

(1) Calc. transformatele Laplace

a)  $2t^2$

$$f(t) = 2t^2 \Rightarrow F(s) = \mathcal{L}\{2t^2\}$$

$$= \int_0^{\infty} 2t^2 \cdot e^{-st} dt$$

$$f(t) = 2t^2 \rightarrow f'(t) = 4t$$

$$g'(t) = e^{-st} \rightarrow g(t) = -\frac{1}{s} e^{-st}$$

$$= \left( 2t^2 \cdot \left(-\frac{1}{s}\right) e^{-st} \right) \Big|_0^{\infty} - \int_0^{\infty} 4t \left(-\frac{1}{s}\right) e^{-st} dt$$

$$- \frac{1}{s} \left( \underbrace{\lim_{t \rightarrow \infty} \frac{2t^2}{e^{st}}}_0 - \underbrace{\lim_{t \rightarrow 0} \frac{2t^2}{e^{st}}}_0 \right) = 0$$

$$+\frac{4}{s} \int_0^{\infty} t \cdot e^{-st} dt$$

$$f(t) = t$$

$$f'(t) = 1$$

$$g'(t) = e^{-st}$$

$$g(t) = -\frac{1}{s} e^{-st}$$

$$= +\frac{4}{s} \left[ \underbrace{t \cdot \left(-\frac{1}{s}\right) e^{-st}}_0 \bigg|_0^{\infty} - \int_0^{\infty} \underbrace{-\frac{1}{s} e^{-st}}_0 dt \right]$$

$$\Rightarrow t \cdot e^{-st} \bigg|_0^{\infty} = \lim_{t \rightarrow \infty} \frac{t}{e^{st}} - \lim_{t \rightarrow 0} \frac{t}{e^{st}} = 0$$

$$\int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \bigg|_0^{\infty} =$$

$$= -\frac{1}{s} \left( \underbrace{\lim_{t \rightarrow \infty} e^{-st}}_0 - \underbrace{\lim_{t \rightarrow 0} e^{-st}}_1 \right) = \frac{1}{s}$$

$$= +\frac{4}{s} \cdot \frac{1}{s^2} = \frac{4}{s^3}$$

$$\mathcal{L}\{t^2\} = \frac{4}{s^3}$$

$$b) 3 \sin(2t) \quad \mathcal{L}\{3 \sin 2t\} = F(s)$$

$$F(s) = \int_0^{\infty} 3 \sin 2t e^{-st} dt$$

$$f(t) = \sin 2t$$

$$f'(t) = 2 \cos 2t$$

$$g(t) = e^{-st}$$

$$g'(t) = -\frac{1}{s} e^{-st}$$

$$= 3 \sin 2t \cdot \left(-\frac{1}{s}\right) e^{-st} \Big|_0^{\infty} =$$

$$- 3 \int_0^{\infty} 2 \cos 2t \cdot \left(-\frac{1}{s}\right) e^{-st} dt$$

$$-\frac{3}{s} \left( \lim_{t \rightarrow \infty} \frac{\sin 2t}{e^{st}} - \lim_{t \rightarrow 0} \frac{\sin 2t}{e^{st}} \right) = 0$$

$$\mathcal{L}\{3 \sin(2t)\} = \frac{6}{s} \int_0^{\infty} \cos 2t e^{-st} dt$$

$$\int_0^{\infty} \cos 2t \cdot e^{-st} dt$$

$$f(t) = \cos 2t$$

$$g'(t) = e^{-st}$$

$$f'(t) = -2 \sin 2t$$

$$g(t) = -\frac{1}{s} e^{-st}$$

$$= -\frac{1}{s} \cos 2t e^{-st} \Big|_0^{\infty} - \frac{2}{s} \int_0^{\infty} \sin 2t e^{-st} dt$$

$$\Rightarrow -\frac{1}{s} \left( \underbrace{\lim_{t \rightarrow \infty} \frac{\cos 2t}{e^{st}}}_0 - \underbrace{\lim_{t \rightarrow 0} \frac{\cos 2t}{e^{st}}}_1 \right) \frac{F(s)}{3}$$

$$\rightarrow \frac{1}{s} - \frac{2 F(s)}{3s} = \frac{3 - 2 F(s)}{3s}$$

$$F(s) = \frac{6}{s} \frac{3 - 2 F(s)}{3s} \Rightarrow$$

$$3s^2 F(s) = 18 - 12 F(s) \quad / : 3$$

$$s^2 F(s) = 6 - 4 F(s)$$

$$F(s) (s^2 + 4) = 6 \Rightarrow \mathcal{L}^{-1} \left\{ \frac{6}{s^2 + 4} \right\} = \frac{6}{s^2 + 4}$$

