(4)

a) $x(k) = \left(\frac{1}{2}\right)^{k}$

$$X(k) = Z\left\{\left(\frac{1}{2}\right)^{k}\right\} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k} =$$

$$= \sum_{k=0}^{\infty} (2z)^{-k} = \underbrace{1 + (2z)^{-1} + (2z)^{-2} + \dots + (2z)^{-n}}_{n \to \infty}$$

$$= 1 \cdot \frac{(2z)^{\overset{\nearrow 0}{-(n+1)}} - 1}{(2z)^{-1} - 1} = \frac{1}{1 - (2z)^{-1}}$$

$$\to \quad Z\{x(k)\} = Z\left\{\left(\frac{1}{2}\right)^{k}\right\} = \frac{1}{1-(2z)^{-1}}$$

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b) $x(k) = k$

$$X(z) = Z\{k\} = \sum_{k=0}^{\infty} k\, z^{-k} =$$

$$= \underbrace{\cancel{0} + z^{-1} + 2z^{-2} + 3z^{-3} + \dots + n z^{-n}}_{n \to \infty}$$

$$X(z) = \underbrace{z^{-1} + 2z^{-2} + 3z^{-3} + \dots + n z^{-n}}_{n \to \infty}$$

$$X(z) = \overbrace{\underbrace{z^{-1} + z^{-2} + z^{-3} + \dots + z^{-n}}_{n \to \infty}}^{X_1(z)} +$$

$$+ \overbrace{\underbrace{1 \cdot z^{-2} + 2 \cdot z^{-3} + \dots + (n-1) z^{-n}}_{n \to \infty}}^{X_2(z)}$$

$$X_1(z) = z^{-1} \cdot \frac{(z^{-n})^{\to \infty} - 1}{z^{-1} - 1} = z^{-1} \cdot \frac{1}{1 - z^{-1}}$$

$$X_2(z) = \underbrace{z^{-2} + 2z^{-3} + \dots + (n-1) z^{-n}}_{n \to \infty}$$

$$\int X_2(z)\, dz = \underbrace{-(z)^{-1} - (z^{-2}) - (z^{-3}) \dots - (z^{-n+1})}_{n \to \infty}$$

$$\int z^{-n}\, dz = \frac{z^{-n+1}}{-n+1} = \left(\frac{1}{n+1}\right)\left(-z^{-n+1}\right)$$

$$\int X_2(z)\, dz = -z^{-1} \cdot \frac{(z^{-n+1})^{\to 0} - 1}{z^{-1} - 1} = \frac{z^{-1}}{z^{-1} - 1}$$

$$\int x_2(z)\,dz = \frac{z^{-1}}{z^{-1}-1} = \frac{\frac{1}{z}}{\frac{1}{z}-1} = \frac{\cancel{\frac{1}{z}}}{\frac{1-z}{\cancel{z}}} =$$

$$= \frac{1}{1-z}$$

$$x_2(z) = \left(\frac{1}{1-z}\right)' = +(1-z)^{-2}$$

$$X(z) = X_1(z) + x_2(z) = \frac{z^{-1}}{1-z^{-1}} + \frac{1}{(1-z)^2}$$

$$\frac{z^{-1}}{1-z^{-1}} = \frac{\cancel{\frac{1}{z}}}{\frac{z-1}{\cancel{z}}} = -\frac{1}{1-z}$$

$$x(z) = \frac{z-1+1}{(1-z)^2} = \frac{z}{(1-z)^2}$$

$$X(z) = \mathcal{Z}\{k\} = \frac{z}{(1-z)^2}$$

$$x(z) = \frac{\cancel{z^2}\,z^{-1}}{\cancel{z^2}\,(1-z^{-1})^2} = \frac{z^{-1}}{(1-z^{-1})^2}$$

c) $x(k) = e^{ak}$

$$X(z) = \mathcal{Z}\{e^{ak}\} = \sum_{k=0}^{\infty} e^{ak} z^{-k}$$

$$X(z) = \sum_{k=0}^{\infty} \left(e^{a} \cdot z^{-1}\right)^{k} =$$

$$= \underbrace{1 + \left(e^{a} \cdot z^{-1}\right)^{2} + \ldots \left(e^{a} \cdot z^{-1}\right)^{n}}_{n \to \infty} =$$

$$= 1 \cdot \frac{\left(e^{a} \cdot z^{-1}\right)^{n+1 \to 0} - 1}{e^{a} \cdot z^{-1} - 1} = \frac{1}{1 - e^{a} z^{-1}}$$

$$X(z) = \mathcal{Z}\{e^{ak}\} = \frac{1}{1 - e^{a} z^{-1}}$$

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d) $x(k) = \begin{cases} 2, & k \geq 3 \\ 0, & k < 3 \end{cases}$

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k} =$$

$$= \underbrace{x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + \ldots x(n) z^{-n}}_{n \to \infty}$$

$$X(z) = \underbrace{0 \cdot z^{0} + \dots\ 0 \cdot z^{-2}}_{0} +$$

$$+ \underbrace{2 \cdot z^{-3} + \dots\ 2 \cdot z^{-n}}_{n \to \infty}$$

$$X(z) = 2\left(z^{-3} \cdot \frac{(z^{-n+2})^{\to 0} - 1}{z^{-1} - 1}\right) =$$

$$X(z) = 2 \frac{z^{-3}}{1 - z^{-1}}$$

$$X(z) = Z\{x(k)\} = \frac{2 z^{-3}}{1 - z^{-1}}$$

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5 $$y(k+1) = (1+\lambda)\, y(k)$$

