Decision-focused Learning in Partial Index Tracking

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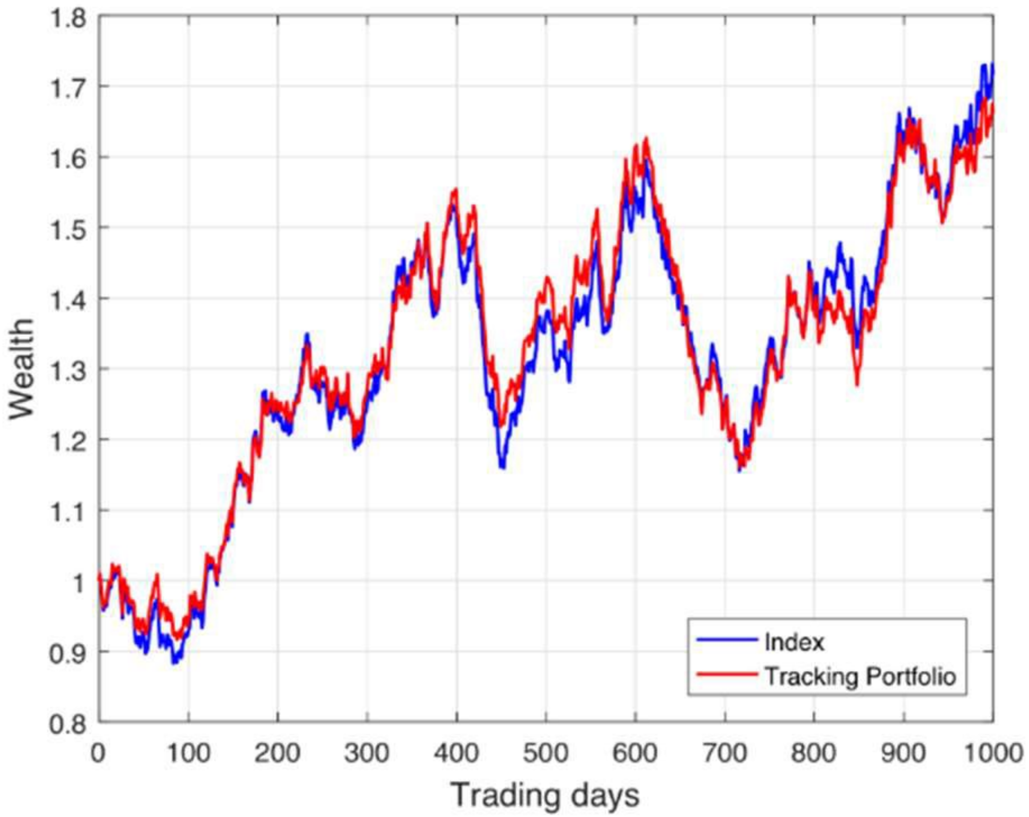
Introduction

