Graph Theory for Operations Research and Management:

Applications in Industrial Engineering

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Chapter 8
Hamiltonian Paths and Cycles

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ABSTRACT

In this chapter, the concepts of Hamiltonian paths and Hamiltonian cycles are discussed. In the first section, the history of Hamiltonian graphs is described, and then some concepts such as Hamiltonian paths, Hamiltonian cycles, traceable graphs, and Hamiltonian graphs are defined. Also some most known Hamiltonian graph problems such as travelling salesman problem (TSP), Kirkman's cell of a bee, Icosian game, and knight's tour problem are presented. In addition, necessary and (or) sufficient conditions for existence of a Hamiltonian cycle are investigated. Furthermore, in order to solve Hamiltonian cycle problems, some algorithms are introduced in the last section.

INTRODUCTION

In graph theory, the concept of Hamiltonian graphs has great importance since a Hamiltonian graph includes a cycle which contains each vertex of that graph. But identifying if a graph includes a Hamiltonian cycle or not, is an NP-complete problem. In this way, some necessary and (or) sufficient conditions for the existence of a Hamiltonian cycle are investigated. Also, some algorithms are proposed to solve this problem.

HISTORICAL BACKGROUND

In 1859, the Irish mathematician Sir William Rowan Hamilton illustrated a mathematical game which is called Icosian. This game was played on a surface of wooden dodecahedron which consists of 20 corners (vertices). Each of these corners was labeled with the name of a city. The objective of the game was to find a cycle in such a way that visits every vertex exactly once and then returns to the starting point (Chartrand et al., 2010).

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HAMILTONIAN PATHS AND CYCLES

Recall that a path can be defined as a nonempty graph, $P = (V, E)$, in which $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$. Therefore, it concatenates vertices sequentially so that connects x_0 and x_k.

A Hamiltonian path or traceable path is one that contains every vertex of a graph exactly once. Also a Hamiltonian cycle is a cycle which includes every vertices of a graph (Bondy & Murty, 2008). Example of Hamiltonian path and Hamiltonian cycle are shown in Figure 1(a) and Figure 1(b) respectively.

A traceable graph is a graph which contains a Hamiltonian path and a Hamiltonian graph is one which contains a Hamiltonian cycle (Bondy & Murty, 2008). Figure 2(a) represents the dodecahedron which is a Hamiltonian graph and Figure 2(b) represents the Hershel graph which is traceable.

Furthermore, a graph is Hamiltonian connected graph if for every pair of vertices the graph is traceable. Also a Hamiltonian decomposition is decomposition such that every subgraph contains a Hamiltonian cycle (Balakrishnan, 1995).

HAMILTONAIN GRAPH PROBLEMS

In this section, some most known traversal problems is introduced which involves finding a cycle passing through all the vertices of a graph.

Figure 1. (a) An example of Hamiltonian path and (b) an example of Hamiltonian cycle

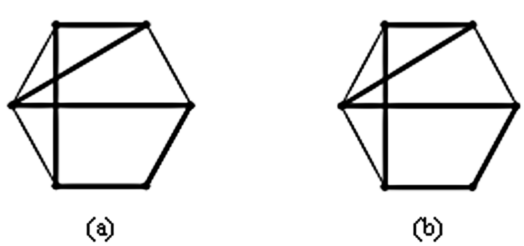
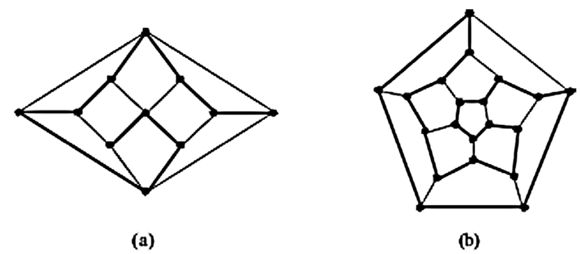


Figure 2. Hamiltonian and traceable graphs: (a) the dodecahedron and (b) the Hershel graph



Knight's Tour Problem

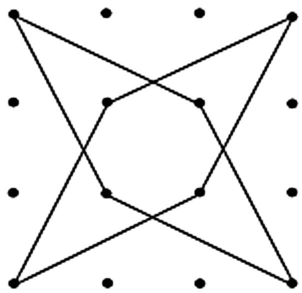
One of the early examples of Hamiltonian graph problems is the Knight's Tour Problem. This problem can be dated back to the 9th century AD. The aim of this problem was to find a sequence of knight's moves on a chessboard, so that every of the 64 squares are visited exactly once and then the knight returns to the originated point (James, 1999).

