GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Lecture 2

Nuclear models: Fermi-Gas Model Shell Model

The basic concept of the Fermi-gas model

The theoretical concept of a Fermi-gas may be applied for systems of weakly interacting fermions, i.e. particles obeying Fermi-Dirac statistics leading to the Pauli exclusion principle ➔

- Simple picture of the nucleus:

— Protons and neutrons are considered as moving freely within the nuclear volume. The binding potential is generated by all nucleons

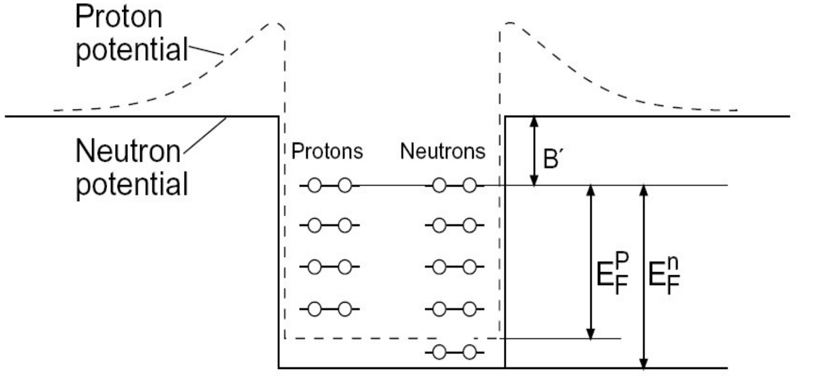
— In a first approximation, these nuclear potential wells are considered as rectangular: it is constant inside the nucleus and stops sharply at its edge

— Neutrons and protons are distinguishable fermions and are therefore situated in two separate potential wells

— Each energy state can be ocupied by two nucleons with different spin projections

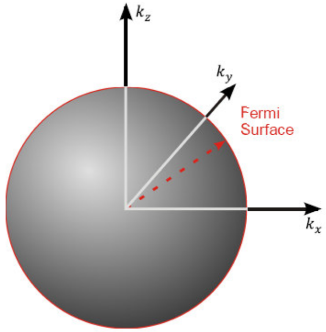
— All available energy states are filled by the pairs of nucleons ➔ no free states , no transitions between the states

— The energy of the highest occupied state is the Fermi energy E_F



— The difference *B‘* between the top of the well and the Fermi level is constant for most nuclei and is just the average binding energy per nucleon *B‘/A* = 7–8 MeV.

Number of nucleon states



Heisenberg Uncertainty Principle: $\Delta x \, \Delta p \geq \frac{1}{2}\hbar$

The volume of one particle in phase space: $2\pi\,\hbar$

The number of nucleon states in a volume V:

$$n = \frac{\int\int d^3r\, d^3p}{(2\pi\hbar)^3} = \frac{V \cdot 4\pi \int_0^{p_{\max}} p^2 dp}{(2\pi\hbar)^3} \qquad (1)$$

At temperature T = 0, i.e. for the nucleus in its ground state, the lowest states will be filled up to a maximum momentum, called the Fermi momentum p_F. The number of these states follows from integrating eq.(1) from 0 to $p_{max}=p_F$:

$$n = \frac{V \cdot 4\pi \int_0^{p_F} p^2 dp}{(2\pi\hbar)^3} = \frac{V \cdot 4\pi\; p_F^3}{(2\pi\hbar)^3 \cdot 3} \quad \Rightarrow \quad \boxed{n = \frac{V \cdot p_F^3}{6\pi^2\hbar^3}} \qquad (2)$$

Since an energy state can contain two fermions of the same species, we can have

Neutrons: $N = \dfrac{V \cdot \left(p_F^n\right)^3}{3\pi^2\hbar^3}$ **Protons:** $Z = \dfrac{V \cdot \left(p_F^p\right)^3}{3\pi^2\hbar^3}$

p_F^n is the fermi momentum for neutrons, p_F^p – for protons

Fermi momentum

Use $R = R_0 \cdot A^{1/3}$ fm, $$V = \frac{4\pi}{3}R^3 = \frac{4\pi}{3}R_0^{\ 3}A$$

