(1) Calc. transformatele Laplace

a) $2t^2$

$$f(t) = 2t^2 \Rightarrow F(s) = \mathcal{L}\{2t^2\}$$

$$= \int_0^\infty 2t^2 \cdot e^{-st}\, dt$$

$$f(t) = 2t^2 \rightarrow f'(t) = 4t$$

$$g'(t) = e^{-st} \rightarrow g(t) = -\frac{1}{s} e^{-st}$$

$$= \underbrace{\left(2t^2 \cdot \left(-\frac{1}{s}\right) e^{-st}\right)\Big|_0^\infty}_{} - \int_0^\infty 4t \left(-\frac{1}{s}\right) e^{-st}\, dt$$

$$-\frac{1}{s}\left(\underbrace{\lim_{t\to\infty} \frac{2t^2}{e^{st}}}_{0} - \underbrace{\lim_{t\to 0} \frac{2t^2}{e^{st}}}_{0}\right) = 0$$

$$+\frac{4}{s}\int_0^\infty t \cdot e^{-st}\,dt$$

$$f(t) = t \qquad f'(t) = 1$$

$$g'(t) = e^{-st} \qquad g(t) = -\frac{1}{s}e^{-st}$$

$$= +\frac{4}{s}\left[\underbrace{t\cdot\left(-\frac{1}{s}\right)e^{-st}\Big|_0^\infty}_{0} - \underbrace{\int_0^\infty -\frac{1}{s}e^{-st}\,dt}\right]$$

$$\Rightarrow t\cdot e^{-st}\Big|_0^\infty = \underbrace{\lim_{t\to\infty}\frac{t}{e^{st}}}_{0} - \underbrace{\lim_{t\to 0}\frac{t}{e^{st}}}_{0} = 0$$

$$\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty =$$

$$= -\frac{1}{s}\left(\underbrace{\lim_{t\to\infty}e^{-st}}_{0} - \underbrace{\lim_{t\to 0}e^{-st}}_{1}\right) = \frac{1}{s}$$

$$= +\frac{4}{s}\cdot\frac{1}{s^2} = \frac{4}{s^3}$$

$$\mathcal{L}\{2t^2\} = \frac{4}{s^3}$$

b) $3\sin(2t)$ $\qquad \mathcal{L}\{3\sin 2t\} = F(s)$

$$F(s) = \int_0^\infty 3\sin 2t \; e^{-st}\,dt$$

$f(t) = \sin 2t \qquad f'(t) = 2\cos 2t$

$g'(t) = e^{-st} \qquad g(t) = -\frac{1}{s}e^{-st}$

$$= 3\sin 2t \cdot \left(-\frac{1}{s}\right)e^{-st}\Big|_0^\infty -$$

$$- 3\cdot\int_0^\infty 2\cos 2t \cdot \left(-\frac{1}{s}\right)e^{-st}\,dt$$

$$-\frac{3}{s}\left(\lim_{t\to\infty}\frac{\sin 2t}{\underbrace{e^{st}}_{\infty}} - \lim_{t\to 0}\frac{\overset{=0}{\sin 2t}}{\underset{=1}{e^{st}}}\right) = 0$$

$$\mathcal{L}\{3\sin(2t)\} = \frac{6}{s}\int_0^\infty \cos 2t \; e^{-st}\,dt$$

$$\int_0^\infty \cos 2t \cdot e^{-st}\, dt$$

$f(t) = \cos 2t \qquad f'(t) = -2\sin 2t$

$g'(t) = e^{-st} \qquad g(t) = -\frac{1}{s} e^{-st}$

$$= -\frac{1}{s}\cos 2t\, e^{-st}\Big|_0^\infty - \frac{2}{s}\underbrace{\int_0^\infty \sin 2t\, e^{-st}\, dt}_{\frac{F(s)}{3}}$$

$$\Rightarrow -\frac{1}{s}\left(\underbrace{\lim_{t\to\infty}\frac{\cos 2t}{e^{st}}}_{0} - \underbrace{\lim_{t\to 0}\frac{\cos 2t}{e^{st}}}_{1}\right)$$

