

# Mathematical Methods — Bound Reference

December 6, 2023

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## Part I

# Theory

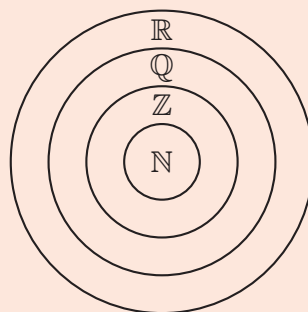
## 1 Functions and relations

### Set notation

- A **set** is a collection of objects called **elements**.
- $x \in A$  means that element  $x$  is a member of set  $A$  (and its counterpart is  $x \notin A$ ).
- $B$ , another set, is a **subset** of set  $A$  if *every element* of  $B$  is also in  $A$ . We write this as  $B \subseteq A$ .
- $\emptyset$  is known as the **empty set**.
- The set of elements that are common to two sets  $A$  and  $B$  is called the **intersection** of  $A$  and  $B$ , and is denoted by  $A \cap B$ . Thus,  $x \in A \cap B \iff x \in A$  and  $x \in B$ .
- Sets  $A$  and  $B$  are **disjoint** if they have no elements in common ( $A \cap B = \emptyset$ ).
- The set of elements that are in  $A$  or in  $B$  (or in *both*) is called the **union** of sets  $A$  and  $B$ , and is denoted by  $A \cup B$ .
- The **set difference** of two sets  $A$  and  $B$  is given by  $A \setminus B = \{x : x \in A, x \notin B\}$ .

### Sets of numbers

- $\mathbb{N} = \{1, 2, 3, \dots\}$  = Counting numbers.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  = Whole numbers.
- $\mathbb{Q} = \left\{p, q \in \mathbb{Z} : \frac{p}{q}\right\}$  = Rational numbers.
- The set of *all the numbers which cannot be represented by ratios of two integers* is called the set of **real numbers**, and is denoted by  $\mathbb{R}$ .
  - Positive real numbers:  $\mathbb{R}^+ = \{x : x > 0\}$
  - Negative real numbers:  $\mathbb{R}^- = \{x : x < 0\}$
  - Real numbers excluding zero:  $\mathbb{R} \setminus \{0\}$



## Interval notation

- Suppose that  $a$  and  $b$  are real numbers, with  $a < b$ .
  - $(a, b) = \{x : a < x < b\}$
  - $[a, b] = \{x : a \leq x \leq b\}$
  - $(a, b] = \{x : a < x \leq b\}$
  - $[a, b) = \{x : a \leq x < b\}$
  - $(a, \infty) = \{x : a < x\}$
  - $[a, \infty) = \{x : a \leq x\}$
  - $(-\infty, b) = \{x : x < b\}$
  - $(-\infty, b] = \{x : x \leq b\}$

When using number lines to represent intervals,

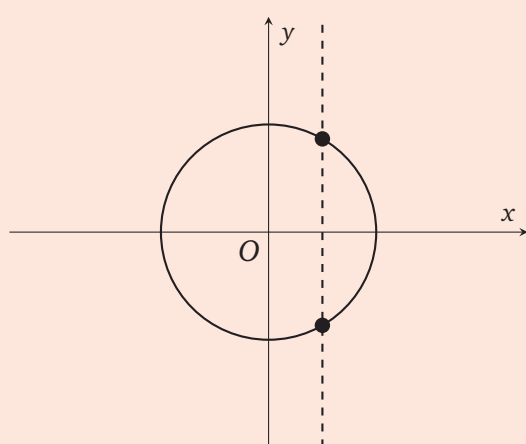
- The 'closed' circle ( $\bullet$ ) indicates that the number is included.
- The 'open' circle ( $\circ$ ) indicates that the number is **not** included.

## Functions VS relations

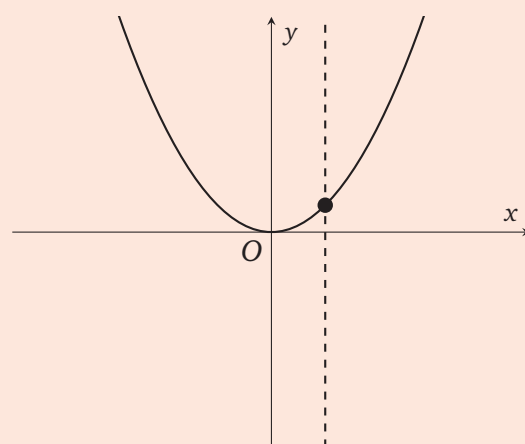
- A **function** is a relation such that for each  $x$ -value there is only one corresponding  $y$ -value. This means that, if  $(a, b)$  and  $(a, c)$  are ordered pairs of a function, then  $b = c$ .
- In other words, a function cannot contain two different ordered pairs with the same first coordinate.

### Vertical line test

- If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph at a maximum of once, then the relation is a function.



$x^2 + y^2 = 1$  is not a function



$y = x^2$  is a function

## Function notation

$$\underline{f : X \rightarrow Y, f(x) = \dots}$$

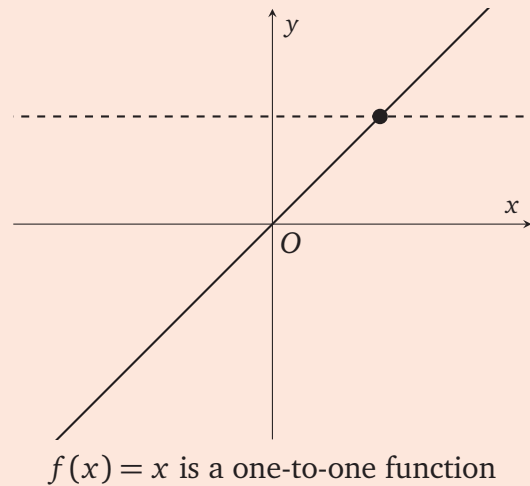
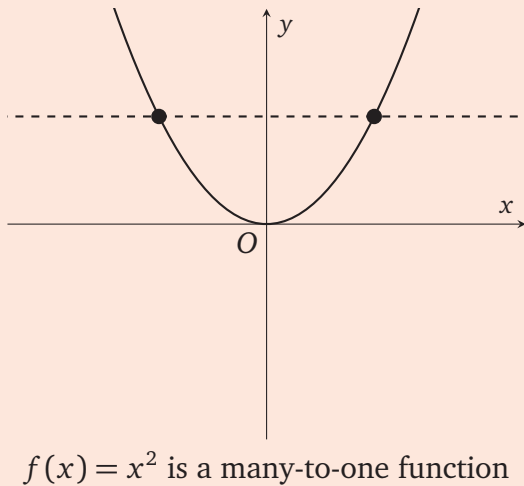
- $f$  is the function name
- $X$  is the **domain** of the function, *i.e.*, the set of values for which the function is defined.
- $Y$  is the **codomain** of the function, *i.e.*, the set of values which the **range** (the range is the set of outputs of the function) of the function falls into.

## Types of functions (many/one-to-one)

- If  $\forall a, b \in \text{dom}(f) : f(a) = f(b) \iff a = b$ , or, to put it another way,  $\forall a, b \in \text{dom}(f) : f(a) \neq f(b) \iff a \neq b$ , then a function is a **one-to-one** function.
- A function that does not satisfy the above condition(s) is a **many-to-one** function.

### Horizontal line test

- If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is a **one-to-one**.



## Parity of functions

- A function is **even** if  $\forall x \in \text{dom}(f) : f(x) = f(-x)$ .
- A function is **odd** if  $\forall x \in \text{dom}(f) : f(-x) = -f(x)$ .
- A function can be **neither odd nor even** (if both of the above statements do not apply).

## Implied/maximal domain

- The **implied** domain (also referred to as the **maximal** domain) of a function is the largest subset of  $\mathbb{R}$  for which the rule for the function is defined.



## Sum and product of functions

- $(f + g)(x) = f(x) + g(x)$  for  $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$ 
  - $\text{dom}(f + g) = \text{dom}(f) \cap \text{dom}(g)$
- $(f - g)(x) = f(x) - g(x)$  for  $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$ 
  - $\text{dom}(f - g) = \text{dom}(f) \cap \text{dom}(g)$
- $(f \cdot g)(x) = f(x) \cdot g(x)$  for  $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$ 
  - $\text{dom}(f \cdot g) = \text{dom}(f) \cap \text{dom}(g)$

### Addition of ordinates (sketching $y = (f + g)(x)$ )

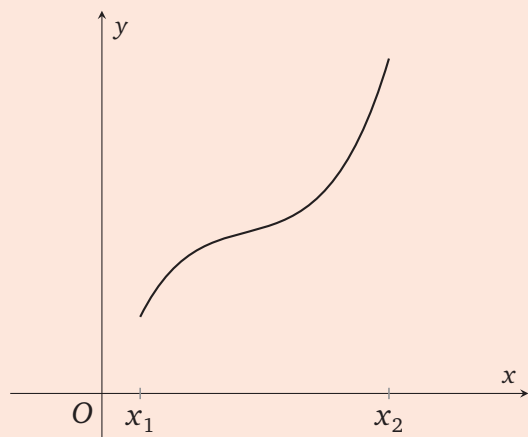
- When  $f(x) = 0$ ,  $(f + g)(x) = g(x)$ .
- When  $g(x) = 0$ ,  $(f + g)(x) = f(x)$ .
- If  $f(x)$  and  $g(x)$  are **both** positive, then  $(f + g)(x) > f(x)$  **and**  $(f + g)(x) > g(x)$ .
- If  $f(x)$  and  $g(x)$  are **both** negative, then  $(f + g)(x) < f(x)$  **and**  $(f + g)(x) < g(x)$ .
- If  $f(x)$  is positive and  $g(x)$  is negative, then  $g(x) < (f + g)(x) < f(x)$ .
- Look for values of  $x$  for which  $f(x) + g(x) = 0$ .

## Composite functions

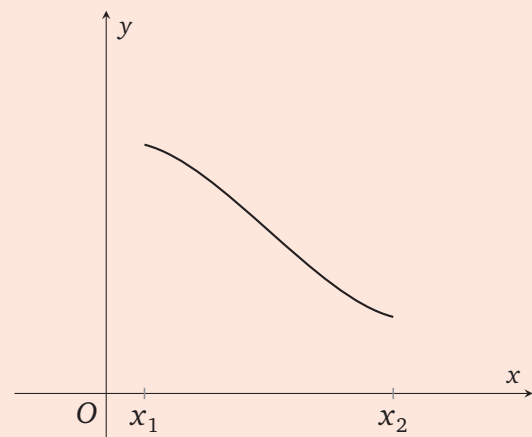
- Given that  $\text{ran}(g) \subseteq \text{dom}(f)$ , we can define the new function  $h$  as a **composition** of  $f$  with  $g$ .
- This is written  $h = f \circ g$  (read ‘composition of  $g$  followed by  $f$ ’) and the rule for  $h$  is given by  $h(x) = f(g(x))$ .
  - $\text{dom}(h) = \text{dom}(g)$
- Generally,  $f \circ g \neq g \circ f$

## Increasing and decreasing functions

- If  $\forall x \in [x_1, x_2] : \underline{f(x_2) > f(x_1)} \mid x_2 > x_1$ , then  $f$  is **strictly increasing** over the interval  $[x_1, x_2]$ .
- If  $\forall x \in [x_1, x_2] : \underline{f(x_2) < f(x_1)} \mid x_2 > x_1$ , then  $f$  is **strictly decreasing** over the interval  $[x_1, x_2]$ .
- These intervals include the values of  $x$  for which  $\frac{df}{dx} = 0$ , but the gradient never changes sign as such.
  - An example would be that the function of  $f : [0, \infty) \rightarrow \mathbb{R}, f(x) = x^2$  is strictly increasing.
- A function that is strictly increasing or decreasing is also a one-to-one function.