In order to simplify the explanation, rather than the usual 8 × 8 chessboard, a 4 × 4 chessboard is considered. Each square on this chessboard is represented by a vertex of a graph and two vertices are joined by an edge if a knight could make a move from the middle of the corresponding squares. Figure 3 illustrates vertices and some of the edges of a 4×4 chessboard.

As it is shown in Figure 3, the corner vertices, all have degree of precisely two and consequently the eight edges do have to be included in any Hamiltonian circuit. This implies that no such Hamiltonian cycle can exist since each of the center four vertices must be visited at least twice. Therefore, it can be concluded that there is no knight's tour for a 4 × 4 chessboard (Aldous & Wilson, 2000).

The solution of standard 8 × 8 chessboard was proposed by Euler in 1759. Also he showed that there is no Knight's Tour for a chessboard with an odd number of squares. Other mathematician

Figure 3. The vertices and edges of a 4 × 4 chessboard



like De Lavernede (1839), Legendre (1830), Pratt (1825), Warndorff (1823), De Moivre and Montmort, Vandermonde (1774), tried to solve the general problem, but no general method have been propose to solve it in last two centuries (Takefuji, 1992).

Kirkman's Cell of a Bee

In 1855 Kirkman discussed the polyhedrons that contain a cycle passing thorough every vertex of a graph. He proved that every polyhedron with even sided faces but an odd number of vertices has no such cycle and represented an example of such a polyhedron which is obtained by "cut in two the cell of a bee" (Gross & Yellen, 2006). This example is shown in Figure 4.

Icosian Game

In 1856 Sir William Rowan Hamilton illustrated a mathematical game on the dodecahedron, which was named Icosian game or Hamilton's puzzle. The aim of this game is to visit every vertex of dodecahedron exactly once and then return to the originated vertex (Gross & Yellen, 2006; James, 1999).

Travelling Salesman Problem

A travelling salesman problem (TSP) is a problem in which a salesman intends to visit a number of cities ($n \geq 3$) exactly once and then return to his originated point. So, with a given journey cost between every pair of cities, he plans so that the cost of traveling is minimized. If cities and possible routes are considered as a vertices and edges of graph respectively, then it can be concluded that the salesman tries to find a minimum-weight Hamilton cycle in a weighted given graph.

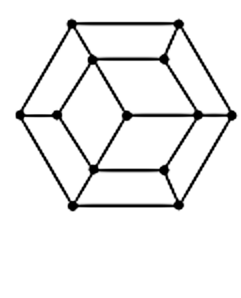
The TSP is an NP-hard problem and no efficient algorithm is known to solve it. Also it has not been proven that whether or not an efficient algorithm for finding at least expensive cycle exists (Asratian et al., 1998; Bollobás, 1998).

NECESSARY AND (OR) SUFFICIENT CONDITIONS FOR EXISTENCE OF HAMILTONIAN CYCLE

Identifying if a graph includes a Hamiltonian cycle or not is a NP-complete problem. However there are some necessary and (or) sufficient conditions for the existence of Hamiltonian cycle. It is evident that every Hamiltonian graph is necessarily 2-connected, because deletion of a vertex consequence that graph includes a Hamiltonian path. As a result, it can be concluded that every Hamiltonian graph includes no cut vertex (Balakrishnan, 1995). In this section the necessary and (or) sufficient conditions for the existence of Hamiltonian cycle are discussed.

Theorem 1: (A necessary condition for existence of Hamiltonian cycle). If G is a Hamiltonian graph and S is a nonempty set of vertices,

Figure 4. Kirkman's "cell of a bee"



then the graph G - S has at the most $|S|$ components. Therefore $C(G - S) \leq |S|$

Furthermore when above equality holds, thus every component of G - S has a Hamiltonian path, also each of the Hamilton cycles belonging to the graph G in each component contains a Hamilton path.

