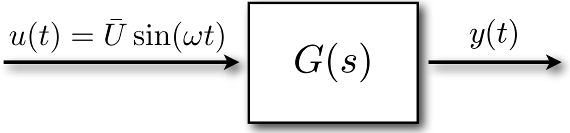
Analiza in domeniul frecventa. Stabilitatea sistemelor

Special thanks to:
A. Bemporad, Automatic Control 1, Lecture Notes, University of Trento, Italy, 2011, http://cse.lab.imtlucca.it/~bemporad/automatic_control_course.html
P. Raica, Systems Theory, Lecture Notes, Technical University of Cluj-Napoca, Cluj-Napoca, 2012, https://cs.utcluj.ro/files/educatie/licenta/2021-2022/22_CALCen_ST.pdf

Frequency response

Definition

The *frequency response* of a linear dynamical system with transfer function $G(s)$ is the complex function $G(j\omega)$ of the real angular frequency $\omega \geq 0$



Theorem

If $G(s)$ is asymptotically stable (poles with negative real part), for $u(t) = \bar{U}\sin(\omega t)$ in steady-state conditions $\lim_{t\to\infty} y(t) - y_{ss}(t) = 0$, where

$$y_{ss}(t) = \bar{U}|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

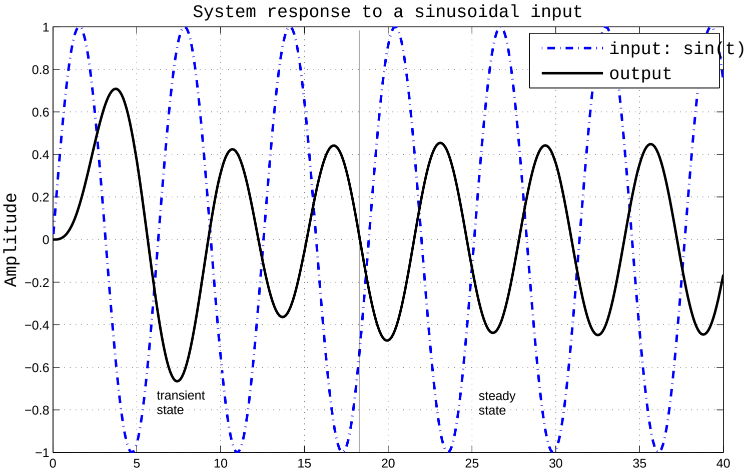
The frequency response $G(j\omega)$ of a system allows us to analyze the response of the system to sinusoidal excitations at different frequencies ω

Introduction

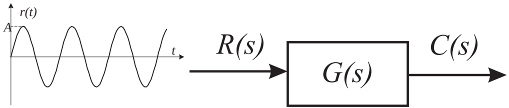
- Analyze the steady-state response of a system with sinusoidal input as the frequency of the sinusoid ω is varied.
- Examine the transfer function $G(j\omega)$ and develop several forms of plotting when ω is varied.

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal and the resulting output signal, for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase angle.

Example


System response to a sinusoidal input
Amplitude
input: sin(t)
output
transient
state
steady
state
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
5
10
15
20
25
30
35
40

Introduction



$$r(t) = Asin\omega t, \quad R(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$G(s) = \frac{m(s)}{\prod_{i=1}^{n}(s + p_i)}$$

$$C(s) = G(s)R(s) = \frac{k_1}{s + p_1} + ... + \frac{k_1}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

Introduction

System is stable $\Rightarrow$ p_i have negative nonzero real parts and the terms corresponding to the poles in $c(t)$ will be zero at steady-state, or:

$$\lim_{t\to\infty} \mathcal{L}^{-1}\left[\frac{k_i}{s+p_i}\right] = 0$$

For $t \to \infty$ (the steady -state):

$$c(t) = \mathcal{L}^{-1}\left[\frac{\alpha s+\beta}{s^2+\omega^2}\right] = \frac{1}{\omega}|A\omega G(j\omega)| sin(\omega t+\varphi)$$

$$c(t) = A|G(j\omega)| sin(\omega t+\varphi)$$

where $\varphi = \angle G(j\omega)$.

First-order system with the transfer function:

Example

$$G(s) = \frac{1}{s+1}$$

The input is sinusoidal: $r(t) = A \sin \omega t$. Calculate the output signal $c(t)$ at steady state.

$$C(s) = R(s)G(s) = \frac{A\omega}{(s^2+\omega^2)(s+1)} = \frac{A\omega}{\omega^2+1}\frac{1}{s+1} + \frac{A\omega}{\omega^2+1}\frac{1-s}{s^2+\omega^2}$$

$$c(t) = \mathcal{L}^{-1}[C(s)] = \frac{A\omega}{\omega^2+1}\mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{1-s}{s^2+\omega^2}\right]$$

$$c(t) = \frac{A\omega}{\omega^2+1}e^{-t} + \frac{A\omega}{\omega^2+1}\mathcal{L}^{-1}\left[\frac{1-s}{s^2+\omega^2}\right]$$

Example

At steady-state:

$$c(t) = \frac{A\omega}{\omega^2+1}\mathcal{L}^{-1}\left[\frac{1-s}{s^2+\omega^2}\right] = \frac{A\omega}{\omega^2+1}\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\frac{1}{\omega} - \frac{s}{s^2+\omega^2}\right]$$

$$c(t) = \frac{A\omega}{\omega^2+1}\left[\frac{1}{\omega}\sin\omega t - \cos\omega t\right]$$

If we replace $s = j\omega$ into $G(s)$ we obtain a complex quantity with the magnitude and phase angle:

$$|G(j\omega)| = \left|\frac{1}{j\omega+1}\right| = \frac{1}{\sqrt{\omega^2+1}}$$

$$\angle G(j\omega) = \varphi = \angle\frac{1}{j\omega+1} = -\arctan\omega$$

Example

$$\tan\varphi = -\omega; \quad \frac{\sin\varphi}{\cos\varphi} = -\omega; \quad \cos\varphi = \frac{1}{\sqrt{\omega^2+1}}$$

By replacing these results into $c(t)$:

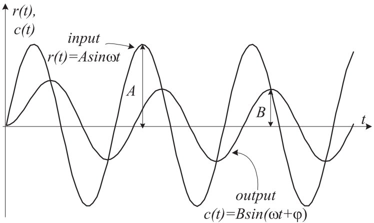
$$c(t) = \frac{A\omega}{\omega^2+1}\frac{\sin\omega t - \omega\cos\omega t}{\omega} =$$

$$= \frac{A}{\omega^2+1}\frac{1}{\cos\varphi}[\sin\omega t\cos\varphi + \sin\varphi\cos\omega t]$$

$$c(t) = \frac{A}{\omega^2+1}\frac{1}{\cos\varphi}\sin(\omega t+\varphi) = \frac{A}{\sqrt{\omega^2+1}}\sin(\omega t+\varphi)$$

$$c(t) = A|G(j\omega)|\sin\left(\omega t + \angle G(j\omega)\right)$$

Introduction

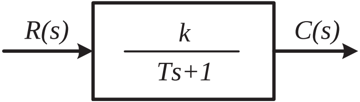


$$|G(j\omega)| = \left|\frac{C(j\omega)}{R(j\omega)}\right|, \quad \angle G(j\omega) = \angle\frac{C(j\omega)}{R(j\omega)}$$

$G(j\omega)$: **the sinusoidal transfer function**.

A negative phase angle is called **phase lag** and a positive phase angle is called **phase lead.**

Example



For the sinusoidal input $r(t) = Asin\omega t$, $c_{ss}(t)$ can be found:

$$G(j\omega) = \frac{k}{jT\omega + 1}$$

$$|G(j\omega)| = \frac{k}{\sqrt{1 + T^2\omega^2}}, \quad \varphi = \angle G(j\omega) = -arctan T\omega$$

Thus, for the input $r(t)$, the steady-state output $c_{ss}(t)$ is:

$$c_{ss}(t) = \frac{Ak}{\sqrt{1 + T^2\omega^2}} sin(\omega t - arctan T\omega)$$

Example

$$u(t) = 5\sin 10t \longrightarrow \boxed{G(s) = \frac{10}{(1+0.1s)^2}} \longrightarrow y(t)$$

- The poles are $p_1 = p_2 = -10$, the system is asymptotically stable
- The steady-state response is

$$y_{ss}(t) = 5|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

where

$$G(j\omega) = \frac{10}{(1+0.1j\omega)^2} = \frac{10}{1+0.2j\omega - 0.01\omega^2}$$

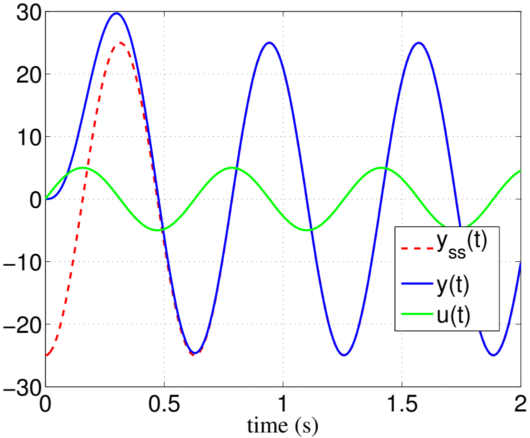
- For $\omega = 10$ rad/s,

$$G(10j) = \frac{10}{1+2j-1} = \frac{5}{j} = -5j$$

- Finally, we get

$$y_{ss}(t) = 5 \cdot 5\sin(10t - \frac{\pi}{2}) = 25\sin(10t - \frac{\pi}{2})$$

Example (cont'd)



$$u(t) = 5\sin(10t) \;\rightarrow\; y_{ss}(t) = 25\sin(10t - \frac{\pi}{2})$$

Bode plot

- The *Bode plot* is a graph of the module $|G(j\omega)|$ and phase $\angle G(j\omega)$ of a transfer function $G(s)$, evaluated in $s = j\omega$

- The Bode plot shows the system's frequency response as a function of ω, for all $\omega > 0$

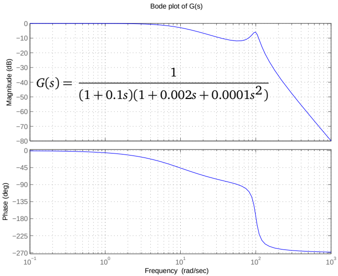
Hendrik Wade Bode

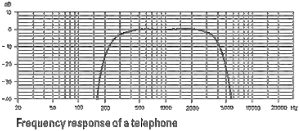
(1905–1982)

Bode plot

Bode magnitude plot

Bode phase plot





Example: frequency response of a telephone, approx. 300–3,400 Hz, good enough for speech transmission

