ABUBAKAR TAFAWA BALEWA UNIVERSITY, BAUCHI
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICAL SCIENCES

MTH111

Elementary Algebra I

Lecture note

Course Outline:

Trigonometric functions*: Radian measure, Trigonometric identities, Sum, Difference and Product formulae, Laws of Sine and Cosine, inverse trigonometric functions, Solution of trigonometric equations.*

Exponential functions: *Definition of* a^x *for any positive number a and any real number* x*, graphs of exponential functions, Laws of Exponents(indices), the number* e*, Natural exponential function.*

Logarithmic functions: *Definition of* $log_a x$ *for any positive number a and any positive real number* x*, graphs of logarithmic functions, Laws of Logarithms, the number* e*, Natural logarithmic function.*

Algebraic functions: *Polynomials, Division algorithms, Long division, Synthetic division, Factor theorem, remainder theorem.*

Rational functions: *Asymptotes, Partial fraction decomposition, Roots of a rational functions, finding the domain.*

Complex numbers: *Representation in the plane, Sum, Difference, Product and Quotient of complex numbers, Modulus and Argument of complex numbers, Complex conjugate and its properties, Polar representation of complex numbers, Unit circle,* n^{th} *root of complex numbers, De Mover's Theorem, Zero of polynomials, Quadratic formula.*

Section A

Trigonometric functions

Course Outline:

Trigonometric functions: Radian measure, Trigonometric identities, Sum, Difference and Product formulae, Laws of Sine and Cosine, inverse trigonometric functions, Solution of trigonometric equations.

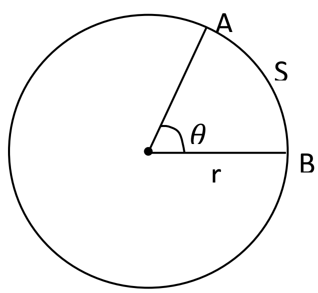
Trigonometric functions:

Measurement of an angle: Angles are generally measured in degree or in radian.

Definition1: A degree is measure of an angle whose vertex is at the center of a circle, and that it intersect an arc equal in length to ($\frac{1}{360}$) of the circumference.

Definition2: A radian is a measure of an angle whose vertex is at the center of a circle, and that it intersect an arc equal in length to the radius of the circle.

Relationship between degree and radian: Let consider a circle ABA with center o, and radius r, and that it subtend an angleθ, and let S be the length of the arc $\widehat{AB}$.



⇨ $$S = \frac{\theta}{360^o} 2\pi r \qquad . . . (1)$$

From the definition of radian we have

$$\theta = \frac{S}{r} rad. \qquad . . . (2)$$

Now if θ= 360^o, then equation(1) reduces to

$$S = 2\pi r \quad . \qquad . . . (3)$$

Substituting (3) in (2) we have

$$360^o = 2\pi \, rad. \qquad . . . (4)$$

From equation(4) we have.

$$1^o = \frac{\pi}{180} rad. \qquad . . . (5)$$

And

$$1 \text{ rad.} = \frac{180^o}{\pi} \qquad . . . (6)$$

Thus equation (5) and (6) form a basis for conversion from degree to radian and vice-versa.

Example

1) Express the following angles in radian.

$i)\ 30^o$ $ii)\ 120^o$ $iii)\ 270^o$ $iv)\ 315^o$

Solution:

$i)\ 1^o = \frac{\pi}{180} rad.$

Multiply both side by 30 we have

$30^o = \frac{\pi}{6} rad.$

$ii)\ 1^o = \frac{\pi}{180} rad.$

Multiply both side by 120 we have

$120^o = \frac{2\pi}{3} rad.$

$iii)\ 1^o = \frac{\pi}{180} rad.$rad.

Multiply both side by 270 we have

$270^o = \frac{3\pi}{2} rad.$

$iv)\ 1^o = \frac{\pi}{180} rad.$

Multiply both side by 315 we have

$315^o = \frac{7\pi}{4} rad.$

2) Express the following angles in degree.

$i)\ \frac{3\pi}{4} rad.$ $ii)\ \frac{\pi}{12} rad.$ $iii)\ \frac{3\pi}{2} rad.$ $iv)\ \frac{5\pi}{4} rad.$

Solution:

$i)\ 1rad = \frac{180^o}{\pi}$

Multiply both side by $\frac{3\pi}{4}$wehave

$\frac{3\pi}{4} rad. = 135^o$

$ii)\ 1rad = \frac{180^o}{\pi}$

Multiply both side by $\frac{\pi}{12}$ we have

$\frac{\pi}{12} rad. = 15^o$

$iii)\ 1rad = \frac{180^o}{\pi}$

Multiply both side by $\frac{3\pi}{2}$ we have

$\frac{3\pi}{2} rad. = 270^o$

$iv)\ 1rad = \frac{180^o}{\pi}$

Multiply both side by $\frac{5\pi}{4}$ we have

$\frac{5\pi}{4} rad. = 225^o$

Trigonometric ratios:

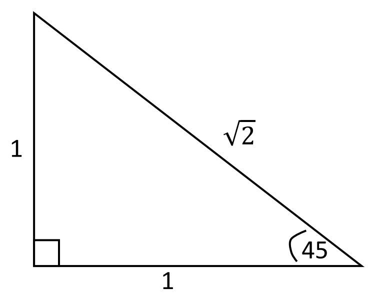
Recall the following:

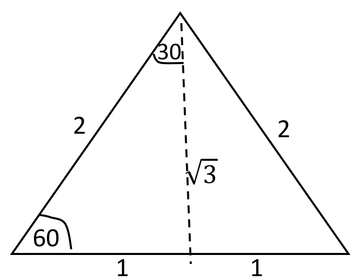
1. Trigonometric functions.
2. SOH, CAH, TOA and CAST, ACTS or ASTC.
3. Pythagoras Theorem.
4. Types of triangle and properties of triangles.

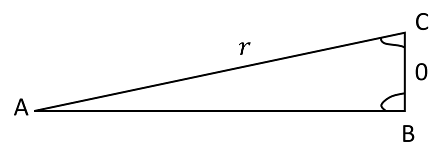
a) Trigonometric ratio some of special angles:

Definition3: Any angle multiple of 15^o including 0^o is called a special angle.

Note: The trigonometric ratios of special angles are frequently used, especially in the areas of Mechanics, Physics and Engineering, and therefore it is useful to have their values in surd form. The ratios however can be obtained exactly from consideration of the triangle as follows.







$\overline{AB}=\overline{AC}, = r$

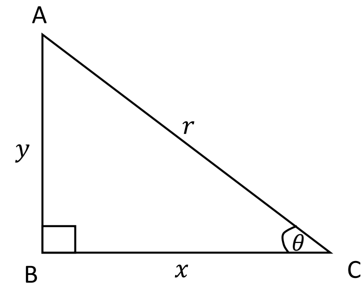
$A\hat{B}C = B\hat{C}A = 90^0$

$C\hat{A}B = 0^0$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0^o	0	1	0
30^o	½	$\sqrt{3}/2$	$1/\sqrt{2}$
45^o	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60^o	$\sqrt{3}/2$	½	$\sqrt{3}$
90^o	1	0	∞

b) Trigonometric ratio of acute angles:

Consider any triangle ABC.



