

$$(4) \quad a) \quad x(k) = \left(\frac{1}{2}\right)^k$$

$$\begin{aligned}
 X(z) &= \mathcal{Z} \left\{ \left(\frac{1}{2}\right)^k \right\} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k} = \\
 &= \sum_{k=0}^{\infty} (2z)^{-k} = 1 + \underbrace{(2z)^{-1} + (2z)^{-2} + \dots + (2z)^{-n}}_{n \rightarrow \infty} \\
 &= \frac{(2z)^0 - 1}{(2z)^{-1} - 1} = \frac{1}{1 - (2z)^{-1}}
 \end{aligned}$$

$$\rightarrow \mathcal{Z} \left\{ x(k) \right\} = \mathcal{Z} \left\{ \left(\frac{1}{2}\right)^k \right\} = \frac{1}{1 - (2z)^{-1}}$$


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$$b) \quad x(k) = k$$

$$\begin{aligned}
 X(z) &= \mathcal{Z} \{ k \} = \sum_{k=0}^{\infty} k z^{-k} = \\
 &= \underbrace{1z^{-1} + 2z^{-2} + 3z^{-3} + \dots + n z^{-n}}_{n \rightarrow \infty}
 \end{aligned}$$

$$X(z) = \underbrace{z^{-1} + 2z^{-2} + 3z^{-3} + \dots + nz^{-n}}_{n \rightarrow \infty}$$

$$X(z) = \underbrace{z^{-1} + z^{-2} + z^{-3} + \dots + z^{-n}}_{n \rightarrow \infty} +$$

$$+ \underbrace{1 \cdot z^{-2} + 2 \cdot z^{-3} + \dots + (n-1)z^{-n}}_{n \rightarrow \infty}$$

$$X_1(z) = z^{-1} \cdot \frac{(z^{-n}) - 1}{z^{-1} - 1} = z^{-1} \cdot \frac{1}{1 - z^{-1}}$$

$$X_2(z) = \underbrace{z^{-2} + 2z^{-3} + \dots + (n-1)z^{-n}}_{n \rightarrow \infty}$$

$$\int X_2(z) dz = - \underbrace{\left( z^{-1} \right) - \left( z^{-2} \right) - \left( z^{-3} \right) \dots - \left( z^{-n+1} \right)}_{n \rightarrow \infty}$$

$$\int z^{-n} dz = \frac{z^{-n+1}}{-n+1} = \left( \frac{1}{n+1} \right) \left( -z^{-n+1} \right)$$

$$\int X_2(z) dz = - z^{-1} \cdot \frac{(z^{-n+1}) - 1}{z^{-1} - 1} = \frac{z^{-1}}{z^{-1} - 1}$$

$$\int x_2(z) dz = \frac{z^{-1}}{z^{-1} - 1} = \frac{\frac{1}{z}}{\frac{1}{z} - 1} = \frac{\frac{1}{z}}{\frac{1-z}{z}} = \frac{1}{1-z}$$

$$x_2(z) = \left( \frac{1}{1-z} \right)' = + (1-z)^{-2}$$

$$X(z) = x_1(z) + x_2(z) = \frac{z^{-1}}{1-z^{-1}} + \frac{1}{(1-z)^2}$$

$$\frac{z^{-1}}{1-z^{-1}} = \frac{\frac{1}{z}}{\frac{z-1}{z}} = -\frac{1}{1-z}$$

$$X(z) = \frac{z^{-1} + 1}{(1-z)^2} = \frac{z}{(1-z)^2}$$

$$X(z) = Z\{k\} = \frac{z}{(1-z)^2}$$

$$X(z) = \frac{\cancel{z} z^{-1}}{\cancel{z}^2 (1-z^{-1})^2} = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$c) \quad x(k) = e^{ak}$$

$$X(z) = \mathcal{Z}\{e^{ak}\} = \sum_{k=0}^{\infty} e^{ak} z^{-k}$$

$$X(z) = \sum_{k=0}^{\infty} (e^a \cdot z^{-1})^k =$$

$$= 1 + \underbrace{(e^a \cdot z^{-1})^1 + \dots + (e^a \cdot z^{-1})^n}_{n \rightarrow \infty}$$

$$= 1 \cdot \frac{(e^a \cdot z^{-1})^{n+1} - 1}{e^a \cdot z^{-1} - 1} = \frac{1}{1 - e^a z^{-1}}$$

$$X(z) = \mathcal{Z}\{e^{ak}\} = \frac{1}{1 - e^a z^{-1}}$$

$$d) \quad x(k) = \begin{cases} 2, & k \geq 3 \\ 0, & k < 3 \end{cases}$$

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k} =$$

$$= \underbrace{x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \dots + x(n)z^{-n}}_{n \rightarrow \infty}$$

$$X(z) = \underbrace{0 \cdot z^0 + \dots + 0 \cdot z^{-2}} +$$

$$+ \underbrace{2 \cdot z^{-3} - \dots - 2 \cdot z^{-n}}_{n \rightarrow \infty}$$

$$X(z) = 2 \left( z^{-3} \cdot \frac{(z^{-n+2})^0 - 1}{z^{-1} - 1} \right) =$$

$$X(z) = 2 \frac{z^{-3}}{1 - z^{-1}}$$

$$X(z) = z \{ x(k) \} = \frac{2 z^{-3}}{1 - z^{-1}}$$


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⑤  $y(k+1) = (1+\alpha) y(k)$