- The frequency ω axis is in *logarithmic scale*
- The module $|G(j\omega)|$ is expressed as *decibel (dB)*

$$|G(j\omega)|_{\mathrm{dB}} = 20\log_{10}|G(j\omega)|$$

- Example: the DC-gain $G(0) = 1$, $|G(j0)|_{\mathrm{dB}} = 20\log_{10} 1 = 0$

MATLAB

```
»bode(G)

»evalfr(G,w)
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Bode form

- To study the frequency response of the system is useful to rewrite the transfer function $G(s)$ in *Bode form*

$$G(s) = \frac{K}{s^h} \frac{\Pi_i(1 + s\tau_i)}{\Pi_j(1 + sT_j)} \frac{\Pi_i\left(1 + \frac{2\zeta_i'}{\omega_{ni}'}s + \frac{1}{\omega_{ni}'^2}s^2\right)}{\Pi_j\left(1 + \frac{2\zeta_j}{\omega_{nj}}s + \frac{1}{\omega_{nj}^2}s^2\right)}$$

- K is the *Bode gain*
- h is the *type* of the system, that is the number of poles in $s = 0$
- T_j (for real $T_j > 0$) is said a *time constant* of the system
- ζ_j is a *damping ratio* of the system, $-1 < \zeta_j < 1$
- ω_{nj} is a *natural frequency* of the system

Bode magnitude plot

$$|G(j\omega)|_{\mathrm{dB}} = 20\log_{10}\left|\frac{K}{(j\omega)^h}\,\frac{\Pi_i(1+j\omega\tau_i)}{\Pi_j(1+j\omega T_j)}\,\frac{\Pi_i\left(1+\frac{2j\zeta_i'\omega}{\omega_{ni}'}-\frac{\omega^2}{\omega_{ni}'^2}\right)}{\Pi_j\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega_{nj}^2}\right)}\right|$$

- Recall the following properties of logarithms:

$$\log\alpha\beta = \log\alpha + \log\beta, \quad \log\frac{\alpha}{\beta} = \log\alpha - \log\beta, \quad \log\alpha^\beta = \beta\log\alpha$$

- Thus we get

$$\begin{aligned}
|G(j\omega)|_{\mathrm{dB}} \quad &= \quad 20\log_{10}|K| - h\cdot 20\log_{10}\omega \\
&+\sum_i 20\log_{10}|1+j\omega\tau_i| - \sum_j 20\log_{10}|1+j\omega T_j| \\
&+\sum_i 20\log_{10}\left|1+\frac{2j\zeta_i'\omega}{\omega_{ni}'}-\frac{\omega^2}{\omega_{ni}'^2}\right| - \sum_j 20\log_{10}\left|1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega_{nj}^2}\right|
\end{aligned}$$

Bode magnitude plot

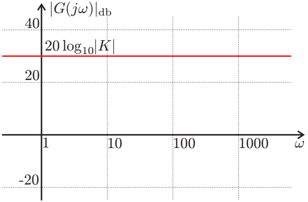
- We can restrict our attention to four basic components only:

$$
\begin{aligned}
|G(j\omega)|_{\mathrm{dB}} \quad &= \quad \underbrace{20\log_{10}|K|}_{\#1} - h \cdot \underbrace{20\log_{10}\omega}_{\#2} \\
&\quad + \sum_i 20\log_{10}|1+j\omega\tau_i| - \sum_j \underbrace{20\log_{10}|1+j\omega T_j|}_{\#3} \\
&\quad + \sum_i 20\log_{10}\left|1+\frac{2j\zeta_i'\omega}{\omega_{ni}'} - \frac{\omega^2}{\omega_{ni}'^2}\right| - \sum_j \underbrace{20\log_{10}\left|1+\frac{2j\zeta_j\omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2}\right|}_{\#4}
\end{aligned}
$$

Bode magnitude plot

- Basic component #1

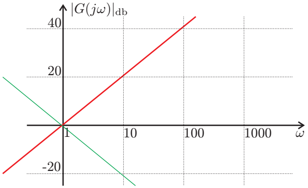
$$20\log_{10}|K|$$



- Basic component #2

$$20\log_{10}|\omega|$$

$$-20\log_{10}|\omega|$$



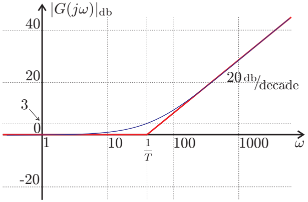
Bode magnitude plot

- Basic component #3

$$20\log_{10}|1 + j\omega T|$$

$|G(j\omega)| \approx 1$ for $\omega \ll \frac{1}{T}$

$|G(j\omega)| \approx \omega T$ for $\omega \gg \frac{1}{T}$

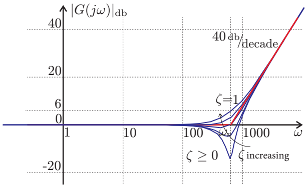


- Basic component #4

$$20\log_{10}\left|1 + 2j\zeta\frac{\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right|$$

$|G(j\omega)| \approx 1$ for $\omega \ll \omega_n$

$|G(j\omega)| \approx \frac{\omega^2}{\omega_n^2}$ for $\omega \gg \omega_n$

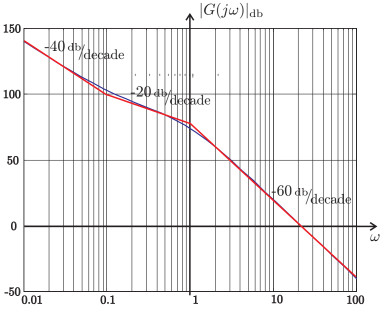


For $\zeta < \frac{1}{\sqrt{2}}$ the peak of the response is obtained at the *resonant frequency* $\omega_{\text{peak}} = \omega_n\sqrt{1 - 2\zeta^2}$

Example

Draw the Bode plot of the transfer function

$$G(s) = \frac{1000(1+10s)}{s^2(1+s)^2}$$



- For $\omega \ll 0.1$: $|G(j\omega)| \approx \frac{1000}{\omega^2}$
- For $\omega = 0.1$: $20\log_{10}\frac{1000}{\omega^2} = 20\log_{10}10^5 = 100$
- For $0.1 < \omega < 1$: effect of zero $s = -0.1$, increase by 20 dB/decade
- For $\omega > 1$: effect of double pole $s = -1$, decrease by 40 dB/decade

Bode phase plot

$$\angle G(j\omega) = \angle\left(\frac{K}{(j\omega)^h}\frac{\Pi_i(1+j\omega\tau_i)}{\Pi_j(1+j\omega T_j)}\frac{\Pi_i\left(1+\frac{2j\zeta'_i\omega}{\omega'_{ni}}-\frac{\omega^2}{\omega'^2_{ni}}\right)}{\Pi_j\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega^2_{nj}}\right)}\right)$$

- Because of the following properties of exponentials

$$\begin{aligned}
&\angle(\rho e^{j\theta}) = \theta \\
&\angle(\alpha\beta) = \angle(\rho_\alpha e^{j\theta_\alpha}\rho_\beta e^{j\theta_\beta}) = \angle(\rho_\alpha\rho_\beta e^{j(\theta_\alpha+\theta_\beta)}) = \theta_\alpha+\theta_\beta = \angle\alpha+\angle\beta \\
&\angle\frac{\alpha}{\beta} = \angle\frac{\rho_\alpha e^{j\theta_\alpha}}{\rho_\beta e^{j\theta_\beta}} = \angle\left(\frac{\rho_\alpha}{\rho_\beta}e^{j(\theta_\alpha-\theta_\beta)}\right) = \theta_\alpha-\theta_\beta = \angle\alpha-\angle\beta
\end{aligned}$$

- we get

$$\begin{aligned}
\angle G(j\omega) \quad = \quad & \angle K - \angle\left((j\omega)^h\right) \\
& + \sum_i \angle(1+j\omega\tau_i) - \sum_j \angle(1+j\omega T_j) \\
& + \sum_i \angle\left(1+\frac{2j\zeta'_i\omega}{\omega'_{ni}}-\frac{\omega^2}{\omega'^2_{ni}}\right) - \sum_j \angle\left(1+\frac{2j\zeta_j\omega}{\omega_{nj}}-\frac{\omega^2}{\omega^2_{nj}}\right)
\end{aligned}$$

Bode phase plot

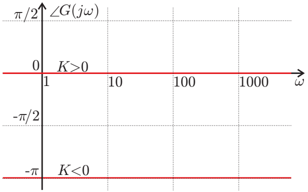
- We can again restrict our attention to four basic components only:

$$\begin{aligned}\angle G(j\omega) &= \underbrace{\angle K}_{\#1} - \underbrace{\angle\left((j\omega)^h\right)}_{\#2} \\ &+ \sum_i \angle(1 + j\omega\tau_i) - \sum_j \underbrace{\angle(1 + j\omega T_j)}_{\#3} \\ &+ \sum_i \angle\left(1 + \frac{2j\zeta_i'\omega}{\omega_{ni}'} - \frac{\omega^2}{\omega_{ni}'^2}\right) - \sum_j \underbrace{\angle\left(1 + \frac{2j\zeta_j\omega}{\omega_{nj}} - \frac{\omega^2}{\omega_{nj}^2}\right)}_{\#4}\end{aligned}$$

Bode phase plot

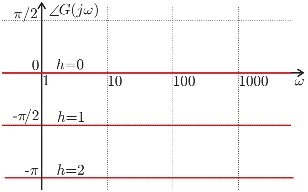
- Basic component #1

$$\angle K = \begin{cases} 0 & \text{for} \quad K > 0 \\ -\pi & \text{for} \quad K < 0 \end{cases}$$



- Basic component #2

$$\angle\left((j\omega)^h\right) = \frac{h\pi}{2}$$



Bode phase plot

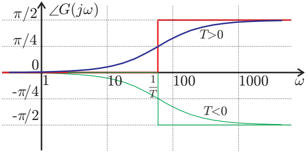
- Basic component #3

$$\angle(1+j\omega T)=\text{atan}(\omega T)$$