The density of nucleons in a nucleus = number of nucleons in a volume V:

$$n = 2 \cdot \frac{V \cdot p_F^3}{6\pi^2\hbar^3} = 2 \cdot \frac{4\pi}{3}R_0^{\ 3}A \cdot \frac{p_F^3}{6\pi^2\hbar^3} = \frac{4A}{9\pi}\frac{R_0^{\ 3}p_F^3}{\hbar^3} \qquad (3)$$

two spin states

Fermi momentum p_F:

$$p_F = \left(\frac{6\pi^2\hbar^3 n}{2V}\right)^{1/3} = \left(\frac{9\pi\hbar^3}{4A}\frac{n}{R_0^{\ 3}}\right)^{1/3} = \left(\frac{9\pi \cdot n}{4A}\right)^{1/3} \cdot \frac{\hbar}{R_0} \qquad (4)$$

After assuming that the proton and neutron potential wells have the same radius, we find for a nucleus with $n=Z=N=A/2$ the Fermi momentum p_F:

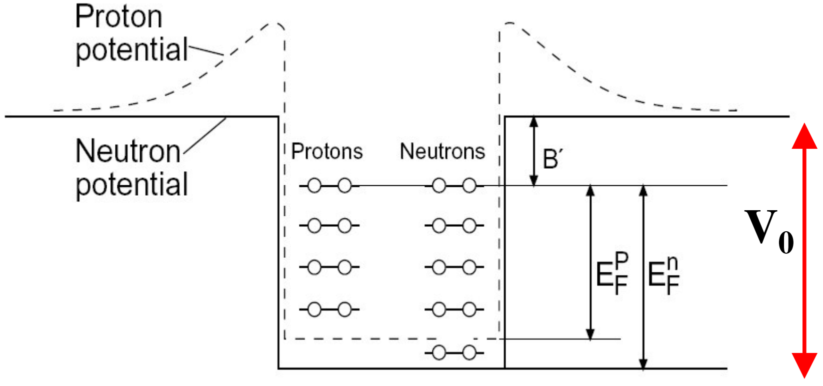
$$p_F = p_F^n = p_F^p = \left(\frac{9\pi}{8}\right)^{1/3} \cdot \frac{\hbar}{R_0} \approx 250\, MeV/c$$

The nucleons move freely inside the nucleus with large momenta.

Fermi energy: $$E_F = \frac{p_F^2}{2M} \approx 33\ MeV$$

M =938 MeV- the mass of nucleon

Nucleon potential



The difference B' between the top of the well and the Fermi level is constant for most nuclei and is just the average binding energy per nucleon B/A = 7–8 MeV.

➔ The depth of the potential V_0 and the Fermi energy are independent of the mass number A:

$$\boxed{V_0 = E_F + B' \approx 40\ MeV}$$

Heavy nuclei have a surplus of neutrons. Since the Fermi level of the protons and neutrons in a stable nucleus have to be equal (otherwise the nucleus would enter a more energetically favourable state through *β*-decay) this implies that the depth of the potential well as it is experienced by the neutron gas has to be larger than of the proton gas (cf Fig.).

Protons are therefore on average less strongly bound in nuclei than neutrons. This may be understood as a consequence of the Coulomb repulsion of the charged protons and leads to an extra term in the potential:

$$V_C = (Z-1)\frac{\alpha \cdot \hbar c}{R}$$

Kinetic energy

The dependence of the binding energy on the surplus of neutrons may be calculated within the Fermi gas model.