- **Partial Index Tracking**



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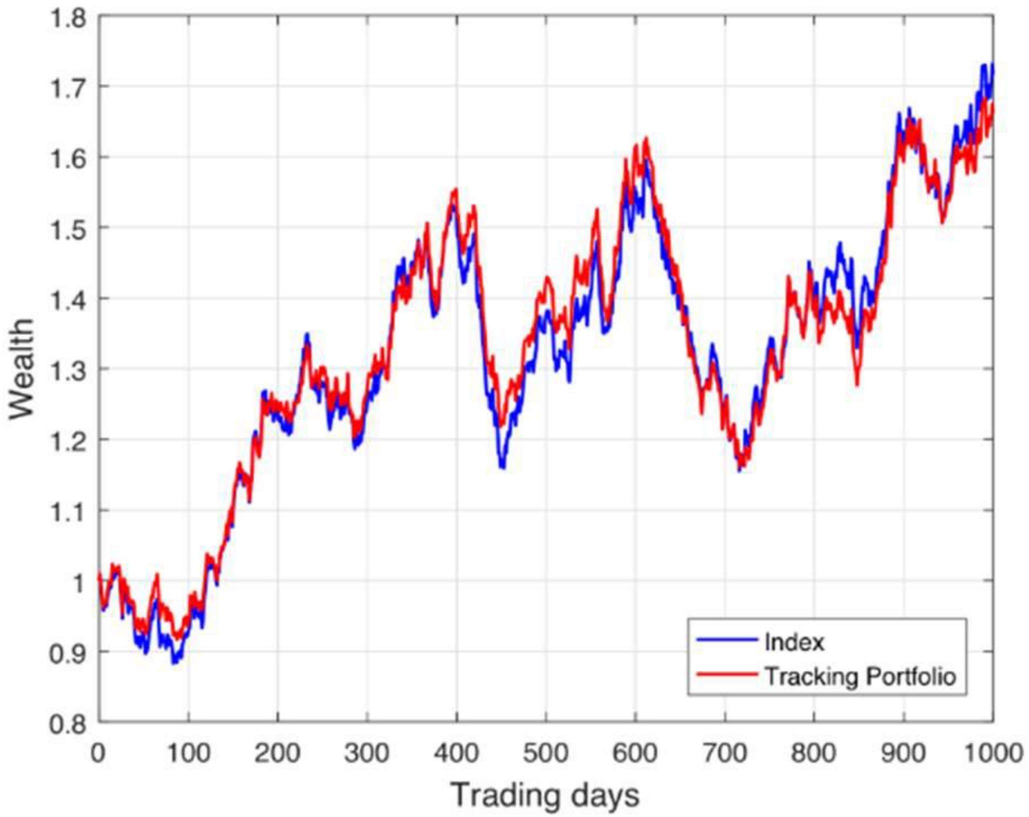

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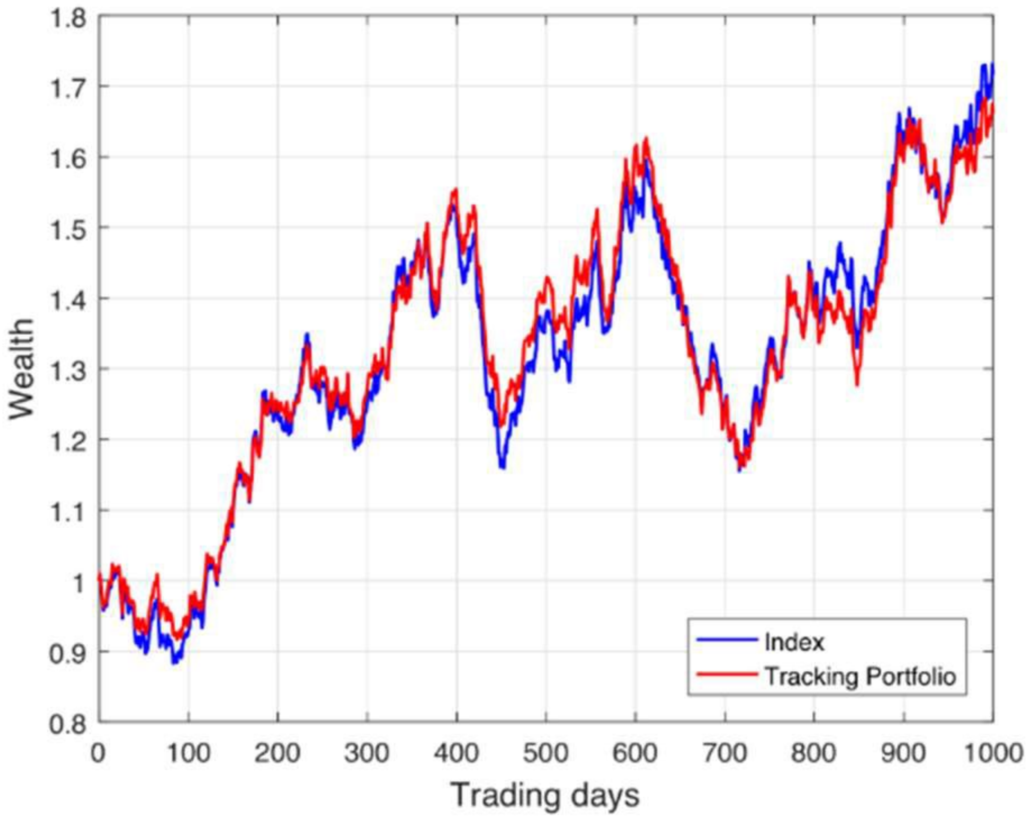

