Decision-focused Learning in Partial Index Tracking

Financial Engineering Lab
Department of Industrial Engineering

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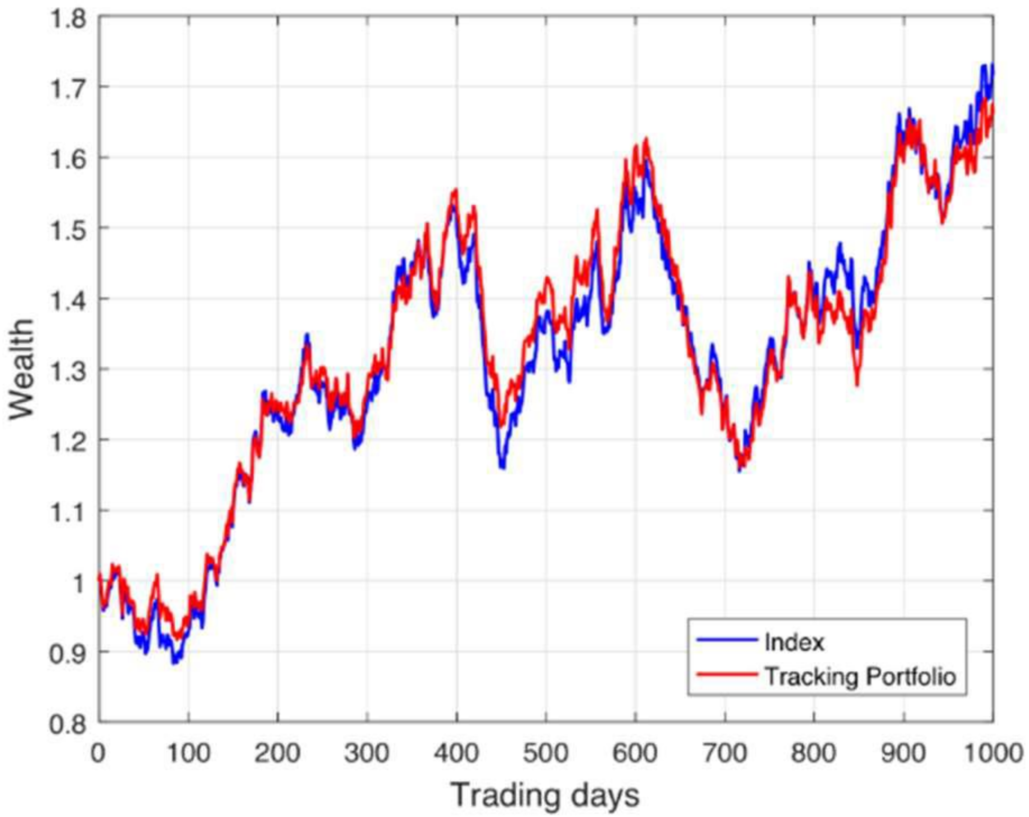
ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY
KAIST
Korea Advanced Institute of Science and Technology