This function is **strictly increasing** over  $[x_1, x_2]$



This function is **strictly decreasing** over  $[x_1, x_2]$

## Inverse functions

- If  $f$  is a **one-to-one function**, then a new function  $f^{-1}$ , called the **inverse** of  $f$ , may be defined by

$$f^{-1}(x) = y \quad \text{if } f(y) = x,$$

for  $x \in \text{ran}(f)$ ,  $y \in \text{dom}(f)$ .

- Domain and range:

- $\text{dom}(f^{-1}) = \text{ran}(f)$
- $\text{ran}(f^{-1}) = \text{dom}(f)$

- Compositions:

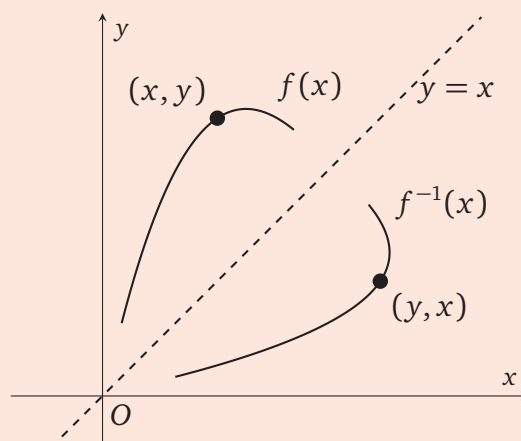
- $\forall x \in \text{dom}(f^{-1}) : (f \circ f^{-1})(x) = x$
- $\forall x \in \text{dom}(f) : (f^{-1} \circ f)(x) = x$

- The point  $(x, y)$  is on the graph of  $f^{-1}$  if and only if the point  $(y, x)$  is on the graph of  $f$ . Thus, the graph of  $f^{-1}$  is a **reflection** of the graph of  $f$  in the line  $y = x$ .

- If  $f$  is strictly increasing, then  $f^{-1}$  is also strictly increasing (and vice versa).

- **At least one** of the intersections of  $f$  and  $f^{-1}$  lie on the line  $y = x$  (if the functions intersect at all, that is), so, to solve for this intersection point, either  $f(x) = x$  or  $f^{-1}(x) = x$  will suffice.

- If “just one intersection point” is needed, then the line  $y = x$  is **tangential** to both functions at that point. This means that the gradients of  $f$ ,  $f^{-1}$ , and  $y = x$  are equal at that point (they are all 1, as the gradient of  $y = x$  is always 1).



## 2 Coordinate geometry

### Solving simultaneous equations

- There are two ways to solve simultaneous equations: **substitution** and **elimination**.

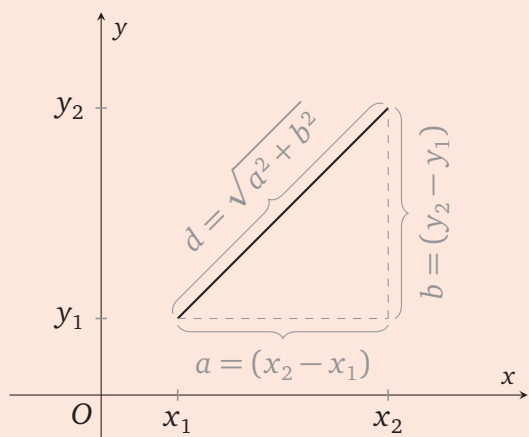
### Linear coordinate geometry

- The following is revision of basic concepts of linear coordinate geometry.
- This is **linear** coordinate geometry, meaning these concepts can only apply to straight lines.

#### Distance between two points

- Let  $d$  be the **distance** between two points  $A(x_1, y_1)$  and  $B(x_2, y_2)$ .

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The Pythagorean theorem comes into play for the derivation of the distance formula, as it is just the hypotenuse of a right-angled triangle formed by the horizontal and vertical components of the line.

#### Midpoint of a line

- The **midpoint**, of a line beginning and ending at points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  respectively is given by the formula:

$$\text{Midpoint} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

#### Gradient of a line

- The **gradient**,  $m$ , of a line going through the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

## Equation of a line

- The equation of a line (in slope-intercept form) with a  $y$ -intercept at the point  $A(0, c)$  is given by the formula:

$$y = mx + c$$

- The equation of a line (in point-slope form) going through the point  $A(x_1, y_1)$  is given by the formula:

$$y - y_1 = m(x - x_1)$$

- The equation of a line (in intercept form) going through the two points  $A(a, 0)$  and  $B(0, b)$  is given by the formula:

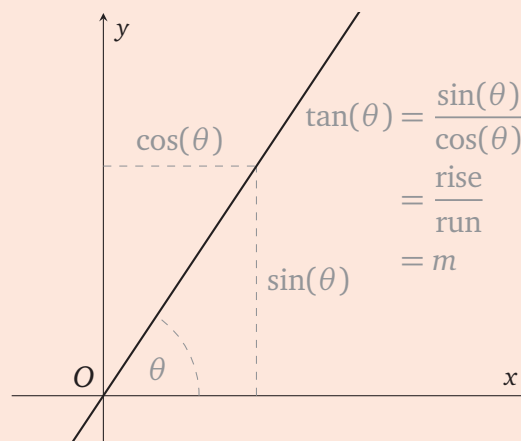
$$\frac{x}{a} + \frac{y}{b} = 1$$

## Tangent of the angle of slope

- For a straight line with gradient  $m$ , the angle of slope is found using:

$$m = \tan(\theta)$$

where  $\theta$  is the angle that the line makes with the positive direction of the  $x$ -axis.



## Perpendicular and parallel lines

- If two straight lines are perpendicular to each other (meet at right angles), the product of their gradients is  $-1$  (unless one is vertical and the other horizontal).

$$m_{\perp} \times m = -1$$

- Parallel lines have the same gradient.

## The geometry of simultaneous linear equations

- There are three cases for a system of two linear equations with two variables.

|               | Example                        | Solutions                           | Geometry                     |
|---------------|--------------------------------|-------------------------------------|------------------------------|
| <i>Case 1</i> | $2x + y = 5$<br>$x - y = 4$    | Unique solution:<br>$x = 3, y = -1$ | Two lines meeting at a point |
| <i>Case 2</i> | $2x + y = 5$<br>$2x + y = 7$   | No solutions                        | Distinct parallel lines      |
| <i>Case 3</i> | $2x + y = 5$<br>$4x + 2y = 10$ | Infinitely many solutions           | Two copies of the same line  |

## Dilations

### ■ Dilation from the $x$ -axis:

- For  $b \in \mathbb{R}^+$ , a dilation of factor  $b$  from the  $x$ -axis is described by the rule:

$$(x, y) \rightarrow (x, by)$$

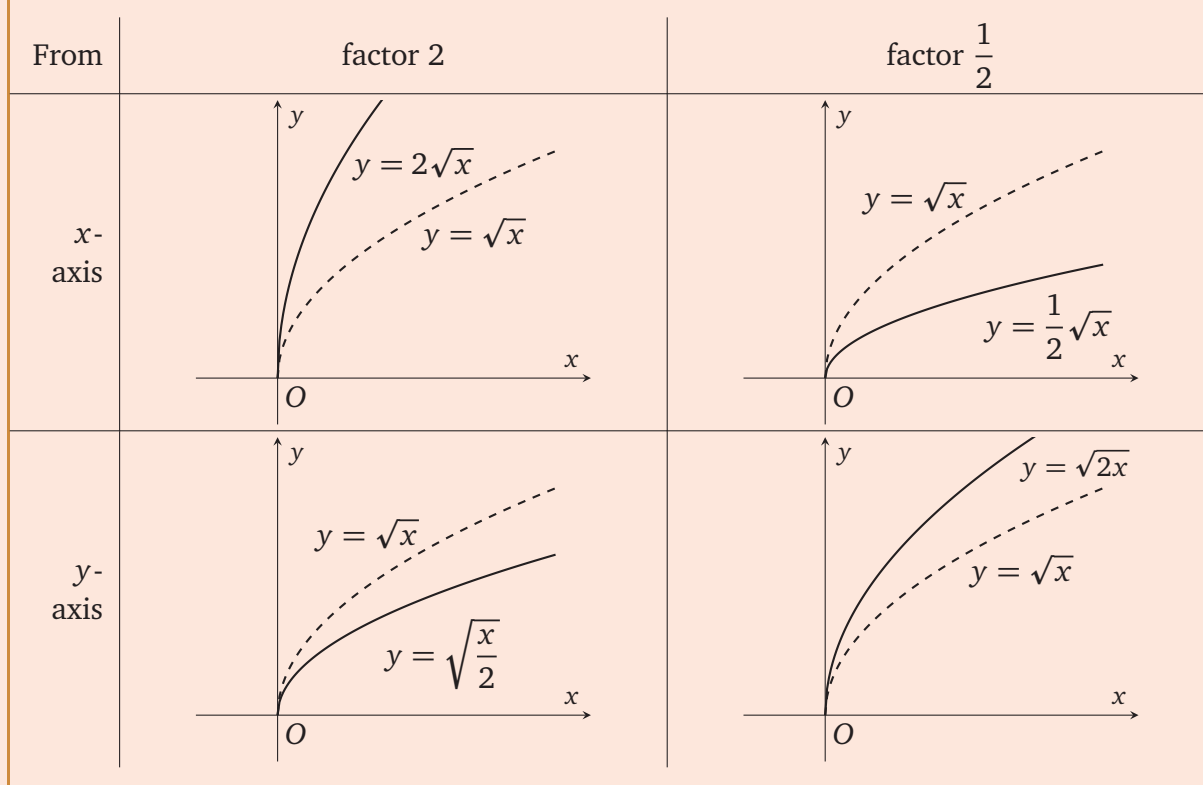
- This means that this dilation can also be applied as such:  $y = b \cdot f(x)$ .

### ■ Dilation from the $y$ -axis:

- For  $a \in \mathbb{R}^+$ , a dilation of factor  $a$  from the  $y$ -axis is described by the rule:

$$(x, y) \rightarrow (ax, y)$$

- This means that this dilation can also be applied as such:  $y = f\left(\frac{x}{a}\right)$ .



## Table of transformations

| Mapping   | $(x, y) \rightarrow$ | $y = f(x) \rightarrow$          |
|---|----------------------|---------------------------------|
| Reflection in the $x$ -axis   | $(x, -y)$            | $y = -f(x)$                     |
| Reflection in the $y$ -axis   | $(-x, y)$            | $y = f(-x)$                     |
| Dilation of factor $a$ from the $y$ -axis                           | $(ax, y)$            | $y = f\left(\frac{x}{a}\right)$ |
| Dilation of factor $b$ from the $x$ -axis                           | $(x, by)$            | $y = bf(x)$                     |
| Reflection in the line $y = x$ (inverse function)                   | $(y, x)$             | $x = f(y)$                      |
| Translation of $h$ units in the positive direction of the $x$ -axis | $(x + h, y)$         | $y = f(x - h)$                  |
| Translation of $k$ units in the positive direction of the $y$ -axis | $(x, y + k)$         | $y - k = f(x)$                  |

## Applying transformations

- $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(x, y) = (ax + h, by + k), \quad a \neq 0, b \neq 0$ 
  - Note that this notation is the same as writing  $(x, y) \rightarrow (ax + h, by + k)$ .
- Denote the **transformed** pair of coordinates (the new ones) as  $(x', y')$ .
- $(x', y') = T(x, y)$   
 $\therefore (x', y') = (ax + h, by + k), \quad a \neq 0, b \neq 0$
- Solve for the original  $x$  and  $y$  to be subbed into the function in question.
 
$$x' = ax + h$$

$$\therefore x = \frac{x' - h}{a}$$

$$y' = bx + k$$

$$\therefore y = \frac{y' - k}{b}$$
- Substitute  $x$  and  $y$  back into the function  $y = f(x)$ .
  - Remember to solve for  $y$  if there is more than one term on that side of the equation.