- **Partial**: use only subset of assets in index
- **Index:** measure the performance of finance market
- **Tracking:** aim to replicate performance of index

Basic Formulation

- **Variables & Functions**

Name	Definition
$n \in \mathbb{N}$	The number of assets
$\hat{r} \in \mathbb{R}^n$	The return of assets **(Uncertain)**
$w \in \mathbb{R}^n$	The weight of portfolio
$I \in \mathbb{R}$	Target Index
$\mathbf{1}$	One vector

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$\mathbf{Card}(\cdot)$	The number of positive component
$\lVert\cdot\rVert_2$	Euclidean 2-norm
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$$s.t \quad \mathbf{1}^\top w = 1 \quad \Longrightarrow \text{Sum of portfolio weight} = 1$$

$$\mathbf{Card}(w) \leq k \quad \Longrightarrow \text{Cardinality constraint}$$

DFL in partial index tracking

- **Challenge**

  - Cardinality constraint makes the problem discrete and not differentiable

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  - In this presentation, we introduce two strategies for this challenge

1) Semi-definite Relaxation Approach

$\min_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2\left(\mathbf{1}^\top W \hat{R} C\right)$
	$\mathbf{Tr}(I_N W) = 1$
	$\mathbf{1}^\top \lvert W \rvert \mathbf{1} \leq k\mathbf{Tr}(W)$
	$W \in S_n^+$

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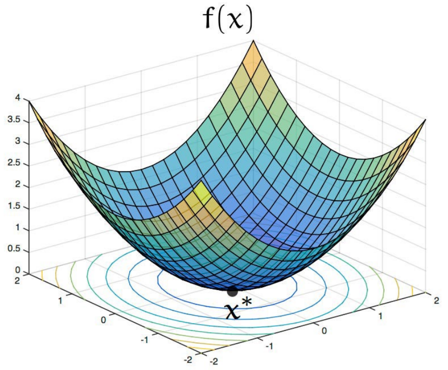
2) Surrogate Loss Approach

$\mathcal{L}_{DFL} \approx$



Semi-definite Relaxation Approach

- With semi-definite relaxation, we can transform the original problem to be convex

Semi-definite Relaxation Approach

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- Then, the CvxpyLayer can be used to obtain the gradients required for DFL

$$
\begin{aligned}
\min_{w \in \mathbb{R}^n} \quad & \|\hat{r}^\top w - I\|^2 \\
s.t \quad & \mathbf{1}^\top w = 1 \\
& \mathbf{Card}(w) \leq k
\end{aligned}
$$

Semi-definite Relaxation

CVXpy

$$
\begin{aligned}
x^\star(\theta) = \underset{x}{\operatorname{argmin}} \quad & f(x;\theta) \\
\text{subject to} \quad & g(x;\theta) \leq 0 \\
& h(x;\theta) = 0
\end{aligned}
$$

PyTorch

TensorFlow

Semi-definite reformulation

- We first introduce semi-definite matrix variable $W = ww^\top \in \mathbb{R}^{n\times n}$ and $\hat{R} = \hat{r}\hat{r}^\top \in \mathbb{R}^{n\times n}$

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- The objective function can be reformulated as (**Tr**: Trace of matrix)

$$\|\hat{r}^\top w - I\|^2 = (\hat{r}^\top w)^2 - 2I(\hat{r}^\top w) + I^2 = \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W\hat{r}) + I^2$$

