_2 = r\sin\phi$$

$$r = \sqrt{y_1^2 + y_2^2}$$

So,

$$L(y_1, y_2) = \frac{e^{-A^2/2\sigma^2}}{2\pi}\int_0^{2\pi} e^{\left\{\frac{Ar}{\sigma^2}\cos(\psi-\phi)\right\}} d\psi$$

$$L(y_1, y_2) = \frac{e^{-A^2/2\sigma^2}}{2\pi} I_0\left(\frac{Ar}{\sigma^2}\right) \cdot 2\pi.$$

where I_0 is the zeroth order modified Bessel function. of the first kind.

For Bayseian Hypothesis:

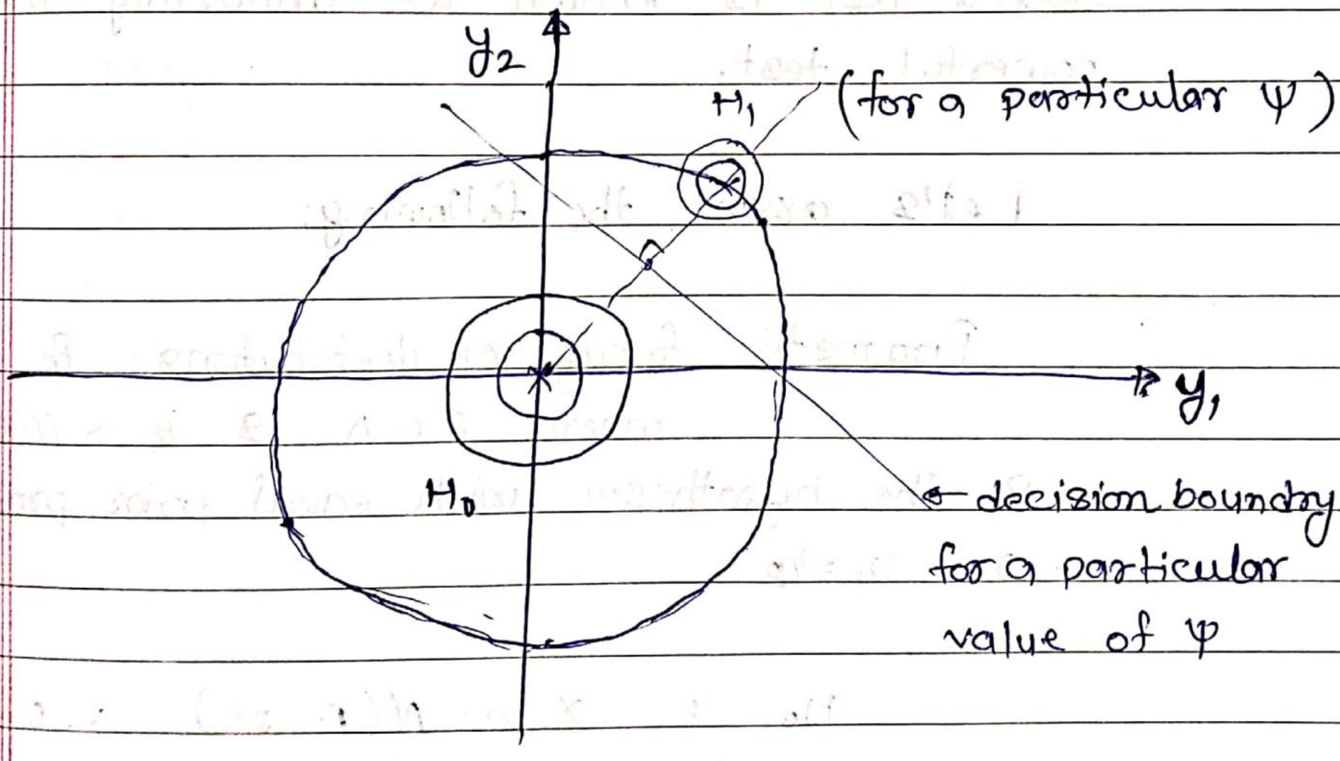
$$L(y_1, y_2) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1} = \tau \text{ (Let's assume)}$$

$$\frac{e^{-A^2/2\sigma^2}}{\cancel{2\pi}} I_0\left(\frac{Ar}{\sigma^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \tau.$$

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\sigma^2}{A}\right) I_0^{-1}\left(\tau \cdot e^{A^2/2\sigma^2}\right)$$

So,

$$\boxed{r \underset{H_0}{\overset{H_1}{\gtrless}} (\sigma^2/A)\, I_0^{-1}\left(\tau \cdot e^{A^2/2\sigma^2}\right)}$$



Uniformly most powerful (UMP) test:

In Neyman-Pearson formulation we assure that the false-alarm probability doesn't exceed a given value 'α'.

Then an ideal test would be the one which maximizes P_D for every $\theta \in \Lambda'$ while assuring $P_F \leq \alpha$ for all θ. Such a test is known as uniformly most powerful test.

Let's assume the following:

Parametric family of distributions: P_θ where $\theta \in \Lambda$ & $P_\theta \sim N(\theta, \sigma^2)$ & the hypothesis with equal prior probabilities $\pi_0 = \pi_1 = 1/2$

$$H_0 : x \sim N(A, \sigma^2) \quad ; \quad \theta = A$$
$$H_1 : x \sim N(\mu, \sigma^2) \quad ; \quad \mu \in (A, \infty)$$

Here A is fixed number.

We can say: $\Lambda_0 = \{A\}$

$\Lambda_1 = \{A, \infty)$

from Neyman - Pearson hypothesis: (Page - 16)

$$P_D = 1 - \phi (\phi^{-1} (1-\alpha) + d)$$

where ϕ is q-fuction.

so, for UMP

$$P_D(\theta) = 1 - \phi (\phi^{-1} (1-\alpha) + \frac{\mu - A}{\sigma})$$

where $\mu \in (A, \infty)$

The above test maximizes P_D for every $\mu \in (A, \infty)$ assuming $P_F \leq \alpha$.