$$\to \frac{1}{s} - \frac{2F(s)}{3s} = \frac{3-2F(s)}{3s}$$

$$F(s) = \frac{6}{s}\ \frac{3-2F(s)}{3s} \Rightarrow$$

$3s^2 F(s) = 18 - 12F(s) \quad /:3$

$s^2 F(s) = 6 - 4F(s)$

$F(s)(s^2+4) = 6 \to \mathcal{L}\{3\sin 2t\} = \frac{6}{s^2+4}$

c) $3e^{-t}$ $\quad \mathcal{L}\{3e^{-t}\} = F(s)$

$$F(s) = \int_0^\infty 3e^{-t} e^{-st} dt = \int_0^\infty 3e^{-(s+1)t} dt$$

$$= \int_0^\infty 3 \cdot -\frac{1}{s+1} \left(e^{-(s+1)t}\right)' dt =$$

$$= -\frac{3}{s+1} e^{-(s+1)t} \Big|_0^\infty =$$

$$= -\frac{3}{s+1} \Big(\underbrace{\lim_{t \to \infty} \frac{1}{e^{(s+1)t}}}_{0} - \underbrace{\lim_{t \to 0} \frac{1}{e^{(s+1)t}}}_{1} \Big) =$$

$$\Rightarrow \mathcal{L}\{3e^{-t}\} = \frac{3}{s+1}$$

d) $t e^{-t}$ $\quad \mathcal{L}\{te^{-t}\} = F(s)$

$$F(s) = \int_0^\infty t e^{-t} e^{-st}\, dt =$$

$$= \int_0^\infty t \; e^{-(s+1)t}\, dt \Rightarrow$$

$f(t) = t$ $\qquad f'(t) = 1$

$g'(t) = e^{-(s+1)t}$ $\qquad g(t) = -\frac{1}{s+1} e^{-(s+1)t}$

$$= -\frac{1}{s+1} t e^{-(s+1)t} \Big|_0^\infty + \frac{1}{s+1} \int_0^\infty e^{-(s+1)t}$$

$$\lim_{t \to \infty} \frac{t}{e^{(s+1)t}} - \lim_{t \to 0} \frac{t}{e^{(s+1)t}} = 0$$

$$\frac{1}{s+1} \int_0^\infty \left(-\frac{1}{s+1}\right) \left(e^{-(s+1)t}\right)' dt =$$

$$= -\frac{1}{(s+1)^2} e^{-(s+1)t} \Big|_0^\infty = \left(-\frac{1}{s+1}\right)^2 \overbrace{\lim_{t \to \infty} \frac{1}{e^{(s+1)t}}}^{0} - \overbrace{\lim_{t \to 0} \frac{1}{e^{t}}}^{1}$$

$$= \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$$

(2) Găsiți transf. Laplace inversă

a) $\dfrac{1}{s^2+2s+1} = \dfrac{1}{(s+1)^2}$

pt. $f(t) = te^{-t} \to \mathcal{L}\{te^{-t}\} = \dfrac{1}{(s+1)^2}$

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b) $\dfrac{(2s+4)}{(s+1)(s^2-2)} = \dfrac{A}{s+1} + \dfrac{B}{s+\sqrt{2}} + \dfrac{C}{s-\sqrt{2}}$

$\Rightarrow A(s+\sqrt{2})(s-\sqrt{2}) + B(s+1)(s-\sqrt{2}) +$
$+ C(s+1)(s+\sqrt{2}) = 2s+4$

$A(s^2-2) + B(s^2+s-\sqrt{2}s-\sqrt{2}) +$
$+ C(s^2+s+\sqrt{2}s+\sqrt{2}) =$
$= (A+B+C)s^2 + s(B - B\sqrt{2} + C + C\sqrt{2})$
$+ (-2)A - B\sqrt{2} + C\sqrt{2} = 2s+4$

$$\begin{cases} A + B + C = 0 \\ B - B\sqrt{2} + C + C\sqrt{2} = 2 \\ -2A - B\sqrt{2} + C\sqrt{2} = 4 \quad / : \sqrt{2} \end{cases}$$

$$-\sqrt{2}A - B + C = 2\sqrt{2} \quad +$$

$$\sqrt{2}A + \sqrt{2}B + \sqrt{2}C = 0$$

$$(\sqrt{2}-1)B + (\sqrt{2}+1)C = 2\sqrt{2} \quad +$$

$$-(\sqrt{2}-1)B + (\sqrt{2}+1)C = 2$$

$$(2\sqrt{2}+2)C = 2\sqrt{2}+2$$

$$\underline{C = 1} \qquad B - B\sqrt{2} + 1 + \sqrt{2} = 2$$

$$(1-\sqrt{2})B = 1 - \sqrt{2}$$

$$\underline{A = -2} \qquad \underline{B = 1}$$

$$\frac{2s+4}{(s+1)(s^2-2)} = \frac{-2}{(s+1)} + \frac{1}{s+\sqrt{2}} + \frac{1}{s-\sqrt{2}}$$

$$-2e^{-t} \qquad e^{-\sqrt{2}t} \qquad e^{\sqrt{2}t}$$

$$f(t) = -2e^{-t} + e^{-\sqrt{2}t} + e^{\sqrt{2}t}$$

(3) Rezolvați ecuațiile diferențiale

a) $y' + y = 2t^2 - 1$; $y(0) = -1$

$$\underbrace{\mathcal{L}\{y'\}}_{s Y(s) - y(0)} + \underbrace{\mathcal{L}\{y\}}_{Y(s)} = \mathcal{L}\{2t^2\} - \mathcal{L}\{1\}$$