$\angle G(j\omega)\approx 0$ for $\omega \ll \frac{1}{T}$

$\angle G(j\omega)\approx \frac{\pi}{2}$ for $\omega \gg \frac{1}{T}$, $T>0$

$$\angle(1-j\omega T)=-\text{atan}(\omega T)$$



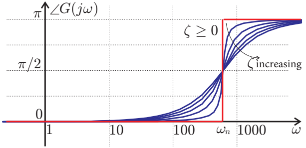
- Basic component #4

$$\angle\left(1+2j\zeta\frac{\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right)$$

$\angle G(j\omega)\approx 0$ for $\omega \ll \omega_n$

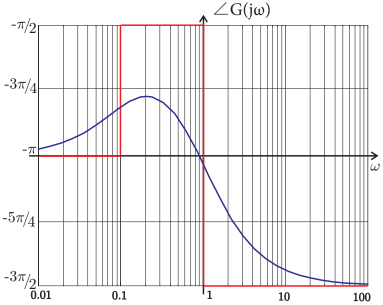
For $\zeta \geq 0$, $\angle G(j\omega)=\frac{\pi}{2}$ for $\omega=\omega_n$,

$\angle G(j\omega)\approx \angle\left(-\frac{\omega^2}{\omega_n^2}\right)=\pi$ for $\omega \gg \omega_n$



Example (cont'd)

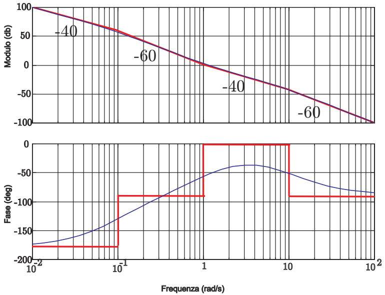
$$G(s) = \frac{1000(1+10s)}{s^2(1+s)^2}$$



- For $\omega \ll 0.1$: $\angle G(j\omega) \approx -\pi$
- For $0.1 < \omega < 1$: effect of zero in $s = -0.1$, add $\frac{\pi}{2}$
- Per $\omega > 1$: effect of double pole in $s = -1$, subtract $2\frac{\pi}{2} = \pi$

Example

$$G(s) = \frac{10(1+s)}{s^2(1-10s)(1+0.1s)}$$



- For $\omega \ll 0.1$: $|G(j\omega)| \approx \frac{10}{\omega^2}$ (slope=-40 dB/dec), $\angle G(j\omega) \approx -\pi$
- For $\omega = 0.1$: $20\log_{10}\frac{10}{\omega^2} = 60$ dB
- For $0.1 < \omega < 1$: effect of unstable pole $s = 0.1$, decrease module by 20 dB/dec, increase phase by $\frac{\pi}{2}$
- For $1 < \omega < 10$: effect of zero $s = -1$, $+20$ dB/dec module, $+\frac{\pi}{2}$ phase
- For $\omega > 10$: effect of pole $s = -10$, -20 dB/dec module, -$\frac{\pi}{2}$ phase

Summary table for drawing asymptotic Bode plots

		magnitude	phase
stable real pole	$T > 0$	-20 dB/dec	$-\pi/2$
unstable real pole	$T < 0$	-20 dB/dec	$\pi/2$
stable real zero	$\tau > 0$	$+20$ dB/dec	$\pi/2$
unstable real zero	$\tau < 0$	$+20$ dB/dec	$-\pi/2$
pair of stable complex poles	$\zeta > 0$	-40 dB/dec	$-\pi$
pair of unstable complex poles	$\zeta < 0$	-40 dB/dec	π
pair of stable complex zeros	$\zeta' > 0$	$+40$ dB/dec	π
pair of unstable complex zeros	$\zeta' < 0$	$+40$ dB/dec	$-\pi$

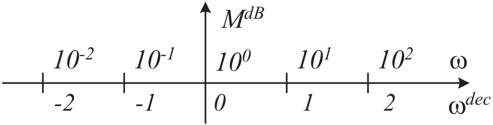
Bode plots (diagrams)

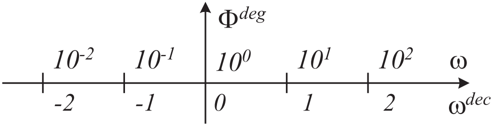
- The sinusoidal transfer function – a complex function of the frequency ω has a magnitude and a phase angle, with frequency as parameter.
- It may be represented by two separate plots: magnitude versus frequency and the phase angle versus frequency.
- Logarithmic plots or Bode plots in the honor of H.W.Bode who used them extensively in his studies of feedback amplifiers.

$$G(j\omega) = |G(\omega)|e^{j\Phi(\omega)}$$

Bode plots (diagrams)

- A Bode diagram:
  - the plot of the log magnitude $|G(j\omega)|^{dB} = M^{dB} = 20log_{10}|G(j\omega)|$.
  - the plot of $\angle G(j\omega)$ against $\omega^{dec} = log_{10}\omega$.
- The phase angle $\phi(\omega)$ in degrees or radians.





The general form of a sinusoidal transfer function is:

$$G(j\omega) = \frac{k \prod_{i=1}^{m_1}(T_i(j\omega)+1) \prod_{p=1}^{m_2}(\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega)+1)}{(j\omega)^n \prod_{l=1}^{n_1}(T_l(j\omega)+1) \prod_{k=1}^{n_2}(\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega)+1)}$$

Bode plots (diagrams)

$$M^{dB} = |G(j\omega)|^{dB} = 20logk + 20\sum_{i=1}^{m_1} log|T_i(j\omega)+1| +$$

$$+20\sum_{p=1}^{m_2} log|(\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega)+1| - 20log|j\omega|^n -$$

$$20\sum_{l=1}^{n_1} log|T_l(j\omega)+1| - 20\sum_{k=1}^{n_2} log|\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega)+1|$$

Bode plots (diagrams)

The phase angle of $G(j\omega)$ can be calculated as:

$$\Phi = \angle(G(j\omega)) = \angle\frac{k}{(j\omega)^n} + \sum_{i=1}^{m_1}\angle(T_i(j\omega)+1) +$$

$$+ \sum_{p=1}^{m_2}\angle((\frac{1}{\omega_p^2}(j\omega)^2 + \frac{2\zeta_p}{\omega_p}(j\omega)+1) - \sum_{l=1}^{n_1}\angle(T_l(j\omega)+1) -$$

$$- \sum_{k=1}^{n_2}\angle(\frac{1}{\omega_k^2}(j\omega)^2 + \frac{2\zeta_k}{\omega_k}(j\omega)+1)$$

Bode plots (diagrams)

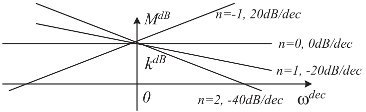
Therefore three different kinds of factors that may occur in a transfer function are as follows:

1. Gain k and the integral or derivative factors $k/(j\omega)^n$, (n can be positive or negative)
2. First-order factors, $(T(j\omega)+1)^{\pm 1}$
3. Quadratic factors $[(\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega)+1)]^{\pm 1}$

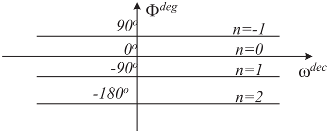
It is possible to construct composite plots for any general form of $G(j\omega)$ by sketching the plot for each factor and adding individual curves graphically.

Bode plots (diagrams)

$$G_1(j\omega) = \frac{k}{(j\omega)^n}$$



$$\begin{aligned} M_1^{dB} &= 20log\left|\frac{k}{(j\omega)^n}\right| \\ &= 20logk - 20nlog\omega \\ &= k^{dB} - 20n\omega^{dec} \end{aligned}$$



= the equation of a straight line which crossed the vertical axis at k^{dB} and has the slope $-20n$ (dB/dec).

The phase angle:

$$\Phi_1 = -n \cdot arctan\frac{\omega}{0} = -90^o \cdot n$$

Bode plots (diagrams)

$$G_2(\omega) = \frac{1}{T(j\omega)+1}$$

$$M_2^{dB} = 20log\left|\frac{1}{Tj\omega+1}\right| = -20log\sqrt{T^2\omega^2+1}$$

For low frequencies $\omega \ll 1/T \;\Rightarrow\; M_2^{dB}|_{\omega\ll} \cong -20log1 = 0\ dB$

For high frequencies $M_2^{dB}|_{\omega\gg} \cong -20log\omega T\ dB = -20\omega^{dec} - 20logT$

The equation of a straight line with the slope -20 (dB/dec)
The *corner frequency* is calculated from:

$$0 = -20\omega_c^{dec} - 20logT \;\Rightarrow\; \omega_c^{dec} = -logT = log\frac{1}{T}$$

Bode plots (diagrams)

The error in the magnitude curve occurs at the corner frequency:

$$M_2^{dB}|_{\omega=\omega_c} = 20log\sqrt{1+T\cdot\frac{1}{T}} = 20log\sqrt{2} \cong 3.03\ dB$$

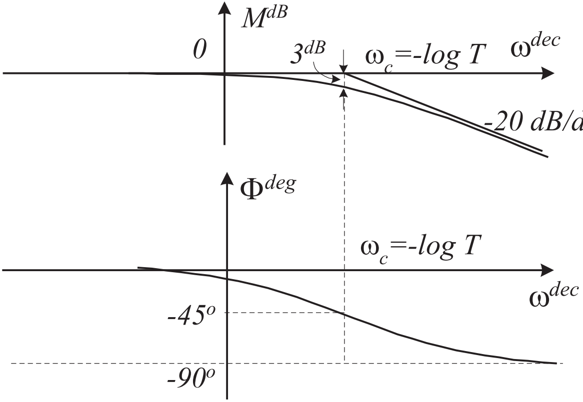
The phase angle Φ_2 of the factor $1/(Tj\omega+1)$ is:

$$\Phi_2 = -arctan\,\omega T, \qquad \Phi_2|_{\omega=0} = 0^o$$

$$\Phi_2|_{\omega=\omega_c} = -arctan\frac{T}{T} = -45^o, \qquad \Phi_2|_{\omega=\infty} = -arctan\,\infty = -90^o$$

Logarithmic plot for $1/(T\ j\ \omega+1)$

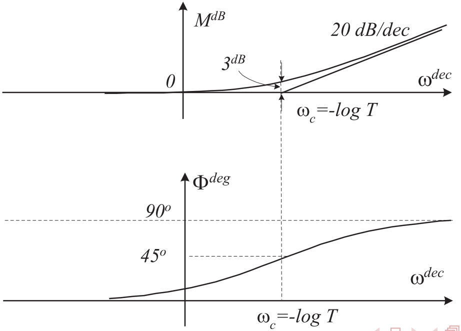
Bode plots (diagrams)



Bode plots (diagrams)