### 3 Polynomial functions

#### Quadratics

- For a quadratic in standard (polynomial) form  $(ax^2 + bx + c)$ ,
  - if  $a > 0$ , then the graph has a **minimum** point.
  - if  $a < 0$ , then the graph has a **maximum** point.
  - the **vertex (turning point)** is the point  $(h, k)$ , where  $h = -\frac{b}{2a}$  and  $k = \frac{4ac - b^2}{4a}$ .
  - the **axis of symmetry** is  $x = h$ , where  $h = -\frac{b}{2a}$ .
  - the quadratic can be written in “turning point form” by **completing the square** for  $x$  using the formula:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

- the solutions to  $ax^2 + bx + c = 0$  can be obtained using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$$

- the **discriminant** ( $\Delta$ ) for a quadratic polynomial is:

$$\Delta = b^2 - 4ac$$

For the equation  $ax^2 + bx + c = 0$ ,

- \* if  $\Delta > 0$ , there are two solutions.
- \* if  $\Delta = 0$ , there is one solution (tangential).
- \* if  $\Delta < 0$ , there are no solutions.

For the equation  $ax^2 + bx + c = 0$  where  $a, b, c \in \mathbb{Q}$ ,

- \* if  $\Delta$  is a perfect square and  $\Delta \neq 0$ , then the equation has two rational solutions.
- \* if  $\Delta = 0$ , then the equation has one rational solution.
- \* if  $\Delta$  is not a perfect square and  $\Delta > 0$ , then the equation has two irrational solutions.

#### Remainder theorem

- When  $P(x)$  is divided by  $\beta x + \alpha$ , the remainder is  $P\left(-\frac{\alpha}{\beta}\right)$ .

## Factor theorem

- For the polynomial  $P(x)$ , if  $P(\alpha) = 0$ , then  $x - \alpha$  is a factor of  $P(x)$ .
- Conversely, if  $x - \alpha$  is a factor of  $P(x)$ , then  $P(\alpha) = 0$ .

More generally:

- For the polynomial  $P(x)$ , if  $\beta x + \alpha$  is a factor of  $P(x)$ , then  $P\left(-\frac{\alpha}{\beta}\right) = 0$ .
- Conversely, if  $P\left(-\frac{\alpha}{\beta}\right) = 0$ , then  $\beta x + \alpha$  is a factor of  $P(x)$ .

## Rational root theorem

- The root of a polynomial function  $P(x)$  such that:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients are integers is of the form:

$$\frac{p}{q}, \text{ where } p = \text{a factor of } a_0 \text{ and } q = \text{a factor of } a_n$$

## Polynomials of degree $n$

- For a polynomial  $P(x)$  of degree  $n$ , there are **at most**  $n$  solutions to the equation  $P(x) = 0$ . Therefore, the graph of  $P(x)$  has **at most**  $n$   $x$ -axis intercepts.
- The graph of a polynomial of even degree may have no  $x$ -axis intercepts: for example,  $P(x) = x^2 + 1$ . But the graph of a polynomial of odd degree must have at least one  $x$ -axis intercept.

## Difference and sum of two variables of the same degree

- $x^2 - y^2 = (x - y)(x + y)$
- $x^3 - y^3 = (x + y)(x^2 - xy + y^2)$
- If  $n$  is odd,
  - $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$
  - $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

## 4 Exponential functions

### Exponential function characteristics

- For  $a \in \mathbb{R}^+ \setminus \{1\}$ , the graph of  $y = a^x$  has the following properties:
  - The  $x$ -axis is an asymptote.
  - The  $y$ -values are always positive.
  - The  $y$ -axis intercept is 1.
  - There is no  $x$ -axis intercept.

### Euler's number — $e$

- Euler's number is defined as follows:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\,828\dots$$

### Index laws

For all positive numbers  $a$  and  $b$  and all real numbers  $x$  and  $y$ :

- |  |                            |                            |                      |
|--|----------------------------|----------------------------|----------------------|
| ■ $a^x \cdot a^y = a^{x+y}$                      | ■ $a^x \div a^y = a^{x-y}$ | ■ $(a^x)^y = a^{xy}$       | ■ $(ab)^x = a^x b^x$ |
| ■ $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ | ■ $a^{-x} = \frac{1}{a^x}$ | ■ $a^x = \frac{1}{a^{-x}}$ | ■ $a^0 = 1$          |

## Logarithms

- For  $a \in \mathbb{R}^+ \setminus \{1\}$ , the **logarithm function** with base  $a$  is defined as follows:

$$a^x = y \iff \log_a(y) = x$$

- Since  $a$  is positive, the expression  $\log_a(y)$  is only defined when  $y$  is positive ( $y > 0$ ).

### Log laws

- |   |  |
|---|--|
| ■ $\log_a(1) = 0$                               | ■ $\log_a(b) = \frac{\ln(b)}{\ln(a)}$                      |
| ■ $\log_a(a) = 1$                               | ■ $\log_a(a^b) = b$  |
| ■ $\log_a(x^b) = b \cdot \log_a(x)$             | ■ $\log_a\left[\left(\frac{1}{a}\right)^n\right] = -n$     |
| ■ $\log_{a^b}(x) = \frac{1}{b} \cdot \log_a(x)$ | ■ $a^{\log_a(b)} = b$                                      |
| ■ $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$ | ■ $\log_a(a) + \log_a(b) = \log_a(ab)$                     |
| ■ $\log_{\frac{1}{a}}(x) = -\log_a(x)$          | ■ $\log_a(a) - \log_a(b) = \log_a\left(\frac{a}{b}\right)$ |

- The graph of  $y = \log_a(x)$  can be obtained from the graph of  $y = \log_b(x)$  by a dilation of factor  $\frac{1}{\log_b(a)}$  from the  $x$ -axis.
- The graph of  $y = a^x$  can be obtained from the graph of  $y = b^x$  by a dilation of factor  $\frac{1}{\log_b(a)}$  from the  $y$ -axis.
- When dividing both sides of an inequality by  $\log_a(x)$  where  $0 < x < 1$ , reverse the inequality as the logarithm will evaluate to negative.

## Exponential growth and decay

- In a situation where the growth/decay of something is exponential, the amount of that thing can be modelled using a function of the form:

$$A(t) = A_0 \cdot e^{kt}$$

where  $A_0$  is the initial quantity at  $t = 0$  (where  $t$  is a variable representing a unit of time) and  $k$  is a constant.

- Growth corresponds to  $k > 0$ .
- Decay corresponds to  $k < 0$ .

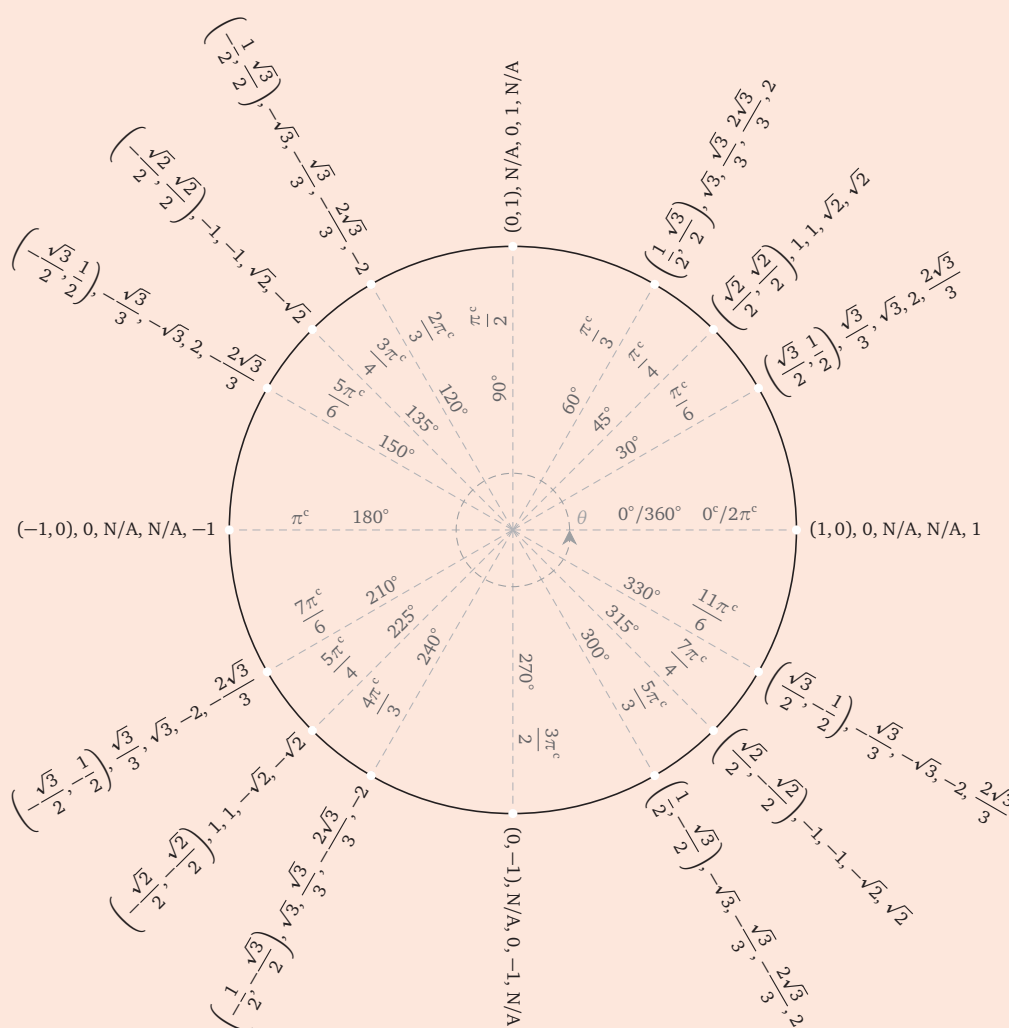
## 5 Circular functions

### Radians and degrees

- One **radian** (written  $1^c$ ) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert between the two, use the following:

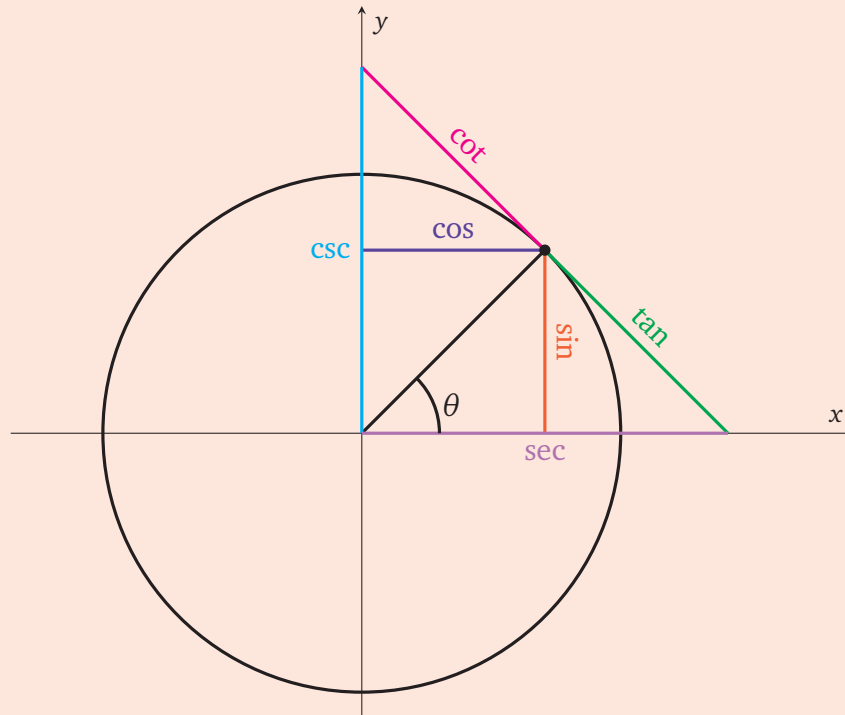
$$1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

### Unit circle



$(x, y), z, a, b, c$ , where:  
 $x = \cos$   
 $y = \sin$   
 $z = \tan = \frac{\sin}{\cos}$   
 $a = \cot = \frac{1}{\tan} = \frac{\cos}{\sin}$   
 $b = \csc = \frac{1}{\sin}$   
 $c = \sec = \frac{1}{\cos}$

## Trigonometric functions as triangles



## Properties of trigonometric functions

- $y = \pm a \sin(nt)$ 
  - The period of  $\frac{2\pi}{n}$ .
  - The amplitude is  $a$ .
  - The range is  $[-a, a]$ .
- $y = \pm a \cos(nt)$ 
  - The period of  $\frac{2\pi}{n}$ .
  - The amplitude is  $a$ .
  - The range is  $[-a, a]$ .
- $y = a \tan(nt)$ 
  - The period of  $\frac{\pi}{n}$ .
  - The vertical asymptotes have equations  $t = \frac{(2k+1)\pi}{2n}$  where  $k \in \mathbb{Z}$ .
  - The axis intercepts are at  $t = \frac{k\pi}{n}$  where  $k \in \mathbb{Z}$ .