Signal detection:

Let's say that we observed a continuous-time waveform which contains one of the two possible signals which are corrupted by noise. Our objective is to decide which signal is present.

Let's take 'n' (finite) samples of the observed waveform:

Now the hypothesis can be defined as:

$$H_0 : Y_K = N_K + S_{0K}$$

vs

$$H_1 : Y_K = N_K + S_{1K}$$

where $K = 1, 2, \ldots n$

Here:

$\underline{Y} = (Y_1, Y_2 \ldots Y_n)^T$ is observed vector.

$\underline{N} = (N_1, N_2 \ldots N_n)^T$ is noise vector

$\underline{S_0} = (S_{01}, S_{02} \ldots S_{0n})^T$ &

$\underline{S_1} = (S_{11}, S_{12} \ldots S_{1n})^T$ are two possible signal vectors.

The transmitted signal can be classified as one of the three basic types:

- They are completely known signals
- Some of their parameters are known
- They are completely unknown.

Let's assume the statistics of the transmitted signals are known at the receiver.:

So the conditional density of our observation vector $\underline{y}$ will be

$$P_{\underline{N}}(\underline{y} - \underline{s}_j)$$

($\underline{y}$ & $\underline{s}_j$ are known)

So the likelihood ratio will be

$$L(\underline{y}) = \frac{E\{P_{\underline{N}}(\underline{y} - \underline{s}_1)\}}{E\{P_{\underline{N}}(\underline{y} - \underline{s}_0)\}}$$

where $P_{\underline{N}}(\underline{y} - \underline{s}_1)$ is the joint distribution of all the samples.

(Assuming that noise $\underline{N}_K$ is not iid, general case)

Here the expectation is for $S_j = s_j$.

(Assuming that the signal is not known, general case)

Detection of the Deterministic Signals
(coherent detection)

Most of the time the transmitted signals $\underline{S_0}$, $\underline{S_1}$ are completely deterministic.
Let's assume $\underline{S_j} = \underline{s_j}$ (known)

So the likelihood ratio will be:

$$L(\underline{y}) = \frac{P_{\underline{N}}(\underline{y} - \underline{S_1})}{P_{\underline{N}}(\underline{y} - \underline{S_0})}$$

$$= \frac{P_{\underline{N}}(y_1 - S_{11}, y_2 - S_{12} - - - - y_n - S_{1n})}{P_{\underline{N}}(y_1 - S_{01}, y_2 - S_{02} - - - - y_n - S_{0n})}$$

Let's assume that the noise samples $N_1, N_2 - - - N_n$ are independent.
So,

$$L(\underline{y}) = \frac{P_{N_1}(y_1 - S_{11}) \cdot P_{N_2}(y_2 - S_{12}) - - - - P_{N_n}(y_n - S_{1n})}{P_{N_1}(y_1 - S_{01}) \cdot P_{N_2}(y_2 - S_{02}) - - - P_{N_n}(y_n - S_{0n})}$$

$$L(\underline{y}) = \prod_{K=1}^{n} L_K(y_K)$$

where:

$$L_k(y_k) = \frac{p_{N_k}(y_k - s_{1k})}{p_{N_k}(y_k - s_{0k})}$$

for Bayesian hypothesis:

$$L(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{1-\pi_0} = \tau$$

$$\log(L(\underline{y})) \underset{H_0}{\overset{H_1}{\gtrless}} \log(\tau)$$

$$\boxed{\sum_{k=1}^{n} \log(L_k(y_k)) \underset{H_0}{\overset{H_1}{\gtrless}} \log(\tau)}$$

Date
Page

Detection of the Deterministic Signals in Gaussian Noise :

Let's assume :

$\underline{S_j}$: known transmitted signals which take values s_j

$\underline{N}$: Gaussian random noise vector with mean vector $\underline{\mu}$ & Covariance matrix Σ_N

$$\Sigma \triangleq E\{(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^T\}$$

$$\& \quad \mu \triangleq E(\underline{X})$$

So the Prob. density function for N samples (In general)

$$P_{\underline{x}}(x) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}} \cdot \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T\Sigma^{-1}(\underline{x}-\underline{\mu})\right\}$$

for the hypothesis 'H_0' & 'H_1' the likelihood ratio will be :

$$L(\underline{y}) = \frac{P_1(\underline{y})}{P_0(\underline{y})}$$

$$= \frac{\frac{1}{(2\pi)^{N/2}|\Sigma_{N1}|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y}-\mu_1)^T \Sigma_{N1}^{-1} (\underline{y}-\mu_1)\right\}}{\frac{1}{(2\pi)^{N/2}|\Sigma_{N0}|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y}-\mu_0)^T \Sigma_{N0}^{-1} (\underline{y}-\mu_0)\right\}}$$

for Bayesian hypothesis testing:

$$L(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

$$\log(L(\underline{y})) \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

so,

$$\frac{1}{2}\left\{(\underline{y}-\mu_0)^T \Sigma_{N0}^{-1} (\underline{y}-\mu_0) - (\underline{y}-\mu_1)^T \Sigma_{N1}^{-1} (\underline{y}-\mu_1)\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

Case : When $\Sigma_0 = \Sigma_1 = \Sigma$

$$\underline{y}^T (\Sigma_0^{-1} - \Sigma_1^{-1}) \underline{y} + (\underline{\mu_1}^T \Sigma_1^{-1} - \underline{\mu_0}^T \Sigma_0^{-1}) \underline{y} \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

$$\left((\mu_1 - \mu_0)^T \Sigma^{-1}\right) \underline{y} \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

Case : When $\mu_1 = \mu_0 = 0$

$$\underline{y}^T (\Sigma_0^{-1} - \Sigma_1^{-1}) \underline{y} \underset{H_0}{\overset{H_1}{\gtrless}} \tau'$$

Detection of Non-coherent Signals (Modulated Sinusoidal Carriers)

Let's assume the signals are:

$$\underline{S}_0(\theta) = 0$$

& $\underline{S}_1(\theta) = \underline{S}(\theta)$

and the hypothesis are:

$$H_0 : \quad Y_K = N_K$$

& $H_1 : \quad Y_K = N_K + S_K(\theta)$

$$N_K \sim \mathcal{N}(0, \sigma^2) \text{ noise}$$

where:

$$S_K(\theta) = a_K \sin[(K-1)\omega_c T_s + \theta]$$

$$K = 1, 2, \ldots n$$

where:

$a_1, a_2 \ldots a_n$ are amplitudes

θ is random phase ~~noise~~ angle

$$\theta \in U[0, 2\pi]$$

ω_c : carrier frequency

T_s : Sampling interval

Assume that the number of samples per cycle of sinusoid is an integer larger than '1'.

