TP 2

Friday 15th November, 2024
joao.candido@hesge.ch
Deadline: Sunday 1st December, 2024, 23:59

1 Simple exercises with matrices

1.1 Practice with Numbers

Give the result and if there is no answer, say so.

1. $\begin{bmatrix} 5 & 2 \\ 3 & 4 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 50 \\ 20 & 30 \\ 30 & 20 \end{bmatrix}$
2. $\begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 5 \\ 40 & 20 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$
5. $\begin{bmatrix} 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
6. $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 5 & 3 & 7 \end{bmatrix}$
7. $\begin{bmatrix} 0 & 1 & 2 \\ 10 & -10 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
8. Note that the first matrix bellow is called a *permutation* matrix. Such matrices are always square and have exactly one 1 in each line and each column. This particular one will permute the rows of the right hand side.
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

9. $5\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$

10. $-1\begin{bmatrix}1 & 2\\3 & 4\\-5 & -6\end{bmatrix}$

11. $\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} - \begin{bmatrix}10 & 20\\30 & 40\\50 & 60\end{bmatrix}$

12. $\begin{bmatrix}2 & 1\\3 & 4\\5 & 6\end{bmatrix}'$

13. $\begin{bmatrix}1 & 6\\3 & 4\\5 & 2\end{bmatrix}^{\mathrm{T}}$

14. $\left(\begin{bmatrix}5 & 2\\3 & 4\\1 & 6\end{bmatrix}^{\mathrm{T}}\right)^{\mathrm{T}}$

1.2 Matrix type

Determine the *most specific* name for each matrix bellow from the following choices: column matrix, diagonal matrix, identity matrix, lower triangular, permutation matrix, row matrix, square matrix, symmetric matrix, upper triangular matrix.

1. $\begin{bmatrix}a & b\\7 & 12\end{bmatrix}$ square matrix

2. $\begin{bmatrix}3\\4\end{bmatrix}$ column matrix

3. $\begin{bmatrix}1 & 2 & 3\end{bmatrix}$ row matrix

4. $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ identity matrix

5. $\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}$ diagonal matrix

6. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ symmetric matrix

7. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ permutation matrix

8. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ upper triangular matrix

9. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$ lower triangular matrix

1.3 Some matrix Operations

We are given a sample of n, k-dimensional, instances, which we organise in the $n \times k$ matrix:

$$\mathbf{X} = \begin{bmatrix} x_{11} & \dots & x_{1k} \\ x_{21} & \dots & x_{2k} \\ \dots & & \dots \\ x_{n1} & \dots & x_{nk} \end{bmatrix}$$

In addition we have the standard sample estimates of the mean, variance and covariance of two variables $\mathbf{x}$, $\mathbf{y}$, from the n-dimensional sample above:

$\mu_{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i$	The sample mean of $\mathbf{x}$
$\sigma^2_{\mathbf{x}} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2$	The sample variance of $\mathbf{x}$
$\sigma_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y)$	The sample covariance of $\mathbf{x}$ and $\mathbf{y}$

You should recognise these quantities in the algebraic expressions given bellow. You should explain in detail each one of the expressions bellow, how does its structure look like? and what does it contain?

1. $\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ matrice nxn of 1/n
2. $\frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\mathbf{x}$ we're doing matrix nxn dot vector nx1
3. $\mathbf{x} - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\mathbf{x}$
4. $\left[\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right]\mathbf{x}$
5. $\left[\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right]\mathbf{1}$
6. $\mathbf{M} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}}$

7. $\mathbf{1}^{\mathrm{T}}\mathbf{x}$
8. $\mathbf{1}^{\mathrm{T}}(\mathbf{M}\mathbf{x})$
9. $\mathbf{M}\mathbf{M}$
10. $\mathbf{x}^{\mathrm{T}}\mathbf{x}$
11. $(\mathbf{M})^{\mathrm{T}}(\mathbf{M})$
12. $\mathbf{x}^{\mathrm{T}}\mathbf{y}$
13. $(\mathbf{M}\mathbf{x})^{\mathrm{T}}(\mathbf{M}\mathbf{y})$
14. $\frac{1}{n-1}(\mathbf{M}\mathbf{x})^{\mathrm{T}}(\mathbf{M}\mathbf{y})$
15. $\mathbf{M}\mathbf{X}$
16. $(\mathbf{M}\mathbf{X})^{\mathrm{T}}(\mathbf{M}\mathbf{X})$