## Trigonometric identities

### Pythagorean identities

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\csc^2(x) - \cot^2(x) = 1$

### Double-angle identities

- $\sin(2x) = 2 \sin(x) \cos(x)$
- $\cos(2x) = 2 \cos^2(x) - 1$
- $\cos(2x) = 1 - 2 \sin^2(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$

### Sum/Difference identities

- $\sin(s + t) = \sin(s) \cos(t) + \cos(s) \sin(t)$
- $\sin(s - t) = \sin(s) \cos(t) - \cos(s) \sin(t)$
- $\cos(s + t) = \cos(s) \cos(t) - \sin(s) \sin(t)$
- $\cos(s - t) = \cos(s) \cos(t) + \sin(s) \sin(t)$
- $\tan(s + t) = \frac{\tan(s) + \tan(t)}{1 - \tan(s) \tan(t)}$
- $\tan(s - t) = \frac{\tan(s) - \tan(t)}{1 + \tan(s) \tan(t)}$

### Product-to-sum identities

- $\cos(s) \cos(t) = \frac{\cos(s - t) + \cos(s + t)}{2}$
- $\sin(s) \cos(t) = \frac{\sin(s + t) + \sin(s - t)}{2}$
- $\sin(s) \sin(t) = \frac{\cos(s - t) - \cos(s + t)}{2}$
- $\cos(s) \sin(t) = \frac{\sin(s + t) - \sin(s - t)}{2}$

### Triple-angle identities

- $\sin(3x) = -\sin^3(x) + 3 \cos^2(x) \sin(x)$
- $\sin(3x) = -4 \sin^3(x) + 3 \sin(x)$
- $\cos(3x) = \cos^3(x) - 3 \sin^2(x) \cos(x)$
- $\cos(3x) = 4 \cos^3(x) - 3 \cos(x)$
- $\tan(3x) = \frac{3 \tan(x) - \tan^3(x)}{1 - 3 \tan^2(x)}$
- $\cot(3x) = \frac{3 \cot(x) - \cot^3(x)}{1 - 3 \cot^2(x)}$

## General solutions

### General solutions for $\sin(x)$

$$\begin{aligned}\sin(\theta) &= \alpha, \alpha \in [-1, 1] \\ \therefore \theta &= 2n\pi + \sin^{-1}(\alpha), n \in \mathbb{Z} \quad \text{or} \\ &= (2n+1)\pi - \sin^{-1}(\alpha), n \in \mathbb{Z}; \\ &= n\pi + (-1)^n \sin^{-1}(\alpha), n \in \mathbb{Z} \quad (\text{concise})\end{aligned}$$

### General solutions for $\cos(x)$

$$\begin{aligned}\cos(\theta) &= \alpha, \alpha \in [-1, 1] \\ \therefore \theta &= 2n\pi \pm \cos^{-1}(\alpha), n \in \mathbb{Z}\end{aligned}$$

### General solutions for $\tan(x)$

$$\begin{aligned}\tan(\theta) &= \alpha, \alpha \in [-1, 1] \\ \therefore \theta &= n\pi + \tan^{-1}(\alpha), n \in \mathbb{Z}\end{aligned}$$

## Period of two trigonometric functions' sum/difference

- For two trigonometric functions  $f$  and  $g$  which are being added to (or subtracted from) each other to produce the function  $h$ , the period of  $h$  is the LCM (lowest common multiple) of the respective periods of  $f$  and  $g$ .

$$\text{Let } f(x) = a \sin(bx + c)$$

$$\text{Let } g(x) = k \cos(mx + n)$$

$$\begin{aligned}\text{Let } h(x) &= f(x) + g(x) \\ &= a \sin(bx + c) + k \cos(mx + n)\end{aligned}$$

$$\text{period}(f) = \frac{2\pi}{b}$$

$$\text{period}(g) = \frac{2\pi}{m}$$

$$\therefore \text{period}(h) = \text{lcm}\left(\frac{2\pi}{b}, \frac{2\pi}{m}\right)$$



## 6 Differentiation

### Average rate of change

- For any function  $y = f(x)$ , the **average rate of change** of  $y$  with respect to  $x$  over the interval  $[a, b]$  is the gradient of the line through the two points  $A(a, f(a))$  and  $B(b, f(b))$ .

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

### Differentiation from first principles

- The **derivative** of the function  $f$  is denoted by  $f'$  and is defined by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- The derivative of a function  $f$  with respect to  $x$  when  $x = a$  is also known as the **instantaneous rate of change** of  $f$  with respect to  $x$  when  $x = a$ .

### Derivative rules

#### Differentiation results

- **Constant function:**  $f(x) = c \implies f'(x) = 0$
- **Multiple:**  $f(x) = k \cdot g(x) \implies f'(x) = k \cdot g'(x)$
- **Sum:**  $f(x) = g(x) + h(x) \implies f'(x) = g'(x) + h'(x)$
- **Difference:**  $f(x) = g(x) - h(x) \implies f'(x) = g'(x) - h'(x)$

## Limits

### Algebra of limits

- **Sum:**  $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- **Multiple:**  $\lim_{x \rightarrow a} (k \cdot f(x)) = k \cdot \lim_{x \rightarrow a} f(x), k \in \mathbb{R}$
- **Product:**  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- **Quotient:**  $\lim_{x \rightarrow a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \lim_{x \rightarrow a} g(x) \neq 0$

### Left and right limits

- If the value of  $f(x)$  approaches the number  $p$  as  $x$  approaches  $a$  from the right-hand side, then it is written as  $\lim_{x \rightarrow a^+} f(x) = p$ .
- If the value of  $f(x)$  approaches the number  $p$  as  $x$  approaches  $a$  from the left-hand side, then it is written as  $\lim_{x \rightarrow a^-} f(x) = p$ .
- For  $\lim_{x \rightarrow a} f(x)$  to exist,  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow a^-} f(x)$  must be equal.

## Continuity of a function

- A function  $f$  is **continuous** at the point  $x = a$  if the following conditions are met:
  - $f(a)$  is defined.
  - $\lim_{x \rightarrow a} f(x) = f(a)$

## Differentiability of a function

- A function  $f$  is said to be differentiable at  $x = a$  if  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  exists.
- If a function is differentiable at a point, then it is also continuous at that point (the same cannot be said for the converse statement).
- An easy way to remember this is that a function is **not differentiable** at a *sharp corner* or a *cusp* (a sharp point where two points meet).

## Tangent line

- The **tangent line** to the graph of the function  $f$  at the point  $(a, f(a))$  is defined to be the line through  $(a, f(a))$  with the gradient  $f'(a)$ .
- The equation of the tangent line to the graph of  $y = f(x)$  at the point  $(a, f(a))$  can be found using the formula:

$$y - f(a) = f'(a) \cdot (x - a)$$

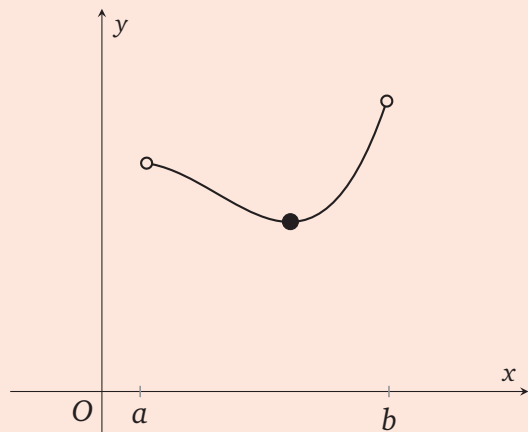
## Normal line

- The **normal line** to the graph of the function  $f$  at the point  $(a, f(a))$  is defined to be the line through  $(a, f(a))$  and is perpendicular to the tangent to the function  $f$  at that point.
- The equation of the tangent line to the graph of  $y = f(x)$  at the point  $(a, f(a))$  can be found using the formula:

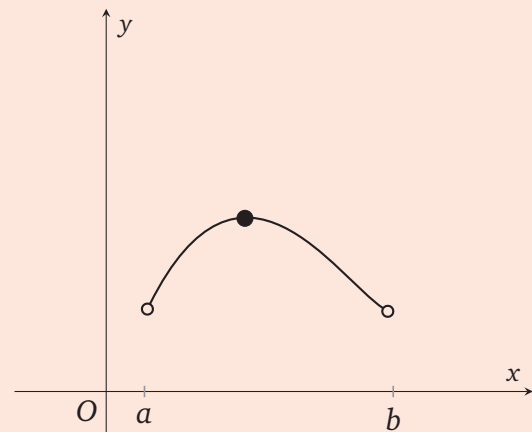
$$y - f(a) = -\frac{1}{f'(a)} \cdot (x - a)$$

## Second derivative of a function (concavity)

- Let  $f$  be a function defined on an interval  $(a, b)$ , and assume that both  $f'(x)$  and  $f''(x)$  exist for all  $x \in (a, b)$ .
- If  $\forall x \in (a, b) : f''(x) > 0$ , then the gradient of the curve  $y = f(x)$  is increase in the interval  $(a, b)$ . The curve is **concave up** (i.e., it has a **local minimum** in the interval  $(a, b)$ ).
- If  $\forall x \in (a, b) : f''(x) < 0$ , then the gradient of the curve  $y = f(x)$  is decrease in the interval  $(a, b)$ . The curve is **concave down** (i.e., it has a **local maximum** in the interval  $(a, b)$ ).
- If  $f''(x) = 0$  for  $x = a$ , then there is a **stationary point of inflection** in the curve  $y = f(x)$  at the point when  $x = a$ .



This function is **concave up** over  $(a, b)$



This function is **concave down** over  $(a, b)$

## 7 Integration

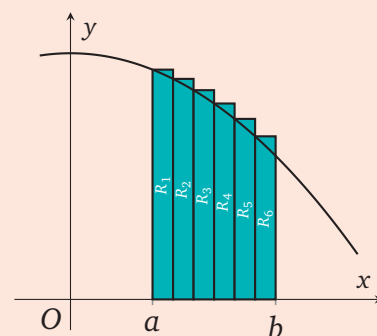
### Estimating the area under a graph

#### Left-endpoint estimate

- The formula for the left-endpoint estimate for a function  $f$  over the domain  $[a, b]$  with rectangles of width  $w$  is as follows:

$$\text{Area}_{\text{est.}} = \sum_{k=1}^{(b-a)/w} w \cdot f(a + w \cdot (k-1))$$

- For a function  $f$  that is...
  - strictly increasing in the domain  $[a, b]$ , the left-endpoint estimate  $\leq$  actual area.
  - strictly decreasing in the domain  $[a, b]$ , the left-endpoint estimate  $\geq$  actual area.