$s Y(s) - y(0) + Y(s)$

$\mathcal{L}\{2t^2\} = \frac{4}{s^3}$ (calculat la p1 a)

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= -\frac{1}{s} \left(\underbrace{\lim_{t \to \infty} \frac{1}{e^{st}}}_{0} - \underbrace{\lim_{t \to 0} \frac{1}{e^{st}}}_{1} \right)$$

$$= + \frac{1}{s}$$

$$s Y(s) - y(0) + Y(s) = \frac{4}{s^3} - \frac{1}{s}$$

$$(s+1) Y(s) + 1 = \frac{4 - s^2}{s^3}$$

$$(s+1)\,Y(s) = \frac{4-s^2}{s^3} - 1 = \frac{4-s^2-s^3}{s^3}$$

$$Y(s) = \frac{4-s^2-s^3}{s^3(s+1)} = \frac{A}{s+1} + \frac{\overset{s^2}{B}}{s} + \frac{\overset{s}{C}}{s^2} + \frac{D}{s^3}$$

$$= \frac{A}{s+1} + \frac{Bs^2 + Cs + D}{s^3} \longrightarrow$$

$$\longrightarrow As^3 + Bs^3 + Cs^2 + Ds + Bs^2 + Cs + D = 4 - s^2 - s^3$$

$$\begin{cases} A + B = -1 \rightarrow A = -4 \\ B + C = -1 \rightarrow B = 3 \\ C + D = 0 \rightarrow C = -4 \\ D = 4 \end{cases}$$

$$Y(s) = \underbrace{-\frac{4}{s+1}}_{F(s)} + \underbrace{\frac{3}{s}}_{G(s)} \underbrace{-\frac{4}{s^2}}_{H(s)} + \underbrace{\frac{4}{s^3}}_{i(s)}$$

$$F(s) = -\frac{4}{s+1} \rightarrow f(t) = -4e^{-t}$$

$$G(s) = \frac{3}{s} \rightarrow g(t) = 3$$

$$H(s) = -\frac{4}{s^2} \rightarrow h(t) = -4t$$

$$i(s) = \frac{4}{s^3} \longrightarrow i(t) = 2t^2$$

$$y(t) = -4e^{-t} + 3 - 4t + 2t^2$$

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b) $y'' + 4y = \sin(2t)$ $\quad y(0) = 1$

$y'(0) = 0$

$$\underbrace{\mathcal{L}\{y''\}}_{\downarrow} + \underbrace{\mathcal{L}\{4y\}}_{4Y(t)} = \underbrace{\mathcal{L}\{\sin 2t\}}_{\frac{2}{s^2+4}}$$

$$s^2 Y(t) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + 4Y(t) = \frac{2}{s^2+4}$$

$$s^2 Y(t) + 4Y(t) - s = \frac{2}{s^2+4}$$

$$(s^2+4)\,Y(t) = \frac{2}{s^2+4} + s = \frac{s^3 + 4s + 2}{s^2+4}$$

$$Y(t) = \frac{s^3+4s+2}{(s^2+4)^2} = \frac{As+B}{(s^2+4)^2} + \frac{Cs+D}{s^2+4}$$

$$As + B + Cs^3 + 4Cs + Ds^2 + 4D = s^3 + 4s + 2$$

$$\begin{cases} C = 1 \\ D = 0 \\ A + 4C = 4 \rightarrow A = 0 \\ B + 4D = 2 \rightarrow B = 2 \end{cases}$$

$$Y(t) = \frac{2}{(s^2+4)^2} + \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{ \frac{2a^3}{(s^2+a^2)^2} \right\} = \sin at - at\cos(at)$$

$a=2$

$$\mathcal{L}^{-1}\left\{ \frac{2\cdot 8}{(s^2+4)^2} \cdot \frac{1}{8} \right\} = \frac{1}{8}(\sin 2t - 2t\cos 2t)$$

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \right\} = \cos 2t$$

$$\boxed{y(t) = (\sin 2t - 2t\cos 2t) + \cos 2t}$$