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- Since I^2 is constant,

$\min_{w\in\mathbb{R}^n}$	$\|\hat{r}^\top w - I\|^2$

$\rightarrow$

$\min_{W\in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W\hat{r})$

Semi-definite reformulation

- Furthermore, the weight sum constraint and cardinality constraint are reformulated as

$$
\begin{aligned}
\min_{w \in \mathbb{R}^n} \quad & \|\hat{r}^\top w - I\|^2 \\
& \mathbf{1}^\top w = 1 \\
& \mathbf{Card}(w) \le k
\end{aligned}
\qquad \longrightarrow \qquad
\begin{aligned}
\min_{W \in S_n^+} \quad & \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W \hat{r}) \\
& \mathbf{Tr}(I_N W) = 1 \\
& \mathbf{Card}(W) \le k^2
\end{aligned}
$$

Semi-definite reformulation

- Finally, $W = ww^{\top}$ is equivalent ($\mathbf{Rank}(W) = 1$ & $W \in S_n^+$ (Positive semi-definite))

$\min_{w \in \mathbb{R}^n}$	$\lVert \hat{r}^{\top} w - I \rVert^2$
	$\mathbf{1}^{\top} w = 1$
	$\mathbf{Card}(w) \leq k$
	$W = ww^{\top}$

$\min_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^{\top} W \hat{r})$
	$\mathbf{Tr}(I_N W) = 1$
	$\mathbf{Card}(W) \leq k^2$
	$\mathbf{Rank}(W) = 1$
	$W \in S_n^+$

Semi-definite relaxation

- Next, Let's relax the reformulated problem. We first drop the Rank constraint.

$\min_{W \in S_n^+}$	$\mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W\hat{r})$
	$\mathbf{Tr}(I_n W) = 1$
	$\mathbf{Card}(W) \leq k^2$
	$\cancel{\mathbf{Rank}(W) = 1}$
	$W \in S_n^+$

Semi-definite relaxation

- Next, by relaxing $\mathbf{Card}(W) \leq k^2$ to $\|w\|_1^2 \leq k\|w\|_2^2$ and transforming to $\mathbf{1}^\top|W|\mathbf{1} \leq k\mathbf{Tr}(W)$,

$$
\begin{array}{|l|}
\hline
\min_{W \in S_n^+} \quad \mathbf{Tr}\left(\hat{R}W\right) - 2I(\mathbf{1}^\top W \hat{r}) \\
\hline
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\hline
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\begin{array}{|l|}
\hline
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DPP-compliant

- When using CvxpyLayer, the formulation must satisfy DPP-compliant
    1. The optimization problem must be convex in variable W → **Already Satisfied**

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DPP-compliant

- When using CvxpyLayer, the formulation must satisfy DPP-compliant
  1. The optimization problem must be convex in variable W $\rightarrow$ **Already Satisfied**
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DFL loss for Partial Index Tracking

- As seen in previous session, the DFL loss is expressed as

$$\mathcal{L}(\hat{\boldsymbol{R}}, \boldsymbol{R}) := \boldsymbol{Regret}(\boldsymbol{w}^*(\hat{\boldsymbol{R}}), \boldsymbol{R})$$

$$= \boldsymbol{f}(\boldsymbol{w}^*(\hat{\boldsymbol{R}}), \boldsymbol{R}) - \boldsymbol{f}(\boldsymbol{w}^*(\boldsymbol{R}), \boldsymbol{R})$$

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Realized decision quality with estimated $\widehat{\boldsymbol{R}}$

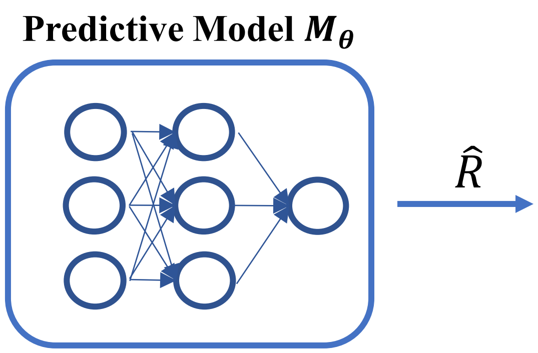
Realized decision quality with ground truth R