First we find the average kinetic energy per nucleon:

$$\langle E \rangle = \frac{\int_0^{E_F} E \cdot \frac{dn}{dE} dE}{\int_0^{E_F} \frac{dn}{dE} dE} = \frac{\int_0^{p_F} E \cdot \frac{dn}{dp} dp}{\int_0^{p_F} \frac{dn}{dp} dp} \qquad \text{where} \qquad \frac{dn}{dp} = Const \cdot p^2$$

$$n = \frac{V \cdot p_F^3}{6\pi\hbar^3}$$

distribution function of the nucleons

$$\langle E_{\mathrm{kin}} \rangle = \frac{\int_0^{p_{\mathrm{F}}} E_{\mathrm{kin}}\, p^2 \mathrm{d}p}{\int_0^{p_{\mathrm{F}}} p^2 \mathrm{d}p} = \frac{3}{5} \cdot \frac{p_{\mathrm{F}}^2}{2M} \approx 20\ \mathrm{MeV}$$

The total kinetic energy of the nucleus is therefore

$$E_{\mathrm{kin}}(N,Z) = N\langle E_{\mathrm{n}} \rangle + Z\langle E_{\mathrm{p}} \rangle = \frac{3}{10M}\left(N \cdot (p_{\mathrm{F}}^{\mathrm{n}})^2 + Z \cdot (p_{\mathrm{F}}^{\mathrm{p}})^2\right)$$

$$E_{\mathrm{kin}}(N,Z) = \frac{3}{10M} \frac{\hbar^2}{R_0^2} \left(\frac{9\pi}{4}\right)^{2/3} \frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \qquad (5)$$

where the radii of the proton and the neutron potential well have again been taken the same.

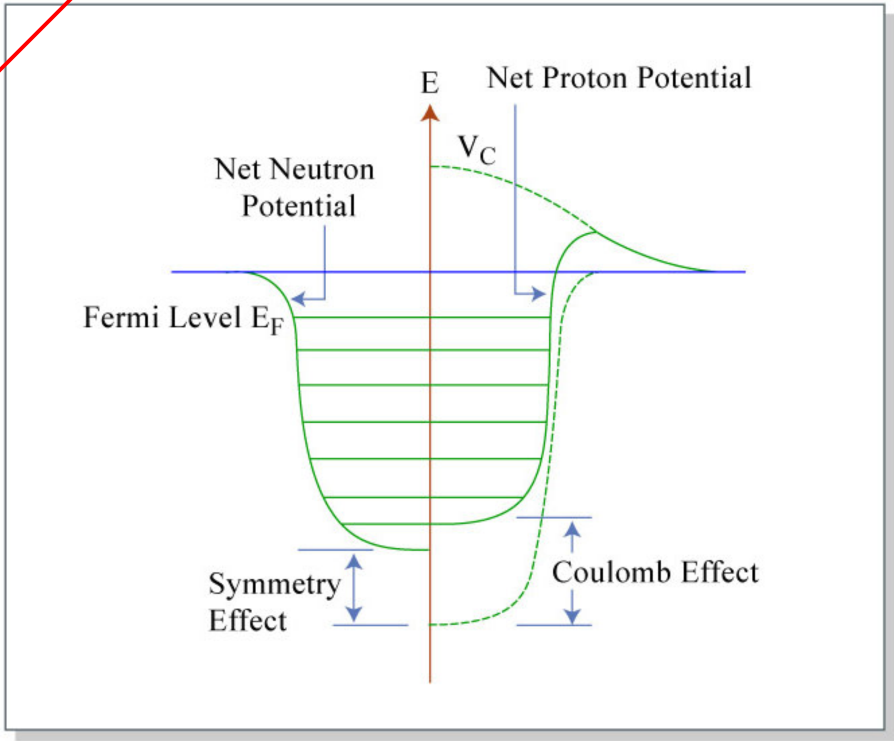
Binding energy

This average kinetic energy has a minimum at $N = Z$ for fixed mass number A (but varying N or, equivalently, Z). Hence the binding energy gets maximal for $N = Z$.