$$c) \quad 3e^{-t} \quad \mathcal{L}\{3e^{-t}\} = F(s)$$

$$F(s) = \int_0^{\infty} 3e^{-t} e^{-st} dt = \int_0^{\infty} 3e^{-(s+1)t} dt$$

$$= \int_0^{\infty} 3 \cdot -\frac{1}{s+1} \left( e^{-(s+1)t} \right)' dt =$$

$$= \frac{3}{s+1} \left[ e^{-(s+1)t} \right]_0^{\infty} =$$

$$= -\frac{3}{s+1} \left( \lim_{t \rightarrow \infty} \frac{1}{e^{(s+1)t}} - \lim_{t \rightarrow 0} \frac{1}{e^{(s+1)t}} \right) =$$

$$\Rightarrow \mathcal{L}\{3e^{-t}\} = \frac{3}{s+1}$$

$$d) \quad t e^{-t} \quad \mathcal{L}\{t e^{-t}\} = F(s)$$

$$F(s) = \int_0^{\infty} t e^{-t} e^{-st} dt =$$

$$= \int_0^{\infty} t e^{-(s+1)t} dt \Rightarrow$$

$$f(t) = t \quad f'(t) = 1$$

$$g'(t) = e^{-(s+1)t} \quad g(t) = -\frac{1}{s+1} e^{-(s+1)t}$$

$$= -\frac{1}{s+1} t e^{-(s+1)t} \Big|_0^{\infty} + \frac{1}{s+1} \int_0^{\infty} e^{-(s+1)t} dt$$

$$\lim_{t \rightarrow \infty} \frac{t}{e^{(s+1)t}} - \lim_{t \rightarrow 0} \frac{t}{e^{(s+1)t}} = 0$$

$$\frac{1}{s+1} \int_0^{\infty} \left(-\frac{1}{s+1}\right) \left(e^{-(s+1)t}\right)' dt =$$

$$= -\frac{1}{(s+1)^2} e^{-(s+1)t} \Big|_0^{\infty} = \left(-\frac{1}{s+1}\right)^2 \overbrace{\lim_{t \rightarrow \infty} \frac{1}{e^{(s+1)t}} - \lim_{t \rightarrow 0} \frac{1}{e^t}}^0$$

$$= \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{t e^{-t}\} = \frac{1}{(s+1)^2}$$

(2) Găsit, transf. Laplace inversă

$$a) \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}$$

$$\text{pt. } f(t) = t e^{-t} \rightarrow \mathcal{L}\{t e^{-t}\} = \frac{1}{(s+1)^2}$$

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$$b) \frac{(2s+4)}{(s+1)(s^2-2)} = \frac{A}{s+1} + \frac{B}{s+\sqrt{2}} + \frac{C}{s-\sqrt{2}}$$

$$\Rightarrow A(s+\sqrt{2})(s-\sqrt{2}) + B(s+1)(s-\sqrt{2}) + C(s+1)(s+\sqrt{2}) = 2s+4$$

$$\begin{aligned} & A(s^2-2) + B(s^2+s-\sqrt{2}s-\sqrt{2}) + \\ & + C(s^2+s+\sqrt{2}s+\sqrt{2}) = \\ & = (A+B+C)s^2 + s(B-B\sqrt{2}+C+C\sqrt{2}) \\ & + (-2)A - B\sqrt{2} + C\sqrt{2} = 2s+4 \end{aligned}$$

$$\begin{cases} A + B + C = 0 \\ B - B\sqrt{2} + C + C\sqrt{2} = 2 \\ -2A - B\sqrt{2} + C\sqrt{2} = 4 \quad / : \sqrt{2} \end{cases}$$

$$-\sqrt{2}A - B + C = 2\sqrt{2} +$$

$$\sqrt{2}A + \sqrt{2}B + \sqrt{2}C = 0$$

$$(\sqrt{2} - 1)B + (\sqrt{2} + 1)C = 2\sqrt{2} +$$

$$-(\sqrt{2} - 1)B + (\sqrt{2} + 1)C = 2$$

$$(2\sqrt{2} + 2)C = 2\sqrt{2} + 2$$

$$\underline{\underline{C = 1}}$$

$$B - B\sqrt{2} + 1 + \sqrt{2} = 2$$

$$(1 - \sqrt{2})B = 1 - \sqrt{2}$$

$$\underline{\underline{A = -2}}$$

$$B = 1$$

$$-2e^{-t} + e^{-\sqrt{2}t}$$

$$e^{\sqrt{2}t}$$

$$\frac{2s+4}{(s+1)(s^2-2)} = \frac{-2}{(s+1)} + \frac{1}{s+\sqrt{2}} + \frac{1}{s-\sqrt{2}}$$