$|AB| = y,\ |AB| = x,\ \ |AB| = r, r > 0.$

⇨$\sin\theta = \frac{y}{r}$,$\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$,

⇨ $\sin^2\theta = \left(\frac{y}{r}\right)^2, \cos^2\theta = \left(\frac{x}{r}\right)^2,\ \tan^2\theta = \left(\frac{y}{x}\right)^2$

⇨$\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, and $\cot\theta = \frac{x}{y}$,

⇨ $\csc^2\theta = \left(\frac{r}{y}\right)^2, \sec^2\theta = \left(\frac{r}{x}\right)^2,\ \cot^2\theta = \left(\frac{x}{y}\right)^2$

According to Pythagoras theorem,

$$x^2 + y^2 = r^2 \qquad ...\ (1)$$

Dividing (1) by r^2 on both sides

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

⇨ $$\boldsymbol{\sin^2\theta + \cos^2\theta = 1} \qquad ...\ (2)$$

Dividing (1) by y^2 on both sides

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \quad ⇨ \quad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

⇨ $$\boldsymbol{\cot^2\theta + 1 = \csc^2\theta} \qquad ...\ (3)$$

And dividing (1) by x^2 on both sides

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2} \quad ⇨ \quad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

⇨ $$\boldsymbol{1 + \tan^2\theta = \sec^2\theta} \qquad ...\ (4)$$

Equations (2), (3) and (4) are called Simple identities, and the relationship that exist between these three equations enable us to obtain the other trigonometric ratios if at least one is known.

Examples:

1) Evaluate $\sin\theta$ and $\tan\theta$, given that $\cos\theta = \frac{12}{13}$ and that θ is acute.

Solution:

$$\cos\theta = \frac{12}{13} \quad \Rightarrow \quad \cos^2\theta = \frac{144}{169}, \quad \Rightarrow \quad \sin^2\theta = 1 - \frac{144}{169} = \frac{25}{169} \quad \Rightarrow \quad \sin\theta = \frac{5}{13}$$

Also $\cos\theta = \frac{12}{13} \quad \Rightarrow \quad \sec\theta = \frac{13}{12} \quad \Rightarrow \quad \sec^2\theta = \frac{169}{144} \quad \Rightarrow \quad \tan^2\theta = \frac{169}{144} - 1 = \frac{25}{144}$

$\Rightarrow \quad \tan\theta = \frac{5}{12}$

2) Evaluate $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{3}{2}$ and that $0 \le \theta \le \frac{\pi}{2}$.

Solution:

$$\tan\theta = \frac{3}{2} \quad \Rightarrow \quad \tan^2\theta = \frac{9}{4}, \quad \Rightarrow \quad \sec^2\theta = \frac{9}{4} + 1 = \frac{13}{4} \quad \Rightarrow \quad \sec\theta = \frac{\sqrt{13}}{2} \Rightarrow \quad \cos\theta = \frac{2}{\sqrt{13}}$$

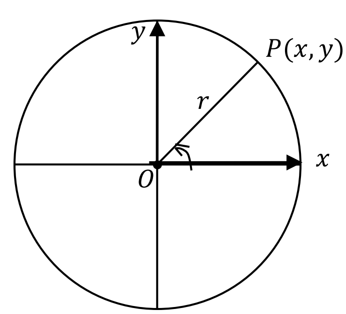
Also $\cos\theta = \frac{2}{\sqrt{13}} \quad \Rightarrow \quad \cos^2\theta = \frac{4}{13} \quad \Rightarrow \quad \sin^2\theta = 1 - \frac{4}{13} \quad \Rightarrow \quad \sin\theta = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$

c) Trigonometric ratio of general angles:

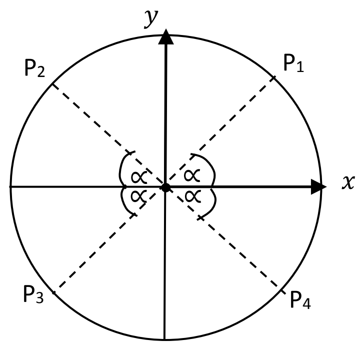
Definition: General angle is an angle of any size, including negative angles.

Note: Here, we shall define the trigonometric ratio in such a way that it will be applicable to angle of any size including negative angles.

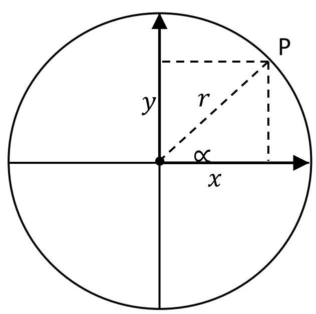
Let (x, y)be the Cartesian coordinate of P, and let $|OP| = r$, and $0 \le \theta \le \frac{\pi}{2}$.



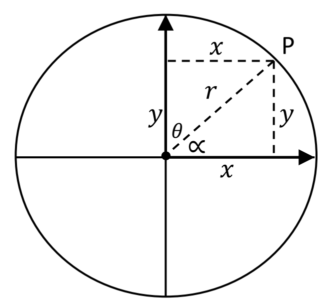
Definition: The acute angle between line |OP| and the horizontal axis (x-axis) is called the basic angle.



$i)$ *Consider the first quadrant:*



(a)



(b)

$a)$ $\sin\alpha = \sin\theta,$ $\cos\alpha = \cos\theta$ and $\tan\alpha = \tan\theta$ . . . (1)

$b)$ $\sin(90^o - \theta) = \sin\alpha,$ $\cos(90^o - \theta) = \cos\alpha$ and $\tan(90^o - \theta) = \tan\alpha$

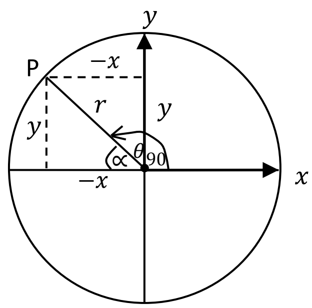
But $\sin\alpha = \frac{y}{r}$ *and* $\cos\theta = \frac{y}{r}$ $\Rightarrow$ $\sin\alpha = \cos\theta$

$\cos\alpha = \frac{x}{r}$ *and* $\sin\theta = \frac{x}{r}$ $\Rightarrow$ $\cos\alpha = \sin\theta$

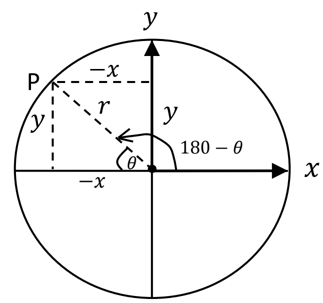
$\tan\alpha = \frac{y}{x}$ *and* $\tan\theta = \frac{x}{y}$ $\Rightarrow$ $\cot\theta = \frac{y}{x}$ $\Rightarrow$ $\tan\alpha = \cot\theta$

$\Rightarrow$ $\boldsymbol{\sin(90^o - \theta) = \cos\theta,\ \cos(90^o - \theta) = \sin\theta,\ \tan(90^o - \theta) = \cot\theta}$ . . . (2)

ii) *Consider the second quadrant:*



(a)



(b)

$a)$ $\sin(90^o+\theta)=\sin\alpha,$ $\cos(90^o+\theta)=-\cos\alpha$ and $\tan(90^o+\theta)=-\tan\alpha$

But $\sin\alpha=\frac{y}{r}$ $\quad and \quad$ $\cos\theta=\frac{y}{r}$ $\Rightarrow$ $\sin\alpha=\cos\theta$

$\cos\alpha=\frac{-x}{r}$ $\quad and \quad$ $\sin\theta=\frac{-x}{r}$ $\Rightarrow$ $\cos\alpha=\sin\theta$