$$20log\left|T\omega+1\right| = 20log\left|\frac{1}{T\omega+1}\right|, \;\; \angle(Tj\omega+1) = -\angle\left(\frac{1}{T\omega+1}\right)$$

Logarithmic plot for $Tj\omega + 1$



Bode plots (diagrams)

$$G_3(j\omega) = \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1}$$

$$M_3^{dB} = -20log\sqrt{(1-\frac{\omega^2}{\omega_n^2})^2 + (2\zeta\frac{\omega}{\omega_n})^2}$$

For low frequencies, $\omega \ll \omega_n \;\Rightarrow\; M_3^{dB}|_{\omega\ll} = -20log1 = 0dB$

For high frequencies $\Rightarrow \;\; M_3^{dB}|_{\omega\gg} = -20log\frac{\omega^2}{\omega_n^2} = -40log\omega - 40log\omega_n\ dB$

The equation is a straight line with the slope $-40\ dB/dec$.
The corner (or break) frequency ω_c:

$$-40log\omega_c - 40log\omega_n = 0, \;\Rightarrow\;\; \omega_c = \omega_n$$

Near the corner frequency ω_c, a resonant peak occurs = the maximum error:

Bode plots (diagrams)

$$M_3^{dB}|_{\omega=\omega_c} = -20log\sqrt{(1-\frac{\omega_n^2}{\omega_n^2})^2 + (2\zeta\frac{\omega_n}{\omega_n})^2}$$

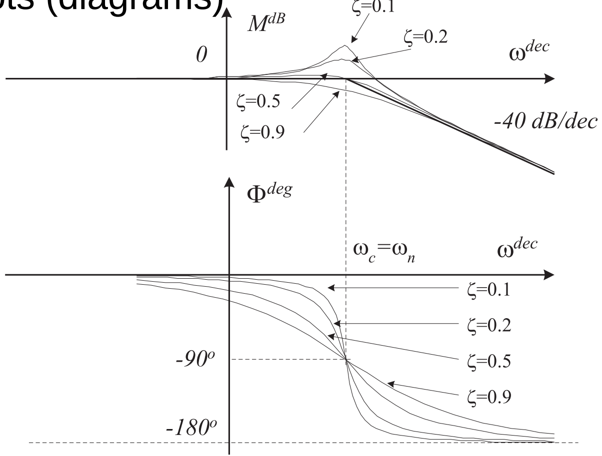
$$= -20log(2\zeta) = -20log2 - 20log\zeta \cong -6^{dB} - \zeta^{dB}$$

The phase angle of the quadratic factor

$$\Phi_3 = \angle\left(\frac{1}{(\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega)+1)}\right) = -arctan\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

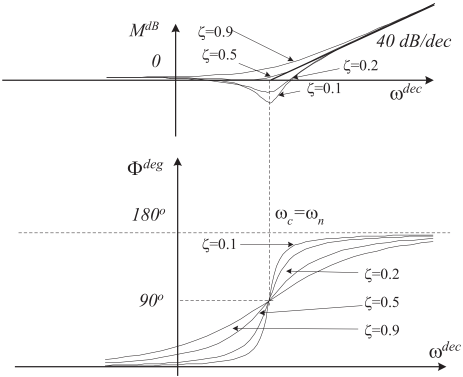
$$\Phi_3|_{\omega=0} = 0, \;\; \Phi_3|_{\omega=\omega_c} = -90^o, \;\; \Phi_3|_{\omega=\infty} = -180^o,$$

Bode plots (diagrams)



Bode plots (diagrams)

$$\frac{1}{\omega_n^2}(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + 1$$



Draw the Bode diagram for the following transfer function:

$$G(s) = \frac{10^3(s+10)}{s(s+1)(s^2+10s+100)} == \frac{10^2(\frac{1}{10}s+1)}{s(s+1)(\frac{1}{100}s^2+\frac{1}{10}s+1)}$$

Example

$$\begin{aligned}
G_1(j\omega) &= \frac{10^2}{j\omega} = \frac{k}{(j\omega)^1};\\
G_2(j\omega) &= \frac{1}{10}(j\omega)+1 = T_1(j\omega)+1;\\
G_3(j\omega) &= \frac{1}{j\omega+1} = \frac{1}{T_2 j\omega+1};\\
G_4(j\omega) &= \frac{1}{\frac{1}{100}(j\omega)^2+\frac{1}{10}(j\omega)+1} = \frac{1}{\frac{1}{\omega_n^2}(j\omega)^2+\frac{2\zeta}{\omega_n}(j\omega)+1}
\end{aligned}$$

Example

The Bode diagram is plotted for every one of these factors and than the curves are added graphically.

$$k = 10^2; \quad T_1 = \frac{1}{10}; \quad T_2 = 1; \;\; \omega_n = 10; \;\; \zeta = 0.5.$$

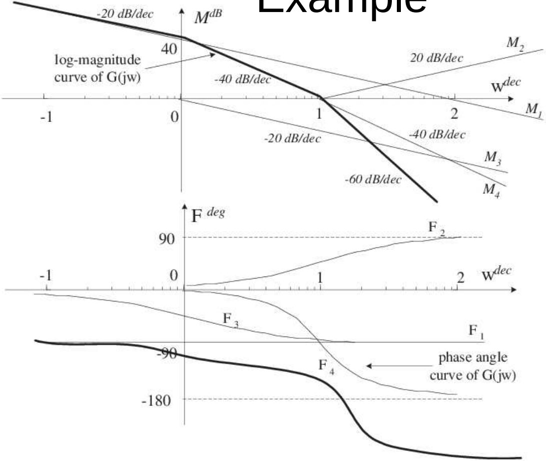
$$k^{dB} = 20 log 10^2 = 40dB, \;\; \Phi_1 = -90^o$$

$$\omega_{c1}^{dec} = -log T_1 = -log \frac{1}{10} = log 10 = 1 \; dec; \;\; \Phi_2 \in [0, 90^o]$$

$$\omega_{c2}^{dec} = -log T_2 = -log 1 = 0 \; dec, \;\; \Phi_3 \in [0, -90^o]$$

$$\omega_{c3} = \omega_n = 10; \quad \omega_{c3}^{dec} = log \; 10 = 1 dec; \quad \Phi_4 \in [0, -180^o]$$

Example



-20 dB/dec
M^dB
40
log-magnitude
curve of G(jw)
-40 dB/dec
M_2
20 dB/dec
w^dec
M_1
-1
0
1
2
-20 dB/dec
-40 dB/dec
M_3
-60 dB/dec
M_4
F deg
90
F_2
-1
0
1
2
w^dec
F_3
F_1
-90
F_4
phase angle
curve of G(jw)
-180

Reading Bode plots

$$|Y(j\omega)| = |G(j\omega)| \cdot |X(j\omega)|$$

- If $|G(j\omega)| > 1$ (or $M^{dB} = |G(j\omega)|^{dB} > 0$), the output will be amplified.
- If $|G(j\omega)| < 1$ (or $M^{dB} = |G(j\omega)|^{dB} < 0$), the output is attenuated.

- If $\Phi > 0$ the output is shifted with a positive angle comparing to the input and the system has phase lead.
- If $\Phi < 0$ the output is shifted with a negative angle with respect to the input (phase lag).

More on damped oscillatory modes

- Let $s = a \pm jb$ be complex poles, $b \neq 0$
- Since $(s-(a-jb))(s-(a+jb)) = (s-a)^2 + b^2 = s^2 - 2as + (a^2+b^2) = (a^2+b^2)(1 - \frac{2a}{a^2+b^2}s + \frac{1}{a^2+b^2}s^2)$, we get

$$\omega_n = \sqrt{a^2+b^2} = |a \pm jb|, \quad \zeta = -\frac{a}{\sqrt{a^2+b^2}} = -\cos\angle(a \pm jb)$$

- Note that $\zeta > 0$ if and only if $a < 0$
- Vice versa, $a = -\zeta\omega_n$, and $b = \omega_n\sqrt{1-\zeta^2}$
- The natural response is

$$Me^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)$$

  where M, ϕ depend on the initial conditions
- The frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called *damped natural frequency*

Zero/pole/gain form and Bode form

- Sometimes the transfer function $G(s)$ is given in *zero/pole/gain form* (ZPK)

$$G(s) = \frac{K'}{s^h} \frac{\Pi_i(s - z_i)}{\Pi_j(s - p_j)} \frac{\Pi_i \left(s^2 + 2\zeta'_i \omega'_{ni} s + {\omega'}_{ni}^2\right)}{\Pi_j \left(s^2 + 2\zeta_j \omega_{nj} s + \omega_{nj}^2\right)}$$

- The relations between Bode form and ZPK form are

$$z_i = -\frac{1}{\tau_i}, \;\; p_j = -\frac{1}{T_i}, \;\; K = K' \frac{\Pi_i(-z_i)\Pi_i {\omega'}_{ni}^2}{\Pi_j(-p_j)\Pi_i \omega_{ni}^2}$$

MATLAB
`»zpk(G)`

Nyquist (or polar) plot

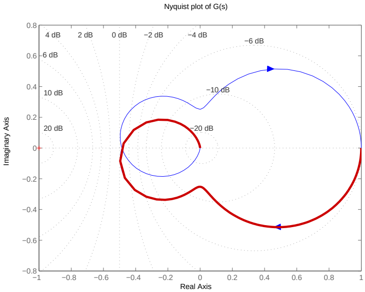
- Let $G(s)$ be a transfer function of a linear time-invariant dynamical system
- The *Nyquist plot* (or *polar plot*) is the graph in **polar** coordinates of $G(j\omega)$ for $\omega \in [0, +\infty)$ in the complex plane
- $G(j\omega) = \rho(\omega)e^{j\phi(\omega)}$, where $\rho(\omega) = |G(j\omega)|$ and $\phi(\omega) = \angle G(j\omega)$
- The Nyquist plot is one of the classical methods used in stability analysis of linear systems
- The Nyquist plot combines the Bode magnitude & phase plots in a single plot

Harry Nyquist

(1889–1976)

Nyquist plot

To draw a Nyquist plot of a transfer function $G(s)$, we can give some hints:



$$G(s) = \frac{1}{(1+0.1s)(1+0.002s+0.0001s^2)}$$

- For $\omega = 0$, the Nyquist plot equals the DC gain $G(0)$
- If $G(s)$ is strictly proper, $\lim_{\omega\to\infty} G(j\omega) = 0$
- In this case the angle of arrival equals $(n_z^- - n_z^+ - n_p^- + n_p^+)\frac{\pi}{2}$, where $n_{z[p]}^{+[-]}$ is the # of zeros [poles] with positive [negative] real part
- $G(-j\omega) = \overline{G(j\omega)}$

A system without any zero with positive real part is called *minimum phase*

Frequency response of continuous-time systems

Frequency response specifications

Consider the standard second order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The closed-loop frequency response is:

$$G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2j\zeta\frac{\omega}{\omega_n}}$$