So the likelihood ratio will be:

$$L(\underline{y}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{P_N(\underline{y} - \underline{s}(\theta))}{P_N(\underline{y})}\, d\theta$$

↳ As the phase is uniformly distributed from 0 to 2π so the expected value.

$$L(\underline{y}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_k - s_k)^2/2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_k)^2/2\sigma^2}}\, d\theta$$

$$L(\underline{y}) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{(2\underline{y}_k s_k(\theta) - s_k^2(\theta))}{2\sigma^2}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{1}{\sigma^2}\left(\sum_{K=1}^{n} y_K s_K(\theta) - \frac{1}{2}\sum_{K=1}^{n} s_K^2(\theta)\right)\right\} d\theta$$

Let's assume:

$$y_c = \sum_{K=1}^{n} a_K y_K \cos[(K-1)\omega_c T_s]$$

$$y_s = \sum_{K=1}^{n} a_K y_K \sin[(K-1)\omega_c T_s]$$

So,

$$\sum_{K=1}^{n} y_K S_K(\theta) = y_c \sin\theta + y_s \cos\theta$$

$(\sin(A+B) = \sin A \cos B + \cos A \sin B)$

&

$$-\frac{1}{2}\sum_{K=1}^{n} S_K^2(\theta) = -\frac{1}{4}\sum_{K=1}^{n} a_K^2 + \frac{1}{4}\sum_{K=1}^{n} a_K^2 \cos(2(K-1)\omega_c T_s + 2\theta)$$

$(\sin^2 A = (1/2) - (1/2)\cos 2A)$

$$\because \sum_{K=1}^{n} \cos(2(K-1)\omega_c T_s + 2\theta) \approx 0$$

(sum of +1's & -1's)

So,

$$-\frac{1}{2}\sum_{K=1}^{n} S_K^2(\theta) = -\frac{n\bar{a}^2}{4}$$

So.

$$L(\underline{y}) = e^{-(n\bar{a}^2/4\sigma^2)} \times \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\frac{1}{\sigma^2}(y_c \sin\theta + y_s\cos\theta)\right\} d\theta$$

$$L(\underline{y}) = e^{-(n\bar{a}^2/4\sigma^2)}\, I_0(r/\sigma^2)$$

where $r = \sqrt{y_c^2 + y_s^2}$

Chernoff & Related Bounds :

Let's assume two hypothesis :

H_0 : $X \sim P_0$ (pmf p_0) ; π_0

vs

H_1 : $X \sim P_1$ (pmf p_1) ; π_1

so the probability of error :

$$P_e = \int \min\{\pi_0 P_0(x), \pi_1 P_1(x)\} dx$$

Sometimes it becomes practically impossible to compute probability of error or P_M, P_D. In those situations it is sufficient to obtain good upper bounds. Chernoff bound gives bound on the performance of likelihood ratio.

Example :

$$P_e^n = \int_{\mathbb{R}} \min\{\pi_0 P_0^n(x), \pi_1 P_1^n(x)\} dx$$

'min of product of 'n' numbers.

$$\min\{a, b\} \leq a^{1-s} b^{s} \quad \text{for } s \in [0,1]$$

Proof :

assume $a \leq b$

$$\min\{a, b\} = a \leq a^{1-s} b^{s}$$

$$\Rightarrow a^{s} \leq b^{s} \quad \text{for } s \in [0,1]$$

Now the prob. of error :

$$P_e = \int_{\mathbb{R}} \min\{\pi_1 P_1(x), \pi_0 P_0(x)\} \, dx$$

$$\leq \int_{\mathbb{R}} (\pi_1 P_1(x))^{s} (\pi_0 P_0(x))^{1-s} \, dx$$

for $s \in [0,1]$

$$= \int_{\mathbb{R}} (\pi_1^{s} \pi_0^{1-s}) (P_0(x)^{s} P_1(x)^{1-s}) \, dx$$

$$\leq \int_{\mathbb{R}} (P_0(x)^{s} P_1(x)^{1-s}) \, dx$$

$$= \int_{\mathbb{R}} \left(\frac{P_0(x)}{P_1(x)} \right)^{s} P_1(x) \, dx$$

$$P_e \leq E[l^{s}(x) \mid H_1]$$

or,

$$P_e \le \int_{R} \left(\frac{P_1(x)}{P_0(x)} \right)^s P_0(x) \, dx$$

$$P_e \le E\left[l^s(x) / H_0 \right]$$

$$\triangleq E_0\left[l^s(x) \right]$$

$$\boxed{P_e \le E\left[l^s(x) / H_0 \right]}$$

Chernoff bound

⇒ Find out an 's' for min of $E[l^s(x)/H_0]$ to get the tightest bound.

Bhattacharayya bound:

when $s = 1/2$. the solution for 'min' known as bhattacharayya bound.

for $s = 1/2$

$$P_e \le \int_{R} \sqrt{P_1(x) \, P_0(x)} \, dx$$

(may not be the tightest bound)

Bhattacharyya Distance & Coefficient

Bhattacharyya coefficient is a measure of the amount of overlap between two statistical samples.

It determines the relative closeness of two samples.