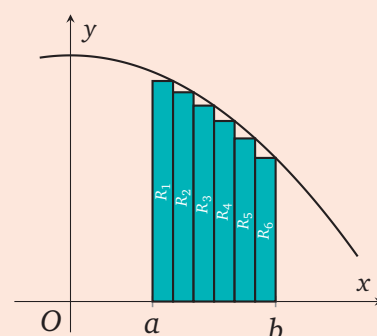


#### Right-endpoint estimate

- The formula for the right-endpoint estimate for a function  $f$  over the domain  $[a, b]$  with rectangles of width  $w$  is as follows:

$$\text{Area}_{\text{est.}} = \sum_{k=1}^{(b-a)/w} w \cdot f(a + wk)$$

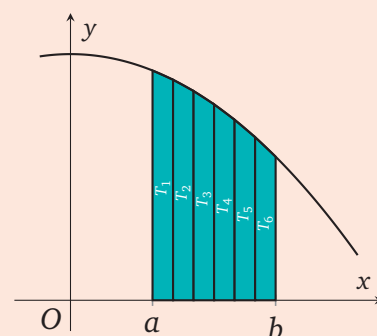
- For a function  $f$  that is...
  - strictly increasing in the domain  $[a, b]$ , the left-endpoint estimate  $\geq$  actual area.
  - strictly decreasing in the domain  $[a, b]$ , the left-endpoint estimate  $\leq$  actual area.



#### Trapezium estimate

- The formula for the right-endpoint estimate for a function  $f$  over the domain  $[a, b]$  with rectangles of width  $w$  is as follows:

$$\text{Area}_{\text{est.}} = \sum_{k=1}^{(b-a)/w} w \cdot [f(a + w \cdot (k-1)) + f(a + wk)]$$



## The fundamental theorem of calculus

$$\frac{d}{dx}[F(x)] = f(x) \implies \int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

- As the constant (+C) cancels out, we normally ignore it and take the antiderivative of  $f$  with  $C = 0$ .

## Antidifferentiation rules

### Antidifferentiation results

- **Sum:**  $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$
- **Difference:**  $\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$
- **Multiple:**  $\int [k \cdot f(x)] dx = k \cdot \int f(x) dx, k \in \mathbb{R}$

### Properties of the definite integral

- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
- $\int_a^a f(x) dx = 0$
- $\int_a^b [k \cdot f(x)] dx = k \cdot \int_a^b f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$

## Signed area

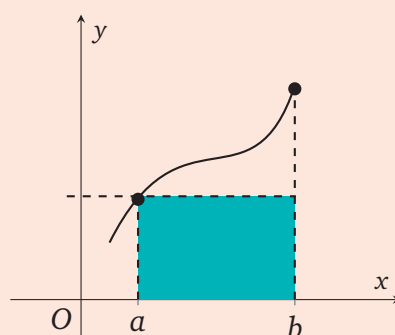
- For any continuous function  $f$  on an interval  $[a, b]$ , the **definite integral**  $\int_a^b f(x) \, dx$  gives the **signed area** enclosed by the graph of  $y = f(x)$  between  $x = a$  and  $x = b$ .
- To get the **unsigned area**, just take the absolute value of the function like so:  $\int_a^b |f(x)| \, dx$ .

## Average value of a function

- The **average value** of a continuous function  $f$  over an interval  $[a, b]$  is:

$$\frac{1}{b-a} \cdot \int_a^b f(x) \, dx$$

- In terms of the graph of  $y = f(x)$ , the average value is the **height of a rectangle** having the same area as the area under the graph for the interval  $[a, b]$  (the interval forms the rectangle's base).



## 8 Probability

### Basic laws of probability

- **Total law of probability:**  $\forall x \subseteq \mathcal{E} : \Pr(X = x) = 1$
- $\Pr(X = x) \geq 0 \iff x \in \mathcal{E}$
- **Addition rule:**  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
- $\Pr(\emptyset) = 0$
- $\Pr(A') = 1 - \Pr(A)$ , where  $A'$  is the complement of  $A$ .

### Mutually exclusive events

- Two events  $A$  and  $B$  are mutually exclusive if:

$$\Pr(A \cap B) = 0$$

- For mutually exclusive events, the addition rule becomes:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

### Probabilities from data

- When the number of trials is sufficiently large, the observed relative frequency of an event  $A$  becomes close to the probability  $\Pr(A)$ . That is,

$$\Pr(A) \approx \frac{\text{number of times } A \text{ occurs}}{\text{number of trials}} \quad \text{for a large number of trials}$$

### Probability tables (Karnaugh maps)

|      | $B$              | $B'$              |           |
|------|------------------|-------------------|-----------|
| $A$  | $\Pr(A \cap B)$  | $\Pr(A \cap B')$  | $\Pr(A)$  |
| $A'$ | $\Pr(A' \cap B)$ | $\Pr(A' \cap B')$ | $\Pr(A')$ |
|      | $\Pr(B)$         | $\Pr(B')$         | $1$       |

### Conditional probability

- The **conditional probability** of an event  $A$ , given that event  $B$  has already occurred, is given by:

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{if } \Pr(B) \neq 0$$

- This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A | B) \cdot \Pr(B)$$

## Law of total probability

- The **law of total probability** states that, in the case of two events  $A$  and  $B$ ,

$$\Pr(A) = \Pr(A | B) \cdot \Pr(B) + \Pr(A | B') \cdot \Pr(B')$$

## Independent events

- For events  $A$  and  $B$  with  $\Pr(A) \neq 0$  and  $\Pr(B) \neq 0$ , the following three conditions are all **equivalent conditions** for the independence of  $A$  and  $B$ :
  - $\Pr(A | B) = \Pr(A)$
  - $\Pr(B | A) = \Pr(B)$
  - $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$
- In the special case that  $\Pr(A) = 0$  or  $\Pr(B) = 0$ , the third condition ( $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ ) still holds since both sides are zero, so events  $A$  and  $B$  are still independent.

## Discrete probability functions

- The probability distribution of  $X$  is a function  $p(x) = \Pr(X = x)$  that assigns a probability to each value of  $X$ . It can be represented by a rule, a table or a graph, and must give a probability  $p(x)$  for every value  $x$  that  $X$  can take.
- For *any* discrete probability function  $p(x)$ , the following two conditions must hold:
  - Each value of  $p(x)$  belongs to the interval  $[0, 1]$ . That is,

$$\forall x \in \text{dom}(p) : 0 \leq p(x) \leq 1$$

- The sum of all the values of  $p(x)$  must be 1. That is,

$$\sum_x p(x) = 1$$

- The sum of the values of values of  $p(x)$  for  $x$  between  $a$  and  $b$  inclusive is written as

$$\sum_{a \leq x \leq b} p(x) = \Pr(a \leq X \leq b)$$



### Expected value

- The **expected value** of a discrete random variable  $X$  is determined by summing the products of each value of  $X$  and the probability that  $X$  takes that value. That is,

$$\begin{aligned} E(X) &= \sum_x [x \cdot \Pr(X = x)] \\ &= \sum_x [x \cdot p(x)] \end{aligned}$$

- The expected value  $E(X)$  may be considered as the long-run average value of  $X$ .
- It is generally denoted by  $\mu$ , and is also called the **mean** of  $X$ .
- $E[g(X)] = \sum_x [g(x) \cdot p(x)]$
- $E(aX + b) = a \cdot E(X) + b$  (for  $a, b$  constant)
  - Generally,  $E[g(X)] \neq g[E(X)]$ , but the linear case is an exception.
- If  $X$  and  $Y$  are two random variables, then  $E(X + Y) = E(X) + E(Y)$

### Variance

- The **variance** of a random variable  $X$  is the measure of the spread of the probability distribution about its mean or expected value  $\mu$ .
- It is defined as:

$$\begin{aligned} \text{Var}(X) &= E[(X - \mu)^2] \\ &= \sum_x [(x - \mu)^2 \cdot \Pr(X = x)] \\ &= \sum_x [(x - \mu)^2 \cdot p(x)] \end{aligned}$$

- Alternatively, the computational formula for calculating variance is as such:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

- It may be considered the long-run average value of the square of the distance from  $X$  to  $\mu$ .
- The variance is denoted using  $\sigma^2$ .
- $\text{Var}(aX + b) = a^2 \cdot \text{Var}(X)$  (for  $a, b$  constant)

## Standard deviation

- The **standard deviation** is defined as the square-root of the variance  $\sigma^2$ . That is,

$$\text{sd}(X) = \sqrt{\text{Var}(X)}$$

- It is usually denoted with  $\sigma$ .

## Bernoulli sequence

- A **Bernoulli sequence** is the name used to describe a sequence of repeated trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success,  $S$ , or a failure,  $F$ .
  - The probability of success on a single trial,  $p$ , is constant for all trials (and thus the probability of failure on a single trial is  $1 - p$ ).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

## Binomial probability distribution

- The number of successes in a Bernoulli sequence of  $n$  trials is called a **binomial random variable** and is said to have a **binomial probability distribution**.
- If the random variable  $X$  is the number of successes in  $n$  independent trials, each with probability of success  $p$ , then  $X$  has a **binomial distribution**, written  $X \sim \text{Bi}(n, p)$  and the rule is

$$\Pr(X = x) = \binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x} \quad x = 0, 1, \dots, n$$

where  $\binom{n}{x} = \frac{n!}{x! \cdot (n-x)!}$

- As the value of  $p$  increases, the graph of the binomial distribution is more skewed to the right (negatively skewed). A value of  $p = 0.5$  makes the peak of the graph of  $y = p(x)$  line up with the midway of the interval  $[0, n]$  of the  $x$ -axis.

## Population parameters for the binomial distribution

- $E(X) = np$
- $\text{Var}(X) = np(1 - p)$

## Probability density functions

- In general, the probability density function  $f$  is a function with domain some interval (e.g., domain  $[c, d]$  or  $\mathbb{R}$ ) such that:

1.  $\forall x \in \text{dom}(f) : f(x) \geq 0$
2. The area under the graph of  $y = f(x)$  is equal to 1.

- If the domain of  $f$  is  $[c, d]$ , then this condition corresponds to  $\int_c^d f(x) dx = 1$ .

- The values of a probability density function  $f$  are not probabilities, and  $f(x)$  may take values greater than 1.
- The probability of any specific value of  $X$  is 0. That is,  $\Pr(X = a) = 0$ .
- It follows that all of the following expressions have the same numerical value:
  - $\Pr(a < X < b)$
  - $\Pr(a \leq X < b)$
  - $\Pr(a < X \leq b)$
  - $\Pr(a \leq X \leq b)$

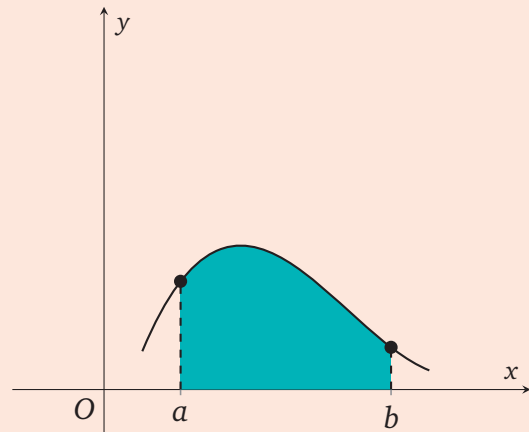
- If  $f$  has the domain  $[c, d]$  and  $a \in [c, d]$ , then  $\Pr(X < a) = \Pr(X \leq a) = \int_c^a f(x) dx$ .