If we expand (5) in the difference $N - Z$ we obtain

$$E_{\text{kin}}(N,Z) = \frac{3}{10M}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(N-Z)^2}{A} + \cdots\right)$$

The first term corresponds to the volume energy in the Weizsäcker mass formula, the second one to the asymmetry energy. The asymmetry energy grows with the neutron (or proton) surplus, thereby reducing the binding energy



Note: this consideration neglected the change of the nuclear potential connected to a change of N on cost of Z. This additional correction turns out to be as important as the change in kinetic energy.

Shell model

Magic numbers: Nuclides with certain proton and/or neutron numbers are found to be exceptionally stable. These so-called magic numbers are

2, 8, 20, 28, 50, 82, 126

— The doubly magic nuclei: ${}^{4}_{2}\mathrm{He}_{2},\quad {}^{16}_{8}\mathrm{O}_{8},\quad {}^{40}_{20}\mathrm{Ca}_{20},\quad {}^{48}_{20}\mathrm{Ca}_{28},\quad {}^{208}_{82}\mathrm{Pb}_{126}$

— Nuclei with magic proton or neutron number have an unusually large number of stable or long lived nuclides.

— A nucleus with a magic neutron (proton) number requires a lot of energy to separate a neutron (proton) from it.

— A nucleus with one more neutron (proton) than a magic number is very easy to separate.

— The first exitation level is very high: a lot of energy is needed to excite such nuclei