with the magnitude and phase angle:

$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}, \quad \varphi = atan\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

Resonant frequency

If $M = |G(j\omega)|$ has a peak value at some frequency, this is called *resonant frequency*.
Max $M =$ max $|G(j\omega)|$ will occur when the denominator is minimum.
The resonant frequency ω_r is:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

for $0 \leq \zeta \leq 0.707$.

$$\zeta \to 0, \quad \omega_r \to \omega_n$$

For $\zeta > 0.707$ there is no resonant peak. The magnitude of $G(j\omega)$ decreases monotonically with ω.

Resonant peak

The magnitude of the resonant peak is obtained by substituting ω_r into M.
For $0 \leq \zeta \leq 0.707$:

$$M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{(}1-\zeta^2)}$$

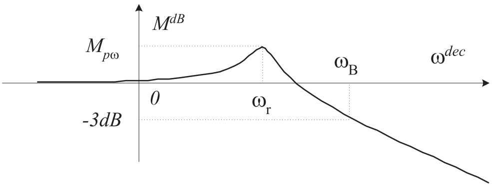
For $0.707 \leq \zeta \leq 1$: $M_r = 1$
If $\zeta \to 0$, $M_r \to \infty$.

The magnitude of the resonant peak - indication of the relative stability of the system. A large M_r indicates the presence of a pair of dominant poles with small damping ratio (undesirable transient response).

Cutoff frequency and bandwidth

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

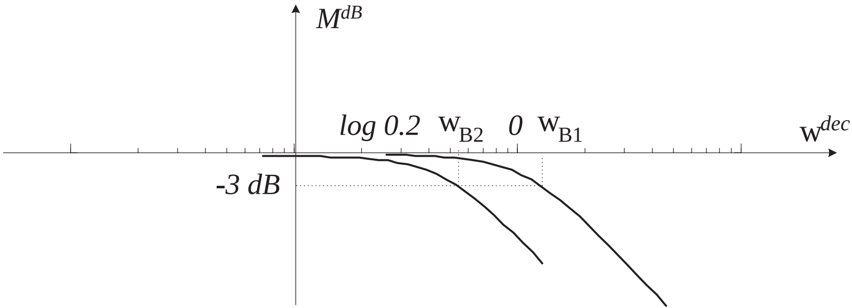
$M_{p\omega} = M_r$ (max) occurs at the *peak* (or *resonant*) frequency ω_r.



The *bandwidth* ω_B is a measure of the system's ability to faithfully reproduce an input signal = the frequency at which the freq. resp. has declined -3 dB from 0dB.

Bandwidth (example)

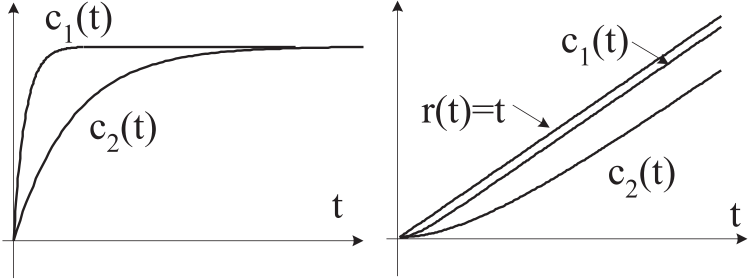
$$T_1(s) = \frac{1}{s+1}, \qquad T_2(s) = \frac{1}{5s+1}$$



System 1 has a larger bandwidth than System 2.

Bandwidth (example)

The system with the larger bandwidth provides the faster step response and the higher fidelity ramp response.



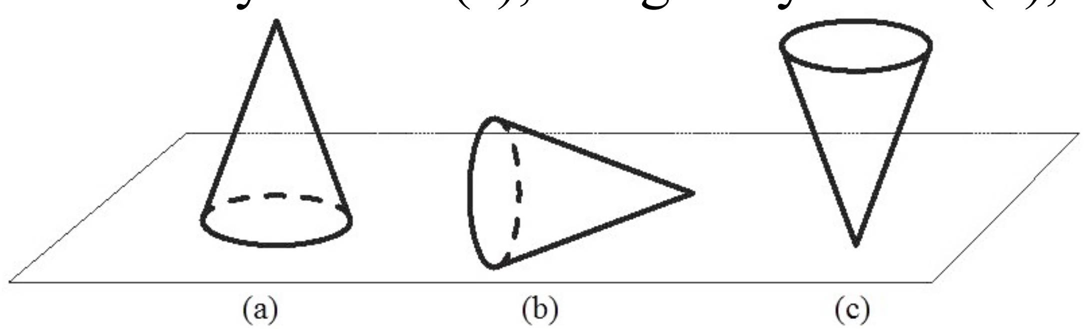
System stability

The quality of a system to be ***stable*** is influenced as an effect of modifying:
- one / several system inputs,
- the system parameters,
- the system structure.

All these causes have repercussions on the situation and regime in which the system was prior to the modification(s) and also effects on the evolution of system output (state).

Knowing that an unstable system will exhibit an erratic and destructive response, it is searched to ensure that a system is stable and exhibits a bounded transient response.

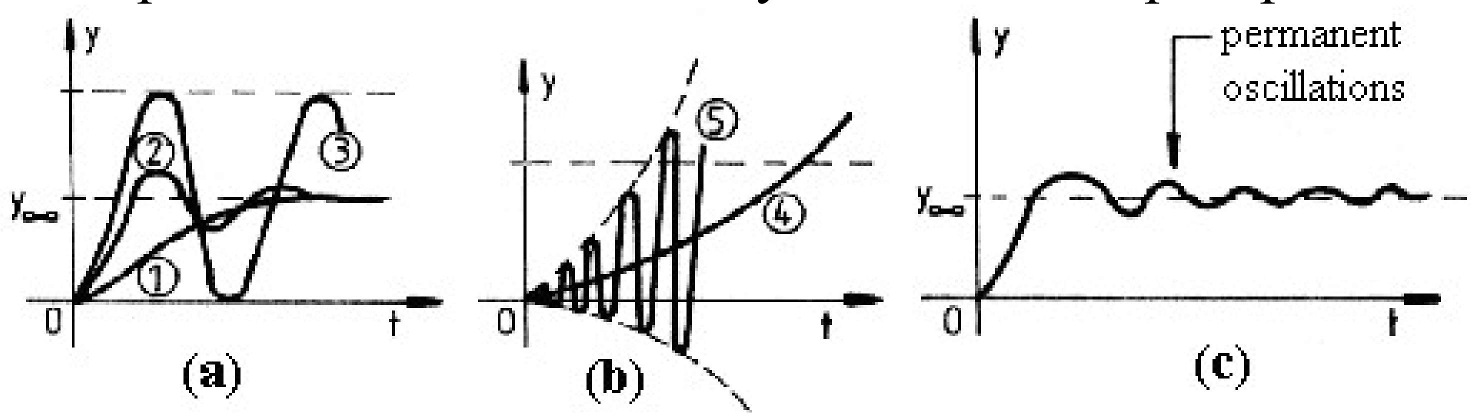
Illustration of cone stability: stable (a), marginally stable (b), unstable (c):



Whether a linear system is stable or unstable is a property of the system itself and does not depend on the input.

A) The external stability of a system or the input-output stability. In the context of the IO-MMs, **BIBO** (**B**ounded **I**nput – **B**ounded **O**utput) stability – under the action of an external cause that acts on the system: applying a bounded variation of system input $u(t)$ makes the system respond with a bounded variation of system output $y(t)$.

Graphical interpretation of BIBO stability in terms of step responses:



B) The internal stability of a system or ***the stability of system state.*** In the context of SS-MMs, the concept of internal stability is referred to as ***stability in the sense of Lyapunov*** or ***Lyapunov stability***: the equilibrium state (or point) $\mathbf{x}_0 = \mathbf{0} \in \Re^n$ of the DS (5.1.1) and accordingly the state $\mathbf{x}_0 = \mathbf{0}$ of the PS characterized by the DS (5.1.1) is a **stable state** if by getting the system out of this state:

$$|\Delta x_i(0_+)| < L_{x0},\ i = 1 \ldots n, \tag{5.1.4}$$

with $L_{x0} > 0$, the system will evolve back as follows after removing the cause:

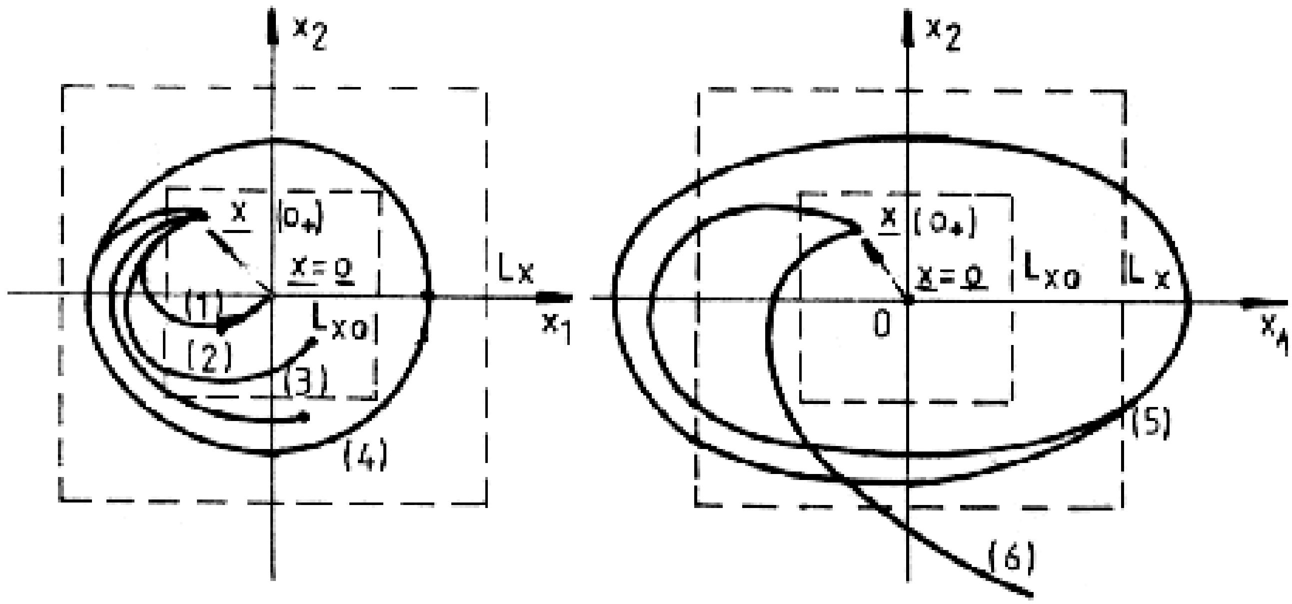
- in the initial stable state $\mathbf{x}_0$ or
- in an acceptable vicinity of this state,

and the resulted state trajectories, $\Delta\mathbf{x}(t)$, $t > 0$, fulfill the condition

$$|\Delta x_i(t)| < L_x,\ i = 1 \ldots n, \tag{5.1.5}$$

where $L_x > 0$, $L_x = f(L_{x0})$ such that $L_x > L_{x0}$.

Graphical interpretation of internal stability using phase trajectories:

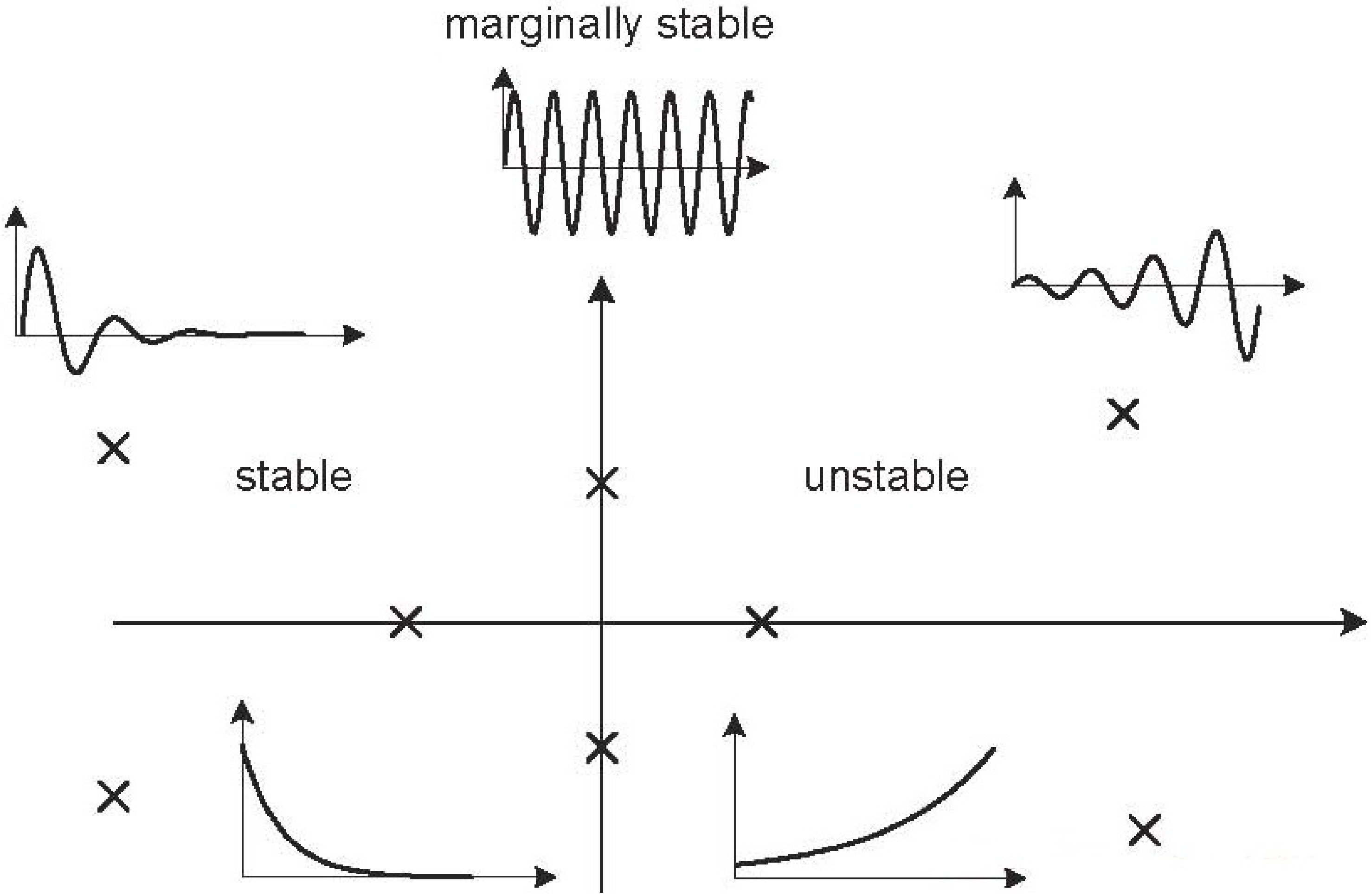


The fundamental stability theorem of continuous-time linear time invariant systems (C-LTIS): a C-LTIS is stable **if and only if** the n roots $s_\nu,\ \nu = 1 \ldots n$, of the characteristic equation (i.e., the system poles) have **strictly negative real parts**, i.e.

$$\mathrm{Re}(s_\nu) < 0,\ \ \nu = 1 \ldots n. \tag{5.2.6}$$

The roots are placed inside the left half-plane of the "s" plane.

Graphical interpretation of fundamental stability theorem in terms of poles positions and their relation to system impulse responses:



Stability criteria for continuous-time linear time invariant systems – for the efficient quantitative assessment of stability avoiding to solve the characteristic equation. Two types of stability criteria based on IO-MMs:

- algebraic criteria,
- frequency criteria, which are usually expressed in terms of graphic-analytic formulations.

The Hurwitz criterion for the assessment of stability of continuous-time linear systems

The Hurwitz criterion is an algebraic criterion that sets – depending on the values of the coefficients of the characteristic equation – the necessary and sufficient conditions for its roots to be placed in the left half-plane of the complex roots plane.

Theorem 3: The **necessary** (but not sufficient) condition for the roots of an algebraic equation

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad (5.3.1)$$

to be placed in the left half-plane of roots plane is that **all coefficients** a_ν of the algebraic equation should be **strictly positive**:

$$a_\nu > 0,\ \nu = 0 \ldots n. \qquad (5.3.2)$$

$\Rightarrow$ If the characteristic equation of a C-LTIS expressed in (5.3.1) **has at least one coefficient** a_ν **that is zero or negative** $\rightarrow$ the system will be **certainly unstable**; therefore, the further application of any stability criterion is useless.

