Applied Dynamical Systems

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5 Symbolic dynamics

5.1 The binary shift

The special case $r = 4$ of the logistic map, or the equivalent $1 - 2x^2$ on $[-1, 1]$ is sometimes called the Ulam map. We note the similarity with the second Tchebyscheff[1] polynomial $T_2(x) = 2x^2 - 1$. In general Tchebyscheff polynomials are defined by the (rather un-polynomial looking) $T_n(x) = \cos n \arccos(x)$, that is, it gives the formula for $\cos nx$ as a polynomial in $\cos x$. This suggests a trigonometric conjugation; for the logistic version we see that if

$$x_{n+1} = 4x_n(1 - x_n)$$

is transformed according to $x_n = \sin^2(\pi y_n/2)$, we find

$$x_{n+1} = 4\sin^2\frac{\pi y_n}{2}(1 - \sin^2\frac{\pi y_n}{2}) = \sin^2(\pi y_n)$$

Thus

$$y_{n+1} = \pm 2y_n \pmod 1$$

If we take the usual arcsin of a positive number, so in the range $[0, \pi/2]$ we find from this

$$y_{n+1} = r\min(y_n, 1 - y_n) = \frac{r}{2}(1 - |2y - 1|)$$

which is called the **tent map**, for $r = 2$.[2] We will keep $r = 2$ for the tent map for the rest of this section. An even simpler related map is the **doubling map** (also called **Bernoulli map**)

$$y_{n+1} = \{2y_n\}$$

where $\{\}$ denotes the fractional part.

Consider the binary representation of the point $y \in [0, 1)$,[3]

$$y = \sum_{j=0}^{\infty} \omega_j 2^{-(j+1)}$$

where the symbols $\omega_j \in \{0, 1\}$. The doubling map just ignores ω_0 and shifts all the other ω_j by one. The tent map does the same, but if $\omega_0 = 1$ all symbols are flipped as well as shifted. In each case the sequence of ω_0 values, which denotes the "rough location" of the point with respect to the partition $\{[0, 1/2), [1/2, 1)\}$ is called the **symbol sequence**.

We see there is (almost) a 1:1 correspondence between y and $\{\omega_j\}$, with the minor exception being the case of trailing repeated 1s, a countable set. Thus the dynamics $\Phi : [0, 1) \to [0, 1)$ is conjugate to the shift $\Sigma : \Omega_2^R \to \Omega_2^R$ where Ω_2^R is the set of "right" sequences of two symbols. The set Ω_2 denotes bi-infinite sequences $j \in \mathbb{Z}$, and is useful for invertible maps. The metric, ie distance between two sequences can be defined as[4]

$$d(\{\omega_j\}, \{\phi_j\}) = 2^{-\min\{|j|:\omega_j \neq \phi_j\}}$$

which then defines a topology in which Σ is continuous. This topological conjugacy between the shift map and doubling or tent maps (and hence also the Ulam map) has some immediate consequences:

- Periodic points are countable and dense.
- There is a dense orbit; this property is called **topological transitivity**
- There are orbits that are neither periodic nor dense.

Example 5.1. *List and concatenate all possible finite symbol sequences* $\{0, 1, 00, 01, 10, 11, 000, \ldots\}$*:*

$$010001101100000101001110010111011 1\ldots$$

Each finite symbol sequence appears infinitely often, so the orbit generated by this sequence is dense.

Example 5.2. *Any aperiodic sequence of* 00 *and* 10 *gives a nowhere dense orbit since there are real numbers with binary expansions containing* 11 *arbitrarily close to any real number.*

This should be compared with the Devaney definition of chaos:[5]

- Periodic points are dense
- The system is topologically transitive
- There is sensitive dependence on initial conditions.

The last condition is that in every neighbourhood of a point, there are initial conditions that eventually separate to a specified distance. It turns out[6] that the last condition follows from the first two. Thus the doubling, tent and Ulam maps are chaotic according to this definition.

We get more specific information about the periodic points — there are clearly 2^n symbol sequences with periods a factor of n for each n, and a dense set of preperiodic

[1] There are other spellings; the initial 'T' makes sense in terms of the usual notation $T_n(x)$

[2] The tent map is also topologically conjugate to the Farey map introduced in chapter 2 using as the conjugation the "Minkowski question mark function". The latter has the property that periodic continued fractions (ie quadratic irrationals) get mapped to periodic binary expansions (ie rationals).

[3] The $j + 1$ is so that j starts at zero, for consistency with the literature for symbolic dynamics.

[4] There are many equivalent metrics used in the literature.