### Visualising a probability density function

- If  $X$  is a continuous random variable with density function  $f$ , then

$$\Pr(a < X < b) = \int_a^b f(x) dx$$

which is the area of the shaded region.



## Computing improper integrals

- If  $\text{dom}(f) = (-\infty, a]$ , then  $\int_{-\infty}^a f(x) \, dx = 1$ . This integral is computed as

$$\lim_{k \rightarrow \infty} \int_{-k}^a f(x) \, dx$$

- If  $\text{dom}(f) = [a, \infty)$ , then  $\int_a^{\infty} f(x) \, dx = 1$ . This integral is computed as

$$\lim_{k \rightarrow \infty} \int_a^k f(x) \, dx$$

- If  $\text{dom}(f) = (-\infty, \infty)$ , then  $\int_{-\infty}^{\infty} f(x) \, dx = 1$ . This integral is computed as

$$\lim_{k \rightarrow \infty} \int_{-k}^k f(x) \, dx$$

## Properties for a continuous probability distribution

### Expected value/mean

- For a continuous random variable  $X$  with probability density function  $f$ , the **mean** or **expected value** of  $X$  is given by

$$E(X) = \int_{-\infty}^{\infty} f(x) \, dx$$

provided the integral exists.

- If  $f(x) = 0$  for all  $x \notin [c, d]$ , then

$$E(X) = \int_c^d f(x) \, dx$$

- This definition is consistent with the definition provided in the “Expected Value” section of the “Population parameters” box. Where appropriate, substitute an integral for the summation symbol and  $f$  in place of  $p$ .

## Percentiles

- The value  $p$  of  $X$  which is the solution of an equation of the form

$$\int_{-\infty}^p f(x) \, dx = q$$

is called a **percentile** of the distribution.

- For example, the 75<sup>th</sup> percentile is the value  $p$  found by taking  $q = 75\% = 0.75$ .

## The median

- The **median** is another measure of centre for a continuous probability distribution.
- The median,  $m$ , of a continuous random variable  $X$  is the value of  $X$  such that

$$\int_{-\infty}^m f(x) \, dx = 0.5$$

- It is also known as the 50<sup>th</sup> percentile.

## Interquartile range

- The **interquartile range** is the range of the middle 50% of the distribution; it is the difference between the 75<sup>th</sup> percentile (also known as Q3) and the 25<sup>th</sup> percentile (also known as Q1).

$$\text{IQR} = b - a$$

where  $a$  and  $b$  are such that

$$\int_{-\infty}^a f(x) \, d(x) = 0.25 \quad \text{and} \quad \int_{-\infty}^b f(x) \, d(x) = 0.75$$

## The variance of a continuous probability distribution

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= E[(X - \mu)^2] \\ &= \int_{-\infty}^{\infty} [(x - \mu)^2 \cdot f(x)] \, dx \end{aligned}$$

## The standard deviation of a continuous probability distribution

$$\text{sd}(X) = \sqrt{\text{Var}(X)}$$

## The probability density function of $aX + b$

- If the probability density function of  $X$  has the rule  $f(x)$ , then the probability density function of  $aX + b$  is  $\frac{1}{a} \cdot f\left(\frac{x-b}{a}\right)$  and is described by the transformation

$$(x, y) \rightarrow \left(ax + b, \frac{y}{a}\right)$$

## The standard normal distribution

- A random variable  $Z$  with the standard normal distribution has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot x^2}$$

- The standard normal distribution has mean  $\mu = 0$  and standard deviation  $\sigma = 1$ .

## Transformations of normal distributions

- If  $X$  is a **normally distributed random variable** with mean  $\mu$  and standard deviation  $\sigma$ , then the probability density function of  $X$  is given by

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2}$$

and

$$\Pr(X \leq a) = \Pr\left(Z \leq \frac{a-\mu}{\sigma}\right)$$

where  $Z$  is the random variable of the standard normal distribution.

- The transformation which maps the graph of a normal distribution with mean  $\mu$  and standard deviation  $\sigma$  to the graph of the standard normal distribution is as follows:

$$(x, y) \rightarrow \left(\frac{x-\mu}{\sigma}, \sigma y\right)$$

- Conversely, the transformation which maps the graph of the standard normal distribution to the graph of a normal distribution with mean  $\mu$  and standard deviation  $\sigma$  is as follows:

$$(x, y) \rightarrow \left(\sigma x + \mu, \frac{y}{\sigma}\right)$$

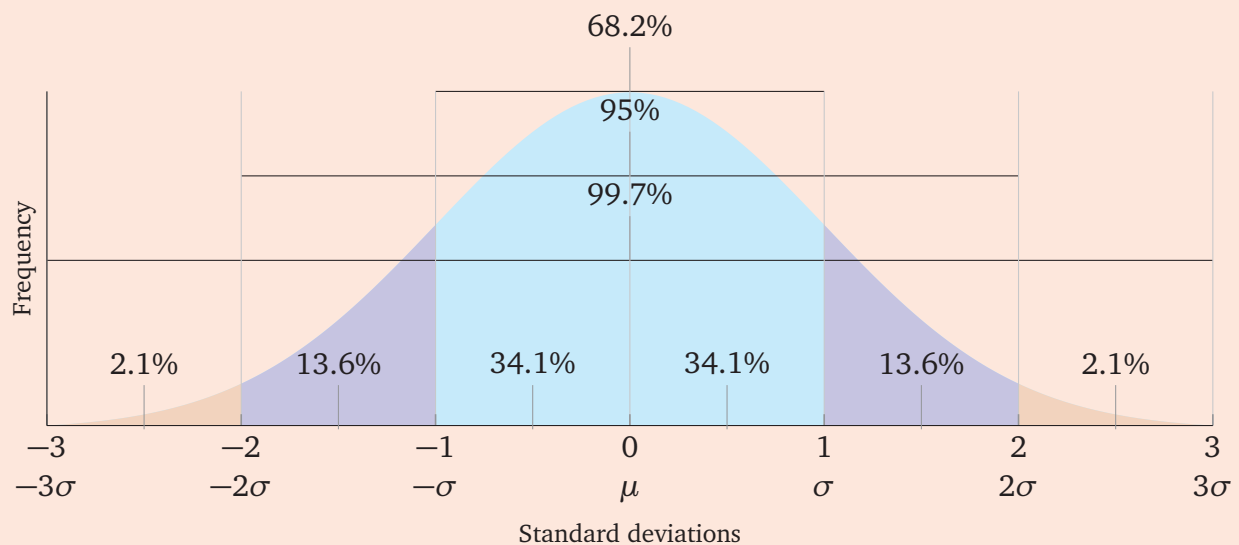
- These transformations are “area preserving”.

## Symmetry properties of the standard normal distribution

- $\Pr(Z > a) = 1 - \Pr(Z \leq a)$
- $\Pr(Z < -a) = \Pr(Z > a)$
- $\Pr(-a < Z < a) = 1 - 2\Pr(Z \geq a)$   
 $= 1 - 2\Pr(Z \leq -a)$

## Empirical formulas

- For a normally distributed random variable, approximately:
  - 68% of values lie within one standard deviation of the mean, which is the interval  $[\mu - \sigma, \mu + \sigma]$ .
  - 95% of values lie within two standard deviation of the mean, which is the interval  $[\mu - 2\sigma, \mu + 2\sigma]$ .
  - 99.7% of values lie within three standard deviation of the mean, which is the interval  $[\mu - 3\sigma, \mu + 3\sigma]$ .



## Normal approximation of a binomial distribution

- If  $n$  is sufficiently large, the binomial random variable  $X$  will be approximately normally distributed, with a mean of  $\mu = np$  and a standard deviation of  $\sigma = \sqrt{np(1-p)}$ .
- One rule of thumb is that  $np > 5$  **and**  $n(1-p) > 5$  for a satisfactory approximation.

## 9 Sampling

### Sample

- A sample of size  $n$  is called a **simple random sample** if it is selected from the population in such a way that every subset of size  $n$  has an equal chance of being chosen as the sample.
- In particular, every member of the population must have an equal chance of being included in the sample.

### Population and sample proportions

- The **population proportion**  $p$  is a **population parameter**; its value is constant. This is also what is used as the value for the probability of success when calculating  $\hat{p}$  from a binomial distribution.

$$p = \frac{\text{number in population with attribute}}{\text{population size}}$$

- The **sample proportion**  $\hat{p}$  is a **sample statistic**; its value is not constant, but varies from sample to sample.

$$\hat{p} = \frac{\text{number in sample with attribute}}{\text{sample size}} = \frac{X}{n}$$

where  $X \sim \text{Bi}(n, p)$ ,  $p$  = probability of a member of the population having the desired attribute.

- Since  $\hat{p}$  varies according to the contents of the random samples, we can consider the sample proportions  $\hat{p}$  as being the values of a random variable, which we will denote by  $\hat{P}$ .

### Hypergeometric distribution

- The **hypergeometric distribution** is a *discrete* probability distribution that describes the probability of  $k$  successes (random draws for which the object drawn has a specified/desired feature) in  $n$  draws (a sample size of  $n$ ), **without replacement** (the next draw is happening from a population size of  $N - 1$ ) from a finite population of size  $N$  that contains exactly  $K$  objects with that feature, wherein each draw is either a success or failure (a Bernoulli trial).
- The probability density function of such a distribution is as described:

$$p_X(k) = \Pr(X = k) = \frac{\binom{K}{k} \cdot \binom{N-K}{n-k}}{\binom{N}{n}}$$

- This is denoted as  $X \sim \text{Hypergeometric}(N, K, n)$ .
- This distribution is converse to the binomial distribution, which describes the probability of  $k$  successes in  $n$  draws *with replacement*.



## Types of distributions for calculating $\hat{p}$

- If the sample is being taken **without replacement**, then we can say that  $\hat{p} = \frac{X}{n}$ , where  $X \sim \text{Hypergeometric}(N, K, n)$  ( $N$  is the population size,  $K$  is the number of members of the population with the desired/specified feature, and  $n$  is the sample size).
  - This is typically done with small, countable population sizes (e.g., marbles in a bag, etc.).
- If the sample is being taken **with replacement**,  $\hat{p} = \frac{X}{n}$ , where  $X \sim \text{Bi}(n, p)$  ( $n$  is the sample size, and  $p$  is the probability of selecting  $x$  member(s) out of the population which possess the desired/specified feature (i.e., a success) (where  $x = 0, 1, \dots, n$ )).
  - This is typically done with large populations consisting of an uncountable number of members (i.e., a country). Normally, this is because you are not given  $N$ , the population size, but just  $p$ , which can be used to work out  $\hat{p}$ .
- The distribution of a statistic which is calculated from a sample (such as the sample proportion) has a special name — it is called a **sampling distribution**.

## Population parameters for the sample

- If we are selecting a random sample of size  $n$  from a *large* population (binomial distribution), then the mean and standard deviation of the sample proportion  $\hat{P}$  are given by:

$$E(\hat{P}) = p \quad \text{and} \quad \text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

- The standard deviation of a sample statistic is called the **standard error**.

## Normal approximation of the sample distribution

- When the sample size  $n$  is *large*, the sample proportion  $\hat{P}$  has an approximately normal distribution, with mean  $\mu = p$  and standard deviation  $\sigma = \sqrt{\frac{p(1-p)}{n}}$ .
  - Approximate the sample distribution to a normal distribution when asked to find  $n$ , the sample size and when given  $p$ , and  $\text{Pr}(\hat{P} > a)$  (or anything of the sort). To do this, you may use the `invNorm(Area,  $\mu$ ,  $\sigma$ )` function on your CAS.

