Prof. Dr. B. Eckhardt
Fachbereich Physik

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Philipps Universität Marburg

Computational Physics I

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3.1. Prandtl-von Karman law

The pressure gradient per mass density, $\Delta p/(\rho L)$ that is required to push a viscous fluid of density ρ with mean speed U down a circular pipe of diameter D is proportional to $U^2/(2D)$. The proportionality factor f is dimensionless and differs for laminar and turbulent flows. For a laminar flow it is given by $64/\mathrm{Re}$ and for a turbulent one by an implicit expression derived by Prandtl and von Karman,

$$\frac{1}{\sqrt{f}} = A \ln\left(\mathrm{Re}\sqrt{f}\right) + B \tag{1}$$

where $A = 0.88$ and $B = -0.81$. The Reynolds number is $\mathrm{Re} = UD/\nu$ is a dimensionless combination of the mean velocity U, the diameter D and the kinemativ viscosity ν.
Obtain the curve f vs. Re for $\mathrm{Re} = 10^2 \ldots 10^8$ for laminar and turbulent flows in a diagram with double logarithmic axes.
To determine f in the turbulent case, solve for f such that $f = g(f)$ with a suitable function g. If $|g'(f)| < 1$, the solution may be determined by iterating $f_{n+1} = g(f_n)$ from some suitable initial conditions.
Can you think of alternative ways of determining the curve?

3.2. Lagrange points

Imagine a small object of mass m moving in the gravitational field of two heavy masses m_1 and m_2 which move around each other on circular orbits. If the force of the small object on the heavy one is neglected (a restricted three-body problem), the Hamiltonian for the slight object moving in the plane of the heavy masses and in a frame of reference that rotates with them is given by

$$H = (p_x^2 + p_y^2)/2 - x p_y + y p_x - \frac{m_1}{\sqrt{(x+m_2)^2 + y^2}} - \frac{m_2}{\sqrt{(x-m_1)^2 + y^2}} \tag{2}$$

Verify the above Hamiltonian (note that units are chosen relative to the total mass of the heavy objects, the radius of the orbit and the frequency of the orbit. The mass of the light particle drops out).

Verify that there are five force equilibria for the light object and calculate them for the earth-sun system, where $m_E = 3 \cdot 10^{-6}$ and $m_S = 1 - m_E$ (in units where the total mass (sun plus earth) is 1 and the radius of the orbit (the astronomical unit) is 1).

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3.3. Residuals

Verify that for the linear matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix} \tag{3}$$

the residual $\mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b}$ is small. Solve the linear system and determine $\mathbf{x}$, say be LU-decomposition or Gaussian elimination. Explain the differences.

3.4. Powers of integers

The sums $S^{(k)}(N) = \sum_{n=1}^{N} n^k$ can be expressed exactly as polynominals in N:

$$S^{(k)}(N) = \sum_{\ell=1}^{k+1} c_\ell N^\ell\,. \tag{4}$$

Evaluating the sums for $N = 1 \ldots k+1$ one can formulate a linear equation from which the coefficients can be determined. Calculate the coefficients, the residual and the condition of the matrix.
Hint: For $k = 7$, the coefficients are $c_8 = 1/8$, $c_7 = 1/2$, $c_6 = 7/12$, $c_4 = -7/24$, $c_2 = 1/12$, and $c_5 = c_3 = c_1 = 0$.