Proof: Suppose that G has a Hamilton cycle C. Thus C obviously includes $|S|$ components at most. Still and also it indicates that G - S includes $|S|$ components at most, since in the graph G, the cycle C is a spanning subgraph. When G - S contains exactly $|S|$ components, then the cycle C will have the same number of components, also the components of C - S will be spanning subgraphs of G - S components. Alternatively, it can be concluded that a Hamiltonian cycle C in each of the components of G - S contains a Hamilton path.

Note that if for every nonempty proper subset S of V equality $C(G - S) \leq |S|$ holds, a graph is called tough graph. Therefore, a nontough graph is not Hamiltonian (Balakrishnan, 1995; Bondy & Murty, 2008).

Figure 5 represents a graph which is not Hamiltonian. It is evident that by removing the set of three vertices pointed out, 4 components remain. So the graph is not tough and it can be concluded that it is non-Hamiltonian.

As mentioned in first chapter, a planar graph is one that has been drawn in the plane in such a way that its edge intersect only at their common end vertices.

Theorem 2: Grinberg's Theorem (A necessary condition for existence of Hamiltonian cycle on the planar graph). If G with a Hamilton cycle C, is a planar graph of order n, and where inside and outside of the Hamiltonian cycle, respectively φ_i' and φ_i'' are the numbers of i-faces, then: $\sum_{i=3}^{n}(i-2)(\varphi_i' - \varphi_i'') = 0$

Proof: Suppose that E' is the subset of $E(G)/E(C)$ which includes internal of C and set $m' = |E'|$. So internal C includes precisely $m' + 1$ faces. Thus: $\sum_{i=1}^{n}\varphi_i' = m' + 1$

So every edge of E' places on the boundary of two faces in internal C, also every edge of C places on the boundary of precisely one face in internal C. Accordingly: $\sum_{i=1}^{n} i\varphi_i' = 2m' + n$

By replacing equation

$$\sum_{i=3}^{n}(i-2)(\varphi_i' - \varphi_i'') = 0,$$

we obtain

$$\sum_{i=1}^{n}(i-2)\varphi_i' = n - 2$$

In addition,

$$\sum_{i=1}^{n}(i-2)\varphi_i'' = n - 2$$

It is evident that equations

$$\sum_{i=1}^{n}(i-2)\varphi_i' = n - 2$$

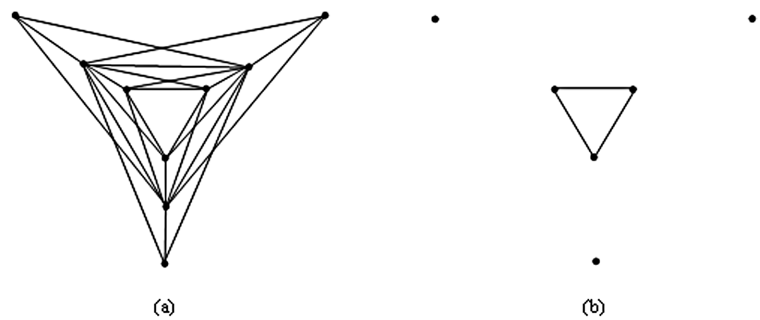
and

$$\sum_{i=1}^{n}(i-2)\varphi_i'' = n - 2$$

results

$$\sum_{i=3}^{n}(i-2)(\varphi_i' - \varphi_i'') = 0.$$

Figure 5. (a) A nontough graph G and (b) the components of G - S



Equation $\sum_{i=3}^{n}\left(i-2\right)\left(\varphi_i'-\varphi_i''\right)=0$ is called Grinberg's identity (Bondy & Murty, 2008). Figure 6 represents the Grinberg graph. With the aid of Grinberg's theorem it is concluded that this graph is non-Hamiltonian.

Theorem 3: Ore's Theorem (A sufficient condition for existence of Hamiltonian cycle). If $G = (V, E)$ be a simple graph with $n \geq 3$ vertices and for every pair of nonadjacent vertices x and y $d(x) + d(y) \geq n$, then graph G is Hamiltonian.