[5] From his textbook, *A first course in chaotic dynamical systems* first published in 1992. This is a popular definition but other definitions are useful in different contexts.

[6] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, Amer. Math. Mon. **99**, 332-334 (1992).

points for each periodic point. Starting from any finite symbol sequence, say 001, we can construct a periodic symbol sequence $\overline{001}$ and hence calculate its corresponding point in the doubling map $y = 0.\overline{001}_2 = \sum_{j=1}^{\infty} 2^{-3j} = 1/7$ where the subscript denotes binary. For the tent map we ensure that the flips are taken into account, giving $y = 0.\overline{001110}_2 = 14\sum_{j=1}^{\infty} 2^{-6j} = 2/9$. Thus the corresponding point in the Ulam map is $x = \sin^2 \pi/9 \approx 0.116978$.

The stability of a periodic point $D\Phi^p = 2^p$ for the doubling map and $\pm 2^p$ for the tent map depending on the parity of the number of 1s in the symbol sequence. We can see that the conjugation relating the tent and Ulam maps preserves this: If $\Psi = h^{-1} \circ \Phi \circ h$ for some conjugating function h, we see that $\Psi^p = h^{-1} \circ \Phi^p \circ h$ and so for a fixed point x of Ψ^p and corresponding $y = h(x)$ of Φ^p we have

$$D\Psi^p|_x = (Dh^{-1}|_y)(D\Phi^p|_y)(Dh|_x) = D\Phi^p|_y$$

since the first and last terms in the product cancel, assuming both are non-zero. Thus we have $\Psi^p = \pm 2^p$ for the Ulam map also, except for the fixed point $x = 0$ (at which the conjugation is singular) which has $D\Psi = 4$.[7]

Remark: The doubling map is particularly bad to simulate directly on a computer, since most software uses a binary representation of real numbers. After a very small number of doublings the result is zero. It is much better to simulate a (pseudo-)random sequence of binary symbols.

Remark: Sequences with trailing 1s are equivalent in the binary representation to others with trailing 0s. They are just pre-images of the two fixed points $\overline{0}$ and $\overline{1}$ which are identified for the doubling map. Thus there are actually 2^{n-1} points of period a factor of n for the doubling map. In contrast, these are all distinct for the tent and Ulam maps.

5.2 Open binary shifts

For the logistic map with $r > 4$ and tent map for $r > 2$ there are intervals around $1/2$ that map out of $[0, 1]$. However the image of the interval $[0, 1/2]$ still includes the whole space $[0, 1]$, as does the image of $[1/2, 1]$. Thus for any point $x \in [0, 1]$ we can construct two preimages $\Phi_0^{-1}(x)$ and $\Phi_1^{-1}(x)$ and hence 2^n preimages of order n, one for each sequence of n symbols. It can be shown (using the Schwarzian derivative property for the logistic map) that this leads to a 1:1 correspondence between the set of points that remain forever in $[0, 1]$ and the binary shift.

Example 5.3. *Consider the case $r = 3$ for the tent map. The intervals $[0, 1/3]$ and $[2/3, 1]$ are each mapped to $[0, 1]$, so that the shift corresponds to the ternary representation*

$$x = \sum_{j=0}^{\infty} 2\omega_j 3^{-j-1}$$

This is the **middle third Cantor set.**

Properties of these sets follow easily from the shift representation: They are uncountable, complete, nowhere dense and totally disconnected (any two points are in different components). Also, the Cantor set itself is structurally stable - perturbing r does not affect any of these properties.

All the periodic orbits remain unstable as r is increased, so we can use the method of **inverse iteration** to locate them: Start at a convenient point (say, $x = 1/2$) and apply a periodic sequence of $\Phi_\omega^{-1}(x)$ until the result converges.

A natural higher dimensional version of the open binary shift is called the Smale horseshoe. If a (roughly rectangular) set is mapped to a "horseshoe" shaped set that covers the full width of the original in two places and for which the original covers the full width of the horseshoe, then the set surviving for infinite time is a Cantor set (labelled as above by the symbol sequence) in the unstable direction and smooth in the stable direction. The set surviving for both positive and negative infinite time is the intersection of two Cantor sets, itself a Cantor set, and labelled by the shift on the full space Ω_2. Again, it is structurally stable.

An important result is that a homoclinic tangle, ie map with a homoclinic point at which the stable and unstable manifolds are transverse, has horseshoe dynamics in a sufficiently high iterate of the map, and hence the full complexity of the binary shift dynamics. Recall Fig. **??**.