- Thus, in index tracking problem, the DFL loss is

$$\mathcal{L}(\hat{R}, R) = \left[\mathbf{Tr}\left(RW^*(\hat{R})\right) - 2(\mathbf{1}^\top W^*(\hat{R})RC)\right] - [\mathbf{Tr}(RW^*(R)) - 2(\mathbf{1}^\top W^*(R)RC)]$$

DFL with semi-definite relaxation

- The process of DFL in Partial Index Tracking



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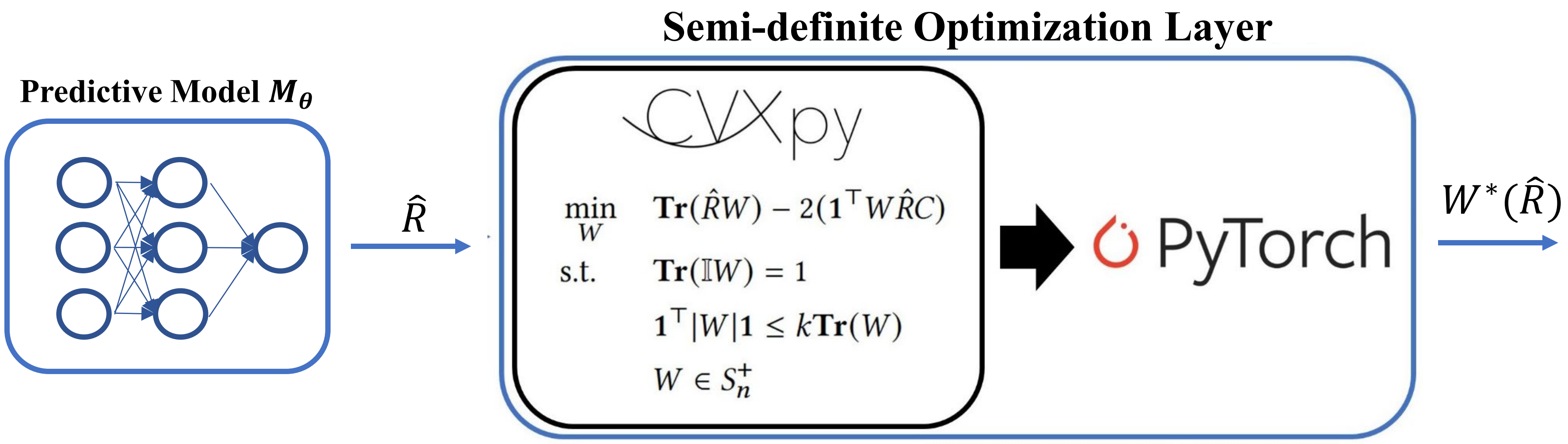

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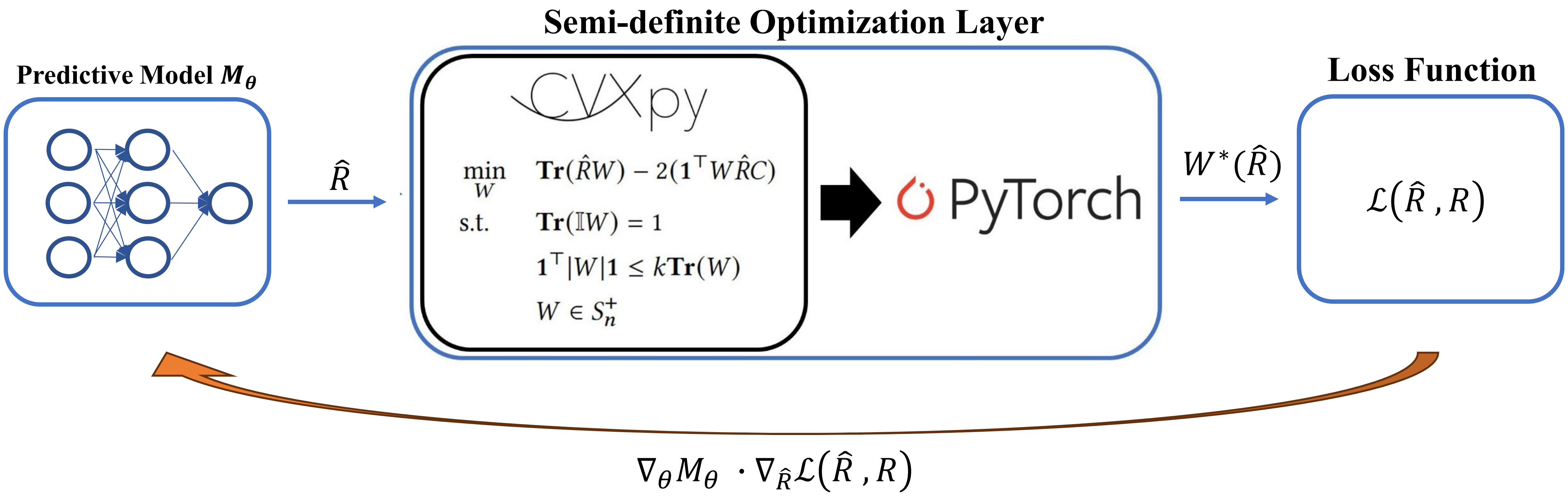