## Inference of the population

### Point estimates

- The value of the sample proportion  $\hat{p}$  can be used to estimate the population proportion  $p$ .
- Since this is a single-valued estimate, it is called a **point estimate** of  $p$ .

### Interval estimates (confidence intervals)

- The value of the sample proportion  $\hat{p}$  obtained from a single sample is going to change from sample to sample.
- What is required is an interval that we are reasonably sure contains the parameter value  $p$ .
- An **interval estimate** for the population proportion  $p$  is called a **confidence interval** for  $p$ .

## Finding confidence intervals

- When the sample size  $n$  is *large* (both  $np$  and  $n(1-p)$  must be larger than 5), the sample proportion  $\hat{P}$  has an approximately normal distribution with  $\mu = p$  and  $\sigma = \sqrt{\frac{p(1-p)}{n}}$ .
- $\therefore Z_{\hat{p}} = \frac{\hat{p} - \mu_{\hat{p}}}{\sigma_{\hat{p}}} = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$ , where  $Z_{\hat{p}}$  is the standard normal variable of the sample distribution  $\hat{P}$ .
- The **standardised**  $a\%$  confidence interval can be found using:

$$\begin{aligned}\Pr(-c < Z_{\hat{p}} < c) &= a, 0 < a < 1 \\ \implies \Pr(Z_{\hat{p}} < c) &= \frac{1-a}{2} + a, 0 < a < 1 \\ &= \frac{a+1}{2}\end{aligned}$$

This is thanks to the symmetry properties of the approximated normal distribution. The `invNorm(Area,  $\mu$ ,  $\sigma$ )` function on your CAS can be used to find the value of  $c$ .

- Remember, the sample proportion  $\hat{p}$  **lies in the middle** of the confidence interval.
- Rearranging this (to the **un-standardised** version), we get the formula given on the formula sheet:

$$C\% \text{ confidence interval} = \left( \hat{p} - k \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}, \hat{p} + k \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}} \right)$$

where  $k$  is such that  $\Pr(-k < Z_{\hat{p}} < k) = \frac{C}{100}$ .

- The **1-prop z interval** function can be used on the CAS to find the un-standardised C.I. (found in Menu  $\rightarrow$  Statistics  $\rightarrow$  Confidence Intervals  $\rightarrow$  1-Prop z interval).

### **$k$ values for confidence intervals**

- 68.2% C.I.:  $k = \text{invNorm}(0.841, 0, 1) = 0.99857627845453 \approx 0.9986$
- 90% C.I.:  $k = \text{invNorm}(0.95, 0, 1) = 1.6448536259066 \approx 1.6449$
- 95% C.I.:  $k = \text{invNorm}(0.975, 0, 1) = 1.9599639859915 \approx 1.9600$
- 99% C.I.:  $k = \text{invNorm}(0.995, 0, 1) = 2.5758293030016 \approx 2.5758$
- 99.7% C.I.:  $k = \text{invNorm}(0.9985, 0, 1) = 2.9677379271247 \approx 2.9677$

### **Margin of error**

- The **distance** between the sample estimate and the endpoints of the confidence interval is called the **margin of error** ( $M$ ).
- For a  $C\%$  confidence interval, the margin of error  $M$  is as such:

$$M = k \cdot \sqrt{\frac{\hat{p} \cdot (1 - \hat{p})}{n}}$$

where  $k$  is the value corresponding to the confidence interval percentage.

- If  $p^*$  is an estimated value for the population proportion  $p$ , then
  - a  $C\%$  confidence interval for a population proportion  $p$  will have margin of error approximately equal to a specified value of  $M$  when the sample size is:

$$n = \left(\frac{k}{M}\right)^2 \cdot p^* \cdot (1 - p^*)$$

where  $M$  is the margin of error and  $k$  is the value associated with the  $C\%$  confidence interval.

## Part II

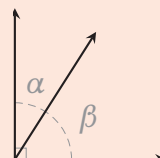
# Extension

## 1 Angle relationships

### Complementary and supplementary angles

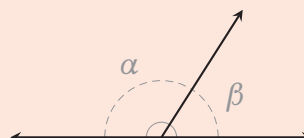
#### Complimentary angles

- In this case, the angles  $\alpha$  and  $\beta$  are complementary, as  $\alpha + \beta = 90^\circ$ .



#### Supplementary angles

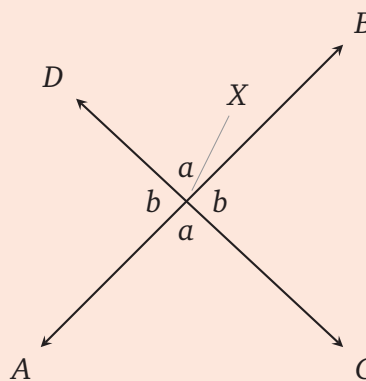
- In this case, the angles  $\alpha$  and  $\beta$  are supplementary, as  $\alpha + \beta = 180^\circ$ .



### Angles formed by intersecting lines

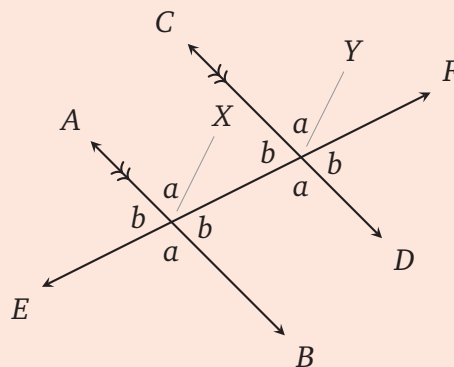
#### Vertically opposite angles

- In this case,  $\angle AXC = \angle DXB$  and  $\angle DXA = \angle BXC$ .



## Angles formed by a transversal

- A **transversal** is a line which crosses two or more lines.
- In this case, the lines  $AB$  and  $CD$  are **parallel**, which is denoted by  $AB \parallel CD$ .
- **Vertically opposite angles:**
  - $\angle CYF = \angle XYD$
  - $\angle YFD = \angle CYX$
  - $\angle AXY = \angle EXB$
  - $\angle AXE = \angle YXB$
- **Alternate interior angles:**
  - $\angle AXY = \angle DYX$
  - $\angle CYX = \angle BXY$
- **Alternate exterior angles:**
  - $\angle FYD = \angle AXE$
  - $\angle FYC = \angle EXB$
- **Corresponding angles:**
  - $\angle FYD = \angle YXB$
  - $\angle FYC = \angle YXA$
  - $\angle EXB = \angle XYD$
  - $\angle EXA = \angle XYC$
- **Same side interior angles (supplementary):**
  - $\angle XYD + \angle YXB = 180^\circ$
  - $\angle AXY + \angle CYX = 180^\circ$



## 2 Counting methods

### Pascal's triangle

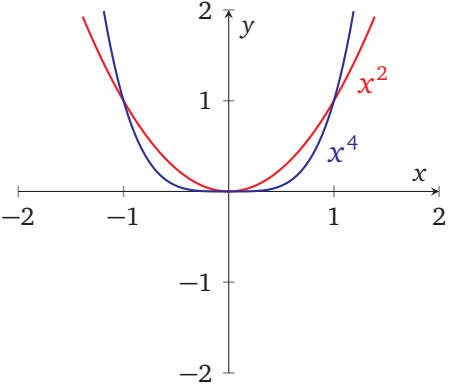
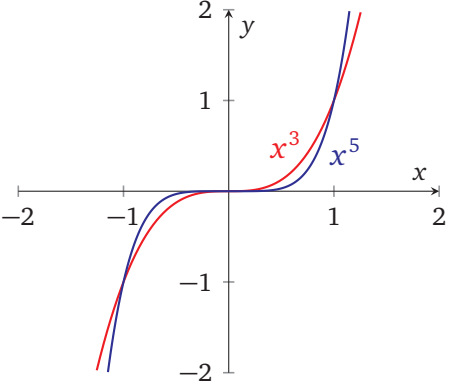
■ Featured below is **pascal's triangle**, in which each row  $n$  and column  $k$  correspond to  $\binom{n}{k}$ .

• Binomial expansion:  $(a + b)^n = \sum_{k=0}^n \binom{n}{k} \cdot a^{n-k} \cdot b^k$  and  $(qa + b)^n = \sum_{k=0}^n \binom{n}{k} \cdot (q \cdot a)^{n-k} \cdot b^k$

| $n$ |   |    |     |     |      |      |      |      |      |      |      |      |     |     |    |    |
|-----|---|----|-----|-----|------|------|------|------|------|------|------|------|-----|-----|----|----|
| 0   | 1 |    |     |     |      |      |      |      |      |      |      |      |     |     |    |    |
| 1   | 1 | 1  |     |     |      |      |      |      |      |      |      |      |     |     |    |    |
| 2   | 1 | 2  | 1   |     |      |      |      |      |      |      |      |      |     |     |    |    |
| 3   | 1 | 3  | 3   | 1   |      |      |      |      |      |      |      |      |     |     |    |    |
| 4   | 1 | 4  | 6   | 4   | 1    |      |      |      |      |      |      |      |     |     |    |    |
| 5   | 1 | 5  | 10  | 10  | 5    | 1    |      |      |      |      |      |      |     |     |    |    |
| 6   | 1 | 6  | 15  | 20  | 15   | 6    | 1    |      |      |      |      |      |     |     |    |    |
| 7   | 1 | 7  | 21  | 35  | 35   | 21   | 7    | 1    |      |      |      |      |     |     |    |    |
| 8   | 1 | 8  | 28  | 56  | 70   | 56   | 28   | 8    | 1    |      |      |      |     |     |    |    |
| 9   | 1 | 9  | 36  | 84  | 126  | 126  | 84   | 36   | 9    | 1    |      |      |     |     |    |    |
| 10  | 1 | 10 | 45  | 120 | 210  | 252  | 210  | 120  | 45   | 10   | 1    |      |     |     |    |    |
| 11  | 1 | 11 | 55  | 165 | 330  | 462  | 462  | 330  | 165  | 55   | 11   | 1    |     |     |    |    |
| 12  | 1 | 12 | 66  | 220 | 495  | 792  | 924  | 792  | 495  | 220  | 66   | 12   | 1   |     |    |    |
| 13  | 1 | 13 | 78  | 286 | 715  | 1287 | 1716 | 1716 | 1287 | 715  | 286  | 78   | 13  | 1   |    |    |
| 14  | 1 | 14 | 91  | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 | 1001 | 364  | 91  | 14  | 1  |    |
| 15  | 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 | 3003 | 1365 | 455 | 105 | 15 | 1  |
| $k$ | 0 | 1  | 2   | 3   | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12  | 13  | 14 | 15 |

3 Base functions

Table 1: Base graphs

| Rule             | Implied domain | Range         | Parity | Graph  | Inverse                  | Asymptote |
|------------------|----------------|---------------|--------|--|--------------------------|-----------|
| $x^n, n$ is even | $\mathbb{R}$   | $[0, \infty)$ | Even   |   | $\sqrt[n]{x}, n$ is even | None      |
| $x^n, n$ is odd  | $\mathbb{R}$   | $\mathbb{R}$  | Odd    |  | $\sqrt[n]{x}, n$ is odd  | None      |