$\tan\alpha=\frac{y}{-x}$ $\quad and \quad$ $\tan\theta=\frac{-x}{y}$ $\Rightarrow$ $\cot\theta=\frac{y}{-x}$ $\Rightarrow$ $\tan\alpha=\cot\theta$

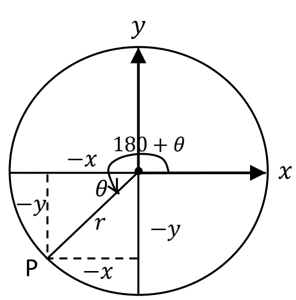
⇨ $\boldsymbol{\sin(90^o+\theta)=\cos\theta}$, $\boldsymbol{\cos(90^o+\theta)=-\sin\theta}$, $\boldsymbol{\tan(90^o+\theta)=-\cot\theta}$ . . . (3)

$b)$ $\sin(180^o-\theta)=\sin\alpha,$ $\cos(180^o-\theta)=-\cos\alpha$ and $\tan(180^o-\theta)=-\tan\alpha$

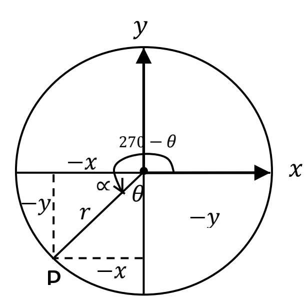
But $\alpha=\theta$

⇨ $\boldsymbol{\sin(180^o-\theta)=\sin\theta}$, $\boldsymbol{\cos(180^o-\theta)=-\cos\theta}$, $\boldsymbol{\tan(180^o-\theta)=-\tan\theta}$ . . (4)

iii) *Consider the third quadrant:*



(a)



(b)

$a)$ $\sin(180^o+\theta)=-\sin\alpha,$ $\cos(180^o+\theta)=-\cos\alpha$ and $\tan(180^o+\theta)=\tan\alpha$

But $\alpha=\theta$

⇨ $\boldsymbol{sin(180^o+\theta)=-sin\,\theta}$, $\boldsymbol{cos(180^o+\theta)=-cos\,\theta}$, $\boldsymbol{tan(180^o+\theta)=tan\,\theta}$ . . *(5)*

$b)$ $\sin(270^o-\theta)=-\sin\alpha,$ $\cos(270^o-\theta)=-\cos\alpha$ and $\tan(270^o-\theta)=\tan\alpha$

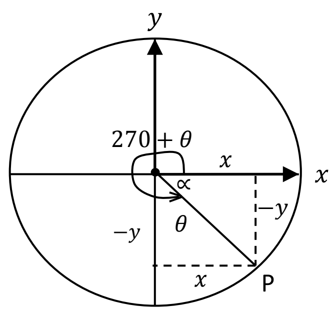
But $\quad \sin\alpha = \frac{-y}{r} \quad and \quad \cos\theta = \frac{-y}{r} \qquad \Rightarrow \qquad \sin\alpha = \cos\theta$

$\cos\alpha = \frac{-x}{r} \quad and \quad \sin\theta = \frac{-x}{r} \qquad \Rightarrow \qquad \cos\alpha = \sin\theta$

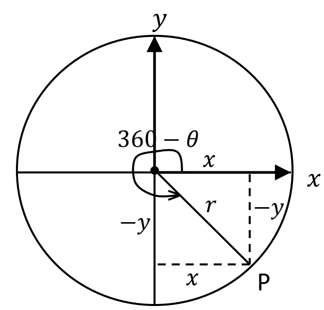
$\tan\alpha = \frac{y}{x} \quad and \quad \tan\theta = \frac{x}{y} \qquad \Rightarrow \qquad \cot\theta = \frac{y}{x} \qquad \Rightarrow \quad \tan\alpha = \cot\theta$

$\Rightarrow \qquad \boldsymbol{sin(270^o - \theta) = -cos\,\theta,\ cos(270^o - \theta) = -sin\,\theta,\ tan(270^o - \theta) = cot\,\theta} \quad ...$ ***(6)***

$iv)$*Consider the fourth quadrant:*



(a)



(b)

$a)\quad \sin(270^o + \theta) = -\sin\alpha, \quad \cos(270^o + \theta) = \cos\alpha$ and $\tan(270^o + \theta) = -\tan\alpha$

But $\quad \sin\alpha = \frac{-y}{r} \quad and \quad \cos\theta = \frac{-y}{r} \qquad \Rightarrow \qquad \sin\alpha = \cos\theta$

$\cos\alpha = \frac{x}{r} \quad and \quad \sin\theta = \frac{x}{r} \qquad \Rightarrow \qquad \cos\alpha = \sin\theta$

$\tan\alpha = \frac{-y}{x} \quad and \quad \tan\theta = \frac{x}{-y} \qquad \Rightarrow \quad \cot\theta = \frac{-y}{x} \qquad \Rightarrow \qquad \tan\alpha = \cot\theta$

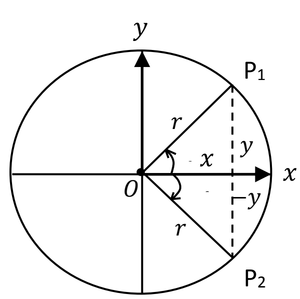
$\Rightarrow \quad \boldsymbol{sin(270^o + \theta) = -cos\,\theta, cos(270^o + \theta) = sin\,\theta,\ tan(270^o + \theta) = -cot\,\theta} \qquad ...$ *(7)*

$b)\quad \sin(360^o - \theta) = \sin\alpha, \quad \cos(360^o - \theta) = -\cos\alpha$ and $\tan(360^o - \theta) = -\tan\alpha$

But $\alpha = \theta$

$\Rightarrow \quad \boldsymbol{sin(360^o - \theta) = -sin\,\theta,\ cos(360^o - \theta) = cos\,\theta,\ tan(360^o - \theta) = -tan\,\theta} \qquad ...$ *(8)*

Negative angles:



$a)$ $\sin\theta = \frac{y}{r}$ and $\sin(-\theta) = \frac{-y}{r}$ and $-\sin\theta = \frac{y}{r}$ ⇨ $\sin(-\theta) = -\sin\theta$

$\sin\theta = \frac{x}{r}$ and $\cos(-\theta) = \frac{x}{r}$ ⇨ $\cos(-\theta) = \cos\theta$

$\tan\theta = \frac{y}{x}$ and $\tan(-\theta) = \frac{-y}{x}$ ⇨ $-\tan\theta = \frac{-y}{x}$ ⇨ $\tan(-\theta) = -\tan\theta$

$\boldsymbol{\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,\ \tan(-\theta) = -\tan\theta}$ . . . *(9)*

From equations (1-8) above and the negative angles it is clear that trigonometric ratio of any angle may be expressed in terms of trigonometric ratio of an acute angle, which means that the table of trigonometric ratios for the angles in the range 0^o and 90^o are quite sufficient.