Surrogate Approach

- The DFL loss function of basic formulation is

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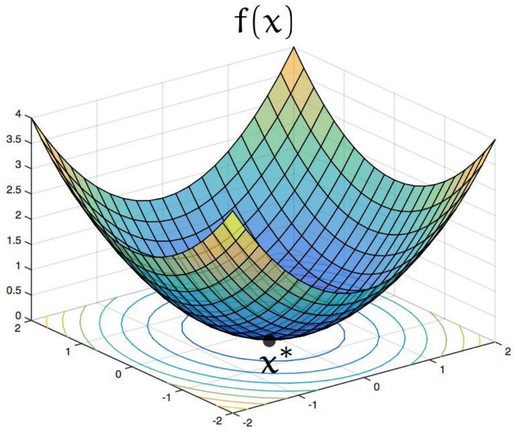
- Surrogate approach aim to learn $\mathcal{L}(\hat{r}, r)$ with another predictive model

r

$\hat{r}$

$\longrightarrow$



$\approx \; \mathcal{L}(r, w^*(\hat{r}))$

DFL with surrogate approach

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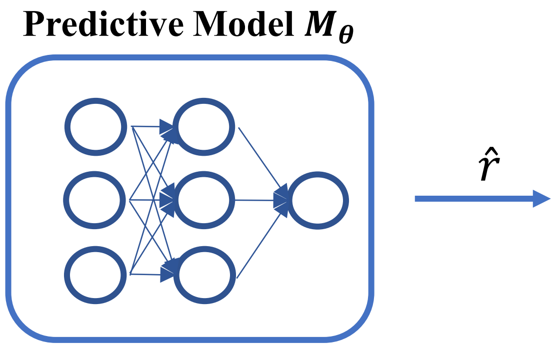

