CS 5050 Advanced Algorithms

Fall Semester, 2017

Asymptotic Notation and Math Review

1 Asymptotic Notation (Growth of Functions)

Note: big-O, big-Ω, big-Θ are required, but small-o and small-ω are optional (but strongly recommended to know).

Let $f(n)$ and $g(n)$ be any two functions of n with $f(n) \geq 0$ and $g(n) \geq 0$ for all n. We have the following definitions.

big-O: $f(n) = O(g(n))$ if and only if there exist a constant $c > 0$ and n_0, such that $f(n) \leq c \cdot g(n)$ for all $n > n_0$.

Examples: $1000 = O(\log n)$, $3n + 5 = O(n)$, $2n^3 - 4n^2 = O(n^3)$, $n^9 = O(2^n)$

Big-O provides an asymptotic upper bound.

big-Ω: $f(n) = \Omega(g(n))$ if and only if there exist a constant $c > 0$ and n_0, such that $f(n) \geq c \cdot g(n)$ for all $n > n_0$.

Examples: $\log n = \Omega(100)$, $3n + 5 = \Omega(n)$, $2n^3 - 4n^2 = \Omega(n^3)$, $n^2 = \Omega(n \log n)$

Big-Ω provides an asymptotic lower bound.

big-Θ: $f(n) = \Theta(g(n))$ if and only if there exist constants $c_1 > 0$, $c_2 > 0$, and n_0, such that $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n > n_0$.

Examples: $n^3 + 2n^2 - 12 = \Theta(n^3)$, $3n + \log n + 5 = \Theta(n)$, $2n^3 - 4n^2 = \Theta(n^3)$

small-o: $f(n) = o(g(n))$ if and only if for any constant $c > 0$, there exists n_0, such that $f(n) < c \cdot g(n)$ for all $n > n_0$.

Examples: $3n \log n + 4 = o(n^2)$, $n^2 = o(n^3)$, $n^6 = o(3^n)$

small-ω: $f(n) = \omega(g(n))$ if and only if for any constant $c > 0$, there exists n_0, such that $f(n) > c \cdot g(n)$ for all $n > n_0$.

Examples: $n^2 = \omega(3n \log n + 4)$, $n^3 = \omega(n^2)$, $3^n = \omega(n^6)$

The following are some properties. To help you understand them intuitively, you may simply think of that you are comparing two real numbers a and b.

- $f(n) = O(g(n)) \Longleftrightarrow g(n) = \Omega(f(n))$

  Think of: $a \leq b \Longleftrightarrow b \geq a$

- $f(n) = \Omega(g(n)) \Longleftrightarrow g(n) = O(f(n))$

  Think of: $a \geq b \Longleftrightarrow b \leq a$

- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

  Think of: $a = b \iff a \leq b$ and $a \geq b$

- $f(n) = o(g(n)) \iff g(n) = \omega(f(n))$

  Think of: $a < b \iff b > a$

- $f(n) = o(g(n)) \implies f(n) = O(g(n))$ and $f(n) \neq \Theta(g(n))$

  Although the reverse is not true in general, you may consider it true for most functions (I believe for all functions you will see in this class).

  Think of: $a < b \iff a \leq b$ and $a \neq b$

2 Mathematics Review

Exponentials: For all real $a > 0$, m, and n,

$a^0 = 1$, $a^1 = a$, $a^{-1} = \frac{1}{a}$, $(a^m)^n = a^{mn}$.

Logarithms: For all real $a > 0$, $b > 0$, $c > 0$, and n,

$\log_c(ab) = \log_c a + \log_c b$, $\log_b a^n = n \log_b a$, $\log_b a = \frac{\log_c a}{\log_c b}$, $\log_b \frac{1}{a} = -\log_b a$, $a = b^{\log_b a}$, $a^{\log_b c} = c^{\log_b a}$.

Arithmetic series:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Sums of squares:

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Geometric series:

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$