Proof: Suppose that the theorem is false. Let H be a non-Hamiltonian graph with $n \geq 3$ vertices so that for every pairs of nonadjacent vertices x and y, $d(x) + d(y) \geq n$. Between every pairs of nonadjacent vertices of H, add as many edges as possible so that the resulting graph G be a maximal non-Hamiltonian graph. With regard to the fact that G is a non-Hamiltonian graph, it can be concluded that vertices v_1 and v_n are not adjacent. Also $d(v_1) + d(v_n) \geq n$ and for $2 \leq i \leq n - 1$, v_1 is adjacent to v_i and v_{i-1} is adjacent to v_n. But $v_1v_2...v_{i-1}v_nv_{n-1}...v_i$ is a Hamiltonian cycle and the contradiction is derived (Wilson, 1996).

Theorem 4: Dirac's Theorem (A sufficient condition for existence of Hamiltonian cycle). If $G = (V, E)$ is a simple graph which has the minimum degree of δ on n vertices, where $\delta \geq n/2$ and $n \geq 3$, thus it can be concluded that G is a Hamiltonian graph.

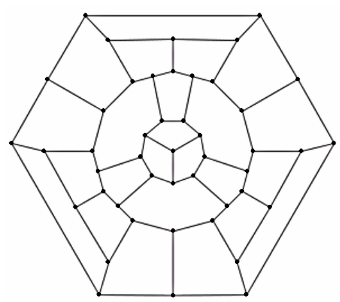
Proof: If the degree of each vertex is at least $n/2$, then degree summation of every pair of vertices is at least n. So according to Ore's theorem, G is Hamiltonian graph (Wilson, 1996).

Lemma 1: Let $G = (V, E)$ be a simple graph with $n \geq 3$ vertices and suppose that for every pair of nonadjacent vertices x and y, the sum of degrees is at least n. Let $G' = G + xy$. Then G is Hamiltonian if and only if G' is.

Proof: If G is Hamiltonian, then G' is too. Conversely, if G' is Hamiltonian, let C be a Hamilton cycle in G'. If $xy \in C$, then $P = C - xy$ is a Hamiltonian path in G. Since $d(u) + d(v) \geq n$, it can be concluded that P has a cycle exchange transforming, so that G has Hamilton cycle, too (Bondy & Murty, 2008).

The closure of a graph G of order n which is denoted by $C(G)$, is the graph that is obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains (Singh, 2010). Figure 7 illustrates the formation of the closure G_1 of graph G.

Figure 6. The Grinberg graph



Lemma 2: Every graph has a unique closure.

Proof: Let G_1 and G_2 are two graphs that are established from a graph G of order n by recursively linking pairs of nonadjacent vertices whose degree sum is at the least n. Suppose that $S = \{e_1, e_2, \ldots, e_k\}$ and $T = \{f_1, f_2, \ldots, f_r\}$ be the sequence of edges added to G in order to obtain G_1 and G_2 respectively. If two sets S and T are not equal, let e_t be a first edge of S which is not included in T and joining vertices x and y. Considering graph H which is obtained by adding edges $e_1, e_2, \ldots, e_{t-1}$. It is evident that H is a subgraph of G_1 and G_2. On the other hand vertices x and y are not adjacent vertices in H and e_t is next edge which is selected in order to included in S. So it can be concluded that the summation of degrees of H is at least n. Also, since H is a subgraph of G_2, it is expected that the sum of degrees of G_2 is at least n. But it is known that the degree sum of any nonadjacent vertices is less than n and the contradiction is derived (Balakrishnan, 1995).

Theorem 5: Bondy and Chvátal's Theorem (A necessary and sufficient condition for existence of Hamiltonian cycle). A graph G which is simple is Hamiltonian if and only if its closure, $C(G)$, is Hamiltonian.

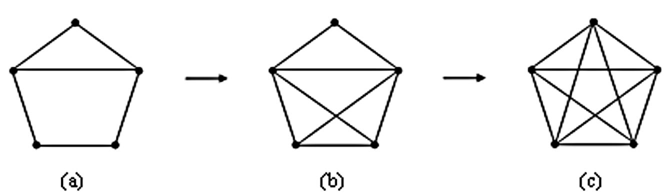
Proof: As mentioned before, the closure of a graph is one which is established by recursively linking pairs of nonadjacent vertices whose degree sum is at least n. Therefore with respect to lemma 1, it can be concluded that a simple graph G is Hamiltonian if and only if its closure is Hamiltonian.