Examples:

1. Express the following trigonometric ratios in terms of the trigonometric ratio of an acute angle. $i)$ $\sin 125^o$ $ii)$ $\cos 225^o$ $iii)$ $\tan 120^o$

Solution:

$i)\ \sin 125^o = \sin(90+35)^o = \cos 35^o$ or $\sin 125^o = \sin(180-55)^o = \sin 55^o$

$ii)$ $\cos 225^o = \cos(180+45)^o = -\cos 45^o$ or $\cos 225^o = \cos(270-45)^o = -\sin 45^o$

$iii)$ $\tan 120^o = \tan(180-60)^o = -\tan 60^o$ or $\tan 120^o = \tan(90+30)^o = -\cot 30^o$

2. Express the following trigonometric ratios in terms of the trigonometric ratio of an acute angle. $i)$ $\csc 240^o$ $ii)$ $\sec 315^o$ $iii)$ $\cot 320^o$

Solution:

$i)$ $\sin 240^o = \sin(270-30)^o = -\cos 30^o$ or $\sin 240^o = \sin(180+60)^o = -\sin 60^o$

⇨ $\csc 240^o = -\sec 30^0$ or $\csc 240^o = -\csc 60^o$

$ii)$ $\cos 315^o = \cos(270+45)^o = \sin 45^o$ or $\cos 315^o = \cos(360-45)^o = \cos 45^o$

⇨ $\sec 315^o = -\csc 45^0$ or $\sec 315^o = -\sec 45^o$

$iii)$ $\tan 320^o = \tan(270+50)^o = -\cot 50^o$ or $\tan 320^o = \tan(360-40)^o = -\tan 40^o$

⇨ $\cot 320^o = -\cot 50^0$ or $\cot 320^o = -\tan 40^o$

3. Express the following trigonometric ratios in terms of the trigonometric ratio of an acute angles.

$i)$ $\sin 110^o$ $ii)$ $\cos 250^o$ $iii)$ $\tan 322^o$ $iv)$ $\sin(-120)^o$ $v)$ $\cos(-160)^o$

Solution:

Do

Examples 4: Without using tables or calculator, find the value of the following

$i)\quad \sin A \ \ and \ \ \cos A\,, \ \ if \ \ \tan A = \frac{3}{4} \qquad 0 < A < \frac{\pi}{2}$

$ii)\quad \sin A \ \ and \ \ \tan A\,, \ \ if \ \ \cos A = \frac{-4}{15} \qquad \pi < A < \frac{3\pi}{2}$

$iii)\quad \cos A \ \ and \ \ \tan A\,, \ \ if \ \ \sin A = \frac{5}{13} \qquad \pi < A < \frac{3\pi}{2}$

Solution:

$i)\quad \tan A = \frac{3}{4} \quad \Rightarrow \quad \tan^2 A = \frac{9}{16} \quad \Rightarrow \quad \sec^2 A = 1 - \frac{9}{16} = \frac{25}{16} \quad \Rightarrow \quad \sec A = \frac{5}{4}$

$\Rightarrow \quad \cos A = \frac{4}{5} \quad \Rightarrow \quad \cos^2 A = \frac{16}{25} \quad \Rightarrow \quad \sin^2 A = 1 - \frac{16}{25} = \frac{9}{25} \quad \Rightarrow \quad \sin A = \frac{3}{5}$

$ii)\quad \cos A = \frac{-4}{15} \quad \Rightarrow \quad \cos^2 A = \frac{16}{225} \quad \Rightarrow \quad \sin^2 A = 1 - \frac{16}{225} = \frac{209}{225} \quad \Rightarrow \quad \sin A = \frac{\sqrt{209}}{15}$

$\cos^2 A = \frac{16}{225} \quad \Rightarrow \quad \sec^2 A = \frac{225}{16} \quad \Rightarrow \quad \tan^2 A = \frac{225}{16} - 1 = \frac{209}{16} \quad \Rightarrow \quad \tan A = \frac{\sqrt{209}}{4}$

$iii)\quad \sin A = \frac{5}{13} \quad \Rightarrow \quad \sin^2 A = \frac{25}{169} \quad \Rightarrow \quad \cos^2 A = 1 - \frac{25}{169} = \frac{144}{169} \quad \Rightarrow \quad \cos A = -\frac{12}{13}$

$\cos^2 A = \frac{144}{169} \quad \Rightarrow \quad \sec^2 A = \frac{169}{144} \quad \Rightarrow \quad \tan^2 A = \frac{25}{144} \quad \Rightarrow \quad \tan A = \frac{5}{12}$

Proving Identity I:

There are no specific guidelines for proving identities, but it is allowed to show that

$i)$ LHS = RHS

$ii)$ RHS = LHS

$iii)$LHS = P and RHS = P $\quad \Rightarrow \quad$ LHS = RHS

Example1: Show that

$i)\quad \left(\frac{1+\sin x}{1+\cos x}\right)\left(\frac{1+\sec x}{1+\csc x}\right) = \tan x \qquad\qquad ii)\quad \frac{\tan x+\cos x}{\sin x} = \sec x + \cot x$

Solution

$$i)\quad \left(\frac{1+\sin x}{1+\cos x}\right)\left(\frac{1+\frac{1}{\cos x}}{1+\frac{1}{\sin x}}\right) = \left(\frac{1+\sin x}{1+\cos x}\right)\left(\frac{\frac{1+\cos x}{\cos x}}{\frac{1+\sin x}{\sin x}}\right) = \left(\frac{1+\sin x}{1+\cos x}\right)\left(\frac{1+\cos x}{\cos x}\right)\left(\frac{\sin x}{1+\sin x}\right) = \left(\frac{\sin x}{\cos x}\right) = \tan x$$

$$ii)\quad \frac{\frac{\sin x}{\cos x}+\cos x}{\sin x} = \frac{\sin x+\cos^2 x}{\cos x} * \frac{1}{\sin x} = \frac{\sin x+\cos^2 x}{\cos x\ \sin x} = \frac{\sin x}{\cos x\ \sin x} + \frac{\cos^2 x}{\cos x\ \sin x} = \frac{1}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \sec x + \cot x$$

Example2: Show that

$i)\quad (\tan x - \sec x)^2 = \frac{1-\sin x}{1+\sin x} \qquad ii)\quad (\sin A + \cos A)(\cot A + \tan A) = \sec A + \csc A$

Solution:

$i)\quad (\tan x - \sec x)^2 = tan^2 x - 2\tan x \sec x + \sec^2 x = sec^2 x - 2\tan x \tan x + tan^2 x$

$$= \frac{1}{cos^2 x} - 2\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{cos^2 x} - 2\left(\frac{\sin x}{\cos^2 x}\right) + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{1-2\sin x + \sin^2 x}{\cos^2 x} = \frac{(1-\sin x)^2}{1-\sin^2 x} = \frac{(1-\sin x)(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{1-\sin x}{1+\sin x}$$

$ii)\quad (\sin A + \cos A)(\cot A + \tan A) = \sin A \cot A + \sin A \tan A + \cos A \cot A + \cos A \tan A$

$$= \sin A\left(\frac{\cos A}{\sin A}\right) + \sin A\left(\frac{\sin A}{\cos A}\right) + \cos A\left(\frac{\cos A}{\sin A}\right) + \cos A\left(\frac{\sin A}{\cos A}\right)$$

$$= \cos A + \frac{\sin^2 A}{\cos A} + \frac{\cos^2 A}{\sin A} + \sin A$$

$$= \frac{\sin A cos^2 A + sin^3 A + cos^3 A + sin^2 A \cos A}{\sin A \cos A}$$

$$= \frac{\sin A(1-sin^2 A) + sin^3 A + cos^3 A + (1-cos^2 A)\cos A}{\sin A \cos A}$$

$$= \frac{\sin A - sin^3 A + sin^3 A + cos^3 A + \cos A - cos^3 A}{\sin A \cos A}$$

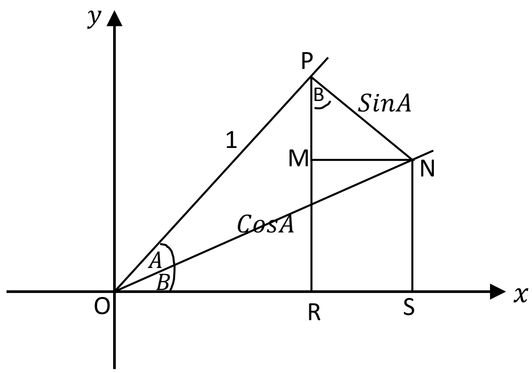
$$= \frac{\sin A + \cos A}{\sin A \cos A} = \frac{\sin A}{\sin A \cos A} + \frac{\cos A}{\sin A \cos A}$$

$$= \frac{1}{\cos A} + \frac{1}{\sin A}$$

$$= \sec A + \csc A$$

Trigonometric ratio of compound angle:

a) *Sum of two angles*



Consider the figure above

In $\Delta OPN, \sin A = \frac{|PN|}{|OP|} = \ |PN|, and\ \cos A = \frac{|ON|}{|OP|} = \ |ON|$

In$\Delta OPR, \sin(A+B) = \frac{|PR|}{|OP|} = \ |PR| = \ |PM| + |MR| = |PM| + |NS|$

In $\Delta PMN, \cos B = \frac{|PM|}{\sin A}$ ⇨ $|PM| = \sin A \cos B$

In $\Delta ONS, \sin B = \frac{|NS|}{\cos A}$ ⇨ $|NS| = \cos A \sin B$

⇨ $\sin(A+B) = \sin A \cos B + \cos A \sin B$ . . . (1)

In $\Delta OPR, \cos(A+B) = \frac{|OR|}{|OP|} = \ |OR| = \ |OS| - |SR| = |OS| - |MN|$

In $\Delta ONS, \cos B = \frac{|OS|}{\cos A}$ ⇨ $|OS| = \cos A \cos B$

In $\Delta PMN, \sin B = \frac{|MN|}{\sin A}$ ⇨ $|MN| = \sin A \sin B$

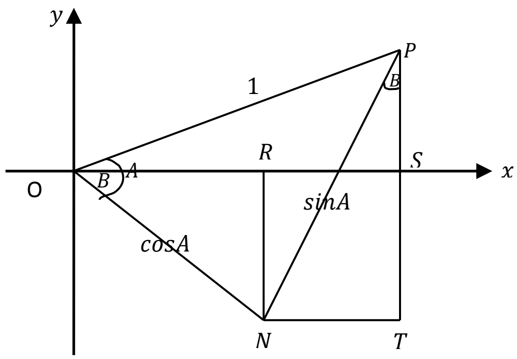
⇨ $\cos(A+B) = \sin A \cos B + \cos A \sin B$ . . . (2)

From (1)and (2) we have that $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

Dividing both the numerator and the denominator by $\cos A \cos B$ we have

$$\tan(A+B) = \frac{\tan B + \tan A}{1 - \tan A \tan B} \quad . . . (3)$$

b) *Difference of two angles.*



Consider the figure above

In $\Delta OPN, \sin A = \frac{|PN|}{|OP|} = |PN|, and \ \cos A = \frac{|ON|}{|OP|} = |ON|$

In $\Delta OPS, \sin(A - B) = \frac{|PS|}{|OP|} = |PS| = |PT| - |TS| = |PT| - |RN|$

In $\Delta NPT, \cos B = \frac{|PT|}{\sin A} \quad \Rightarrow |PT| = \sin A \cos B$

In $\Delta ORN, \sin B = \frac{|RN|}{\cos A} \quad \Rightarrow |RN| = \cos A \sin B$

$\Rightarrow \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$ . . . (4)

In $\Delta OPS, \cos(A - B) = \frac{|O|}{|OP|} = |OS| = |OR| + |RS| = |OR| + |NT|$

In $\Delta ORN, \cos B = \frac{|OR|}{\cos A} \quad \Rightarrow |OR| = \cos A \cos B$

In $\Delta PNT, \sin B = \frac{|NT|}{\sin A} \quad \Rightarrow |NT| = \sin A \sin B$

$\Rightarrow \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$ . . . (5)

From (4) and (5) we have that $\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

Dividing both the numerator and the denominator by $\cos A \cos B$ we have

$$\tan(A - B) = \frac{\tan B - \tan A}{1 + \tan A \tan B} \qquad . . . (6)$$

Example1: Without using tables or calculator, find the value of

$i)$ $\cos 15^o$ $\qquad ii)$ $\sin 105^o$ $\qquad iii)$ $\tan 75^o$

Solution:

$i)$ $\cos 15^o = \cos(45 - 30)^o = \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$

$$= \left(\frac{1}{\sqrt{2}} * \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} * \frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

$ii)$ $\sin 105^o = \sin(60+45)^o = \sin 60^o \cos 45^o + \cos 60^o \sin 45^o$

$$= \left(\frac{\sqrt{3}}{2} * \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} * \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

$iii)$ $\tan 75^o = \tan(45+30)^o = \dfrac{\tan 45^o + \tan 30^o}{1 - \tan 45^o \tan 30^o} = \dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = 2+\sqrt{3}$

Example2: Given that $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, Find $\sin(A+B)$ and $\cos(A+B)$ If

$i)$ A and B are both acute. $ii)$ A is obtuse and B is acute

Solution:

$\sin(A+B) = \sin A^o \cos B^o + \cos A^o \sin B^o$ and

$\cos(A+B)^o = \cos A^o \cos B^o + \sin A^o \sin B^o$

$i)$ $\sin A = \frac{4}{5}$ ⇨ $\sin^2 A = \frac{16}{25}$ ⇨ $\cos^2 A = 1 - \frac{16}{25} = \frac{9}{25}$ ⇨ $\cos A = \frac{3}{5}$ and

$\cos B = \frac{12}{13}$ ⇨ $\cos^2 B = \frac{144}{169}$ ⇨ $\sin^2 B = 1 - \frac{144}{169}$ ⇨ $\sin B = \frac{5}{13}$

⇨ $\sin(A+B) = \left(\frac{4}{5} * \frac{12}{13}\right) + \left(\frac{3}{5} * \frac{5}{13}\right) = \frac{63}{65}$ and

$\cos(A+B) = \left(\frac{3}{5} * \frac{12}{13}\right) + \left(\frac{4}{5} * \frac{5}{13}\right) = \frac{16}{65}$

$ii)$ $\sin A = \frac{4}{5}$ ⇨ $\sin^2 A = \frac{16}{25}$ ⇨ $\cos^2 A = 1 - \frac{16}{25} = \frac{9}{25}$ ⇨ $\cos A = -\frac{3}{5}$ and

$\cos B = \frac{12}{13}$ ⇨ $\cos^2 B = \frac{144}{169}$ ⇨ $\sin^2 B = 1 - \frac{144}{169} = \frac{25}{169}$ ⇨ $\sin B = \frac{5}{13}$

⇨ $\sin(A+B) = \left(\frac{4}{5} * \frac{12}{13}\right) + \left(-\frac{3}{5} * \frac{5}{13}\right) = \frac{33}{65}$ and

$\cos(A+B) = \left(-\frac{3}{5} * \frac{12}{13}\right) + \left(\frac{4}{5} * \frac{5}{13}\right) = -\frac{56}{65}$

Exercise:

1) Given that $\sin A = \frac{\sqrt{5}}{5}, and\ \tan B = \frac{1}{2}$, find $\cos(A+B)$ and $\tan(A+B)$ if A and B are both acute

2) If $\dfrac{\tan\theta + \tan 3\theta}{1 - \tan\theta \tan 3\theta} = -1$, find $\tan\theta$ in surd form.