Example 5.4. *The* **Henon map**, $(x, y) \to (1 - ax^2 + y, bx)$,[8] *has a good example of a Smale horseshoe. For parameters $a = 12$, $b = 0.8$ it has the form shown in Fig. 1, leading to a Cantor set of points that never escape. For other parameter values, such as the original $a = 1.4$, $b = 3$ it behaves like the logistic map for $r < 4$, having an attractor of a fixed point or a fractal. It is closely related to the logistic map, but less well understood and also an active subject of research.*

Example 5.5. *A billiard system consisting of three circular scatterers with a "non-eclipsing" condition (no scatterer intersects the convex hull of the others) has a trapped set with complete binary symbolic dynamics, with symbols denoting which of the other two scatterers is encountered next.*[9]

[7] The argument can often be reversed - if all the periodic points of two hyperbolic dynamical systems have the same spectra (eigenvalues of $D\Phi^p$), they can often be shown to have a smooth conjugation. See for example Thm 20.4.3 in A. Katok and B. Hasselblatt, "Introduction to the modern theory of dynamical systems." (Cambridge University Press, 1997).

[8] Other trivial variations of the equations can be found in the literature

[9] The dynamics of this system was studied rigorously in A. Lopes and R. Markarian, Siam J. Appl. Math. **56** 651-680 (1996). But it had appeared previously in the physics literature — see chaosbook.org of Cvitanovic et al, where it is called three disk pinball

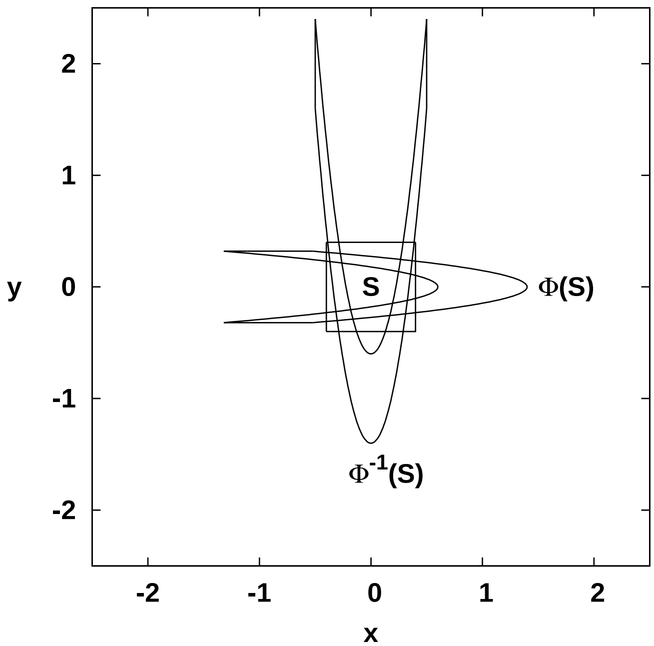

Figure 1: The Henon map, $a = 12$ and $b = 0.8$

If there are both expanding and contracting directions, as in these two dimensional examples, we cannot use inverse iteration to locate the periodic orbits numerically. In boundary value problems in ODEs there are two common approaches: Shooting and relaxation.[10] A shooting method involves finding an approximate initial condition, evolving the system to the end point, and checking the final boundary condition (here, that it is equal to the initial condition). In chaotic systems this is problematic since orbits are often exponentially unstable. Thus we usually need an approximation to the whole orbit, either by running a long trajectory and looking for near recurrences (for example if the periodic orbit is embedded in an attractor) or using known symbolic dynamics. Then we can refine it using one of the following methods:

Damped Multipoint Newton method For a cycle of length p we seek a zero of the function $F : \mathbb{R}^p \to \mathbb{R}^p$

$$F\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_p \end{pmatrix} = \begin{pmatrix} x_1 - \Phi(x_p) \\ x_2 - \Phi(x_1) \\ \dots \\ x_p - \Phi(x_{p-1}) \end{pmatrix}$$

The multidimensional Newton formula, found by taking the Taylor expansion around the zero to linear order, gives

$$(DF)(\mathbf{x}_{n+1} - \mathbf{x}_n) = -\gamma F(\mathbf{x})$$

where a damping parameter $0 < \gamma \leq 1$ is added by hand to increase the basin of attraction; the usual Newton method is $\gamma = 1$. Here we have (writing $\Phi'(x_k) = \Phi'_k$)

$$\begin{pmatrix} 1 & & & -\Phi'_p \\ -\Phi'_1 & 1 & & \\ & \dots & \dots & \\ & & -\Phi'_{p-1} & 1 \end{pmatrix}\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \dots \\ \Delta x_p \end{pmatrix} = -\gamma \begin{pmatrix} F_1 \\ F_2 \\ \dots \\ F_p \end{pmatrix}$$

