Numerical Solution for the Line Curvature

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1 Formular

The expression of curvature:

$$\kappa = \frac{\ddot{\vec{r}} \times \dot{\vec{r}}}{\left|\dot{\vec{r}}\right|^3} \tag{1}$$

In 2D:

$$\kappa = \frac{x''y' - x'y''}{((x')^2 + (y')^2)^{3/2}} \tag{2}$$

With three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, to estimate the curvature, we firstly fit the three point to a 2D expression of parametric equation:

$$\begin{cases} x = a_1 + a_2 t + a_3 t^2 \\ y = b_1 + b_2 t + b_3 t^2 \end{cases} \tag{3}$$

With upper and lower limit of t_a and t_b, we can apply the three points

$$(x, y)|_{t=-t_a} = (x_1, y_1) \tag{4}$$

$$(x, y)|_{t=0} = (x_2, y_2) \tag{5}$$

$$(x, y)|_{t=t_b} = (x_3, y_3) \tag{6}$$

to the parametric equation:

$$\begin{cases} x_1 &= a_1 & -a_2 t_a + a_3 t_a^2 \\ x_2 &= a_1 & \\ x_3 &= a_1 & +a_2 t_b + a_3 t_b^2 \end{cases} \tag{7}$$

and

$$\begin{cases} y_1 &= b_1 & -b_2 t_a + b_3 t_a^2 \\ y_2 &= b_1 & \\ y_3 &= b_1 & +b_2 t_b + b_3 t_b^2 \end{cases} \tag{8}$$

Rewrite the Equation (7,8) into matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -t_a & t_a^2 \\ 1 & 0 & 0 \\ 1 & t_b & t_b^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{9}$$

and

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & -t_a & t_a^2 \\ 1 & 0 & 0 \\ 1 & t_b & t_b^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{10}$$

The equation can be solved by directly inverse the matrix:

$$A = M^{-1}X \tag{11}$$

$$B = M^{-1}Y \tag{12}$$

So we have (a_1, a_2, a_3) and (b_1, b_2, b_3), with which we can derive the parametric equation of the curve. The derivation of the curve is:

$$x' = \left.\frac{dx}{dt}\right|_{t=0} = a_2$$

$$x'' = \left.\frac{d^2x}{dt^2}\right|_{t=0} = 2a_3$$

$$y' = \left.\frac{dy}{dt}\right|_{t=0} = b_2$$

$$y'' = \left.\frac{d^2y}{dt^2}\right|_{t=0} = 2b_3$$

Back to Equantion (2), we have:

$$\kappa = \frac{x''y' - x'y''}{((x')^2 + (y')^2)^{3/2}} = \frac{2(a_3b_2 - a_2b_3)}{(a_2^2 + b_2^2)^{3/2}} \tag{13}$$

2 Usage

See `https://github.com/Pjer-zhang/PJCurvature` for detail