The assessment of the stability of C-LTIS with the coefficients $a_\nu > 0,\ \nu = 0 \ldots n$, starts with testing if all coefficients fulfill the conditions (5.3.2).

- if not $\rightarrow$ the system will certainly be unstable;
- if yes $\rightarrow$ second step: building **the Hurwitz matrix** $\mathbf{H} \in \Re^{n \times n}$:

$$
\mathbf{H} = \begin{array}{c}
\begin{array}{ccccc} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_{n-1} & \end{array} \\
\left[\begin{array}{ccccc}
a_{n-1} & a_{n-3} & a_{n-5} & \ldots\ 0 & 0 \\
a_n & a_{n-2} & a_{n-4} & \ldots\ 0 & 0 \\
0 & a_{n-1} & a_{n-3} & \ldots\ 0 & 0 \\
0 & a_n & a_{n-2} & \ldots\ 0 & 0 \\
\ldots & & & & \\
0 & 0 & 0 & \ldots\ a_0 & 0 \\
0 & 0 & 0 & \ldots\ a_1 & 0 \\
0 & 0 & 0 & \ldots\ a_2 & a_0
\end{array} \right]
\end{array} . \tag{5.3.3}
$$

The following determinants will be next computed on the basis of the Hurwitz matrix **H**:

- the Hurwitz determinant, i.e. the determinant of the matrix **H**, and
- the leading principal minors of **H**:
-

$$\det(\mathbf{H}_n) = \det(\mathbf{H}) = a_0 \det(\mathbf{H}_{n-1}), \ldots, \det(\mathbf{H}_2) = \det\left(\begin{bmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{bmatrix}\right), \det(\mathbf{H}_1) = a_{n-1}. \qquad (5.3.4)$$

Formulation of the Hurwitz stability criterion: the condition for a C-LTIS with the characteristic equation (5.3.1) to be ***stable*** is that **the Hurwitz determinant and all its leading principal minors should be strictly positive**.

Steps proceeded to apply the Hurwitz criterion:

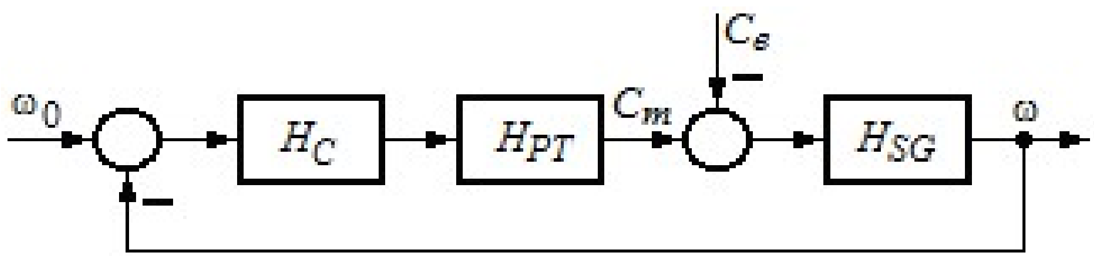
- the system whose stability is analyzed is separated and its IO-MM or SS-MM are derived;
- the characteristic equation is expressed in terms of (5.3.1); the system order, n, must be kept;
- the requirements Theorem 3 are tested, i.e. the condition (5.3.2) is tested for all coefficients; if **the condition (5.3.2) is fulfilled**, then:
- the Hurwitz matrix $\mathbf{H}$ is built and the Hurwitz determinant, $\det(\mathbf{H}_n)$, and the leading principal minors of the Hurwitz matrix, i.e. $\det(\mathbf{H}_1)$, $\det(\mathbf{H}_2)$ … and $\det(\mathbf{H}_{n-1})$ are computed;
- the following stability condition is tested for all these minors:

$$\det(\mathbf{H}_i) > 0, \; i = 1 \ldots n. \tag{5.3.5}$$

If (5.3.5) is fulfilled, then the system is stable. If not, the system is unstable.

Since the formulation of the criterion does not concern the system type, namely open-loop or closed-loop → the Hurwitz criterion is applied in the same manner irrespective of system type. The computations are based on the characteristic polynomial $\Delta(s)$ of the system whose stability is analyzed.

Example 5.2 (lecture material): Let us consider the simplified control system structure of the angular speed (frequency) ω control system of a hydrogenerator:



The t.f.s of the blocks:

- controller (C):

$$H_C(s) = k_C \frac{1+8s}{1+20s},$$

- pipeline/penstock-turbine (PT):

$$H_{PT}(s) = \frac{1-4s}{1+2s},$$

- synchronous generator (SG):

$$H_{SG}(s) = \frac{1}{\alpha + 7s},$$

with α ∈ (0.1, 1.2], which characterizes the strength of connection of SG to the power system, and $k_C > 0$. It is required:

Tasks:

1. Considering α = 1, the domain of values of $k_C > 0$ for which the system is stable should be determined, where $k_{C\max}$ is the upper limit of that domain.
2. Setting $k_{C0} = k_{C\max}/4$, it should be analyzed if for α = 0.2 the control system is or is not stable and the lower limit of k_C for which the system remains stable should be computed.

Solution: 1. For α = 1, the characteristic equation of the control system:

$$\Delta(s) = 1 + H_0(s),$$

where

$$H_0(s) = H_C(s)H_{PT}(s)H_{SG}(s) = \frac{k_C(1+8s)(1-4s)}{(1+20s)(1+2s)(1+7s)};$$

$\rightarrow$ the characteristic equation:

$$\Delta(s) = 280s^3 + (194 - 32k_C)s^2 + (29 + 4k_C)s + 1 + k_C = 0.$$

The necessary stability conditions (5.3.2) specified in Theorem 3 are imposed:

$$194 - 32k_C > 0,$$
$$29 + 4k_C > 0,$$
$$1 + k_C > 0.$$

Since $k_C > 0$, this system of inequations leads to

$$k_C \in (0, 6.0625).$$

The Hurwitz matrix (n=3) is next built $\rightarrow$ the Hurwitz determinant:

$$\det(\mathbf{H}) = \det(\mathbf{H}_3) = \begin{vmatrix} 194-32k_C & 1+k_C & 0 \\ 280 & 29+4k_C & 0 \\ 0 & 194-32k_C & 1+k_C \end{vmatrix}.$$

The stability conditions are imposed as follows:

$$\det(\mathbf{H}_1) = 194-32k_C > 0 \Rightarrow k_C < 194/32 \Leftrightarrow k_C < 6.0625; \tag{1}$$

$$\det(\mathbf{H}_2) = \begin{vmatrix} 194-32k_C & 1+k_C \\ 280 & 29+4k_C \end{vmatrix} > 0 \Leftrightarrow (194-32k_C)(29+4k_C) - 280(1+k_C) > 0$$

$$\Rightarrow k_C \in (-8.3668, 4.9919); \tag{2}$$

$$\det(\mathbf{H}_3) = \begin{vmatrix} 194-32k_C & 1+k_C & 0 \\ 280 & 29+4k_C & 0 \\ 0 & 194-32k_C & 1+k_C \end{vmatrix} > 0 \Leftrightarrow (1+k_C)\det(\mathbf{H}_2) > 0$$

$$\Leftrightarrow k_C \in (-1,0) \cap (-8.3668, 4.9919)$$
$$\Rightarrow k_C \in (-1, 4.9919). \qquad (3)$$

Concluding, since $k_C > 0$, the intersection of this condition and the conditions (1), (2) and (3) leads to the domain of values of k_C, namely D_{k_C}, for which the system is stable:

$$D_{k_C} = (0, 4.9919),$$

with

$$k_{C\max} = 4.9919.$$

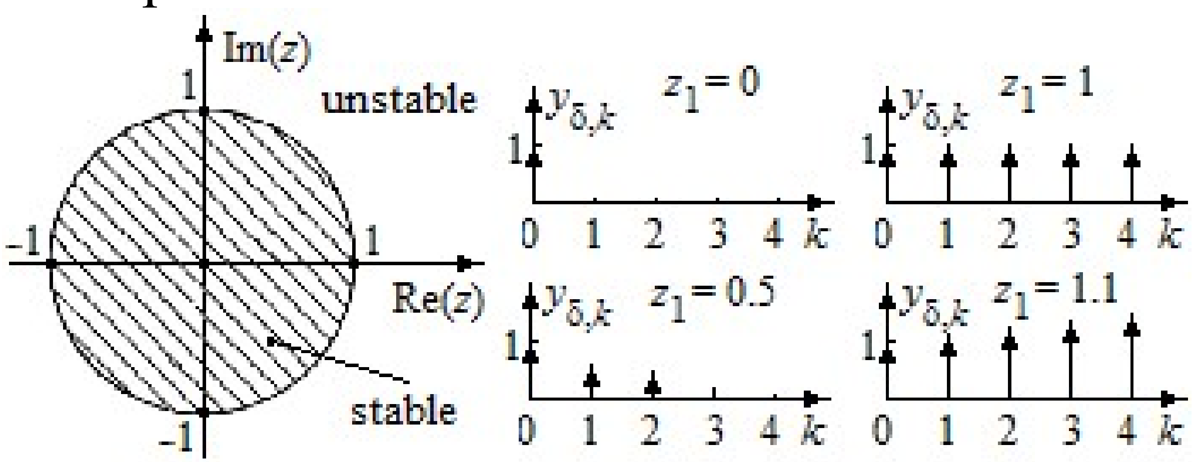
Task 2 – lecture material.

Fundamental stability theorem of discrete-time linear time invariant systems (D-LTIS): a D-LTIS is stable **if and only if** the n roots $z_\nu,\ \nu = 1...n$, of the characteristic equation (i.e., the system poles) have **modulus less than one**, i.e.

$$|z_\nu| < 1,\ \nu = 1...n. \tag{5.4.5}$$

The roots are placed inside the unit disk of the "z" plane.

Graphical interpretation of fundamental stability theorem of D-LTIS using unit impulse sequence responses:



Criteria for discrete-time linear time invariant systems – similar to C-LTIS:

- algebraic criteria: extension of Hurwitz criterion to discrete-time systems (lecture material) + criteria specific to D-LTIS (e.g., Jury),
- frequency (pulsation) criteria, usually expressed in terms of grapho-analytical formulations.

The Jury criterion for the assessment of stability of discrete-time linear systems

Accepting that the t.f. $H(z)$ of D-LTIS is known, the characteristic equation