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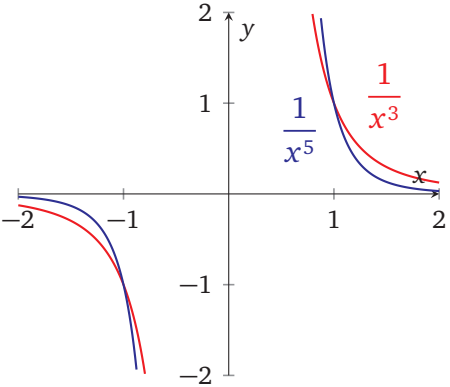
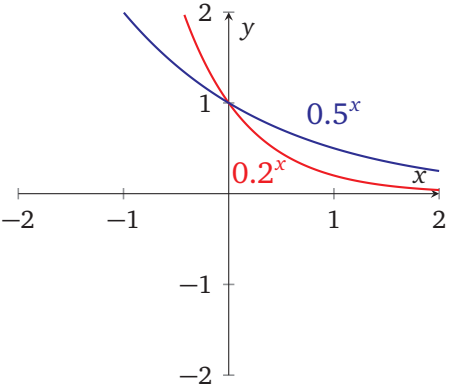
Table 1: Base graphs (Continued)

| Rule                               | Implied domain               | Range          | Parity | Graph   | Inverse                     | Asymptote          |
|------------------------------------|------------------------------|----------------|--------|---|-----------------------------|--------------------|
| $\frac{1}{x}$                      | $\mathbb{R} \setminus \{0\}$ | $\mathbb{R}$   | Odd    | <p>A Cartesian coordinate system showing two hyperbolas. The blue curve is labeled <math>\frac{1}{2x}</math> and the red curve is labeled <math>\frac{1}{x}</math>. Both curves have the x-axis and y-axis as asymptotes. The x-axis is labeled with -2, -1, 1, 2. The y-axis is labeled with -2, -1, 1, 2.</p>                                   | $\frac{1}{x}$               | $y = 0$<br>$x = 0$ |
| $\frac{1}{x^n}, n \text{ is even}$ | $\mathbb{R} \setminus \{0\}$ | $\mathbb{R}^+$ | Even   | <p>A Cartesian coordinate system showing two curves. The blue curve is labeled <math>\frac{1}{x^4}</math> and the red curve is labeled <math>\frac{1}{x^2}</math>. Both curves are symmetric about the y-axis and have the x-axis and y-axis as asymptotes. The x-axis is labeled with -2, -1, 1, 2. The y-axis is labeled with -2, -1, 1, 2.</p> | $\pm \frac{1}{\sqrt[n]{x}}$ | $y = 0$<br>$x = 0$ |

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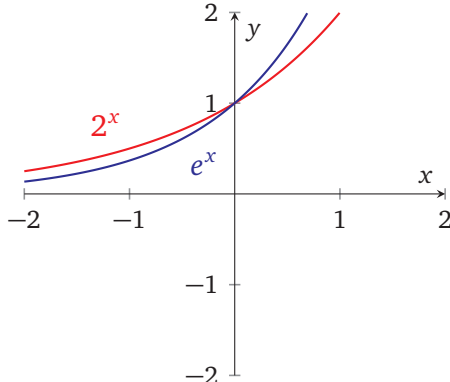
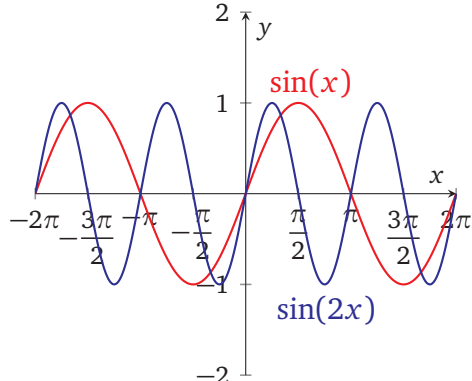


Table 1: Base graphs (Continued)

| Rule                              | Implied domain               | Range                        | Parity | Graph   | Inverse                 | Asymptote          |
|-----------------------------------|------------------------------|------------------------------|--------|---|-------------------------|--------------------|
| $\frac{1}{x^n}, n \text{ is odd}$ | $\mathbb{R} \setminus \{0\}$ | $\mathbb{R} \setminus \{0\}$ | Odd    |  <p>The graph shows two curves on a Cartesian coordinate system. The x-axis ranges from -2 to 2, and the y-axis ranges from -2 to 2. A blue curve, labeled <math>\frac{1}{x^5}</math>, and a red curve, labeled <math>\frac{1}{x^3}</math>, both pass through the points (-1, -1) and (1, 1). They both approach the y-axis (x=0) as y goes to positive or negative infinity and the x-axis (y=0) as x goes to positive or negative infinity. The blue curve is steeper than the red curve for  x  &gt; 1.</p> | $\frac{1}{\sqrt[n]{x}}$ | $y = 0$<br>$x = 0$ |
| $a^x, 0 < a < 1$                  | $\mathbb{R}$                 | $\mathbb{R}^+$               | None   |  <p>The graph shows two exponential decay curves on a Cartesian coordinate system. The x-axis ranges from -2 to 2, and the y-axis ranges from -2 to 2. A blue curve, labeled <math>0.5^x</math>, and a red curve, labeled <math>0.2^x</math>, both pass through the point (0, 1). They both approach the x-axis (y=0) as x goes to positive infinity. The blue curve is above the red curve for all x. The red curve decreases more rapidly as x increases.</p>   | $\sqrt[a]{x}$           | $y = 0$            |

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Table 1: Base graphs (Continued)

| Rule         | Implied domain | Range          | Parity | Graph  | Inverse        | Asymptote |
|--------------|----------------|----------------|--------|--|----------------|-----------|
| $a^x, a > 1$ | $\mathbb{R}$   | $\mathbb{R}^+$ | None   |   | $\sqrt[a]{x}$  | $y = 0$   |
| $\sin(x)$    | $\mathbb{R}$   | $[-1, 1]$      | Odd    |  | $\sin^{-1}(x)$ | None      |

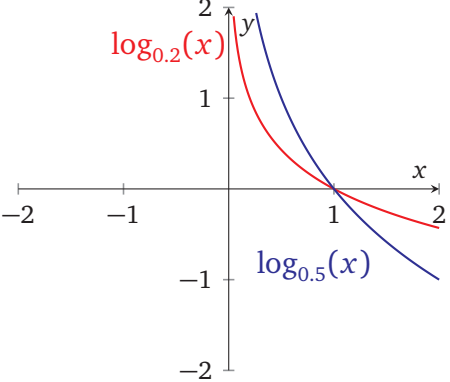
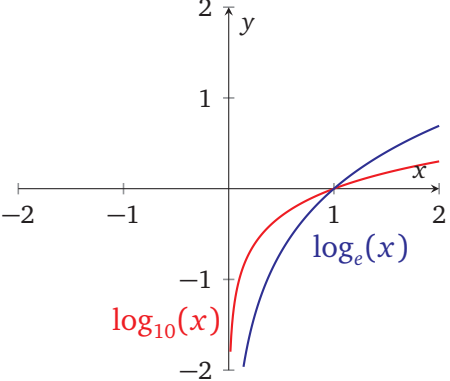
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Table 1: Base graphs (Continued)

| Rule      | Implied domain                                 | Range        | Parity | Graph  | Inverse        | Asymptote                 |
|-----------|--|--------------|--------|--|----------------|---------------------------|
| $\cos(x)$ | $\mathbb{R}$                                   | $[-1, 1]$    | Even   | <p>The graph shows two cosine functions on a coordinate plane. The x-axis is labeled from <math>-2\pi</math> to <math>2\pi</math> with major ticks at <math>-\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi</math>. The y-axis is labeled from <math>-2</math> to <math>2</math> with major ticks at <math>-2, -1, 0, 1, 2</math>. A red curve represents <math>\cos(x)</math> and a blue curve represents <math>\cos(2x)</math>. Both curves are symmetric about the y-axis. The red curve has a period of <math>2\pi</math>, while the blue curve has a period of <math>\pi</math>.</p>   | $\cos^{-1}(x)$ | None                      |
| $\tan(x)$ | $x \neq \frac{(2n-1)\pi}{2}, n \in \mathbb{Z}$ | $\mathbb{R}$ | Odd    | <p>The graph shows two tangent functions on a coordinate plane. The x-axis is labeled from <math>-2\pi</math> to <math>2\pi</math> with major ticks at <math>-\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi</math>. The y-axis is labeled from <math>-2</math> to <math>2</math> with major ticks at <math>-2, -1, 0, 1, 2</math>. Vertical asymptotes are shown as dashed lines at <math>x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}</math>. A red curve represents <math>\tan(x)</math> and a blue curve represents <math>\tan(2x)</math>. Both curves are symmetric about the origin. The red curve has a period of <math>\pi</math>, while the blue curve has a period of <math>\frac{\pi}{2}</math>.</p> | $\tan^{-1}(x)$ | $x = \frac{(2n-1)\pi}{2}$ |

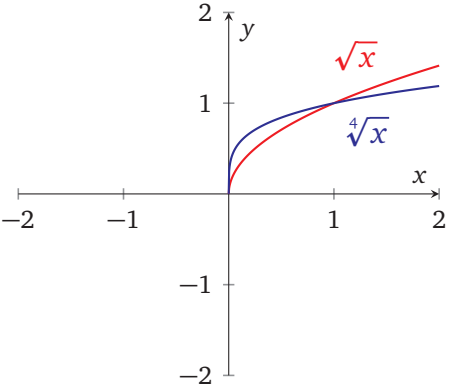
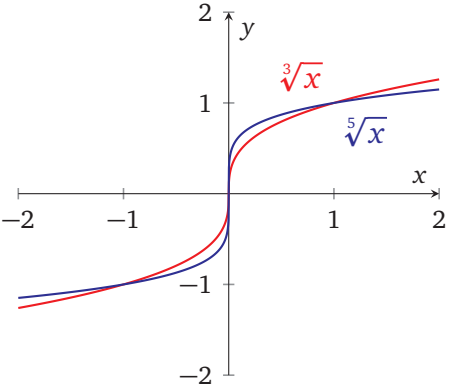
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Table 1: Base graphs (Continued)

| Rule                        | Implied domain | Range        | Parity | Graph  | Inverse          | Asymptote |
|-----------------------------|----------------|--------------|--------|--|------------------|-----------|
| $\log_a(x),$<br>$0 < a < 1$ | $\mathbb{R}^+$ | $\mathbb{R}$ | None   |   | $a^x, 0 < a < 1$ | $y = 0$   |
| $\log_a(x), a > 1$          | $\mathbb{R}^+$ | $\mathbb{R}$ | None   |  | $a^x, a > 1$     | $y = 0$   |

Continued on next page

Table 1: Base graphs (Continued)

| Rule                             | Implied domain            | Range          | Parity | Graph   | Inverse | Asymptote |
|----------------------------------|---------------------------|----------------|--------|---|---------|-----------|
| $\sqrt[n]{x}, n \text{ is even}$ | $\mathbb{R}^+ \cup \{0\}$ | $\mathbb{R}^+$ | None   |  <p>The graph shows two curves in the first quadrant of a Cartesian coordinate system. The x-axis ranges from -2 to 2, and the y-axis ranges from -2 to 2. A red curve, labeled <math>\sqrt{x}</math>, and a blue curve, labeled <math>\sqrt[4]{x}</math>, both start at the origin (0,0) and increase as x increases. The red curve is above the blue curve for x &gt; 1, and below it for 0 &lt; x &lt; 1. They intersect at (1,1).</p>  | $x^n$   | None      |
| $\sqrt[n]{x}, n \text{ is odd}$  | $\mathbb{R}$              | $\mathbb{R}$   | None   |  <p>The graph shows two curves passing through the origin in a Cartesian coordinate system. The x-axis ranges from -2 to 2, and the y-axis ranges from -2 to 2. A red curve, labeled <math>\sqrt[3]{x}</math>, and a blue curve, labeled <math>\sqrt[5]{x}</math>, both pass through the origin (0,0) and (1,1). They also pass through (-1,-1). The red curve is above the blue curve for x &gt; 1, and below it for 0 &lt; x &lt; 1. They intersect at (1,1) and (-1,-1).</p> | $x^n$   | None      |