Row reduction gives

$$\begin{pmatrix} 1 & & & -\Phi'_p \\ & 1 & & -\Phi'_p\Phi'_1 \\ & & \dots & \dots \\ & & & 1 - \Phi'_p\Phi'_1 \dots \Phi'_{p-1} \end{pmatrix}\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \dots \\ \Delta x_p \end{pmatrix} = -\gamma \begin{pmatrix} F_1 \\ F_2 + \Phi'_1 F_1 \\ \dots \\ F_p + \Phi'_{p-1}F_{p-1} + \dots + \Phi'_{p-1}\dots\Phi'_1 F_1 \end{pmatrix}$$

from which the solution may be found by dividing through by the last diagonal element and back substituting. Note that the matrix manipulations have been done explicitly - we need only store the vectors used in the intermediate steps.

Variational method Write an action function such as $S = |F|^2$ and minimise using standard multidimensional minimisation routines.[11] In the open billiard example, there is a natural action given by the sum of the path lengths.

5.3 Subshifts

The period three window of the logistic map has an infinite set of unstable orbits, including all the periodic orbits

— and a quantum version had been considered rigorously in S. Sjöstrand, Duke Math. J. **60** 1-57 (1990). The latter proposed what is now called a fractal Weyl law, relating the fractal properties of the classical trapped set to the distribution of quantum resonances.

[10] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical Recipes* (Cambridge University Press, 1992) gives advice on which of these to try first: “Shoot first, and only then relax.” But if the shooting is chaotic, this may not be the best strategy.

[11] However standard, routines for multidimensional minimisation are not guaranteed to work unless you know a lot about your system.

from the original bifurcation cascade, that persist without any bifucations in this parameter region. The period three points delineate two regions, roughly $0.15 < x < 0.5$ which belongs to the left branch of the map (symbol '0') and $0.5 < x < 0.95$ which belongs to the right branch (symbol '1'). Points in '0' map to '1' while points in '1' may map either to '0' or '1'. Thus we have a symbolic dynamics in which only some of the possible transitions occur; here we specifically exclude the sequence '00.'

If there are a finite[12] number of exclusion rules, this is called a **subshift of finite type** or **Topological Markov chain**. This will occur in a one-dimensional map which is expanding ($\Phi'(x) > 1$) if there is a partition of the space for which each element (corresponding to a symbol) is mapped to a union of elements (modulo boundary points); this is called a **Markov partition**[13] In the case of the logistic map, the derivative $\Phi'(x)$ is not always less than one, so the conjugacy with a symbolic system needs justification, and clearly fails for some of the stable orbits.

Such a system can be represented as a directed graph with adjacency matrix A with entries zero or one to denote whether a transition is possible, and a symbol space

$$\Omega_A = \{\omega \in \Omega_n | (A)_{\omega_n \omega_{n+1}} = 1 \quad \text{for} \quad n \in \mathbb{Z}\}$$

It is easy to show that the number of possible paths of length m from symbols i to j are given by the entry $(A^m)_{ij}$ of the matrix A^m. In particular, the number of periodic points of length m is the trace of A^m.

We can get from i to j iff $(A^m)_{ij} > 0$ for some $m \geq 0$, and write $i \to j$, or j is **accessible** from i. We have $i \to i$ automatically. If $i \to j$ and $j \to i$ then i and j **communicate**; this is an equivalence relation, so partitions the symbols into disjoint classes. On the other hand, if there is a j so that $i \to j$ but $j \not\to i$ then i is **inessential**. If all symbols are essential, ie there is a single communication class, the system[14] is **irreducible** and topologically transitive. In this case we also have a dense set of periodic orbits and hance chaos in the sense of Devaney.

The **period** of a symbol i is the greatest common divisor of the times m at which the dynamics can return to i, that is, when $(A^m)_{ii} \neq 0$, and infinite if $A^m_{ii} = 0$ for all $m > 0$. For example if the directed graph is bipartite, all states have even periods. The period is constant on all communication classes. If all symbols have period one, the system is **aperiodic**. If it is both irreducible and aperiodic, then for all sufficiently large m, all entries of A^m are positive, and the system satisfies a stronger property that topological transitivity:

Definition 5.6. *A system is* **topologically mixing** *if for any two open sets U, V, $\Phi^t(U) \cap V$ is nonempty for all sufficiently large t.*

Clearly topological mixing implies topological transitivity.

In this case we can use

Theorem 5.7. *Perron-Frobenius theorem: For a matrix A with non-negative entries, such that some power A^m has all positive entries, there is an eigenvector with positive entries with corresponding eigenvalue real, positive, simple and greater in magnitude that all other eigenvalues.*

Finally the growth of symbol sequences and periodic orbits are both controlled by this largest eigenvalue of A: If A is irreducible and aperiodic we have

$$\lim_{n\to\infty} \frac{1}{n} \ln \sum_{i,j} (A^n)_{ij} = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{i} (A^n)_{ii} = \ln \lambda_{\max}$$

where $\lambda_{\max}$ is the largest eigenvalue.

For a general dynamical system we can define

Definition 5.8. *Let $N(\epsilon, T)$ be the smallest number of points x_k such that for any $x \in X$ we have $|\Phi^t(x) - \Phi^t(x_k)| < \epsilon$ for all $0 \leq t < T$ and some k. Then the* **topological entropy** *is*

$$h_{top} = \lim_{T\to\infty} \limsup_{\epsilon\to 0} \frac{1}{T} \log N(\epsilon, T)$$

The base of the logarithm is arbitrary, often given as 2. The topological entropy is invariant under topological conjugacy, and in the case of an irredicible and aperiodic symbolic system is given by $\log \lambda_{\max}$.

If the largest eigenvalue of a matrix A is unique and simple, as in the irreducible and aperiodic case, it may be found with the **power method**: Apply A repeatedly to an arbitrary positive vector and normalise. The normalisation constant will converge exponentially to $\lambda_{\max}$ at a rate determined by the spectral gap (difference in magnitude between the largest and next largest eigenvalue(s)). The method does not require any reduction of the matrix, and hence can be used with very large sparse matrices.[15]

The map $\Phi_\beta(x) = \{\beta x\}$ is called the **beta-transformation** (Renyi 1957). For β an integer, we have

[12]There are some simply defined generalisations with an infinite number of rules, such as the even shift, in which each 0 is followed by an even number of 1's. This is not a subshift of finite type, but is in a larger category called sofic shifts, represented by directed graphs in which the same symbol may appear in more than one place. So here we allow transitions $0 \to 0$, $0 \to 1$ $1 \to 1'$, $1' \to 1$, $1' \to 0$ and disallow all others. Many typically encountered shifts of infinite type from dynamical systems do not have a simple representation, however.

[13]Markov partitions (with a more involved definition) are also used in higher dimensional dynamics; note that the boundaries can be fractal, see eg Arnoux, Pierre, and Shunji Ito. "Pisot Substitutions and Rauzy fractals." Bulletin of the Belgian Mathematical Society Simon Stevin **8** 181-208 (2001).

[14]"system" depending on context refers to any of the matrix A, the topological Markov chain, the directed graph, the symbolic dynamics, and the original dynamical system.

[15]It is reputedly used in Google PageRank and Twitter Who To Follow algorithms.

a (full) shift on β symbols as in the previous section. For other values, dividing the unit interval using multiples of β^{-1} gives the "greedy" representation of a number in inverse powers of β:

$$x = \sum_{j=0}^{\infty} \omega_j \beta^{-(j+1)}$$

For some algebraic values of β, the boundary of the final partition element, 1 maps onto a multiple of β^{-1} and we have a Markov partition.

Example 5.9. *$\beta = g = (1+\sqrt{5})/2$, the golden ratio. $\Phi_\beta(1) = \{g\} = (\sqrt{5}-1)/2 = g^{-1}$ and so we have the transitions $0 \to 0$, $0 \to 1$, $1 \to 0$ analogous to the period three window of the logistic map. The transition matrix is*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The higher powers are given by

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where F_n is the Fibonacci number, satisfying $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. This recurrence may be solved explicitly to find

$$F_n = \frac{1}{\sqrt{5}}(g^n - (-g)^{-n})$$

Thus the number of fixed points of order n is $P_n = F_{n+1} + F_{n-1}$. Note that because a matrix satisfies its own characteristic equation we have

$$A^2 - A - I = 0$$

Multiplying by an arbitrary power of A and taking the trace we have

$$P_n = P_{n-1} + P_{n-2}$$

which may be solved together with $P_1 = 1$, $P_2 = 3$ without determining A^n for general n directly. Finally, note that as with the doubling map, the symbolic dynamics is not quite 1:1: The discontinuity $x = g^{-1}$ has two symbol sequences $1\overline{0}$ and $\overline{01}$; similarly for its preimages.

Example 5.10. *Another system with this symbolic dynamics is given by the doubling map $x \to \{2x\}$ but enforcing escape for any x with symbol sequence 11, corresponding to the points $x \in [3/4, 1]$.*