Bhattacharyya coefficient $BC(p,q)$

$$BC(p,q) = \sum_{x \in X} \sqrt{p(x)\, q(x)}$$

$$BC(p,q) = \int \sqrt{p(x)\, q(x)}\, dx$$

Bhattacharyya distance:

$$D_B(p,q) = -\ln\left(BC(p,q)\right)$$

Intution from Bhattacharyya Bound:

$$P_e \leq \int \sqrt{P_0(x)\, P_1(x)}\, dx$$

When the separation between two pdf increases, Bhattacharyya coefficient decreases. Bhattacharyya coefficient becomes max. when both the pdf's are same.

$$\left\{\begin{array}{l}\text{More separation in} \\ \text{pdfs } P_0(x) \text{ \& } P_1(x)\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{less Bhattacharyya} \\ \text{coeff. } \int\sqrt{P_0(x) P_1(x)}\,dx\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{tighter bound} \\ \text{on prob. of} \\ \text{error.}\end{array}\right\}$$

Example:

$x = -10 : 0.01 : 10;$

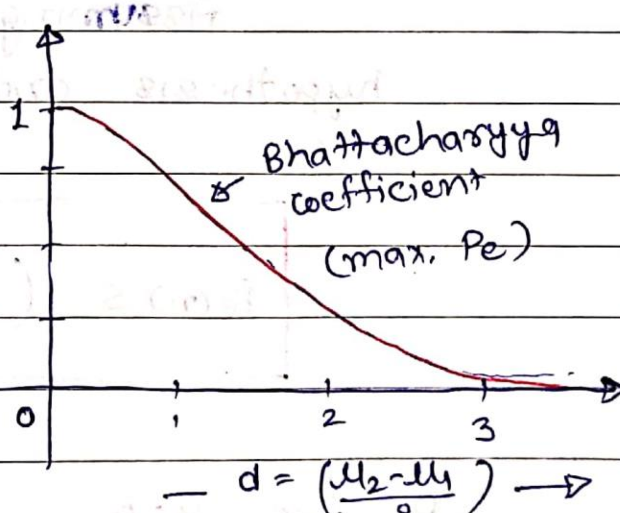
$\mu_1 = 1;$

$\mu_2 = 1 : 0.5 : 8;$

$d = (\mu_2 - \mu_1)/2;$

$\sigma = 1$

$P_1(x), P_2(x) \sim N(\mu, \sigma^2)$



Bhattacharyya Bound is not the best bound on Prob. of error. It is just one of the chernoff bound when $s = 1/2$. 's' can be tuned to get the best upper bound on prob. of error.

Chernoff Bound for 'n' samples:

So the prob of error:

$$P_e = \int_{\mathbb{R}} \min\{\pi_1 P_1^{(n)}(x), \pi_0 P_0^{(n)}(x)\} dx$$

$$P_e \leq \int_{\mathbb{R}} (\pi_1^{s} \pi_0^{1-s}) (P_1^{(n)}(x))^{s} (P_0^{(n)}(x))^{1-s} dx$$

$$P_e \leq E_0 [(l^{(n)}(x))^{s}]$$

Assuming $x_1, x_2 --- x_n$ are iid under each hypothesis and are copies of 'x'.

So,

$$\boxed{P_e(n) \leq (E_0 [(l(x))^{s}])^{n}}$$

⇒ for $s=0$

$$P_e \leq \int P_0(x) dx = 1$$

⇒ for $s=1$

$$P_e \leq \int P_1(x) dx = 1$$

$\Rightarrow$ for $s \in (0,1) \Rightarrow l^s(x) < 1$.

$\Rightarrow$ for $n \to \infty$.

$$\left(E_0[l^s(x)]\right)^n \to 0$$

for large number of 'n' the prob. of error goes to zero.

Error Exponent :

$$= -\frac{1}{n} \log(P_e(n))$$

As :

$$P_e(n) \leq \left(E_0[l^s(x)]\right)^n$$

$$\Rightarrow \log P_e(n) \leq n \log\left(E_0[l^s(x)]\right)$$

$$\Rightarrow \frac{1}{n} \log P_e(n) \leq \log\left(E_0[l^s(x)]\right)$$

$$\boxed{-\frac{1}{n} \log P_e(n) \geq -\log\left(E_0[l^s(x)]\right)}$$

" $-\log\left(E_0[l^s(x)]\right)$ is chernoff distance."

Chernoff Bound for Location testing in Gaussian Noise:

Let the Hypothesis are

$$H_0 : Y = \mathcal{N}(0, \sigma^2)$$
$$H_1 : Y = \mu + \mathcal{N}(0, \sigma^2)$$

Let's assume prior probabilities are equal $\pi_0 = \pi_1 = 1/2$.

so, Simple hypothesis testing:

$$L(y) = \frac{P_1(y)}{P_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(y-\mu)^2/2\sigma^2\right)}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-y^2/2\sigma^2\right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

decision rule will be:

$$\boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu}{2}}$$

Now for 'n' realization:

Let's assume $Y_1, Y_2 \dots Y_n$ are iid.

$$l^{(n)}(y) = \frac{P_1^n(y)}{P_0^n(y)} = \prod_{k=1}^{n} \frac{P_1(y_k)}{P_0(y_k)}$$

Decision Rule:

$$\sum_{k=1}^{n} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{n\mu}{2}$$

Let:

$$T = \sum_{k=1}^{n} Y_k \sim N(n\mu, n\sigma^2)$$

{as, $y_1, y_2 \dots Y_n$ are iid with distribution $\sim N(\mu, \sigma^2)$}

Now for chernoff bound:

$$\therefore E_0\left[\left(l^{(n)}(x)\right)^s\right] = E_0\left\{\left[l^{(s)}(x)\right]^n / H_0\right\}$$

$$= E\left\{\left(\prod_{k=1}^{n} \left(L(y_k)\right)^s / H_0\right)\right\}$$

classmate

Date

Page

As: $L(y_K) = e^{-\frac{(\mu^2 - 2\mu y_K)}{2\sigma^2}}$

So.

$$(L(y_K))^s = e^{-\frac{(\mu^2 s - 2\mu s y_K)}{2\sigma^2}}$$

$y_1, y_2 \dots y_n$ are iid gaussian

So.