$\lambda = 0.1$

$y(0) = 100$

$$\underbrace{Z\{y(k+1)\}} = (1+\lambda)\, \overbrace{Z\{y(k)\}}^{Y(z)}$$

$$z\left[Y(z) - y(0)\right] = (1+\lambda)\, Y(z)$$

$$z\left[Y(z)-y_0\right]=(1+r)\,Y(z)$$

$$z\,Y(z)-y_0\,z=(1+r)\,Y(z)$$

$$Y(z)\left(z-(r+1)\right)=y_0\,z$$

$$Y(z)=\frac{y_0\,z}{z-(r+1)}=\frac{y_0\,\cancel{z}}{\cancel{z}\left(1-(1+r)z^{-1}\right)}$$

$$Y(z)=\frac{y_0}{1-(1+r)z^{-1}}$$

$$y(k)=\mathcal{Z}^{-1}\left\{\frac{y_0}{1-(1+r)z^{-1}}\right\}=$$

$$=y_0\,(1+r)^k \qquad \left(100\cdot(1{,}1)^k\right)$$

$$y(k)=y_0\,(1+r)^k$$

(6) $y(k+1) - 3y(k) = 4^k \quad y(0)=2$

$$\underbrace{\mathcal{Z}\{y(k+1)\}} - 3\mathcal{Z}\{y(k)\} = \underbrace{\mathcal{Z}\{4^k\}}$$

$$z\left[Y(z) - y_0\right] - 3Y(z) = \frac{1}{1-4z^{-1}}$$

$$(z-3)\,Y(z) = z y_0 + \frac{1}{1-4z^{-1}}$$

$$Y(z) = \frac{\frac{1-4z^{-1}}{\cancel{z}}\, y_0 + \frac{1}{1-4z^{-1}}}{z-3} =$$

$$= \frac{zy_0 - 4y_0 + 1}{(z-3)(1-4\cancel{z}^{-1})} = \frac{y_0 + (1-4y_0)z^{-1}}{(1-3z^{-1})(1-4z^{-1})}$$

$$\frac{A}{1-3z^{-1}} + \frac{B}{1-4z^{-1}} = \frac{y_0 + (1-4y_0)z^{-1}}{(1-3z^{-1})(1-4z^{-1})}$$

$$A(1-4z^{-1}) + B(1-3z^{-1}) = y_0 + (1-4y_0)z^{-1}$$

$$\begin{cases} A + B = y_0 & / \cdot 4 \quad + \\ -4A - 3B = 1 - 4y_0 & + \end{cases}$$

$$4A + 4B = 4y_0$$

$$\boxed{B = 1} \longrightarrow \boxed{A = y_0 - 1}$$

$$Y(z) = \frac{y_0 - 1}{1 - 3z^{-1}} + \frac{1}{1 - 4z^{-1}}$$

$$y(k) = Z^{-1}\{Y(z)\} =$$

$$= (y_0 - 1) \cdot 3^k + 4^k$$

$$\boxed{y(k) = (y_0 - 1) \cdot 3^k + 4^k}$$

(7) $\begin{cases} x(k+1) - y(k) = 0 \\ y(k+1) + x(k) = 0 \end{cases}$, $x(0) = 1$, $y(0) = 0$

$$\begin{cases} \mathcal{Z}\{x(k+1)\} - \mathcal{Z}\{y(k)\} = 0 \\ \mathcal{Z}\{y(k+1)\} + \mathcal{Z}\{x(k)\} = 0 \end{cases}$$

$$\begin{cases} z[X(z) - X_0] - Y(z) = 0 \\ z[Y(z) - Y_0] + X(z) = 0 \end{cases} +$$

$$(z+1)X(z) - x_0 + (z-1)Y(z) - y_0 = 0$$

$$Y(z) = \frac{x_0 + y_0 - (z+1)X(z)}{(z-1)}$$

$$Y(z) = \frac{1 - (z+1)(X(z))}{(z-1)}$$

$$\frac{z}{z-1}\left[1 - (z+1)X(z)\right] + X(z) = 0$$

$$\frac{z}{z-1} - \frac{z(z+1)}{z-1} X(z) + X(z) = 0$$

$$z - z(z+1)X(z) + (z-1)X(z) = 0$$

$$z - z^2 X(z) - \cancel{z X(z)} + \cancel{z X(z)} - X(z) = 0$$

$$z - (z^2+1) X(z) = 0$$

$$X(z) = \frac{z}{z^2+1} = \frac{z^{-1}}{1+z^{-2}}$$

$$\sin \omega T = 1 \qquad \longrightarrow \qquad \omega T = \frac{\pi}{2}$$
$$\cos \omega T = 0$$

$$x(k) = Z^{-1}\left\{ \frac{z^{-1}}{1+z^{-2}} \right\} =$$

$$= \sin \omega k T = \sin \frac{\pi}{2} k$$

$$\boxed{x(k) = \sin \frac{\pi}{2} k}$$

$$\begin{cases} z\left[X(z)-x_0\right]-Y(z)=0 \\ z\left[Y(z)-y_0\right]+X(z)=0 \end{cases}$$

$$zX(z)-Y(z)=z$$

$$X(z)+zY(z)=0 \quad /\cdot(-z)$$

$$-zX(z)-z^2Y(z)$$

$$-(z^2+1)Y(z)=z$$

$$Y(z)=-\frac{z}{z^2+1}=-\frac{z^{-1}}{1+z^{-2}}$$

$$\sin\omega t=1 \qquad \rightarrow \omega t=\frac{\pi}{2}$$

$$\cos\omega t=0$$

$$y(k)=Z^{-1}\{Y(z)\}=>$$

$$\boxed{y(k)=-\sin\frac{\pi}{2}k}$$