Introduction

- **Partial Index Tracking**



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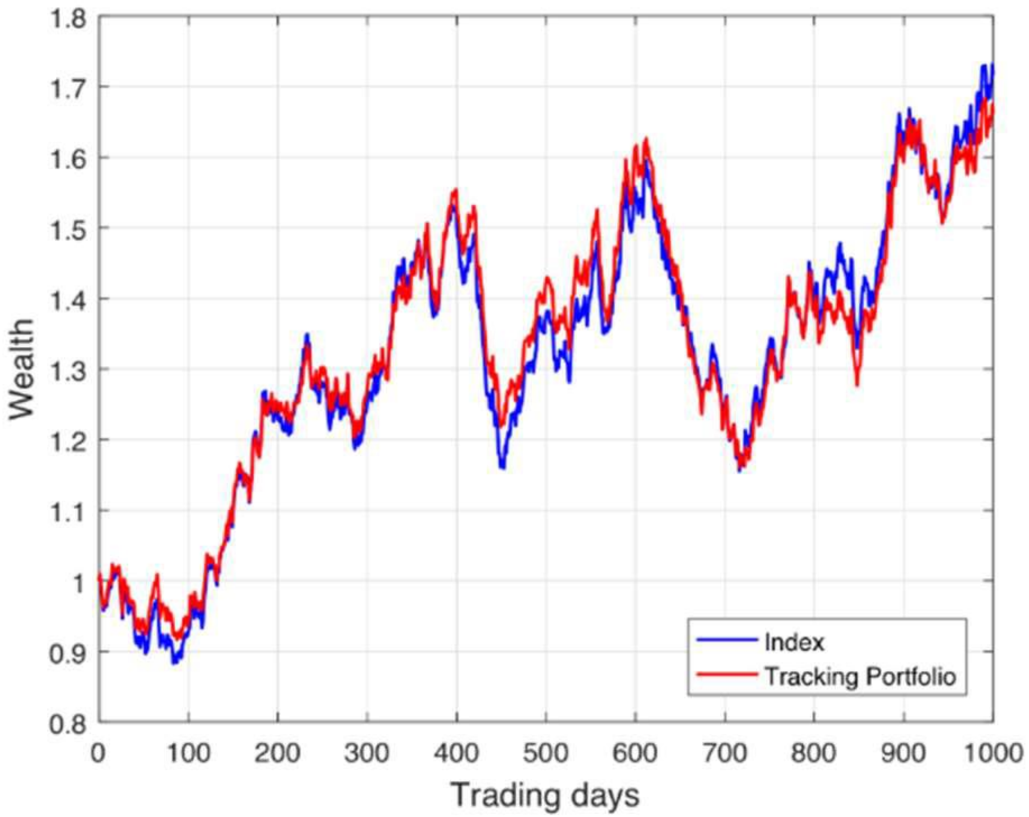

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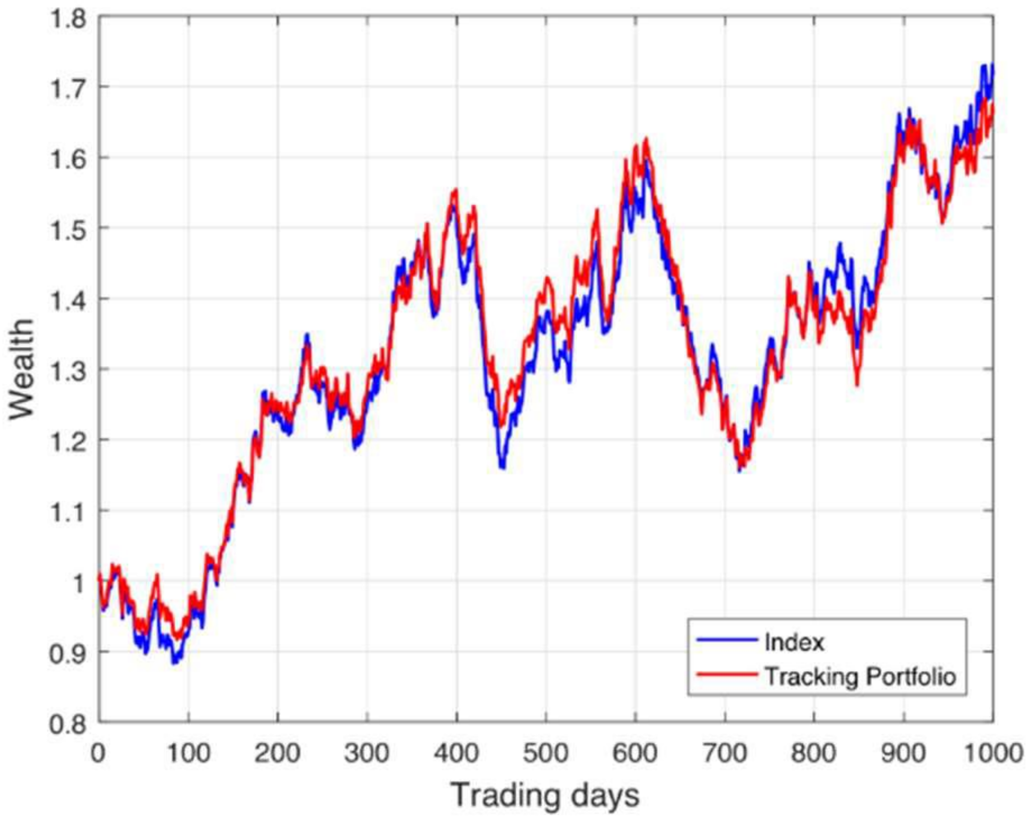