$$(L(y_K))^s = (L(y))^s = e^{-\frac{(\mu^2 s - 2\mu s y)}{2\sigma^2}}$$

So. Now:

$$= E\left\{ \prod_{K=1}^{n} \left((L(y_K))^s / H_0 \right) \right\}$$

$$= E\left\{ \prod_{K=1}^{n} \left\{ e^{-\frac{(\mu^2 s - 2\mu s y)}{2\sigma^2}} / H_0 \right\} \right\}$$

$$= E\left\{ \prod_{K=1}^{n} \left(e^{-\frac{(\mu^2 s - 2\mu s y)}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-y^2/2\sigma^2} \right) \right\}$$

$$= E\left\{ \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y^2 - 2\mu s y + \mu^2 s)}{2\sigma^2}} \right)^n \right\}$$

Chernoff Distance:

$$D_c(P_0, P_1; s) = -\log\left(E_0[\,l^{s}(x)\,]\right)$$

so,

$$= -\log\left\{E\left(\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y^2 - 2\mu s y + \mu^2 s)}{2\sigma^2}}\right)\right\}$$

$$= -\log\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y^2 - 2\mu s y + \mu^2 s^2)}{2\sigma^2}} \cdot e^{-\frac{\mu^2 s(1-s)}{2\sigma^2}}\, dy\right\}$$

$$= -\log\left\{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y-\mu s)^2}{2\sigma^2}}\, dy \cdot e^{-\frac{\mu^2 s(1-s)}{2\sigma^2}}\right\}$$

$$= -\log\left\{e^{-\frac{\mu^2 s(1-s)}{2\sigma^2}}\right\}$$

so,

$$\boxed{D_c(P_0, P_1; s) = \frac{\mu^2 s(1-s)}{2\sigma^2}}$$

For max. of $D_c(P_0, P_1; s)$ for $0 \le s \le 1$

$s = 1/2$,

$$\boxed{D(P_0, P_1) = \frac{\mu^2}{8\sigma^2}}$$

classmate
Date
Page

Multiple Hypothesis Testing
(M-ary Hypothesis)

In many problems we required to distinguish between more than two hypothesis.

Let's assume that there are M possible hypothesis:

$$H_0, H_1, \dots H_{M-1}.$$

The expected Bayes risk is:

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} P(H_i / H_j)\, P(H_j)$$

Let's say that complete region is partitioned into M disjoint regions.

$$\{ \Gamma_i \mid i = 1, 2 \dots M \}$$

such that:

$$\delta(y) = i \quad \text{if } y \in \Gamma_i$$

(MPE) Minimum probability of error rule

$$j^* = \arg\max_{i = 0, 1 \dots M-1} \pi_i P_i(y)$$

Now the Bayes risk or Pe :

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \int_{\Gamma_i} P(y/H_j)\, P(H_j)\, dy$$

$$= \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \int_{\Gamma_i} P(H_j/y)\, P(y)\, dy$$

(Bayes rule)

$$R = \sum_{i=0}^{M-1} \int_{\Gamma_i} \sum_{j=0}^{M-1} P(H_j/y)\, P(y)\, dy$$

We should choose the hypothesis that minimizes

$$\sum_{j=0}^{M-1} P(H_j/y) \quad \text{over} \quad i = 0, 1 \ldots M-1$$

Now the decision rule that minimizes Pe :

$$= \sum_{j=0,\, j\neq i}^{M-1} P(H_j/y) = \sum_{j=0}^{M-1} P(H_j/y) - P(H_i/y)$$

By maximizing $P(H_i/y)$, Prob. of error can be minimized.

to decide Hypothesis H_K the minimum Pe decision rule: $P(H_K/y) > P(H_i/y)$
$i \neq K$

So the M-ary maximum a posteriori probability (MAP) decision rule :

$$\boxed{P(y/H_k) > P(y/H_i)} \quad i \neq k$$

M-ary maximum likelihood (ML) decision rule.

Example:

Let's assume the following hypothesis

$H_0 \; : \; y[n] = 1 + \mathcal{N}(0, \sigma^2)$

$H_1 \; : \; y[n] = 2 + \mathcal{N}(0, \sigma^2)$

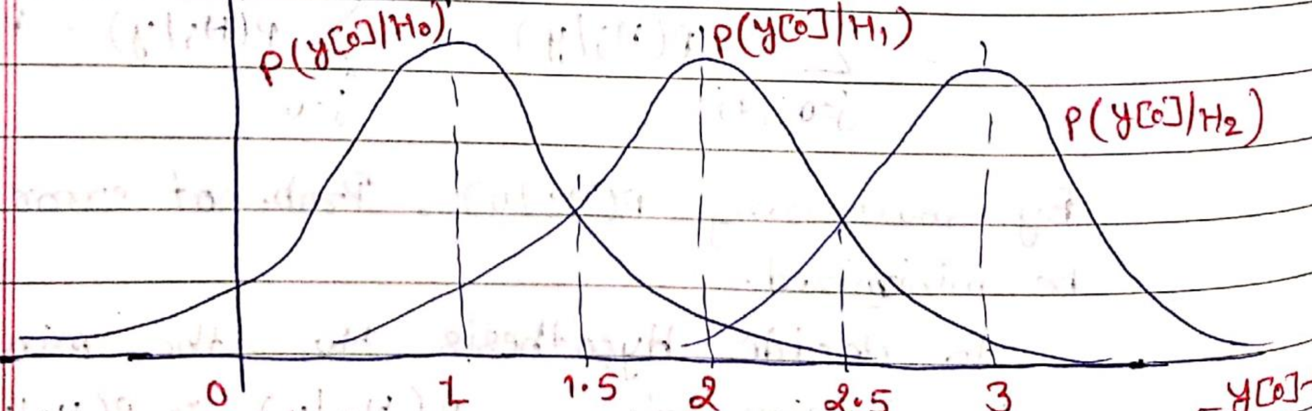
$H_2 \; : \; y[n] = 3 + \mathcal{N}(0, \sigma^2)$

$n = 0, 1 \ldots N-1$

Also assume that prior probabilities are equal :

$\pi_0 = \pi_1 = \pi_2 = 1/3$

or $P(H_0) = P(H_1) = P(H_2) = 1/3$.