3) Verify the following

$i)$ $\cos(90 - A) = -\sin A$ $ii)$ $\sin(270 - A) + \sin(270 + A) = -2\cos A$

Trigonometric ratio of multiple angles:

From the expression for compound angles we have that

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ . . . (1)

$\cos(A+B) = \cos A \cos B + \sin A \sin B$ . . . (2)

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ . . . (3)

Without any loss of generality we set $B = A$

⇨ Eqns. (1),(2) & (3) become

$\sin 2A = 2 \sin A \cos A$ . . . (4)

$$\cos 2A = \left.\begin{matrix} \cos^2 A - \sin^2 A \\ 2\,\mathrm{Cos}^2 A - 1 \\ 1 - 2\sin^2 A \end{matrix}\right\} \quad . . . (5)$$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$ . . . (6)

Equations (4), (5), & (6) are called Multiple angles.

Example1: Show that

$i)\ \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

Solution:

$$RHS = \frac{1 - \frac{\sin^2 A}{cos^2 A}}{1 + \frac{\sin^2 A}{cos^2 A}} = \frac{cos^2 A - sin^2 A}{cos^2 A} * \frac{cos^2 A}{sin^2 A + cos^2 A}$$

$$= \frac{cos^2 A - sin^2 A}{sin^2 A + cos^2 A}$$

$$= cos^2 A - sin^2 A$$

$$= \cos 2A = LHS$$

Example2: Given that $\cos 2A = \frac{3}{5}, find\ \tan A$

Solution:

$\cos 2A = \frac{3}{5}$

$\cos 2A = 2\cos^2 A - 1$

⇨ $2\cos^2 A - 1 = \frac{3}{5}$

⇨ $10\cos^2 A = 8$

⇨ $\cos^2 A = \frac{4}{5}$

⇨ $\sec^2 A = \frac{5}{4}$

⇨ $\tan^2 A = \frac{5}{4} - 1 = \frac{1}{4}$

⇨ $\tan A = \pm\frac{1}{2}$

Example3: Show that

$\cos(A-B)\cos(A+B) - \sin(A-B)\sin(A+B) = \cos 2A$

Proof

$\cos(A-B) = \cos A\cos B + \sin A\sin B$,

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

⇨ $\cos(A-B)\cos(A+B) = \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ . . . (1)

$\sin(A-B) = \sin A\cos B - \cos A\sin B$,

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

⇨ $\sin(A-B)\sin(A+B) = \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$ . . . (2)

From (1) and (2) we have

$$\begin{aligned}
&\cos(A-B)\cos(A+B) - \sin(A-B)\sin(A+B) \\
&= (\cos^2 A\cos^2 B - \sin^2 A\sin^2 B) - (\sin^2 A\cos^2 B - \cos^2 A\sin^2 B) \\
&= (\cos^2 B + \sin^2 B)\cos^2 A - (\sin^2 B + \cos^2 B)\sin^2 A \\
&= \cos^2 A - \sin^2 A \\
&= cos\,2A
\end{aligned}$$

The factor formulae:

From the expression for compound angles we have that

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ . . . (1)

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$ . . . (2)

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$ . . . (3)

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$ . . . (4)

Now we set $A + B = x \;\; and \; A - B = y$

⇨ $x + y = 2A$ ⇨ $A = \frac{x+y}{2}$, *and*

$x - y = 2B$ ⇨ $B = \frac{x-y}{2}$

Substituting for $A \; and \; B$ in equation (1),(2),(3) and (4) we have

$\sin x + \sin y = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$ . . . (5)

$$\sin x - \sin y = 2\cos(\tfrac{x+y}{2})\sin(\tfrac{x-y}{2}) \qquad ...(6)$$

$$\cos x + \cos y = 2\cos(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}) \qquad ...(7)$$

$$\cos x - \cos y = -2\sin(\tfrac{x+y}{2})\sin(\tfrac{x-y}{2}) \qquad ...(8)$$

Equations (5),(6),(7),and (8) are called the factor formulae.

Example1: Verify the following identity

$i)$ $\cos A + \cos 3A + \cos 5A + \cos 7A = 4\cos 4A\cos 2A\cos A$

$ii)$ $\frac{\cos 2A+\cos 5A+\cos 8A}{\sin 2A+\sin 5A+\sin 8A} = \cot 5A$

Solution:

$i)$ $\cos A + \cos 3A + \cos 5A + \cos 7A = (\cos A + \cos 3A) + (\cos 5A + \cos 7A)$

$$= 2\cos\left(\frac{4A}{2}\right)\cos\left(\frac{-2A}{2}\right) + 2\cos\left(\frac{12A}{2}\right)\cos\left(\frac{-2A}{2}\right)$$

$$= 2\cos(2A)\cos(A) + 2\cos(6A)\cos(A)$$

$$= 2\cos A\,(\cos 2A + \cos 6A)$$

$$= 2\cos A\left[2\cos\left(\frac{8A}{2}\right)\cos\left(\frac{-4A}{2}\right)\right]$$

$$= 2\cos A\,[2\cos 4A\cos 2A]$$

$$= 4\cos 4A\cos 2A\cos A$$

$ii)$ $\frac{\cos 2A+\cos 5A+\cos 8A}{\sin 2A+\sin 5A+\sin 8A} = \frac{(\cos 2A + \cos 8A) + \cos 5A}{(\sin 2A + \sin 8A) + \sin 5A}$

$$= \frac{2\cos 5A\cos 3A+\cos 5A}{2\sin 5A\cos 3A+\sin 5A}$$

$$= \frac{2[\cos 3A+1]\cos 5A}{2[\sin 3A+1]\sin 5A}$$

$$= \frac{\cos 5A}{\sin 5A} = \cot 5A$$

Example2: Express the product as a sum or difference

$i)$ $\sin 7t \sin 3t$ $ii)$ $\cos 6t \cos(-4t)$

Solution:

$i)$ $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$\Rightarrow -\frac{1}{2}(\cos x - \cos y) = \sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$ . . . (1)

$\Rightarrow -\frac{1}{2}(\cos x - \cos y) = \sin(7t)\sin(3t)$ . . . (2)

From (1) and (2) we have

$\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}) = \sin(7t)\sin(3t)$

$\Rightarrow$ $\frac{x+y}{2} = 7t \; and \; \frac{x-y}{2} = 3t$

$\Rightarrow$ $x + y = 14t \;\; and$

$x - y = 6t$

$(x+y) + (x-y) = 20t$

$\Rightarrow$ $2x = 20t$ $\Rightarrow x = 10t$

$(x+y) - (x-y) = 8t$

$\Rightarrow$ $2y = 8t$ $\Rightarrow$ $y = 4t$

Substituting for $x \;\; and \;\; y$ in (2) we have

$-\frac{1}{2}(\cos 10t - \cos 4t) = \sin(7t)\sin(3t)$

$ii)$ $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\Rightarrow$ $\frac{1}{2}(\cos x + \cos y) = \cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$ . . . (1)

$\Rightarrow$ $\frac{1}{2}(\cos x + \cos y) = \cos(6t)\cos(-4t)$ . . . (2)