— The doubly magic nuclei have a spherical form

➔ nucleons are arranged into complete shells within the atomic nucleus

Excitation energy for magic nuclei

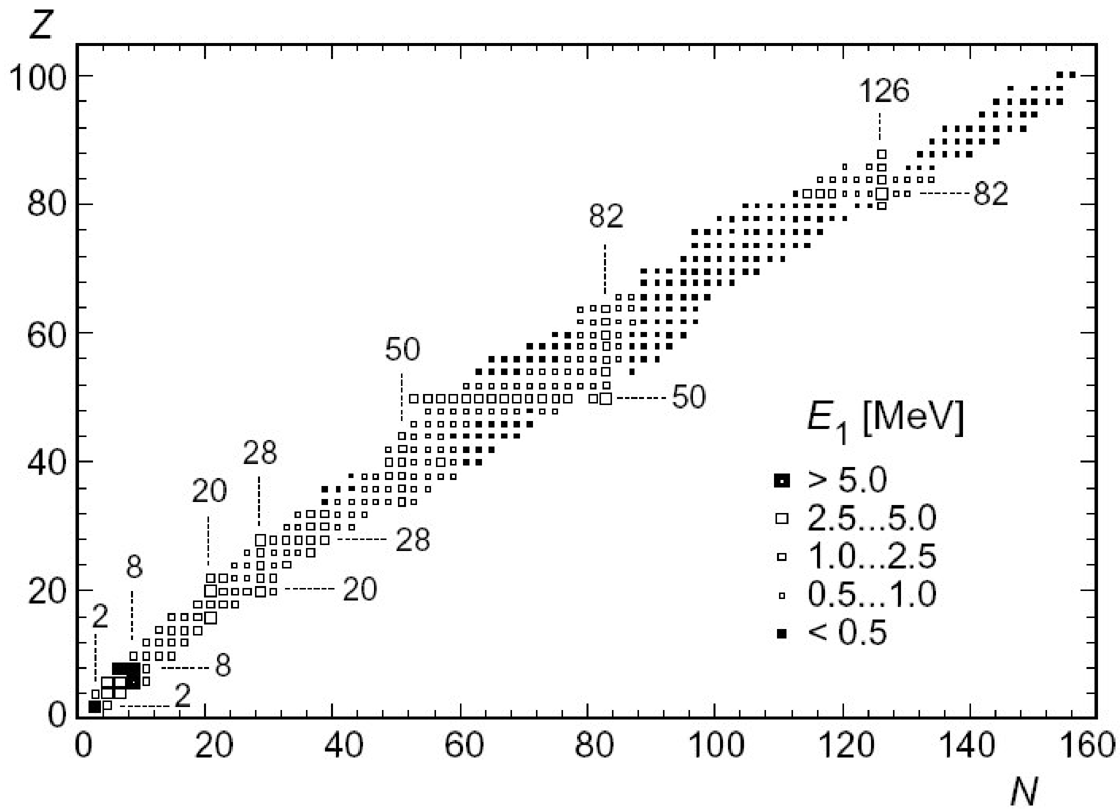


Fig. 17.5. The energy E_1 of the first excited state of even-even nuclei. Note that it is particularly big for nuclei with "magic" proton or neutron number. The excited states generally have the quantum numbers $J^P = 2^+$. The following nuclei are exceptions to this rule: ${}^{4}_{2}\mathrm{He}_2$, ${}^{16}_{8}\mathrm{O}_8$, ${}^{40}_{20}\mathrm{Ca}_{20}$, ${}^{72}_{32}\mathrm{Ge}_{40}$, ${}^{90}_{40}\mathrm{Zr}_{50}$ (0^+), ${}^{132}_{50}\mathrm{Sn}_{82}$, ${}^{208}_{82}\mathrm{Pb}_{126}$ (3^-) and ${}^{14}_{6}\mathrm{C}_8$, ${}^{14}_{8}\mathrm{O}_6$ (1^-). E_1 is small further away from the "magic" numbers – and is generally smaller for heavier nuclei (data from [Le78]).

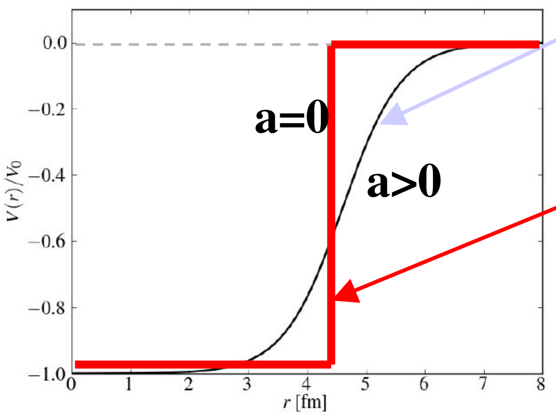
Nuclear potential

The energy spectrum is defined by the nuclear potential
➔ solution of Schrödinger equation for a realistic potential

The nuclear force is very short-ranged => the form of the potential follows the density distribution of the nucleons within the nucleus:

- **for very light nuclei (A < 7), the nucleon distribution has Gaussian form (corresponding to a harmonic oscillator potential)**
- **for heavier nuclei it can be parameterised by a Fermi distribution. The latter corresponds to the Woods-Saxon potential**

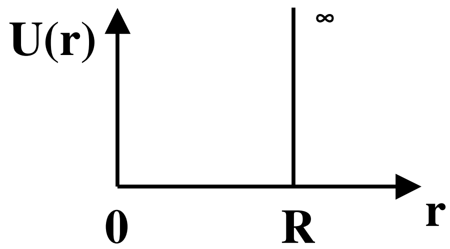
1) Woods-Saxon potential: $$U(r) = -\frac{U_0}{1+e^{\frac{r-R}{a}}}$$



2) a➔0: approximation by the rectangular potential well :

$$U(r) = \begin{cases} -U_0, & r < R \\ 0, & r \geq R \end{cases}$$

e.g. 3) approximation by the rectangular potential well with infinite barrier energy :



$$U(r) = \begin{cases} 0, & r < R \\ \infty, & r \geq R \end{cases}$$

Schrödinger equation

Schrödinger equation: $$\hat{H}\Psi = E\Psi \qquad (1)$$

Single-particle Hamiltonian operator: $$\hat{H} = -\frac{\hbar\nabla^2}{2M} + U(r) \qquad (2)$$

Eigenstates: Ψ(r) - wave function
Eigenvalues: E - energy
U(r) is a nuclear potential – spherically symmetric ➔

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Angular part: $$\hat{\lambda} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$-\hbar^2\hat{\lambda} = \hat{L}^2$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\hat{L}^2}{\hbar^2}$$