$$\Delta(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 \text{ with } a_n > 0 \tag{5.5.5}$$

is involved in building **the array for Jury's stability** test (also called **the Jury array**):

Row	z^0	z^1	z^2	$\ldots z^{n-k} \ldots$	z^{n-2}	z^{n-1}	z^n
1 2	a_0 a_n	a_1 a_{n-1}	a_2 a_{n-2}	$\ldots a_{n-k} \ldots$ $\ldots a_k \ldots$	a_{n-2} a_2	a_{n-1} a_1	a_n a_0
3 4	b_0 b_{n-1}	b_1 b_{n-2}	b_2 b_{n-3}	$\ldots b_{n-k} \ldots$ $\ldots b_k \ldots$	b_{n-2} b_1	b_{n-1} b_0	– –
5 6	c_0 c_{n-2}	c_1 c_{n-3}	c_2 c_{n-4}	$\ldots c_{n-k} \ldots$ $\ldots c_k \ldots$	c_{n-2} c_0	– –	– –
…	…	…	…	…	–	–	–
$2n-5$ $2n-5$	p_0 p_3	p_1 p_2	p_2 p_1	p_3 p_0	– –	– –	– –
$2n-3$	q_0	q_1	q_2	–	–	–	–

The elements of the even-numbered rows are the elements of the preceding row in reverse order.

The elements of the odd-numbered rows are computed in terms of

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix},\ c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix},\ d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \ldots,$$

$$q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix},\ q_1 = \begin{vmatrix} p_0 & p_2 \\ p_3 & p_1 \end{vmatrix},\ q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}. \tag{5.5.6}$$

Formulation of the Jury stability criterion: the D-LTIS with the characteristic polynomial (5.5.4) is **stable** (i.e., all roots are placed inside the unit disk) if and only if the following n+1 **conditions** are fulfilled (with $a_n > 0$):

$\Delta(1) > 0,$ (1)

$\Delta(-1) > 0$ if n is even, (2)

< 0 if n is odd,

$|a_0| < a_n,$ (3)

$|b_0| > |b_{n-1}|,$ (4)

$|c_0| > |c_{n-2}|,$ (5)

$|d_0| > |d_{n-3}|,$ (6)

…

$|q_0| > |q_2|.$ (n+1)

Remarks: 1. Although building the array seems to be heavy, gaining experience after few applications makes it rather simple.

2. The coefficients b_k are not related to the coefficients in the nominator of the system t.f.

3. For a second-order system, the array contains only one row.

4. As in the case of the Hurwitz criterion, the Jury criterion has also the shortcoming of not giving information on the stability degree of the system. Since the number of inequalities is rather high, it is difficult to conduct a stability analysis that depends on one or more system parameters.

Steps to apply the Jury criterion:

- ♦ the system whose stability is analyzed is separated, its characteristic polynomial $\Delta(z)$ is expressed and the system order n is identified;
- ♦ $\Delta(1)$ and $\Delta(-1)$ are computed and the conditions (1), (2) and (3) are tested;
- ♦ if one of the conditions (1), (2) and (3) is not satisfied $\rightarrow$ the criterion is stopped and the system is unstable;
- ♦ otherwise, the Jury array is built and the rest of $(n-2)$ conditions are tested one by one; if one of the conditions is not satisfied, the criterion is stopped and the system is unstable; otherwise, the system is stable.

Example 5.4 (lecture material): Conduct the stability analysis of the discrete-time linear system with the t.f.

$$H(z) = \frac{11z^2 - 3z + 0.5}{z^3 + 3z^2 + 4z + 0.5}.$$

Solution: The characteristic polynomial of the system is

$$\Delta(z) = z^3 + 3z^2 + 4z + 0.5,$$

with $n = 3$ and $a_3 = 1 > 0$. The first three stability conditions are tested:

$$\Delta(1) = 8.5 > 0, \tag{1}$$

$$\Delta(-1) = -1.5 < 0 \ (n = 3 \text{ is odd}), \tag{2}$$

$$|a_0| = 0.5 < a_3 = 1. \tag{3}$$

Since all these conditions are satisfied, the Jury array is built:

Row	z^0	z^1	z^2	z^3
1	0.5 (a_0)	4 (a_1)	3 (a_2)	1 (a_3)
2	1 (a_3)	3 (a_2)	4 (a_1)	0.5 (a_0)
3	–0.75 (b_0)	–1 (b_1)	–2.5 (b_2)	–
4	–2.5 (b_2)	–1 (b_1)	–0.75 (b_0)	–

Its elements are:

$$b_0 = \begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix} = a_0^2 - a_3^2 = -0.75,$$

$$b_1 = \begin{vmatrix} a_0 & a_2 \\ a_3 & a_1 \end{vmatrix} = a_0 a_1 - a_2 a_3 = -1,$$

$$b_2 = \begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix} = a_0 a_2 - a_1 a_3 = -2.5.$$

The last, namely fourth stability condition (n+1 = 4) is next tested:
$|b_0| = 0.75 < |b_2| = 2.5.$ (4)

Since this condition is not satisfied $\rightarrow$ the system is unstable.

The same conclusion can be reached if the system poles are computed, for example, using Matlab: $z_1 = -0.139$, $z_{2,3} = -1.43 \pm j1.25$. These poles are placed outside the unit disk, which indicates that the system is unstable.

Nyquist criterion

- Consider a transfer function $G(s) = N(s)/D(s)$ under unit static feedback $u(t) = -(y(t) - r(t))$
- As $y(t) = G(s)(-y(t) + r(t))$, the closed-loop transfer function from $r(t)$ to $y(t)$ is

$$W(s) = \frac{G(s)}{1 + G(s)} = \frac{N(s)}{D(s) + N(s)}$$

Nyquist stability criterion

The number N_W of closed-loop unstable poles of $W(s)$ is equal to the number N_R of clock-wise rotations of the Nyquist plot around $s = -1 + j0$ plus the number N_G of unstable poles of $G(s)$. $[N_W = N_R + N_G]$

Corollary: simplified Nyquist criterion

For open-loop asympt. stable systems $G(s)$ ($N_G = 0$), the closed-loop system $W(s)$ is asymptotically stable if and only if the Nyquist plot $G(j\omega)$ does not encircle clock-wise the critical point $-1 + j0$. $[N_W = N_R]$

Nyquist stability criterion

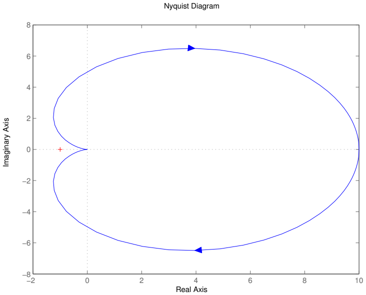
Proof:

- Follows by the *Argument principle*: "The polar diagram of $1+G(s)$ has a number of clock-wise rotations around the origin equal to the number of zeros of $1+G(s)$ (=roots of $N(s)+G(s)$) minus the number of poles of $1+G(s)$ (=roots of $D(s)$) of $1+G(s)$"
- The poles of $W(s)$ are the zeros of $1+G(s)$ □

- Note: the number of *counter-clockwise* encirclements counts as a negative number of clockwise encirclements
- Use of Nyquist criterion for open-loop stable systems:
  1. draw the Nyquist plot of $G(s)$
  2. count the number of clockwise rotations around $-1+j0$
  3. if $-1+j0$ is not encircled, the closed-loop system $W(s)=G(s)/(1+G(s))$ is stable
- The Nyquist criterion is limited to SISO systems

Example

Consider the transfer function $G(s) = \dfrac{10}{(s+1)^2} = \dfrac{10}{s^2+2s+1}$ under static output feedback $u(t) = -(y(t) - r(t))$



MATLAB