Corollary 1: (A sufficient condition for existence of Hamiltonian cycle). If the graph G is simple with no less than three vertices and the closure of the graph is a complete graph, thus the graph G is Hamiltonian (Bondy & Murty, 2008).

Theorem 6: Chvátal's Theorem (A necessary condition for existence of Hamiltonian cycle). Consider the graph G to be simple with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \ldots \leq d_n$ and $n \geq 3$. If there is $k < n/2$ such that $d_k \leq k$ and $d_{n-k} \geq n - k$, thus the graph G is a Hamiltonian graph.

Proof: Consider G as a simple graph which is satisfying the given condition. Let $d'(x)$ is the degree of vertex x in $C(G)$ and suppose $C(G)$ is not a complete graph. Consider two nonadjacent vertices x and y in the closure so that $d'(y) \leq d'(x)$ and degree sum of these two

Figure 7. The closure of a graph



vertices be as large as possible. It is evident that the degree sum is less than n, because they are nonadjacent vertices.

Consider $S = \{z \in V - v \mid z$ and x are nonadjacent vertices in $C(G)\}$ and also $T = \{z \in V - y \mid z$ and y are nonadjacent vertices in $C(G)\}$. Obviously, $|T| = (n - 1) - d'(y)$ and $|S| = (n - 1) - d'(x)$. As we mentioned before y and x are selected in a manner that $d'(y) + d'(x)$ be as large as possible. Let $d'(y) = k$. Since $d'(y) + d'(x) < n$ and $|S| = (n - 1) - d'(x)$, it can be concluded that $|S| \geq k$. On the other hand, $d'(x) < n - k$ and $|T| < n - k$. Therefore the number of vertices of the closure with the minimum degree k is at least k and the number of vertices of the closure with degree less than $n - k$ is at least $n - k$.

Since G is spanning subgraph of $C(G)$, it can be concluded that $d_k \leq k$ and $d_{n-k} < n - k$ which results $k < n/2$. This gives the contradiction. Therefore, $C(G)$ must be complete under these conditions and then C is a Hamilton graph (Diestel, 2006).

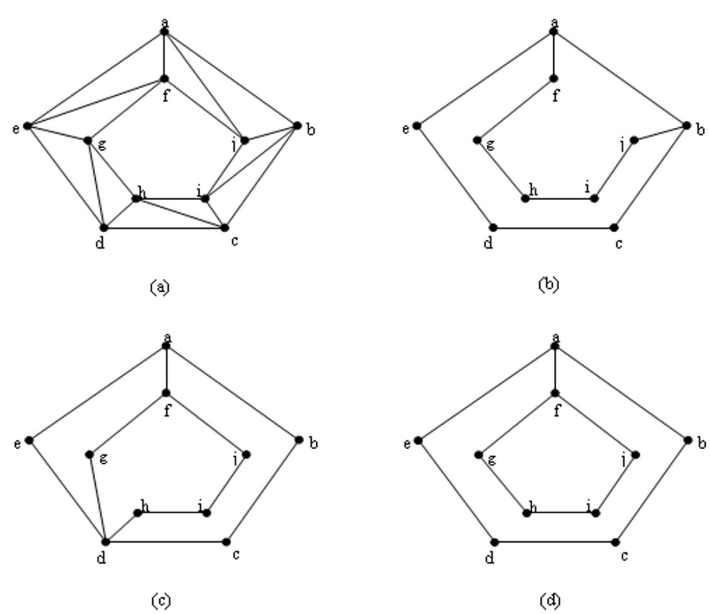
Theorem 7: Let $G = (A \mid B, E)$ be a bipartite Hamiltonian graph. Then, the number of vertices in two parts is equal.

Proof: As mentioned before, a Hamiltonian graph is one which contains a Hamiltonian cycle. According to definition of bipartite graph, each edge in G connects a vertex in A with a vertex in B. Therefore, any cycle alternately passes through a vertex in A and a vertex in B.

Suppose $G = (A \mid B, E)$ be a bipartite graph. Consider $|A| = m$ and $|B| = n$ where $m > n$. Let G contains a Hamiltonian cycle C. Consider $u \in$ B as a starting vertex of cycle C. After $2n$ edges are traversed, the cycle is returned to u again, and all the vertices of B are visited but there are still $m - n$ vertices in A which are not visited. Therefore, contradiction is derived and it can be concluded that number of vertices in two parts is equal.