2 Vector independence, basis

1. What is the definition of linear dependence? How can we show that a set of vectors are linearly dependent?
   (a) is the vector (0,0) linearly dependent? explain your answer.
   (b) Are the vectors (1,5) and (2,10) linearly dependent? Why?
   (c) Show that the vectors $\mathbf{x} = (1, 1, 0)$, $\mathbf{y} = 0, 1, 2)$, and $\mathbf{z} = (3, 1, -4)$ are linearly dependent by finding the scalars α, β, γ such that $\alpha x + \beta y + \gamma z = 0$
   (d) Same as the previous for the vectors $\mathbf{x} = (1, 4, 5)$, $\mathbf{y} = (1, 2, 3)$, and $\mathbf{z} = (2, 2, 4)$.
2. What is the definition of linear independence? How can we show that a set of vectors are linearly independent?
   (a) Show that the vectors $(1, 2)$ and $(3, 4)$ are linearly independent.
   (b) Same as above for the vectors $(3, 4), (5, 2)$.
3. Let $\mathbf{w} = (1, 1, 0, 0), \mathbf{x} = (1, 0, 1, 0), \mathbf{y} = (0, 0, 1, 1)$, and $\mathbf{z} = (0, 1, 0, 1)$.
   (a) We can show that $\{w, x, y, z\}$ is not a spanning set for $\mathbb{R}^4$ by finding a vector $\mathbf{u}$ in $\mathbb{R}^4$ such that $\mathbf{u} \notin \mathrm{span}\{\mathbf{w}, x, y, z\}$. One such vector is $\mathbf{u} = (1, 2, 3, a)$ where a is any number except:...
   (b) Show that $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set of vectors by finding scalars α, γ, δ, such that $\alpha\mathbf{w} + \mathbf{x} + \gamma\mathbf{y} + \delta\mathbf{z} = \mathbf{0}$ Answer: $\alpha = \ldots$, $\gamma = \ldots$, $\delta = \ldots$

(c) Show that $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set by writing $\mathbf{z}$ as a linear combination of $\mathbf{w}, \mathbf{x}$, and $\mathbf{y}$. Answer: $\mathbf{z} = \ldots \mathbf{w} + \ldots \mathbf{x} + \ldots \mathbf{y}$.

4. Let $\mathbf{u} = (\lambda, 1, 0)$, $\mathbf{v} = (1, \lambda, 1)$, and $\mathbf{w} = (0, 1, \lambda)$. Find all values of λ which make $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, a linearly dependent subset of $\mathbb{R}^3$.

5. Let $\mathbf{u} = (2, 0, -1), \mathbf{v} = (3, 1, 0)$, and $\mathbf{w} = (1, -1, c)$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that c is not equal to .... Show how you find c.

6. Let $\mathbf{u} = (1, -1, 3)$, $v = (1, 0, 1)$, and $\mathbf{w} = (1, 2, c)$ where $c \in R$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that c is not equal to .... Show how you find c.

7. Check whether the set of vectors $\mathbf{x} = (1, 1, 1), \mathbf{y} = (0, 2, 1), \mathbf{z} = (3, 1, 0)$ is a basis of $\mathbb{R}^3$. If yes find the representation of the vector $\mathbf{w} = (3, 1, 1)$ in the basis given by $\mathbf{x}, \mathbf{y}, \mathbf{z}$ vectors by solving the system $\mathbf{w} = a_1\mathbf{x} + a_2\mathbf{y} + a_3\mathbf{z}$. What is the representation (coordinates) of $\mathbf{w}$ in the new basis?

8. Convert the $\mathbf{x} = (1, 1, 1), \mathbf{y} = (0, 2, 1), \mathbf{z} = (3, 1, 0)$ to an orthonormal basis, $\mathbf{x}', \mathbf{y}', \mathbf{z}'$, using the Gramm-Schmid algorithm. Show how you can compute the representation of the vector $\mathbf{w} = (3, 1, 1)$ in the orthnormal basis without solving the linear system $\mathbf{w} = a_1'\mathbf{x}' + a_2'\mathbf{y}' + a_3'\mathbf{z}'$.

3 Convolution

In this exercise, we will implement the convolution for signals in 1 and 2 dimensions. Please, follow the notebook present with this pdf (tp_convolution_rotation.ipynb) and fill the missing parts (denoted by "your code here") on the notebook directly. Question are present in the notebook, you should answer them in your report.

4 Rotation

On the same notebook as before (tp_convolution_rotation.ipynb), implement the rotation section and answer the question in your report.