L – operator for the orbital angular momentum

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \qquad (3)$$

Eigenstates: Y_{lm} – spherical harmonics

Radial part

The wave function of the particles in the nuclear potential can be decomposed into two parts: a radial one $\Psi_1(r)$, which only depends on the radius r, and an angular part $Y_{lm}(\theta,\varphi)$ which only depends on the orientation (this decomposition is possible for all spherically symmetric potentials):

$$\Psi(r,\theta,\varphi) = \Psi_1(r)\cdot \mathrm{Y}_{lm}(\theta,\varphi) \qquad (4)$$

From (4) and (1) =>

$$\left[-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\hat{L}^2}{2M}\right]\Psi_1(r)\cancel{\mathrm{Y}_{lm}(\theta,\varphi)} = E\ \Psi_1(r)\cancel{\mathrm{Y}_{lm}(\theta,\varphi)}$$

=> eq. for the radial part:

$$\left[-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]\Psi_1(r) = E\ \Psi_1(r) \qquad (5)$$

Substitute in (5):

$$\Psi_1(r) = \frac{R(r)}{r}$$

$$-\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}R(r) = E\ R(r) \qquad (6)$$

Constraints on E

Eq. for the radial part: $$\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\left[E-\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]R(r)=0 \qquad (7)$$

From (7) ➔

1) Energy eigenvalues for orbital angular momentum *l*:

E:		
l=0	***s***	
l=1	***p***	**states**
l=2	***d***	
l=3	***f***	
…		

2) For each *l*: -*l*< *m* <*l* => (2*l*+1) projections *m* of angular momentum.
The energy is independent of the *m* quantum number, which can be any integer value between ± *l*. Since nucleons also have two possible spin directions, this means that the *l* levels are 2·(2*l*+1) times degenerate if a spin-orbit interaction is neglected.

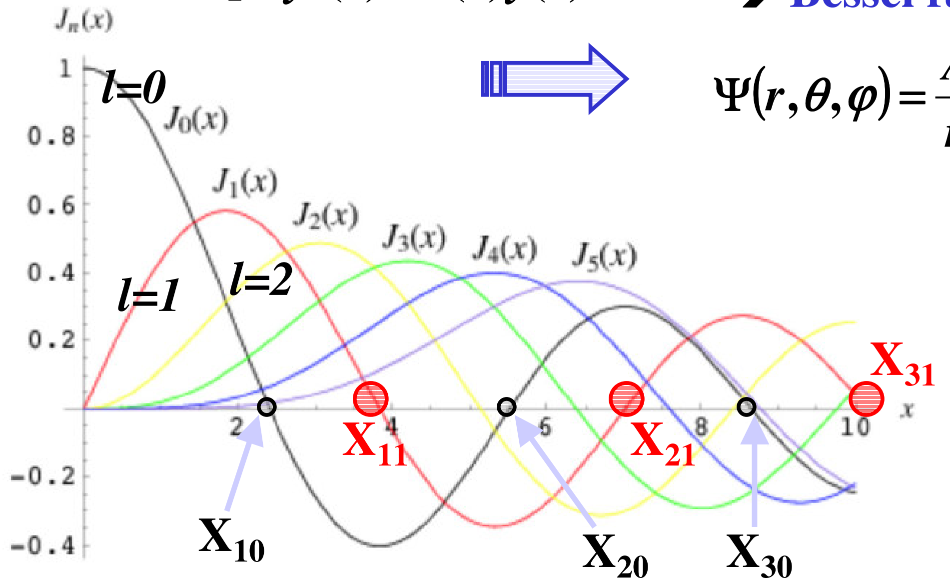
3) The parity of the wave function is fixed by the spherical wave function Y_{ml} and reads $(-1)^l$: $$\hat{P}\,\Psi(r,\theta,\varphi)=P\,\Psi_1(r)\cdot \Upsilon_{lm}(\theta,\varphi)=(-1)^l\;\Psi_1(r)\cdot \Upsilon_{lm}(\theta,\varphi)$$