Decision Rules:

for hypothesis H_0

$$P(y|H_0) > P(y|H_i)$$

or $$P_0(y) > \{P_1(y), P_2(y)\}$$

Similarly:

for hypothesis H_1 : $P_1(y) > \{P_0(y), P_2(y)\}$

& for hypothesis H_2 : $P_2(y) > \{P_0(y), P_1(y)\}$

so,

H_0 : if $\bar{y} < 1.5$

H_1 : if $1.5 < \bar{y} < 2.5$

H_2 : if $\bar{y} > 2.5$

Now the prob. of error.

$$P_e = \underset{\bar{y} < 1.5}{\Pi_0 P_0(x)} + \underset{1.5 < \bar{y} < 2.5}{\Pi_1 P_1(x)} + \underset{\bar{y} > 2.5}{\Pi_2 P_2(x)}$$

so.

$$P_e = \frac{1}{3}\left\{ Q\left(\frac{1.5-1}{\sqrt{\sigma^2/N}}\right) + Q\left(\frac{2.5-2}{\sqrt{\sigma^2/N}}\right) + 1 - Q\left(\frac{1.5-2}{\sqrt{\sigma^2/N}}\right) + 1 - Q\left(\frac{2.5-3}{\sqrt{\sigma^2/N}}\right) \right\}$$

so, $$\boxed{P_e = \frac{2}{3} + \frac{4}{3} Q\left(\frac{0.5}{\sqrt{\sigma^2/N}}\right)}$$

Signal Detection when they are orthogonal:

Let's assume the hypothesis are:

$$H_j : \underline{Y} = \underline{S_j} + \underline{N} \quad \text{for } j = 0, 1 \ldots M-1$$
$$S_j \in \mathbb{R}^n$$

s.

$$s_i^T s_j = \begin{cases} E, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Signals are orthogonal and have equal power.

$$\|S_0\|^2 = \|S_1\|^2 = \cdots \|S_{M-1}\|^2$$

s. $\underline{N} \sim \mathcal{N}(0, \sigma^2 I)$

(It became M-ary decision problem where transmitted signals are orthogonal)

Now:

Distribution of $Y \sim N(s_j, \sigma^2 I)$

Condition pdf of Y under hypothesis 'j'

$$P_j(y) = \frac{1}{(2\pi\sigma^2)^{M/2}} \exp\left\{ -\|\underline{y} - s_j\|^2 / 2\sigma^2 \right\}$$

And the decision rule will be:

$$j^*(y) = \arg\max_{j=0,1\cdots M-1} \pi_j \, P_j(y)$$

Assuming hypothesis with equal probabilities

$$j^*(y) = \arg\max_{j=0,1\cdots M-1} P_j(y)$$

$$= \max\{P_0(y), P_1(y), \cdots P_{M-1}(y)\}$$

$$= \min\{\|y-s_0\|^2, \|y-s_1\|^2, \cdots \|y-s_{M-1}\|^2\}$$

$$= \min\{-ys_0, -ys_1, \cdots -ys_{M-1}\}$$

$$= \max\{ys_0, ys_1, \cdots ys_{M-1}\}$$

$$\boxed{j^*(y) = \max\{s_j^T y\}}$$

Now Let $T_j = s_j^T y$, $j = 0, 1 \cdots M-1$

So. Prob. of 'No Error' will be
(For hypothesis H_0)
$= \text{Prob.}(T_1 \leq u, T_2 \leq u \cdots T_{M-1} \leq u \mid T_0 = u)$

Now Let's calculate Prob. { No Error }

$$1 - P_e = \frac{1}{M} \sum_{j=0}^{M-1} \text{Prob}(H_j) \cdot \text{Prob.}\{\text{No error} \mid H_j\}$$

As all hypothesis are symmetrical so.

$$= \text{Prob.}\{\text{No error} \mid H_0\} \cdot \text{Prob.}(H_0)$$

So. Now:

$$= \int_{\mathbb{R}} \Pr\{\text{No error} \mid H_0, T_0 = u\} \, P_{T_0 \mid H_0}(u) \, du$$

As we know

$$\Pr\{\text{No error} \mid T_0 = u, H_0\}$$

$$= \Pr\{T_1 \leq u, T_2 \leq u \ldots T_{M-1} \leq u \mid T_0 = u, H_0\}$$

$$= \left(\Pr(T_1 \leq u \mid H_0)\right)^{M-1}$$

As all are symmetrical

Now we need to calculate

$$\Pr(T_1 \leq u \mid H_0)$$

Now:

Mean: $E[T_j | H_0] = E[s_j^T y | H_0]$

$$= s_j^T s_0 = \begin{cases} \mathcal{E}, & j=0 \\ 0, & j \neq 0 \end{cases}$$

Covariance Matrix of 'T':

when $i \neq 0$ & $j \neq 0$

$$\Sigma_{ij} = E[T_i T_j | H_0] = E[s_i^T y \, y^T s_j | H_0]$$

$$= s_i^T E[y y^T | H_0] s_j$$

$$= s_i^T [\sigma_N^2 I] s_j \qquad \begin{cases} \text{when } i=j\neq 0 \\ 0, \text{otherwise} \end{cases}$$

$$= \sigma_N^2 s_i^T s_j$$

$$\Sigma_{ij} = \sigma_N^2 \mathcal{E}$$

when $i=0$ & $j=0$

$$\Sigma_{00} = E[(T_0 - \mathcal{E})^2] \qquad \{\sigma_x^2 = E[(x-E(x))^2]$$

$$= E[s_0^T (y-s_0)(y-s_0)^T s_0]$$

$$\{\text{As } T_0 - \mathcal{E} = s_0^T y - s_0^T s_0\}$$

$$= \sigma_N^2 s_0^T s_0$$

$$\Sigma_{00} = \sigma_N^2 \mathcal{E}$$

So.

$$\boxed{\Sigma_{ij} = \sigma_N^2 \mathcal{E}}$$

Now for

$$Pr(T_1 \le u \mid H_0)$$

$$T_1 \sim \mathcal{N}(0, \sigma_N^2 E)$$

So,

$$Pr\{T_1 \le u \mid H_0\} = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi \sigma_N^2 E}} \cdot e^{-t^2/2(\sigma_N^2 E)} \, dt$$

Let $z = t/\sigma_N \sqrt{E}$

So

$$= \int_{-\infty}^{u/\sigma_N\sqrt{E}} \frac{1}{\sqrt{2\pi \sigma_N^2 E}} \, e^{-z^2/2} \, dt$$

$$Pr\{T_1 \le u \mid H_0\} = \phi\left(\frac{u}{\sigma_N \sqrt{E}}\right) \quad \{\phi \rightarrow \text{CDF}\}$$

Now:

$$\left(Pr\{T_1 \le u \mid H_0\}\right)^{M-1} = \left(\phi\left(\frac{u}{\sigma_N\sqrt{E}}\right)\right)^{M-1}$$

& we also know:

$$Pr_{T_0/H_0}(u) = \frac{1}{\sqrt{2\pi \sigma_N^2 E}} \, e^{-(u-E)^2/2\sigma_N^2 E}$$

So Now:

$$1 - P_e = \int_{\mathbb{R}} \left(\phi\left(\frac{u}{\sigma_N \sqrt{E}}\right)\right)^{M-1} \frac{1}{\sqrt{2\pi \sigma_N^2 E}} e^{-(u-E)^2/2\sigma_N^2 E} \, du$$

Let $z = \dfrac{u-E}{\sigma_N \sqrt{E}} \Rightarrow u = z\sigma_N\sqrt{E} + E$

$$\boxed{\int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{\left(\frac{x_0-\mu}{\sigma}\right)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy}$$

So,

$$= \int_{-\infty}^{\infty} \left(\phi\left(z + \frac{\sqrt{E}}{\sigma_N}\right)\right)^{M-1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$1 - P_e = E\left[\left(\phi\left(z + \frac{\sqrt{E}}{\sigma_N}\right)\right)^{M-1}\right]$$

$$Pr\{\text{No error}\} = E\left[\left(\phi\left(z + \frac{\sqrt{E}}{\sigma_N}\right)\right)^{M-1}\right]$$

$\dfrac{\sqrt{E}}{\sigma_N}$ is SNR. As the SNR increases

Prob. of 'no error' also increases.

Nonparametric and Robust Detection:

Till now we have considered hypothesis testing and signal detection problems under variety of assumptions. One of the assumption is that the probability distribution of the data is known under each hypothesis.

In practical situations it is unrealistic to assume that these distribution are known exactly. In such cases the two philosophies that can be applied are nonparametric and robust detection.

Nonparametric Detection:

When the number of parameters is so large that just few of them can not be chosen

or

the distribution of test statistic is largely insensitive to have exact knowledge of data distribution.

The Sign test (Non parametric detection):

Suppose that we have a sequence $Y_1, Y_2 \cdots Y_n$ of iid observations.
Let's assume:

H_0 : $p = 1/2$ $\quad p = P(Y_1 > 0)$
H_1 : $1/2 < p < 1$

$\begin{cases} H_0: & Y_K \text{ has median at zero} \\ H_1: & \text{median of } Y_K \text{ is greater than zero} \end{cases}$

Class of the distribution:

$\mathcal{P}_0$: $\{ P_0 : P((0,\infty)) = 1/2 \}$
$\mathcal{P}_1$: $\{ P_1 : 1/2 < P((0,\infty)) < 1 \}$

$\begin{cases} \text{Prob. that } Y_K > 0 \text{ is } 1/2 \quad vs \\ \text{Prob. that } Y_K > 0 \text{ is } (1/2, 1) \end{cases}$

Let's choose a distribution Q_1 in P_1 wher Q_1 has has density q_1.

$$P_r\{ x \in A / x \sim Q_1 \} = Q_1(A)$$

Now :

$$q_1^+(x) = \begin{cases} q_1(x) & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

divide $q_1(x)$ into two different pdfs

$$q_1^-(x) = \begin{cases} 0 , & x > 0 \\ q_1(x) , & \text{otherwise} \end{cases}$$

Now the density function:

Represent $q_0(x)$ in terms of $q_1^+(x)$ & $q_1^-(x)$ (See they are normalized)

$$q_0(x) = \frac{q_1^+(x)}{2\int_0^\infty q_1(t)\,dt} + \frac{q_1^-(x)}{2\int_{-\infty}^0 q_1(t)\,dt}$$

Now consider simple hypothesis pair:

Now θ_0 & θ_1 have pdf q_0 & q_1 and both are related (so LLR has become easy)

$$H_0' : Y_k \sim \theta_0 , \quad k = 1,2 \ldots n$$
$$H_1' : Y_k \sim \theta_1 , \quad k = 1,2 \ldots n.$$

Likelihood ratio test will be

$$L(y) = \frac{q_1(y)}{q_0(y)} = \prod_{k=1}^{n} \frac{q_1(y_k)}{q_0(y_k)}$$

for $x > 0$

$$q_0(x) = \frac{q_1^+(x)}{2\int_0^\infty q_1(t)\,dt} = \frac{q_1(x)}{2\,\theta_1^+}$$

$$\Rightarrow \quad \frac{q_1(x)}{q_0(x)} = 2\,\theta_1^+$$

where :

$$\theta_1^+ = \int_0^\infty q_1(x)\,dx$$

for $x \leq 0$

$$q_0(x) = \frac{q_1^-(x)}{2\int_{-\infty}^0 q_1(t)\,dt} = \frac{q_1(x)}{2(1-\theta_1^+)}$$

$$\Rightarrow \quad \frac{q_1(x)}{q_0(x)} = 2(1-\theta^+)$$

So,

$$L(y) = \prod_{k=1}^{n} \frac{q_1(y_k)}{q_0(y_k)}$$

where

$$\frac{q_1(y_k)}{q_0(y_k)} = \begin{cases} 2\,\theta_1^+ , & y_k > 0 \\ 2(1-\theta_1^+) , & y_k \leq 0 \end{cases}$$

$$L(y) = 2^n (\theta_1^+)^{t(y)} (1-\theta_1^+)^{n-t(y)}$$

where :

$t(y)$: no. of y_k's which are positive

$$L(y) = 2^n (1-\theta_1^+)^n \left(\frac{\theta_1^+}{1-\theta_1^+}\right)^{t(y)}$$

As we know from hypothesis
$$\frac{1}{2} < \theta_1^+ < 1$$

So, $$\frac{\theta_1^+}{1-\theta_1^+} > 1$$

that concludes that $L(y)$ is an increasing function of '$t(y)$'

⇒ When the $t(y)$ is : half of the y_k's are positive
$$H_0' : t(y) \sim B(n, 1/2)$$
$t(y)$ will be binomial distribution.

⇒ When $1/2 < t(y) < 1$: more than half of the y_k's are positive
$$H_1' : t(y) \sim B(n, \theta_1^+)$$
So,

$$H_0' : t(y) \sim B(n, 1/2)$$
$$H_1' : t(y) \sim B(n, \theta_1^+)$$

Now we can apply our hypothesis test here.

Now for Neyman - Pearson hypothesis Let's assume the prob. of false-alarm is α.

so.

$$\text{Prob}(\text{Detect } H_i' / H_0') = \frac{\sum_{K=\tau}^{n} {}^{n}C_K}{2^n}$$

$$\Rightarrow 2^{-n} \sum_{K=\tau+1}^{n} \frac{\lfloor n}{\lfloor n-K \; \lfloor K} = \alpha.$$

where Probability of detection willbe.

$$= \underset{\text{for } K>\tau}{\text{Prob}(\text{Detect } H_i' / H_i')} + \underset{\text{for } K=\tau}{\text{Prob}(\text{Detect } H_i' / H_i')}$$

$$= \sum_{K=\tau+1}^{n} {}^{n}C_K \, p^K (1-p)^{n-K} + \gamma \frac{\lfloor n}{\lfloor n-\tau \; \lfloor \tau} p^{\tau} (1-p)^{n-\tau}$$

or.

$$\boxed{\begin{aligned} P_D &= \sum_{K=\tau+1}^{n} {}^{n}C_K \, p^K (1-p)^{n-K} \\ P_F &= \frac{1}{2^n} \sum_{K=\tau+1}^{n} {}^{n}C_K. \end{aligned}}$$

Example : The sign test :

Drive a sign detector that uses nine observations and ensures a probability of false alarm prob. of 0.1 for detecting a positive signal A in presence of zero mean Gaussian noise and analyze it's performance for probability of data being positive of 0.75.

Let's assume the hypothesis are

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$
$$n = 1, 2 \ldots 9$$

Given that prob. of false-alarm is 0.1, so.

$$P_{FA} \Rightarrow \frac{1}{2^9} \sum_{K=\tau+1}^{9} {}^9C_K \leq 0.1$$

$$\Rightarrow \sum_{K=\tau+1}^{9} {}^9C_K \leq 51.2$$

A8:

$^9C_9 = 1$; $^9C_8 = 9$; $^9C_7 = 36$; $^9C_6 = 84$

So, $\sum_{K=6+1}^{9} {}^9C_K = 1+9+36 \leq 51.2$

So, $\tau = 6$.

for $\tau = 6$, Probability of false alarm will be

$P_{FA} = \frac{1}{2^9} \sum_{K=7}^{9} {}^9C_K = 0.089$

8. Probability of detection will be

$P_D = \sum_{K=7}^{9} {}^9C_K \cdot p^K (1-p)^{9-K}$

for $p = 0.75$

$P_D = 0.601$

Robust Detection:

Between two extreme situations (i. when complete distribution is available, ii when very little information is known) in which reasonabley accurate model is available.

For example:

H_0 : $X \sim P_0$ such that $P_0(x) = q_0(x)$

H_1 : $X \sim P_1$ such that $P_1(x) = (1-\epsilon)q_1(x) + \epsilon r(x)$

where $q_0(x)$ & $q_1(x)$ are known distributions & $r(x)$ is an unknown distribution. $\{\epsilon \in [0,1]\}$

Let the actual distribution of received signal is of the form:

$$(1-\epsilon)P_j + \epsilon M_j \quad ; \quad j = 0, 1 \ldots$$

where: P_j is nominal distribution

M_j is unknown distribution.

Now so.

H_0 : $y \sim (1-\epsilon)P_0 + M_0$

H_1 : $y \sim (1-\epsilon)P_1 + M_1$

Situations where robust detection can be used:

- External interference is present for fraction ϵ of the time
- Noise occurs with probability ϵ.
- Sensor faults or measurement errors.

The likelihood ratio $\frac{P_1(y)}{P_0(y)}$ is unbounbed

$\{\epsilon (0,\infty)\}$ & M_0 & M_1 can be present in such a way that it may cause $P_1(y) >> P_0(y)$ or $P_0(y) >> P_1(y)$. The likelihood ratio is not bounbed from above and from below. A relatively small deviation in the model might result in substantial performance loss in this situation.

Let's take $Q_0 \in P_0$ & $Q_1 \in P_1$ as least-favorable distributions.

The pdf of Q_0 & Q_1 are such as:

$$q_0(y_K) = \begin{cases} (1-\epsilon)\, P_0(y_K), & \text{if } P_1(y_K) < c'' P_0(y_K) \\ \frac{1-\epsilon}{c''} P_1(y_K), & \text{if } P_1(y_K) \geq c'' P_0(y_K) \end{cases}$$

&

$$q_1(y_K) = \begin{cases} (1-\epsilon)\, P_1(y_K), & \text{if } P_1(y_K) > c' P_0(y_K) \\ c'(1-\epsilon)\, P_0(y_K), & \text{if } P_1(y_K) \leq c' P_0(y_K) \end{cases}$$

Here: $0 < c' < 1$

& $1 < c'' < \infty$

Now the likelihood ratio:

$$\frac{q_1(y_K)}{q_0(y_K)} = \begin{cases} c' & \text{if } l(y_K) < c' \\ l(y_K) & \text{if } c' \leq l(y_K) \leq c'' \\ c'' & \text{if } l(y_K) > c'' \end{cases}$$