$$\alpha = 0.1$$

$$y(0) = 100$$

$$z \{ y(k+1) \} = (1+\alpha) \overbrace{z \{ y(k) \}}^{Y(z)}$$

$$z[Y(z) - y(0)] = (1+\alpha) Y(z)$$

$$z [Y(z) - y_0] = (1+r) Y(z)$$

$$z Y(z) - y_0 z = (1+r) Y(z)$$

$$Y(z) (z - (r+1)) = y_0 z$$

$$Y(z) = \frac{y_0 z}{z - (r+1)} = \frac{y_0 z}{z(1 - (1+r)z^{-1})}$$

$$Y(z) = \frac{y_0}{1 - (1+r)z^{-1}}$$

$$y(k) = z^{-1} \left\{ \frac{y_0}{1 - (1+r)z^{-1}} \right\} =$$

$$= y_0 (1+r)^k \quad (100 \cdot (1,1)^k)$$

$$y(k) = y_0 (1+r)^k$$

$$(6) \quad y(k+1) - 3y(k) = 4^k \quad y(0)=2$$

$$\underline{z\{y(k+1)\} - 3z\{y(k)\} = \underline{z\{4^k\}}}$$

$$z[y(z) - y_0] - 3Y(z) = \frac{1}{1 - 4z^{-1}}$$

$$(z - 3)Y(z) = zy_0 + \frac{1}{1 - 4z^{-1}}$$

$$Y(z) = \frac{1 - 4z^{-1}}{z - 3} y_0 + \frac{1}{1 - 4z^{-1}}$$

$$= \frac{zy_0 - 4y_0 + 1}{(z - 3)(1 - 4z^{-1})} = \frac{y_0 + (1 - 4y_0)z^{-1}}{(1 - 3z^{-1})(1 - 4z^{-1})}$$

$$\frac{A}{1 - 3z^{-1}} + \frac{B}{1 - 4z^{-1}} = \frac{y_0 + (1 - 4y_0)z^{-1}}{(1 - 3z^{-1})(1 - 4z^{-1})}$$

$$A(1-4z^{-1}) + B(1-3z^{-1}) = \gamma_0 + (1-4\gamma_0)z^{-1}$$

$$\begin{cases} A + B = \gamma_0 & | \cdot 4 \quad + \\ -4A - 3B = 1 - 4\gamma_0 & + \end{cases}$$

$$4A + 4B = 4\gamma_0$$

$$B = 1$$

$\rightarrow$

$$A = \gamma_0 - 1$$

$$Y(z) = \frac{\gamma_0 - 1}{1 - 3z^{-1}} + \frac{1}{1 - 4z^{-1}}$$

$$y(k) = \mathcal{Z}^{-1} \{ Y(z) \} =$$

$$= (\gamma_0 - 1) \cdot 3^k + 4^k$$

$$y(k) = (\gamma_0 - 1) \cdot 3^k + 4^k$$



$$(7) \begin{cases} x(k+1) - y(k) = 0 \\ y(k+1) + x(k) = 0 \end{cases}, \quad \begin{matrix} x(0) = 1 \\ y(0) = 0 \end{matrix}$$

$$\begin{cases} z \{ x(k+1) \} - z \{ y(k) \} = 0 \\ z \{ y(k+1) \} + z \{ x(k) \} = 0 \end{cases}$$

$$\begin{cases} z [X(z) - X_0] - Y(z) = 0 \\ z [Y(z) - Y_0] + X(z) = 0 \end{cases}$$

$$(z+1)X(z) - X_0 + (z-1)Y(z) - Y_0 = 0$$

$$Y(z) = \frac{X_0 + Y_0 - (z+1)X(z)}{(z-1)}$$

$$Y(z) = \frac{1 - (z+1)X(z)}{(z-1)}$$

$$\frac{z}{z-1} [1 - (z+1)X(z)] + X(z) = 0$$

$$\frac{z}{z-1} - \frac{z(z+1)}{z-1} X(z) + X(z) = 0$$

$$z - z(z+1) X(z) + (z-1) X(z) = 0$$

$$z - z^2 X(z) - \cancel{z X(z)} + \cancel{z X(z)} - X(z) = 0$$

$$z - (z^2 + 1) X(z) = 0$$

$$X(z) = \frac{z}{z^2 + 1} = \frac{z^{-1}}{1 + z^{-2}}$$

$$\begin{aligned} \sin \omega T &= 1 \\ \cos \omega T &= 0 \end{aligned} \rightarrow \omega T = \frac{\pi}{2}$$

$$X(k) = Z^{-1} \left\{ \frac{z^{-1}}{1 + z^{-2}} \right\} =$$

$$= \sin \omega k T = \sin \frac{\pi}{2} k$$

$$x(k) = \sin \frac{\pi}{2} k$$

$$\begin{cases} z[X(z) - x_0] - Y(z) = 0 \\ z[Y(z) - y_0] + X(z) = 0 \end{cases}$$

$$z X(z) - Y(z) = z$$

$$X(z) + z Y(z) = 0 \quad / + z$$

$$-z X(z) - z^2 Y(z)$$

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$$-(z^2 + 1) Y(z) = z$$

$$Y(z) = -\frac{z}{z^2 + 1} = -\frac{z^{-1}}{1 + z^{-2}}$$

$$\sin \omega t = 1 \quad \rightarrow \omega t = \frac{\pi}{2}$$

$$\cos \omega t = 0$$

$$y(k) = z^{-1} \{ Y(z) \} = 0$$

$$y(k) = -\sin \frac{\pi}{2} k$$