From (1) and (2) we have

$\cos(\frac{x+y}{2})\cos(\frac{x-y}{2}) = \cos(6t)\cos(-4t)$

$\Rightarrow$ $\frac{x+y}{2} = 6t \; and \; \frac{x-y}{2} = -4t$

$\Rightarrow$ $x + y = 12t \;\; and \;\; x - y = -8t$

$\Rightarrow$ $(x+y) + (x-y) = 4t \;\; and \;\; (x+y) - (x-y) = 20t$

$\Rightarrow$ $2x = 4t \;\; and \;\; 2y = 20t$

$\Rightarrow$ $x = 2t \;\; and \;\; y = 10t$

Substituting for $x \;\; and \;\; y$ in (2) we have

$\frac{1}{2}(\cos 2t - \cos 10t) = \cos(6t)\cos(-4t)$

Half angles:

$$\sin A = \sin\left(\frac{A}{2}+\frac{A}{2}\right) = \sin\frac{A}{2}\cos\frac{A}{2} + \cos\frac{A}{2}\sin\frac{A}{2} = 2\sin\frac{A}{2}\cos\frac{A}{2} \qquad . . . (1)$$

$$\cos A = \cos\left(\frac{A}{2}+\frac{A}{2}\right) = \cos\frac{A}{2}\cos\frac{A}{2} - \sin\frac{A}{2}\sin\frac{A}{2} = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$

$$\Rightarrow \quad \cos A = \left.\begin{matrix} \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\ 2\cos^2\frac{A}{2} - 1 \\ 1 - 2\sin^2\frac{A}{2} \end{matrix}\right\} \qquad . . . (2)$$

And

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \qquad . . . (3)$$

From (2) we have

$$\cos A = 1 - 2\sin^2\frac{A}{2}$$

$$\Rightarrow \quad \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} \qquad . . . (4)$$

Also from (2) we have

$$\cos A = 2\cos^2\frac{A}{2} - 1$$

$$\Rightarrow \quad \cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} \qquad . . . (5)$$

And from (4), and (5) we have

$$\Rightarrow \quad \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+cosA}} = \frac{1-\cos A}{\sin A} \qquad . . . (6)$$

Equations (4), (5), and (6) are called half angles formulae.

t-formulae:

From half angles we have that

$$\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

Dividing both sides by $\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)$ we have

$$\sin A = \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)}$$

Dividing both the numerator and the denominator by $\cos^2\left(\frac{A}{2}\right)$ we have

$$\sin A = \frac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} \qquad . \ . \ . \ (1)$$

Also from the definition of half angles we have

$$\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$

Dividing both sides by $\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)$ we have

$$\cos A = \frac{\cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)}$$

Dividing both the numerator and the denominator by $\cos^2\left(\frac{A}{2}\right)$ we have

$$\cos A = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)} \qquad . \ . \ . \ (2)$$

Also from half angles we have

$$\tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)} \qquad . \ . \ . \ (3)$$

Now we set $\tan\left(\frac{A}{2}\right) = t$ in equations (1), (2), and (3) we have

$$\sin A = \frac{2t}{1 - t^2} \qquad . \ . \ . \ (4)$$

$$\cos A = \frac{1 - t^2}{1 + t^2} \qquad . \ . \ . \ (5)$$

$$\tan A = \frac{2t}{1 - t^2} \qquad . \ . \ . \ (6)$$

Equations (4), (5), and (6) are called the ***t-formulae***

Example1: Given that, $\tan\left(\frac{A}{2}\right) = \csc A - \sin A$, show that $\tan^2\left(\frac{A}{2}\right) = -2 \pm \sqrt{5}$

Proof:

$\tan\left(\frac{A}{2}\right) = \csc A - \sin A$, - Given

⇨ $\tan\left(\frac{A}{2}\right) = \frac{1}{\sin A} - \sin A$,

⇨ $t = \frac{1+t^2}{2t} - \frac{2t}{1+t^2}$

⇨ $t = \frac{(1+t^2)^2 - 4t^2}{2t(1+t^2)}$

⇨ $t^4 + 4t^2 - 1 = 0$

⇨ $(t^2)^2 + 4t^2 - 1 = 0$

⇨ $t^2 = -2 \pm \sqrt{5}$ ⇨ $\tan^2\left(\frac{A}{2}\right) = -2 \pm \sqrt{5}$

Hence the proof

Example2: Given that $\tan A = \frac{4}{3}$, calculate the possible values of $\tan\left(\frac{A}{2}\right)$

Solution:

$\tan A = \frac{4}{3}$, and

$\tan A = \frac{2t}{1 - t^2}$

⇨ $\frac{2t}{1 - t^2} = \frac{4}{3}$

⇨ $4t^2 + 6t - 4 = 0$

⇨ $t = \frac{1}{2}$ or $t = -2$

Solution of triangle:

From geometry, is made that the solution of triangles is uniquely determine when

i) Two angles and one side are known (AAS)

ii) One angle and two sides are known(ASS)

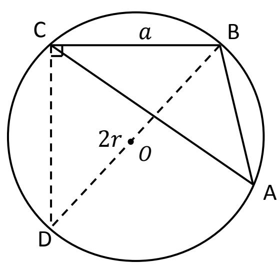
iii) Three sides are known (SSS)

Note: We denote the angles of a triangle by upper case letters, and the sides by lower case letters.

Sine formulae:

Statement:- In any triangle ABC, with the corresponding sides $a, b, and\ c$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$



Proof:

Let $2r$ be the diameter of the circle ABCD with centre O.

Draw a diameter BOD and join CD.

In ΔBDC,

$D\hat{C}B = 90^o$ (Angle in a semi circle),

$B\hat{D}C = B\hat{A}C$ (Angle in the same segment),

⇨ $\sin D = \sin A$

But $\sin D = \frac{a}{2r}$

⇨ $\sin A = \frac{a}{2r}$

⇨ $\frac{a}{\sin A} = 2r$ . . . (1)

Draw a diameter BOD and join AD.

We have that, $B\hat{A}D = 90^o$ (Angle in a semi circle),

$A\widehat{D}B = A\hat{C}B$ (Angle in the same segment),

⇨ $\sin D = \sin C$

But $\sin D = \frac{c}{2r}$⇨$\sin C = \frac{c}{2r}$

⇨ $\frac{c}{\sin C} = 2r$ . . . (2)

Draw a diameter COD and join AD.

We have that, $D\hat{A}C = 90^o$ (Angle in a semi circle),

$A\hat{C}D = D\hat{B}A$ (Angle in the same segment),

⇨ $\sin C = \sin B$

But $\sin C = \frac{b}{2r}$ ⇨ $\sin B = \frac{b}{2r}$

⇨ $\frac{b}{\sin B} = 2r$ . . . (3)

From (1), (2), (3) we have that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

Hence the proof

Cosine formulae

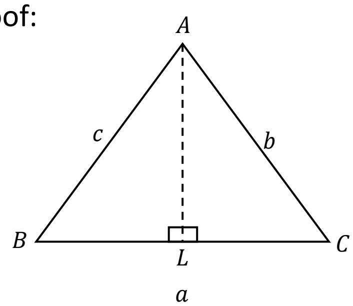
Statement:- In any triangle ABC with the corresponding sides $a, b, and\ c$

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Proof:



In ΔABC, draw a line from A perpendicular to BC and to meet BC at L

⇨ $BC = BL + LC$

⇨ $a = c\cos B + b\cos C$ . . . (1)

In ΔABC, draw a line from B perpendicular to AC and to meet AC at N

⇨ $AC = AN + NC$

⇨ $b = c\cos A + a\cos C$ . . . (2)

In ΔABC, draw a line from C perpendicular to AB and to meet AB at p

⇨ $AB = AP + PB$

⇨ $c = a\cos B + b\cos A$ . . . (3)

Now, we multiply equation (1) by $-a$, equation (2) by b and equation (3) by c and add the three we have.

$$a^2 = b^2 + c^2 - 2bc\cos A \quad . . . (4)$$

Now, we multiply equation (1) by a, equation (2) by $-b$ and equation (3) by c and add the three we have.