- **Partial**: use only subset of assets in index
- **Index:** measure the performance of finance market
- **Tracking:** aim to replicate performance of index

Basic Formulation

- **Variables & Functions**

Name	Definition
$n \in \mathbb{N}$	The number of assets
$\hat{r} \in \mathbb{R}^n$	The return of assets **(Uncertain)**
$w \in \mathbb{R}^n$	The weight of portfolio
$I \in \mathbb{R}$	Target Index
$\mathbf{1}$	One vector

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DFL in partial index tracking

- **Challenge**

  - Cardinality constraint makes the problem discrete and not differentiable

DFL in partial index tracking

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  - Cardinality constraint makes the problem discrete and not differentiable

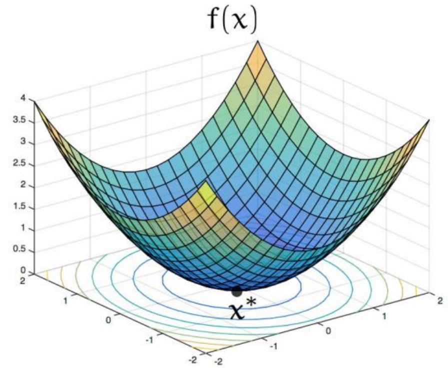
  - In this presentation, we introduce two strategies for this challenge

1) Semi-definite Relaxation Approach

$\min\limits_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2\left(\mathbf{1}^\top W \hat{R} C\right)$
	$\mathbf{Tr}(I_N W) = 1$
	$\mathbf{1}^\top \lvert W \rvert \mathbf{1} \le k\mathbf{Tr}(W)$
	$W \in S_n^+$

2) Surrogate Loss Approach

$\mathcal{L}_{DFL} \approx$



Semi-definite Relaxation Approach

- With semi-definite relaxation, we can transform the original problem to be convex

- Then, the CvxpyLayer can be used to obtain the gradients required for DFL

$$\begin{aligned} \min_{w \in \mathbb{R}^n} \quad & \|\hat{r}^\top w - I\|^2 \\ s.t \quad & \mathbf{1}^\top w = 1 \\ & \mathbf{Card}(w) \leq k \end{aligned}$$

Semi-definite Relaxation

$$\begin{aligned} x^\star(\theta) = \quad & \underset{x}{\text{argmin}} \ f(x;\theta) \\ \text{subject to} \quad & g(x;\theta) \leq 0 \\ & h(x;\theta) = 0 \end{aligned}$$

Semi-definite reformulation

- We first introduce semi-definite matrix variable $W = ww^\top \in \mathbb{R}^{n\times n}$ and $\hat{R} = \hat{r}\hat{r}^\top \in \mathbb{R}^{n\times n}$

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- The objective function can be reformulated as (**Tr**: Trace of matrix)

$$\|\hat{r}^\top w - I\|^2 = (\hat{r}^\top w)^2 - 2I(\hat{r}^\top w) + I^2 = \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W\hat{r}) + I^2$$

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- Since I^2 is constant,

$$\min_{w\in\mathbb{R}^n} \quad \|\hat{r}^\top w - I\|^2 \quad \longrightarrow \quad \min_{W\in S_n^+} \quad \mathbf{Tr}\big(\hat{R}W\big) - 2I(\mathbf{1}^\top W\hat{r})$$