```
»G=tf(10,[1 2 1])
»nyquist(G)
```

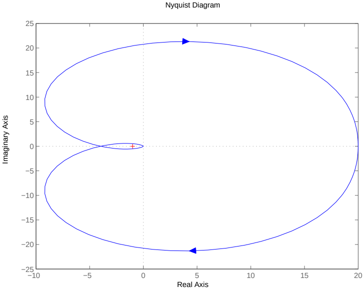
- The closed-loop transfer function is

$$W(s) = \frac{G(s)}{1+G(s)}$$

- The Nyquist plot of $G(s)$ does not encircle $-1+j0$
- By Nyquist criterion, $W(s)$ is asymptotically stable

Example

Consider the transfer function $G(s) = \dfrac{10}{s^3 + 3s^2 + 2s + 1}$ under unit output feedback $u(t) = -2(y(t) - r(t))$



MATLAB

```
»G=tf(10,[1 3 2 1]); K=2
»nyquist(K*G)
»L=feedback(G,K)
»pole(L)
  -3.8797
   0.4398 + 2.2846i
   0.4398 - 2.2846i
```

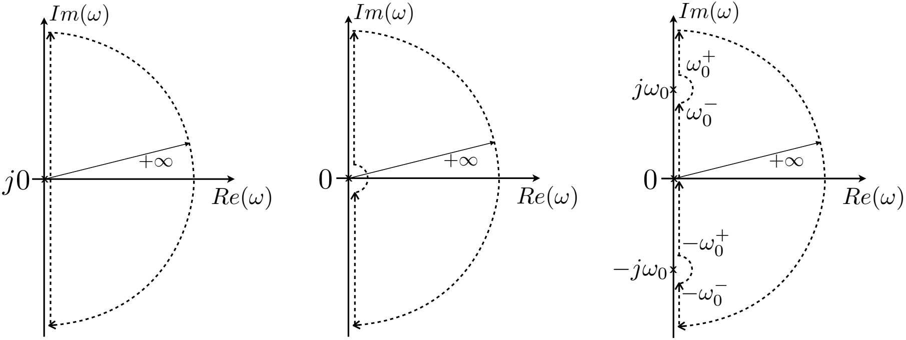
- The closed-loop transfer function is

$$W(s) = \frac{2G(s)}{1 + 2G(s)}$$

- The Nyquist plot of $2G(s)$ encircles $-1 + j0$ twice
- By Nyquist criterion, $W(s)$ is unstable

Dealing with imaginary poles

- The Nyquist plot is generated by a closed curve, the *Nyquist contour*, rotating clock-wise from $0 - j\infty$ to $0 + j\infty$ and back to $0 - j\infty$ along a semi-circle of radius ∞, avoiding singularities of $G(j\omega)$
- We distinguish three types of curves, depending on the number of poles on the imaginary axis:



no poles on imaginary axis

pole(s) in $s = 0$ (counted as stable)

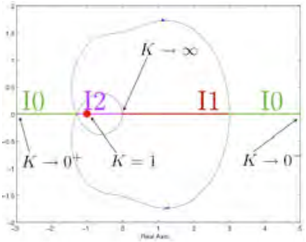
pole(s) in $s = \pm j\omega$ (counted as stable)

Stability analysis of static output feedback

- Under static output feedback $u(t) = -K(y(t) - r(t))$, the closed-loop transfer function from $r(t)$ to $y(t)$ is

$$W(s) = \frac{KG(s)}{1 + KG(s)}$$

- The number of encirclements of $-1 + j0$ of $KG(s)$ is equal to the number of encirclements of $-\frac{1}{K} + j0$ of $G(s)$
- To analyze closed-loop stability for different values of K is enough to draw Nyquist plot of $G(s)$ and move the point $-\frac{1}{K} + j0$ on the real axis



$$G(s) = \frac{60}{(s^2 + 2s + 20)(s + 1)}$$

I0=no unstable closed-loop poles

I2 =two unstable closed-loop poles

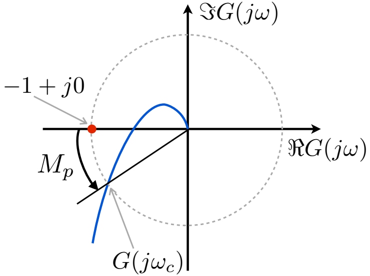
I1=an unstable closed-loop pole

Phase margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let ω_c be the *gain crossover frequency*, that is $|G(j\omega_c)| = 1$
- To avoid encircling the point $-1 + j0$, we want the phase $\angle G(j\omega_c)$ as far away as possible from $-\pi$
- The *phase margin* is the quantity

$$M_p = \angle G(j\omega_c) - (-\pi)$$

- If $M_p > 0$, unit negative feedback control is asymptotically stabilizing
- For robustness of stability we would like a large positive phase margin

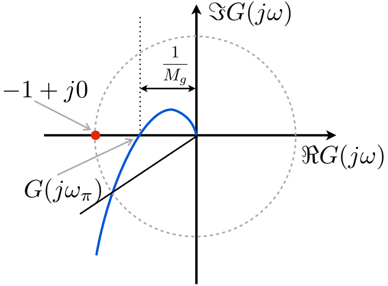


Gain margin

- Assume $G(s)$ is open-loop asymptotically stable
- Let ω_π be the *phase crossover frequency* such that $\angle G(j\omega_\pi) = -\pi$
- To avoid encircling the point $-1 + j0$, we want the point $G(j\omega_\pi)$ as far as possible away from $-1 + j0$.
- The *gain margin* is the inverse of $|G(j\omega_\pi)|$, expressed in dB

$$M_g = 20 \log_{10} \frac{1}{|G(j\omega_\pi)|} = -|G(j\omega_\pi)|_{dB}$$

- For robustness of stability we like to have a large gain margin

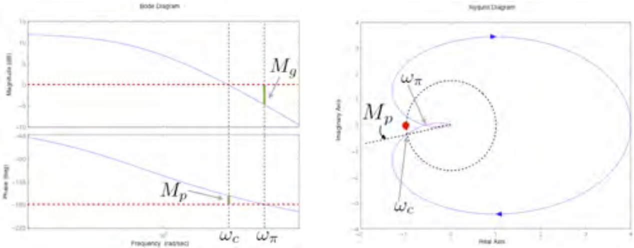


Sometimes the gain margin is defined as

$$M_g = \frac{1}{|G(j\omega_\pi)|}$$

and therefore $G(j\omega_\pi) = -\frac{1}{M_g} + j0$

Stability analysis using phase and gain margins

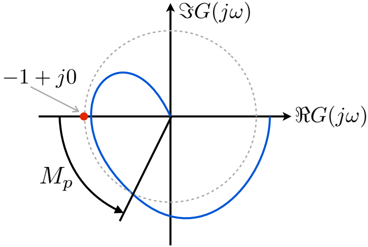


$$G(s) = \frac{8}{s^3 + 4s^2 + 4s + 2}$$

- The phase and gain margins help assessing the degree of robustness of a closed-loop system against uncertainties in the magnitude and phase of the process model
- They can be also applied to analyze the stability of dynamic output feedback laws $u(t) = C(s)(y(t) - r(t))$, by looking at the Bode plots of the loop function $C(j\omega)G(j\omega)$ (see next lecture on "loop shaping")

Stability analysis using phase and gain margins

- However in some cases phase and gain margins are not good indicators, see the following example



- The system is characterized by a large phase margin, but the polar plot is very close to the point $(-1, 0)$
- The system is not very robust to model uncertainties changing $G(s)$ from its nominal value
- In conclusion, phase and gain margins give good indications on how the loop function should be modified, but the complete Bode and Nyquist plots must be checked to conclude about closed-loop stability

A frequency domain stability criterion relates the stability of a closed-loop system to the *open-loop frequency response* and open-loop pole location.

Our problem is the determination of stability from frequency data plots, for the *closed-loop system*.

Consider the closed-loop control system shown in Figure 2.

Stability analysis in the frequency domain

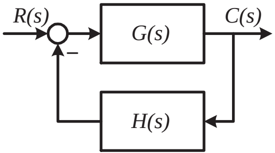


Figure : Closed-loop system

Stability analysis in the frequency domain

- The closed-loop transfer function:

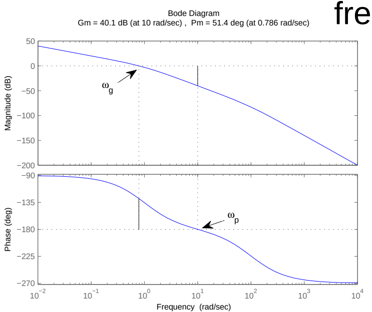
$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

- The open loop transfer function: $G(s)H(s) = GH(s)$.
- A point on the imaginary axis $s = j\omega$ will be a solution of the characteristic equation (i.e. the system is critically stable) if $|GH(j\omega)| = 1$ and $\angle GH(j\omega) = \pm 180^o$.
- We have access to $|GH(j\omega)|$ and $\angle GH(j\omega)$ from a Bode plot : determine the imaginary axis crossings by finding the frequencies ω (if any) on the plot that satisfy the conditions:

$$|GH(j\omega)| = 1 \quad \text{and} \quad \angle GH(j\omega) = \pm 180^o$$

Stability analysis in the frequency domain

- **Gain crossover frequency**: This is the frequency ω_g such that $|GH(j\omega_g)| = 1$ (or equivalently, $20\log_{10}|GH(j\omega_g)| = M^{dB}(\omega_g) = 0$).
- **Phase crossover frequency**: This is the frequency ω_p such that $\angle GH(j\omega_p) = \pm 180^o$.

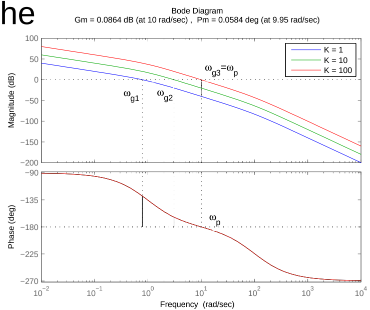


Example. Consider a closed-loop system having the open-loop transfer function:

$$GH(s) = \frac{K}{s(s+1)(\frac{1}{100}s+1)}$$

For $K = 1,\ 10,\ 100$, the open-loop Bode plot is:

Stability analysis in the frequency domain



Relative stability

- **Gain margin**: is the amount by which K can be multiplied before $|KGH(j\omega_p)| = 1$, or $20log|KGH(j\omega_p)| = M^{dB}(\omega_p) = 0dB$ (i.e. the gain crossover frequency and phase crossover frequencies coincide). In other words, the *gain margin* is the reciprocal of the gain $|GH(j\omega)|$ at the frequency at which the phase angle reaches -180^o. The gain margin indicates how much the gain can be increased before the system becomes unstable.
- **Phase margin**: is the amount by which the phase at ω_g exceeds -180^o.

Relative stability

- **Gain margin** K_g:

$$K_g = \frac{1}{|GH(j\omega_p)|}, \quad \text{for} \quad \angle GH(j\omega_p) = -180^o$$

or, in log scale:

$$K_g^{dB} = -M^{dB}(\omega_p)$$

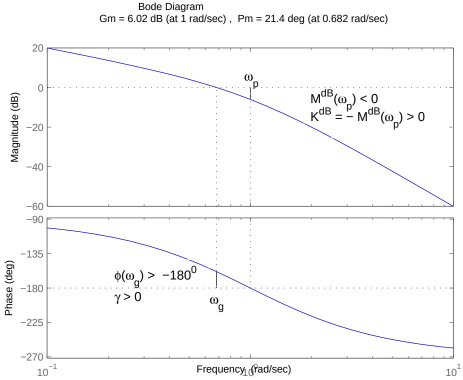
- **Phase margin,** γ:

$$\gamma = 180^o + \angle GH(j\omega_g), \quad \text{for} \quad M(\omega_g) = 1, \quad \text{or} \quad M^{dB}(\omega_g) = 0$$

- For a stable closed-loop system: $K_g^{dB} > 0$, $\gamma > 0$

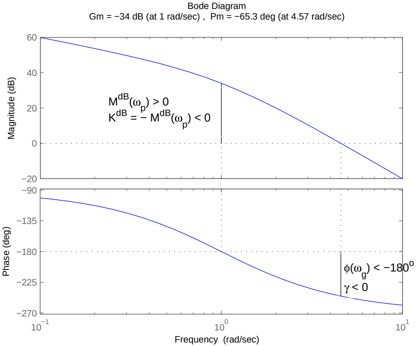
Consider a **closed-loop** control system having the **open-loop** transfer function: $GH(s) = \frac{1}{s(s+1)^2}$. The Bode plot for the open loop system:

Stable closed-loop system



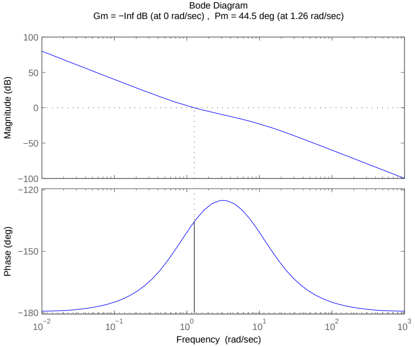
Consider a **closed-loop** control system having the **open-loop** transfer function $GH(s) = \frac{100}{s(s+1)^2}$ The Bode plot for the open loop system:

Unstable closed-loop system



The open-loop transfer function: $GH(s) = \frac{10k(s+1)}{s^2(s+10)}$. Bode plot for $k = 1$:

Stable closed-loop system



Infinite gain margin!

Thank you very much for your attention!