$\Rightarrow$ ***s,d,..* - even states; *p,f,…* - odd states**

Main quantum number *n*

Eq. for the radial part: $$\frac{\hbar^2}{2M}\frac{d^2R(r)}{dr^2}+\left[E-\frac{\hbar^2}{r^2}\frac{l(l+1)}{2M}\right]R(r)=0 \qquad (7)$$

Solution of differential eq: $y''(r)+\lambda(r)y(r)=0$ ➔ **Bessel functions $j_l(kr)$**



$$\Psi(r,\theta,\varphi)=\frac{A}{r}j_l(kr)\cdot Y_{lm}(\theta,\varphi)$$

$$k^2=\frac{2ME}{\hbar^2}$$

Boundary condition for the surface, i.e. at r=R: $\Psi(R,\theta,\varphi)=0$

➔ restrictions on k in Bessel functions: $j_l(kr)=0$

⇒ main quantum number *n* – corresponds to nodes of the Bessel function : X_{nl}

$$k\cdot R(r)=X_{nl} \qquad k^2R^2=X_{nl} \qquad \frac{2ME}{\hbar^2}R^2=X_{nm}^2 \qquad (8)$$

Shell model

Thus, according to Eq. (8) : $\boxed{E_{nl} = \frac{X_{nl}^2 \hbar^2}{2MR^2}} \Leftrightarrow E_{nl} = Const \cdot X_{nl}^2$ (9)

Nodes of Bessel function

Energy states are quantized ➔ **structure of energy states E_{nl}**

$l=0 \quad s-states \quad j_0$

$n=1 \quad X_{10} = 3.14$

$n=2 \quad X_{20} = 6.28$

$n=3 \quad X_{30} = 9.42$

$l=1 \quad p-states \quad j_1$

$n=1 \quad X_{11} = 4.49$

$n=2 \quad X_{21} = 7.72$

$l=2 \quad d-states \quad j_2$

$n=1 \quad X_{12} = 5.76$

$l=3 \quad f-states \quad j_3$

$n=1 \quad X_{13} = 6.99$

$l=4 \quad g-states \quad j_4$

$n=1 \quad X_{14} = 8.1$

state	$E_{nl} = C \cdot X_{nl}^2$	*degeneracy*	*states with* $E \le E_{nl}$
$1s$	$E_{1s} = C \cdot 9.86$	2 $2\cdot(2l+1)$	2
$1p$	$E_{1p} = C \cdot 20.2$	6	8
$1d$	$E_{1d} = C \cdot 33.2$	10	18
$2s$	$E_{2s} = C \cdot 39.5$	2	20
$1f$	$E_{1f} = C \cdot 48.8$	14	34
$2p$	$E_{2p} = C \cdot 59.7$	6	40
$1g$	$E_{1g} = C \cdot 64$	18	58

First 3 magic numbers are reproduced, higher – not!

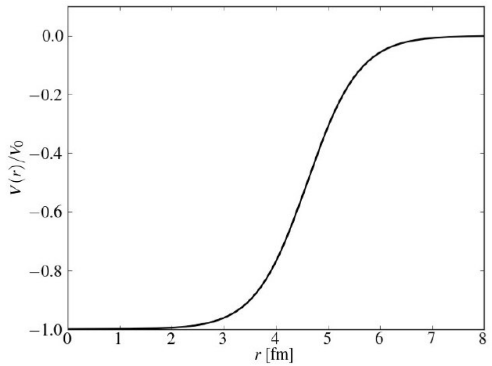
2, 8, 20, 28, 50, 82, 126

Note: here for U(r) = rectangular potential well with infinite barrier energy

Shell model with Woods-Saxon potential

Woods-Saxon potential:

$$U(r) = -\frac{U_0}{1+e^{\frac{r-R}{a}}}$$



N	0	1	2	2	3	3	4	4	4	$\cdots$
$n\ell$	1s	1p	1d	2s	1f	2p	1g	2d	3s	$\cdots$
Degeneracy	2	6	10	2	14	6	18	10	2	$\cdots$
States with $E \leq E_{n\ell}$	2	8	18	20	34	40	58	68	70	$\cdots$

The first three magic numbers (2, 8 and 20) can then be understood as nucleon numbers for full shells.This simple model does not work for the higher magic numbers. For them it is necessary to include spin-orbit coupling effects which further split the *nl* shells.