Semi-definite reformulation

- Furthermore, the weight sum constraint and cardinality constraint are reformulated as

$$\min_{w \in \mathbb{R}^n} \quad \|\hat{r}^\top w - I\|^2 \quad \longrightarrow \quad \min_{W \in S_n^+} \quad \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W \hat{r})$$

$$\mathbf{1}^\top w = 1 \quad \longrightarrow \quad \mathbf{Tr}(I_N W) = 1$$

$$\mathbf{Card}(w) \le k \quad \longrightarrow \quad \mathbf{Card}(W) \le k^2$$

Semi-definite reformulation

- Finally, $W = ww^\top$ is equivalent ($\mathbf{Rank}(W) = 1$ & $W \in S_n^+$)

$$
\begin{aligned}
\min_{w \in \mathbb{R}^n} \quad & \|\hat{r}^\top w - I\|^2 \\
& \mathbf{1}^\top w = 1 \\
& \mathbf{Card}(w) \le k \\
& W = ww^\top
\end{aligned}
$$

$$
\begin{aligned}
\min_{W \in S_n^+} \quad & \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W \hat{r}) \\
& \mathbf{Tr}(I_N W) = 1 \\
& \mathbf{Card}(W) \le k^2 \\
& \mathbf{Rank}(W) = 1 \\
& W \in S_n^+
\end{aligned}
$$

Semi-definite relaxation

- Next, Let's relax the reformulated problem. We first drop the Rank constraint.

$\min_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W \hat{r})$
	$\mathbf{Tr}(I_n W) = 1$
	$\mathbf{Card}(W) \le k^2$
	$\cancel{\mathbf{Rank}(W) = 1}$
	$W \in S_n^+$

Semi-definite relaxation (cont.)

- Next, by relaxing $\mathbf{Card}(W) \le k^2$ to $\|w\|_1^2 \le k\|w\|_2^2$ and transforming to $\mathbf{1}^\top |W| \mathbf{1} \le k\mathbf{Tr}(W)$,

$$
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DPP-compliant

- When using CvxpyLayer, the formulation must satisfy DPP-compliant
  1. The optimization problem must be convex in variable W → **Already Satisfied**
  2. The optimization problem must be linear in parameter $\hat{R}$ → **Use $\boldsymbol{I} = \hat{\boldsymbol{r}}^{\top}\mathbf{C}$ where $\boldsymbol{C}$ is market cap weight**

$\min_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^{\top}W\hat{r})$
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→

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DFL loss for Partial Index Tracking

- As seen in previous session, the DFL loss is expressed as

$$\mathcal{L}(\widehat{\boldsymbol{R}}, \boldsymbol{R}) := \boldsymbol{Regret}(\boldsymbol{w}^*(\widehat{\boldsymbol{R}}), \boldsymbol{R})$$

$$= \boldsymbol{f}(\boldsymbol{w}^*(\widehat{\boldsymbol{R}}), \boldsymbol{R}) - \boldsymbol{f}(\boldsymbol{w}^*(\boldsymbol{R}), \boldsymbol{R})$$

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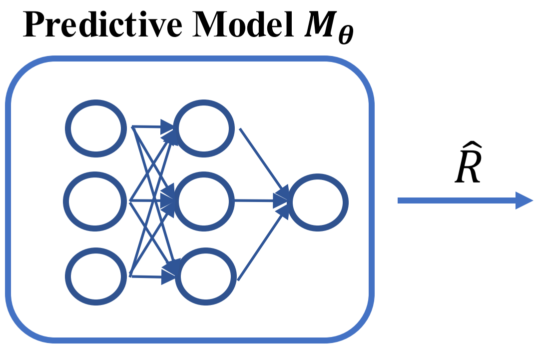
- Thus, in index tracking problem, the DFL loss is

$$\mathcal{L}(R, W^*(\hat{R})) = \mathbf{Tr}(RW^*(\hat{R})) - 2(\mathbf{1}^\top W^*(\hat{R})RC)$$

DFL with semi-definite relaxation

- The process of DFL in Partial Index Tracking



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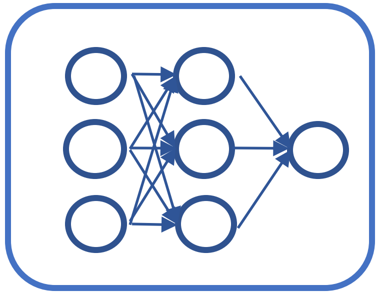
Semi-definite Optimization Layer

Predictive Model M_θ



$\hat{R}$

CVXpy

$$
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PyTorch

$W^*(\hat{R})$

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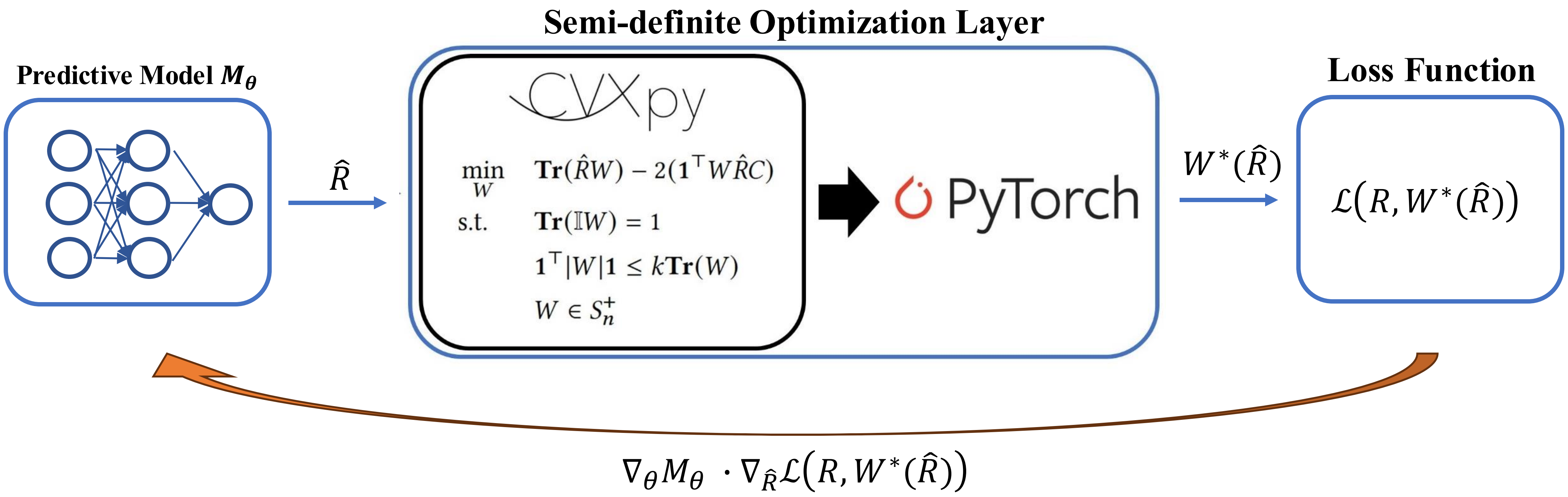

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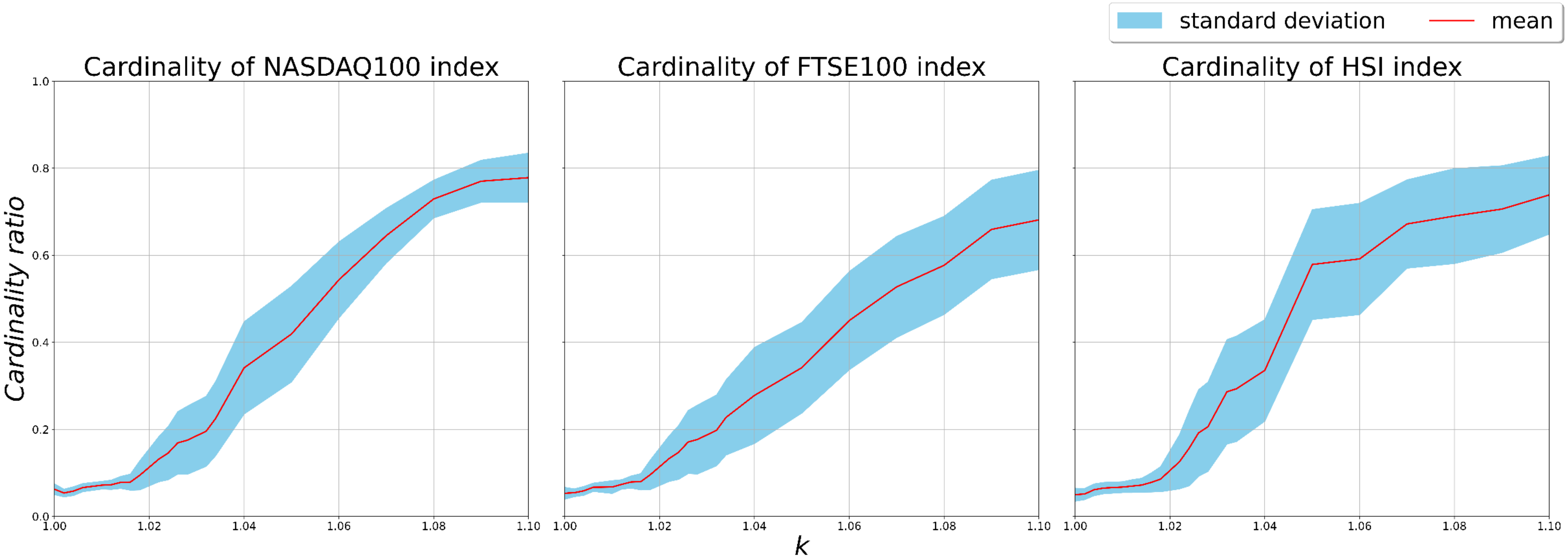

Surrogate Approach

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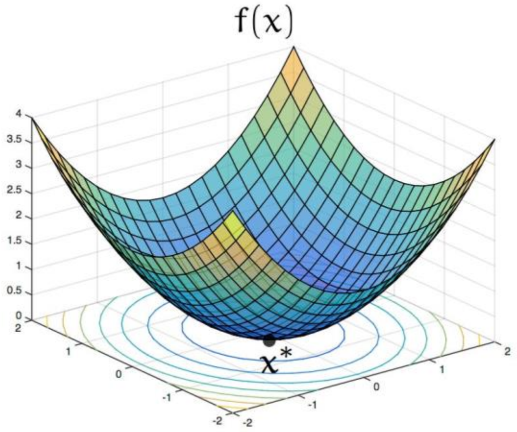
- Surrogate approach aim to learn $\mathcal{L}\big(r, w^*(\hat{r})\big)$ with another predictive model

r

$\hat{r}$ $\longrightarrow$



$\approx \mathcal{L}\big(r, w^*(\hat{r})\big)$

DFL with surrogate approach

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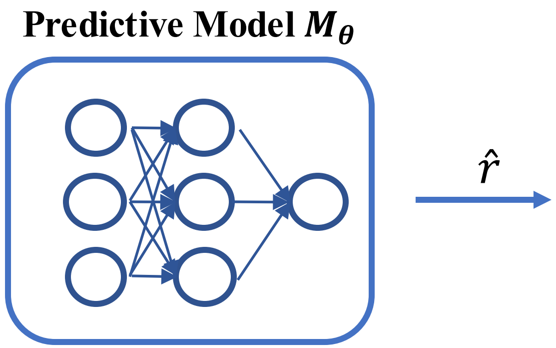

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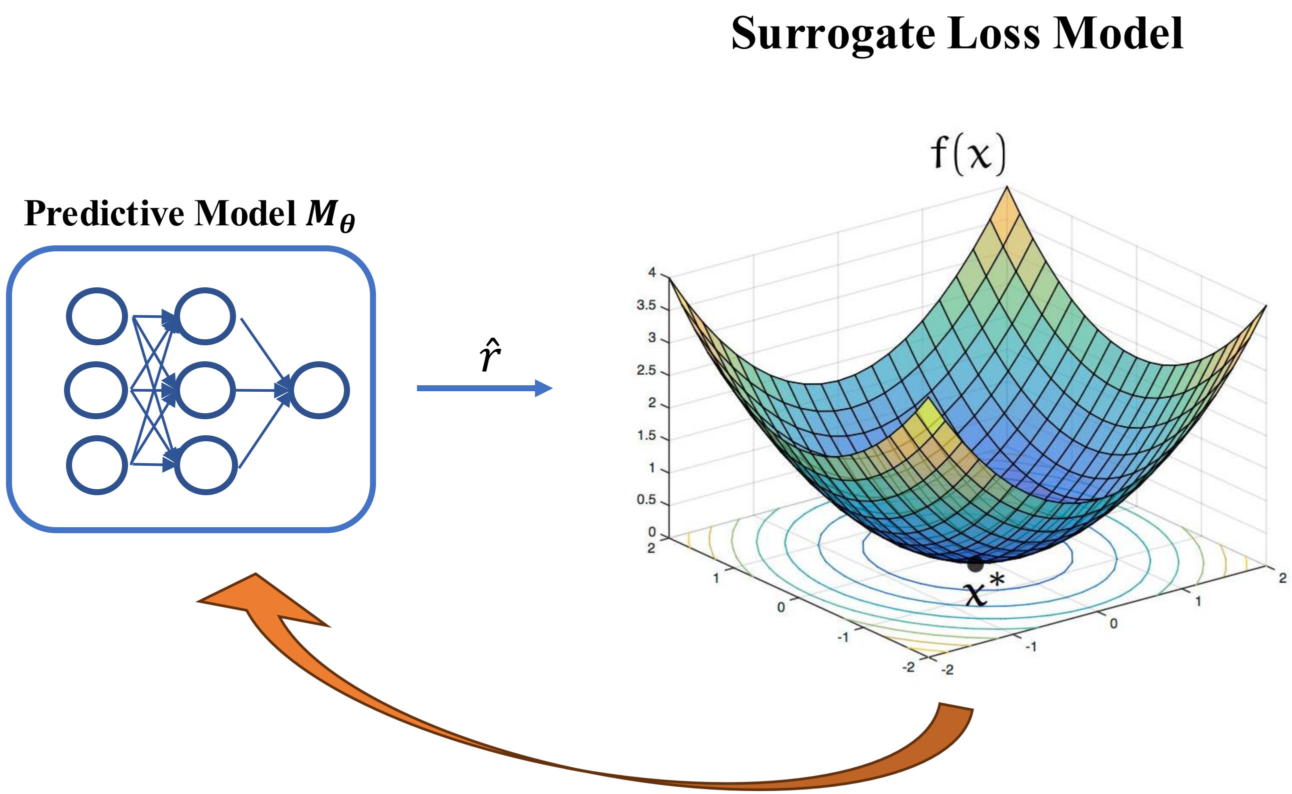

$$\nabla_\theta M_\theta \cdot \nabla_{\hat{r}} \textbf{Surrogate}(r, \hat{r})$$

Hands-on Exercise

- Partial Index Tracking problem with KOSPI index
  - For simplicity, with top 30 assets

Data

- Closed price and Market cap weight of KOSPI30 (2015/01/02 ~ 2014/12/30)

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Cumulative return over the past period

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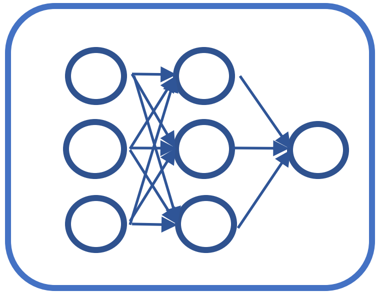

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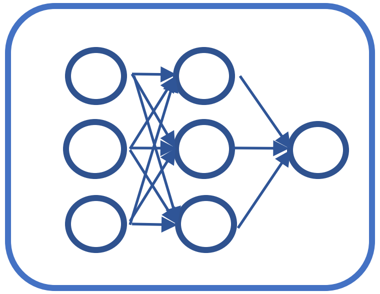

Future cumulative return

Hands-on Exercise

- Let's implement partial index tracking directly with Colab!

Notebook Link

Data Link

Thank you for listening!