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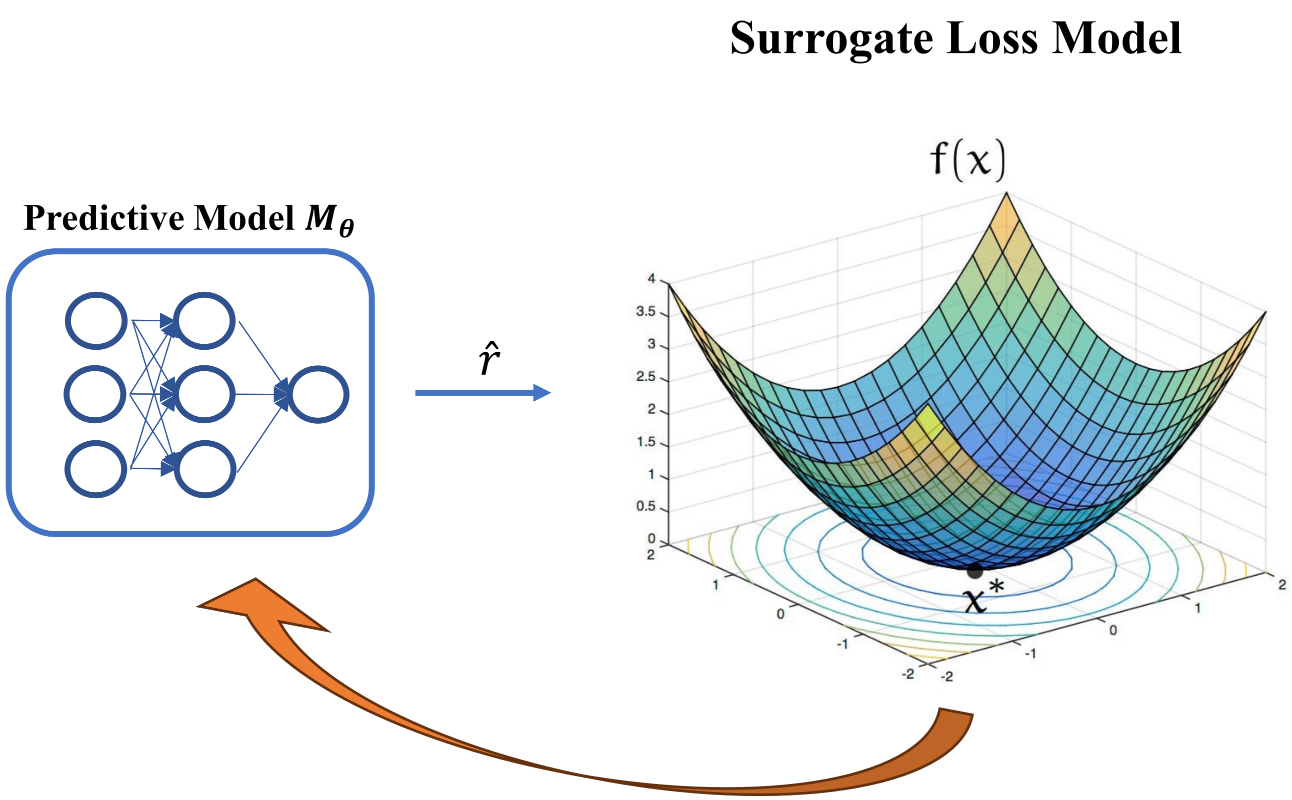

$$\nabla_\theta M_\theta \cdot \nabla_{\hat{r}} \mathbf{Surrogate}(\hat{r}, r)$$

Hands-on Exercise

- Partial Index Tracking problem with KOSPI index
  - For simplicity, with top 30 assets

Data

- Closed price and Market cap weight of KOSPI30 (2015/01/02 ~ 2024/12/30)

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Features

Return before rebalancing day

Cumulative return over the past period

Mean and variance of past returns

Mean and variance of past prices

Current return

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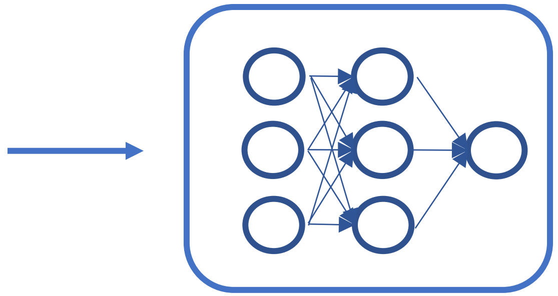

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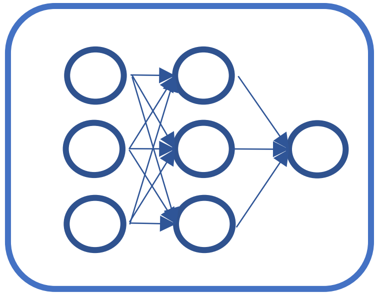

Future cumulative return

Hands-on Exercise

- Let's implement partial index tracking directly with Colab!

Notebook & Data Link

Thank you for listening!