Spin-orbit interaction

Introduce the spin-orbit interaction V_{ls} – a coupling of the spin and the orbital angular momentum:

$$\hat{H} = -\frac{\hbar\nabla^2}{2M} + U(r) + \hat{V}_{ls}$$

$$\left[-\frac{\hbar\nabla^2}{2M} + U(r) + \hat{V}_{ls}\right]\Psi(r,\theta,\varphi) = E\ \Psi(r,\theta,\varphi)$$

$$\left[-\frac{\hbar\nabla^2}{2M} + U(r)\right]\Psi(r,\theta,\varphi) = \left(E - V_{ls}\right)\ \Psi(r,\theta,\varphi)$$

where $\hat{V}_{ls}\Psi(r,\theta,\varphi) = V_{ls}\Psi(r,\theta,\varphi)$

Eigenstates **eigenvalues**

spin-orbit interaction: $\hat{V}_{ls} = C_{ls}(\vec{l},\vec{s})$

total angular momentum: $\vec{j} = \vec{l} + \vec{s}$

$$\vec{j}\cdot\vec{j} = (\vec{l}+\vec{s})(\vec{l}+\vec{s}) = \vec{l}^{\,2} + \vec{s}^{\,2} + 2\vec{l}\vec{s}$$

$$\boxed{\vec{l}\cdot\vec{s} = \frac{1}{2}(\vec{j}^{\,2} - \vec{l}^{\,2} - \vec{s}^{\,2})}$$

Spin-orbit interaction

$$C_{ls}\cdot\frac{1}{2}(\vec{j}^{\,2}-\vec{l}^{\,2}-\vec{s}^{\,2})\Psi(r,\theta,\varphi)=V_{ls}\Psi(r,\theta,\varphi)$$

$$\boxed{V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[j(j+1)-l(l+1)-s(s+1)\right]}$$

Consider:

$$j=l+\frac{1}{2}: \qquad V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[(l+\frac{1}{2})(l+\frac{1}{2})-l^2-l-\frac{1}{2}\cdot\frac{3}{2}\right]=\underline{C_{ls}\frac{\hbar^2}{2}l}$$

$$j=l-\frac{1}{2}: \qquad V_{ls}=C_{ls}\frac{\hbar^2}{2}\left[(l-\frac{1}{2})(l+\frac{1}{2})-l^2-l-\frac{1}{2}\cdot\frac{3}{2}\right]=\underline{-C_{ls}\frac{\hbar^2}{2}(l+1)}$$

This leads to an energy splitting ΔE_{ls} which linearly increases with the angular momentum as

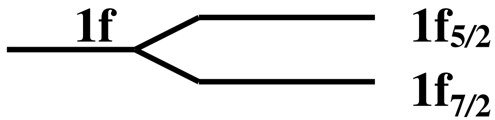
$$\Delta E_{ls}=\frac{2l+1}{2}\langle V_{ls}\rangle$$

It is found experimentally that V_{ls} is negative, which means that the state with $j=l+1/2$ is always energetically below the $j=l-1/2$ level.

Spin-orbit interaction

The total angular momentum quantum number $j = l \pm 1/2$ of the nucleon is denoted by an extra index j: nl_j

e.g. the 1f state splits into a $1f_{7/2}$ and a $1f_{5/2}$ state



The *nlj* level is $(2j + 1)$ times degenerate

➔ Spin-orbit interaction leads to a sizeable splitting of the energy states which are indeed comparable with the gaps between the *nl* shells themselves.

Magic numbers appear when the gaps between successive energy shells are particularly large.

2, 8, 20, 28, 50, 82, 126

Single particle energy levels: