(i) *h contains one point $p \in S$ on its boundary,*
(ii) *h contains two or more points of S on its boundary.*
The number of type (i) candidates is $O(n)$, and they can be found in $O(n)$ time.

The number of type (ii) candidates is clearly quadratic. Because the number of type (i) candidates is much smaller than this, we treat them in a brute-force way: for each of the $O(n)$ half-planes we compute their continuous measure in constant time, and their discrete measure in $O(n)$ time. This way the maximum of the discrepancies of these half-planes can be computed in $O(n^2)$ time. For the type (ii) candidates we must be more careful when computing the discrete measures. For this we need some new techniques. In the remainder of this chapter we introduce these techniques and we show how to use them to compute all discrete measures in $O(n^2)$ time. We can then compute the discrepancy of these half-planes in constant time per half-plane, and take the maximum. Finally, by comparing this maximum to the maximum discrepancy of the type (i) candidates we find the discrepancy of S. This leads to the following theorem.

Theorem 8.2 *The half-plane discrepancy of a set S of n points in the unit square can be computed in $O(n^2)$ time.*

8.2 Duality

A point in the plane has two parameters: its x-coordinate and its y-coordinate. A (non-vertical) line in the plane also has two parameters: its slope and its intersection with the y-axis. Therefore we can map a set of points to a set of lines, and vice versa, in a one-to-one manner. We can even do this in such a way that certain properties of the set of points translate to certain other properties for the set of lines. For instance, three points on a line become three lines through a point. Several different mappings that achieve this are possible; they

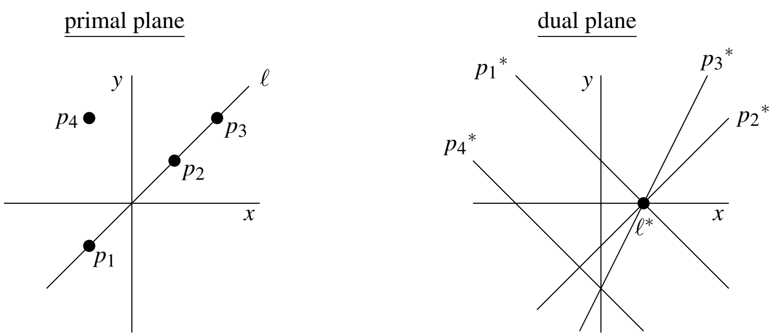


Figure 8.2
An example of duality

are called *duality transforms*. The image of an object under a duality transform is called the *dual* of the object. A simple duality transform is the following. Let $p := (p_x, p_y)$ be a point in the plane. The dual of p, denoted p^*, is the line defined as

$$p^* := (y = p_x x - p_y).$$

The dual of a line $\ell : y = mx + b$ is the point p such that $p^* = \ell$. In other words,

$$\ell^* := (m, -b).$$

The duality transform is not defined for vertical lines. In most cases vertical lines can be handled separately, so this is not a problem. Another solution is to rotate the scene so that there are no vertical lines.

We say that the duality transform maps objects from the *primal plane* to the *dual plane*. Certain properties that hold in the primal plane also hold in the dual plane:

Observation 8.3 *Let p be a point in the plane and let ℓ be a non-vertical line in the plane. The duality transform $o \mapsto o^*$ has the following properties.*
- *It is incidence preserving: $p \in \ell$ if and only if $\ell^* \in p^*$.*
- *It is order preserving: p lies above ℓ if and only if ℓ^* lies above p^*.*

Figure 8.2 illustrates these properties. The three points p_1, p_2, and p_3 lie on the line ℓ in the primal plane; the three lines $p_1{}^*$, $p_2{}^*$, and $p_3{}^*$ go through the point ℓ^* in the dual plane. The point p_4 lies above the line ℓ in the primal plane; the point ℓ^* lies above the line $p_4{}^*$ in the dual plane.

The duality transform can also be applied to other objects than points and lines. What would be the dual of a line segment $s := \overline{pq}$, for example? A logical choice for s^* is the union of the duals of all points on s. What we get is an infinite set of lines. All the points on s are collinear, so all the dual lines pass through one point. Their union forms a *double wedge*, which is bounded by the duals of the endpoints of s. The lines dual to the endpoints of s define two double wedges, a left-right wedge and a top-bottom wedge; s^* is the left-right wedge. Figure 8.3 shows the dual of a segment s. It also shows a line ℓ intersecting s, whose dual

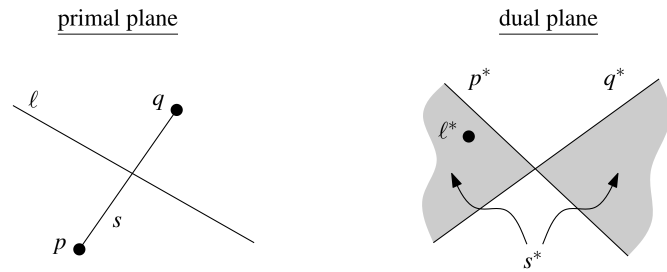
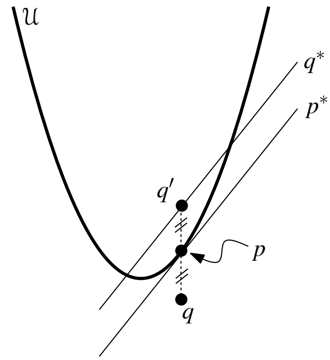


Figure 8.3
The dual transform applied to a line segment

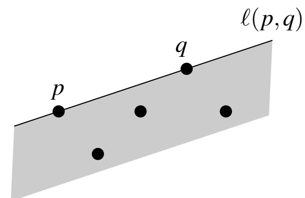
ℓ^* lies in s^*. This is not a coincidence: any line that intersects s must have either p or q above it and the other point below it, so the dual of such a line lies in s^* by the order preserving property of the dual transform.

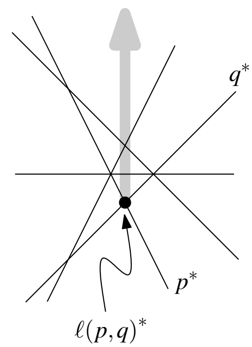
The dual transform defined above has a nice geometric interpretation. Let $\mathcal{U}$ denote the parabola $\mathcal{U}:\ y = x^2/2$. Let's first look at the dual of a point p that lies on $\mathcal{U}$. The derivative of $\mathcal{U}$ at p is p_x, so p^* has the same slope as the tangent line of $\mathcal{U}$ at p. As a matter of fact, the dual of a point $p \in \mathcal{U}$ *is* the tangent line at p, because the intersection of the tangent with the y-axis is $(0, -p_x^2/2)$. Now suppose that a point q does not lie on $\mathcal{U}$. What is the slope of q^*? Well, any two points on the same vertical line have duals with equal slope. In particular, q^* is parallel to p^*, where p is the point that lies on $\mathcal{U}$ and has the same x-coordinate as q. Let q' be the point with the same x-coordinate as q (and as p) such that $q'_y - p_y = p_y - q_y$. The vertical distance between the duals of points with the same x-coordinate is equal to the difference in y-coordinates of these points. Hence, q^* is the line through q' that is parallel to the tangent of $\mathcal{U}$ at p.



When you think about duality for a few minutes you may wonder how duality can be useful. If you can solve a problem in the dual plane, you could have solved it in the primal plane as well by mimicking the solution to the dual problem in the primal plane. After all, the primal and dual problems are essentially the same. Still, transforming a problem to the dual plane has one important advantage: it provides a new perspective. Looking at a problem from a different angle can give the insight needed to solve it.

Let's see what happens when we consider the discrepancy problem in the dual plane. In the previous section we were left with the following problem: Given a set S of n points, compute the discrete measure of every half-plane bounded by a line through two of the points. When we dualize the set S of points we get a set $S^* := \{p^* : p \in S\}$ of lines. Let $\ell(p,q)$ denote the line through two points $p,q \in S$. The dual of this line is the intersection point of the two lines $p^*, q^* \in S^*$. Consider the open half-plane bounded by and below $\ell(p,q)$. The discrete measure of this half-plane is the number of points strictly below $\ell(p,q)$. This means that in the dual plane we are interested in the number of lines strictly above $\ell(p,q)^*$. For the closed half-plane below $\ell(p,q)$ we must also take the lines through $\ell(p,q)^*$ into account. Similarly, for the half-plane bounded by and above $\ell(p,q)$ we are interested in the number of lines below $\ell(p,q)^*$. In the next section we study sets of lines, and we give an efficient algorithm to compute the number of lines above every intersection point, through every intersection point, and below every intersection point. When we apply this algorithm to S^* we get all the information we need to compute the discrete measure of all half-planes bounded by lines through two points in S.





There is one thing that we should be careful about: two points in S with the same x-coordinate dualize to lines with the same slope. So the line through these points does not show up as an intersection in the dual plane. This makes sense, because the dual transform is undefined for vertical lines. In our application this calls for an additional step. For every vertical line through at least two points, we must determine the discrete measures of the corresponding half-planes. Since there is only a linear number of vertical lines through two (or more) points in S, the discrete measures for these lines can be computed in a brute-force manner in $O(n^2)$ time in total.

8.3 Arrangements of Lines

Let L be a set of n lines in the plane. The set L induces a subdivision of the plane that consists of vertices, edges, and faces. Some of the edges and faces

11.4* Convex Hulls and Half-Space Intersection

In Chapter 8 we have met the concept of duality. The strenth of duality lies in that it allows us to look at a problem from a new perspective, which can lead to more insight in what is really going on. Recall that we denote the line that is the dual of a point p by p^*, and the point that is the dual of a line ℓ by ℓ^*. The duality transform is incidence and order preserving: $p \in \ell$ if and only if $\ell^* \in p^*$, and p lies above ℓ if and only if ℓ^* lies above p^*.

Let's have a closer look at what convex hulls correspond to in dual space. We will do this for the planar case. Let P be a set of points in the plane. For technical reasons we focus on its *upper convex hull*, denoted $\mathcal{UH}(P)$, which consists of the convex hull edges that have P *below* their supporting line—see the left side of Figure 11.4. The upper convex hull is a polygonal chain that connects the leftmost point in P to the rightmost one. (We assume for simplicity that no two points have the same x-coordinate.)

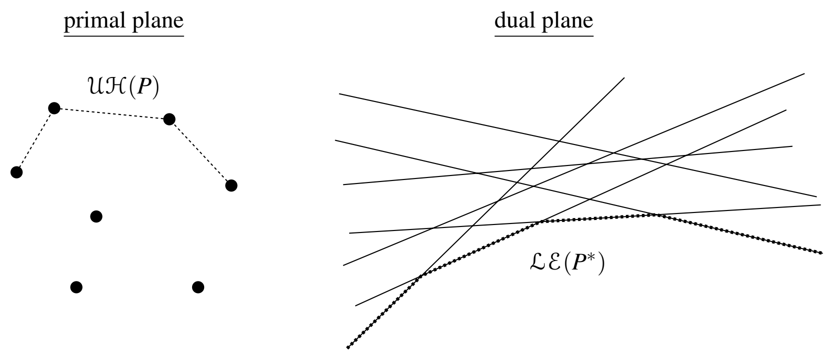


Figure 11.4
Upper hulls correspond to lower envelopes

When does a point $p \in P$ appear as a vertex of the upper convex hull? That is the case if and only if there is a non-vertical line ℓ through p such that all other points of P lie below ℓ. In the dual plane this statement translates to the following condition: there is a point ℓ^* on the line $p^* \in P^*$ such that ℓ^* lies below all other lines of P^*. If we look at the arrangement $\mathcal{A}(P^*)$, this means that p^* contributes an edge to the unique bottom cell of the arrangement. This cell is the intersection of the half-planes bounded by a line in P^* and lying below that line. The boundary of the bottom cell is an x-monotone chain. We can define this chain as the minimum of the linear functions whose graphs are the lines in P^*. For this reason, the boundary of the bottom cell in an arrangement is often called the *lower envelope* of the set of lines. We denote the lower envelope of P^* by $\mathcal{LE}(P^*)$—see the right hand side of Figure 11.4.

The points in P that appear on $\mathcal{UH}(P)$ do so in order of increasing x-coordinate. The lines of P^* appear on the boundary of the bottom cell in order of decreasing slope. Since the slope of the line p^* is equal to the x-coordinate of p, it follows that the left-to-right list of points on $\mathcal{UH}(P)$ corresponds exactly to the right-to-left list of edges of $\mathcal{LE}(P^*)$. So the upper convex hull of a set of points is essentially the same as the lower envelope of a set of lines.

Let's do one final check. Two points p and q in P form an upper convex hull edge if and only if all other points in P lie below the line ℓ through p and q. In the dual plane, this means that all lines r^*, with $r \in P \setminus \{p,q\}$, lie above the intersection point ℓ^* of p^* and q^*. This is exactly the condition under which $p^* \cap q^*$ is a vertex of $\mathcal{LE}(P^*)$.

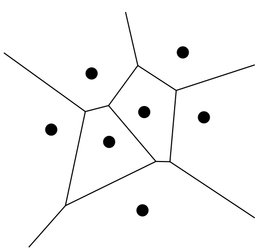
What about the *lower convex hull* of P and the *upper envelope* of P^*? (We leave the precise definitions to the reader.) By symmetry, these concepts are dual to each other as well.

We now know that the intersection of *lower half-planes*—half-planes bounded from above by a non-vertical line—can be computed by computing an upper convex hull, and that the intersection of *upper half-planes* can be computed by computing a lower convex hull. But what if we want to compute the intersection of an arbitrary set H of half-planes? Of course, we can split the set H into a set H_+ of upper half-planes and a set H_- of lower half-planes, compute $\bigcup H_+$ by computing the lower convex hull of ${H_+}^*$ and $\bigcup H_-$ by computing the upper convex hull of ${H_-}^*$, and then compute $\bigcap H$ by intersecting $\bigcup H_+$ and $\bigcup H_-$.

But is this really necessary? If lower envelopes correspond to upper convex hulls, and upper envelopes correspond to lower convex hulls, shouldn't then the intersection of arbitrary half-planes correspond to full convex hulls? In a sense, this is true. The problem is that our duality transformation cannot handle vertical lines, and lines that are close to vertical but have opposite slope are mapped to very different points. This explains why the dual of the convex hull consists of two parts that lie rather far apart.

It is possible to define a different duality transformation that allows vertical lines. However, to apply this duality to a given set of half-planes, we need a point in the intersection of the half-planes. But that was to be expected. As long as we do not want to leave the Euclidean plane, there cannot be any general duality that turns the intersection of a set of half-planes into a convex hull, because the intersection of half-planes can have one special property: it can be empty. What could that possibly correspond to in the dual? The convex hull of a set of points in Euclidean space is always well defined: there is no such thing as "emptiness." (This problem is nicely solved if one works in oriented projective space, but this concept is beyond the scope of this book.) Only once you know that the intersection is not empty, and a point in the interior is known, can you define a duality that relates the intersection with a convex hull.

We leave it at this for now. The important thing is that—although there are technical complications—convex hulls and intersections of half-planes (or half-spaces in three dimensions) are essentially dual concepts. Hence, an algorithm to compute the intersection of half-planes in the plane (or half-spaces in three dimensions) can be given by dualizing a convex-hull algorithm.



11.5* Voronoi Diagrams Revisited

In Chapter 7 we introduced the Voronoi diagram of a set of points in the plane. It may come as a surprise that there is a close relationship between planar Voronoi