$$f(t) = -2e^{-t} + e^{-\sqrt{2}t} + e^{\sqrt{2}t}$$



(3) Rezolvați ecuațiile diferențiale

a)  $y' + y = 2t^2 - 1$  ;  $y(0) = -1$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{2t^2\} - \mathcal{L}\{1\}$$

$$sY(s) - y(0) + Y(s)$$

$$\mathcal{L}\{2t^2\} = \frac{4}{s^3} \quad (\text{calculat la p19})$$

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty}$$

$$= -\frac{1}{s} \left( \lim_{t \rightarrow \infty} \frac{1}{e^{st}} - \lim_{t \rightarrow 0} \frac{1}{e^{st}} \right)$$

$$= +\frac{1}{s}$$

$$sY(s) - y(0) + Y(s) = \frac{4}{s^3} - \frac{1}{s}$$

$$(s+1)Y(s) + 1 = \frac{4 - s^2}{s^3}$$

$$(s+1) Y(s) = \frac{4-s^2}{s^3} - 1 = \frac{4-s^2-s^3}{s^3}$$

$$Y(s) = \frac{4-s^2-s^3}{s^3(s+1)} = \frac{A}{s+1} + \frac{\frac{3}{s^2}}{s} + \frac{\frac{1}{s}}{s^2} + \frac{1}{s^3}$$

$$= \frac{A}{s+1} + \frac{Bs^2 + Cs + D}{s^3} \rightarrow$$

$$\rightarrow As^3 + Bs^3 + Cs^2 + Ds + Bs^2 + Cs + D = 4 - s^2 - s^3$$

$$\begin{cases} A+B = -1 \rightarrow A = -4 \\ B+C = -1 \rightarrow B = 3 \\ C+D = 0 \rightarrow C = -4 \\ D = 4 \end{cases}$$

$$Y(s) = \underbrace{-\frac{4}{s+1}}_{F(s)} + \underbrace{\frac{3}{s}}_{G(s)} - \underbrace{\frac{4}{s^2}}_{H(s)} + \underbrace{\frac{4}{s^3}}_{i(s)}$$

$$F(s) = -\frac{4}{s+1} \rightarrow f(t) = -4e^{-t}$$

$$G(s) = \frac{3}{s} \rightarrow g(t) = 3$$

$$H(s) = -\frac{4}{s^2} \rightarrow h(t) = -4t$$

$$i(s) = \frac{4}{s^3} \rightarrow i(t) = 2t^2$$

$$y(t) = -4e^{-t} + 3 - 4t + 2t^2$$


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$$b) \quad y'' + 4y = \sin(2t) \quad y(0) = 1$$

$$y'(0) = 0$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{4y\} = \mathcal{L}\{\sin 2t\}$$

$\downarrow$ 
 $4y(t)$ 
 $\frac{2}{s^2+4}$

$$s^2 y(t) - \underbrace{1}_{1} y(0) - \underbrace{0}_{0} y'(0) + 4y(t) = \frac{2}{s^2+4}$$

$$s^2 y(t) + 4y(t) - 1 = \frac{2}{s^2+4}$$

$$(s^2+4)y(t) = \frac{2}{s^2+4} + 1 = \frac{s^3 + 4s + 2}{s^2+4}$$

$$y(t) = \frac{s^3 + 4s + 2}{(s^2+4)^2} = \frac{As+B}{(s^2+4)^2} + \frac{Cs+D}{s^2+4}$$

$$As+B + Cs^3 + 4Cs + Ds^2 + 4D = s^3 + 4s + 2$$

$$\left\{ \begin{array}{l} C = 1 \\ D = 0 \\ A + 4C = 4 \rightarrow A = 0 \\ B + 4D = 2 \rightarrow B = 2 \end{array} \right.$$

$$Y(s) = \frac{2}{(s^2+4)^2} + \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1} \left\{ \frac{2a^3}{(s^2+a^2)^2} \right\} = \sin at - at \cos(at)$$

$$a=2$$

$$\mathcal{L}^{-1} \left\{ \frac{2 \cdot 8}{(s^2+4)^2} \cdot \frac{1}{8} \right\} = \frac{1}{8} (\sin 2t - 2t \cos 2t)$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} = \cos 2t$$

$$y(t) = (\sin 2t - 2t \cos 2t) + \cos 2t$$