Figure 4 represents a bipartite graph with odd number of vertices. Therefore, it can be concluded that the graph cannot be a Hamiltonian.

Figure 8. The lack of relationship between Eulerian and Hamiltonian graphs



Theorem 8: Let G be a complete bipartite graph ($G = K_{m,n}$). G is Hamiltonian if and only if $m = n > 1$.

Proof: It is evident that $K_{1,1}$ is not a Hamiltonian graph. According to theorem 7, a bipartite graph is not Hamiltonian if $m \neq n$. Since the degree of each vertex of $K_{n,n}$ is n, according to Ore's theorem it can be concluded that G is Hamiltonian.

HAMILTONIAN GRAPHS AND EULERIAN GRAPHS RELATIONSHIP

As mentioned, Eulerian graphs are simply identified in polynomial time because the necessarily and sufficiency condition is all nodes have even degree however identifying whether a graph is Hamiltonian or not is a NP-complete decision problem. Figure 8 illustrates the lack of relationship between Eulerian and Hamiltonian graphs. Figure 8(a) represents a graph which contains Hamiltonian cycle a → b → c → d → e → f → g → h → i → j → a and Eulerian cycle a → b → c → d → e → a → f → g → h → i → j → f → e → g → d → h → c → i → b → j → a. Figure 8(b) depicts a graph which has a Hamiltonian cycle a → f → g → h → i → j → b → c → d → e → a but has no Eulerian cycle. Figure 8(c) presents a graph which includes an Eulerian cycle a → e → d → g → f → j → i → h → d → c → b → a → f but no Hamiltonian cycle and Figure 8(d) indicated the graph with no Eulerian and Hamiltonian cycle.

HAMILTONIAN CYCLE ALGORITHMS

As mentioned before, the problem whether there is a cycle passing all the vertices of a given graph is a known instance of NP-complete problems (Skiena, 2008). One of the simplest ways to check the existence of Hamiltonian cycle is checking all the possible n! permutations of vertices. However, there is no efficient algorithm for general graphs that obtains Hamiltonian cycle (Riaz & Sikander Hayat Khiyal, 2006). Rubin (1972) proposes a search procedure that finds some or all of the Hamiltonian cycle of undirected or directed graph. Bollobás et al. (1987) develop a polynomial time algorithm that may find the Hamiltonian cycle of undirected graph. Bertossi (1983), Keil (1985), Manacher et al. (1990), Panda and Dos (2003) and Ibarra (2009) present algorithms that search Hamiltonian cycle of proper interval graphs. Wang (2001) presents FT-HAMIL algorithm that investigate a Hamiltonian cycle in a folded n-cube. Deĭneko et al. (2006) develop an exact algorithm which finds the Hamiltonian cycle of planar graphs. Riaz and Sikander Hayat Khiyal (2006) propose a polynomial time algorithm that may find Hamiltonian cycle of undirected graph. Sárközy (2009) proposes an NC^4-algorithm for η-Chvátal graphs. Kumar (2010) proposes an algorithm for Hamiltonian cycle which is based on restricted backtracking. Hung and Change (2011) develop $O(n)$-time certifying algorithm that solves Hamiltonian cycle problem of interval graphs. Eshragh et al. (2011) develop a hybrid algorithm that solves the Hamiltonian cycle problem. This algorithm was constructed by synthesizing Cross Entropy method and Markov decision processes.

CONCLUSION

In this chapter, the concept of Hamiltonian path and Hamiltonian cycle are investigated. Also necessary and (or) sufficient conditions for existence of Hamiltonian cycle are discussed. Furthermore, some algorithms are introduced to solve Hamiltonian cycle.

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KEY TERMS AND DEFINITIONS

Hamiltonian-Connected Graph: Is a graph that contains a Hamiltonian path between every pairs of vertices.

Hamiltonian Cycle: Is a cycle that contains every vertex of graph.

Hamiltonian Decomposition: Is a decomposition such that every subgraph contains a Hamiltonian cycle.

Hamiltonian Graph: Is a graph that contains a Hamiltonian cycle.

Hamiltonian Path (Traceable Path): Is a path that contains every vertex of graph.

Traceable Graph: Is a graph that contains a Hamiltonian path.