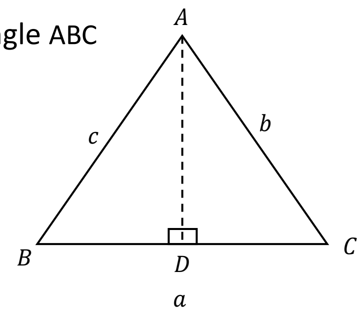
$$b^2 = a^2 + c^2 - 2ac\cos B \quad . . . (5)$$

Now, we multiply equation (1) by a, equation (2) by b and equation (3) by $-c$ and add the three we have.

$$c^2 = a^2 + b^2 - 2ab\cos C \quad . . . (6)$$

Area of a triangle:

Consider triangle ABC



Let ∇ denote the area of the triangle ABC

⇨ $\nabla = \frac{1}{2} * base * height$`

Draw a line from A perpendicular to BC and to meet BC at D we have in ΔABD

$\sin B = \frac{AD}{c}$ ⇨ $AD = c \sin B$

⇨ $\nabla = \frac{1}{2} ac \sin B$ . . . (1)

Draw a line from B perpendicular to AC and to meet AC at D we have in ΔBCD

$\sin C = \frac{BD}{a}$ ⇨ $BD = a \sin C$

⇨ $\nabla = \frac{1}{2} ba \sin C$ . . . (2)

Draw a line from perpendicular to AB and to meet AB at D we have in ΔCAD

$\sin A = \frac{CD}{b}$ ⇨ $CD = b \sin A$

⇨ $\nabla = \frac{1}{2} cb \sin A$ . . . (3)

From sine formulae we have

⇨ $\frac{a}{\sin A} = 2r$ ⇨ $\sin A = \frac{a}{2r}$

Substituting for $\sin A$ in (3) we have

$\nabla = \frac{1}{4r} abc$ . . . (4)

Multiply both sides of (3) by 4 we have

$4\nabla = 2cb \sin A$ . . . (5)

Square (5) on both sides we have

$16\nabla^2 = 4c^2b^2 \sin^2 A$ . . . (6)

From cosine formulae we have

$a^2 = b^2 + c^2 - 2bc \cos A$

⇨ $b^2 + c^2 - a^2 = 2bc \cos A$ . . . (7)

Squaring (7) on both sides we have

$$(b^2 + c^2 - a^2)^2 = 4c^2b^2\cos^2 A \qquad . . . (8)$$

Adding (6) and (8) we have

$$16\nabla^2 + (b^2 + c^2 - a^2)^2 = 4c^2b^2[\sin^2 A + \cos^2 A]$$

⇨ $16\nabla^2 = 4c^2b^2 - (b^2 + c^2 - a^2)^2$

⇨ $16\nabla^2 = (2bc)^2 - (b^2 + c^2 - a^2)^2$

⇨ $16\nabla^2 = (2bc - [b^2 + c^2 - a^2])(2bc + [b^2 + c^2 - a^2])$

⇨ $16\nabla^2 = (2bc - b^2 - c^2 + a^2)(2bc + b^2 + c^2 - a^2)$

⇨ $16\nabla^2 = (a + c - b)(a + b - c)(b + c - a)(a + b + c)$

Now we let $a + b + c = 2S$ ⇨ $S = \frac{1}{2}(a + b + c)$

$a + c - b + 2b = 2S$ ⇨ $a + c - b = 2(S - b)$

$a + b - c + 2c = 2S$ ⇨ $a + b - c = 2(S - c)$

$b + c - a + 2a = 2S$ ⇨ $b + c - c = 2(S - a)$

⇨ $16\nabla^2 = 2S * 2(S - a) * 2(S - b) * 2(S - c)$

⇨ $16\nabla^2 = 16S(S - a)(S - b)(S - c)$

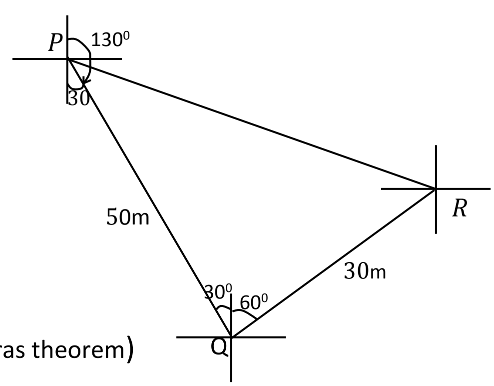
⇨ $\nabla^2 = S(S - a)(S - b)(S - c)$

⇨ $\nabla = \sqrt{S(S - a)(S - b)(S - c)}$ . . . (9)

Equation (9) is called Hero's formulae.

Example1: P, Q and R are on the same horizontal level, the bearing of Q from P is 150^o and the bearing of R from Q is 060^o, if |PQ|= 50m, and |QR|= 30m. find the area of ΔPQR .

Solution:



$q^2 = 50^2 + 30^2$ (Pythagoras theorem)

⇨ $q \cong 58m$

⇨ $S = \frac{1}{2}(50 + 30 + 58) = 69m$

⇨ $\nabla = \sqrt{69(39)(11)(19)} \cong 750m^2$

Example2: A man travel from village A on the bearing of 060^o to village B which is 20km away, from village B he travel to village C on the bearing of 195^o. If village C is directly east of village A. find the solution of the triangle ABC.

Example3: Three villages P, Q and R are such that the distance between P and Q is 50km and the distance between P and R is 90km, if the bearing of Q from P is $075^o$and the bearing of R from P is 310^o. Find the solution of the triangle PQR

Example4: In any triangle ABC, Show that

$i)\ \sin(90 + A) = -\cos(B + C)$ $\quad\quad$ $ii) \sin\left(\frac{A+B}{2}\right) = \cos(\frac{C}{2})$

Proof:

$i)\quad \sin(90 + A) = \cos A$ . . . (1)

But $A + B + C = 108 \quad ⇨ \quad A = 108 - (B + C)$

⇨ $\cos A = \cos[180 - (B + C)]$

⇨ $\cos A = -\cos(B + C)$ . . . (2)

From (1) and (2) we have that

$$\sin(90 + A) = -\cos(B + C)$$

Hence the proof

$ii)\quad A + B + C = 180^o$

⇨ $A + B = 180^o - C$

⇨ $\left(\frac{A+B}{2}\right) = \left(90^o - \frac{C}{2}\right)$

⇨ $\sin\left(\frac{A+B}{2}\right) = \sin\left(90^o - \frac{C}{2}\right)$

⇨ $\sin\left(\frac{A+B}{2}\right) = \cos\left(\frac{C}{2}\right)$

Hence the proof

Example5: In any triangle ABC show that

$i) \cos A + \cos(B - C) = 2\sin B \sin C$ $\quad\quad$ $ii) \cos\left(\frac{C}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)$

Proof:

$i)\quad \cos A + \cos(B - C)$ . . . (1)

But $A + B + C = 180^o$

⇨ $A = 180^o - (B + C)$

⇨ $\cos A = \cos[180^o - (B + C)] = -\cos(B + C)$

Substituting for $\cos A$ in equation (1) we have

$-\cos(B + C.) + \cos(B - C)$

Expanding and collecting like terms (or use factor formulae) we have

$\cos(B - C.) - \cos(B + C) = 2\sin B \sin C$

⇨ $\cos A + \cos(B - C) = 2\sin B \sin C$

Hence the proof

$ii)$ $\cos\frac{C}{2} + \sin\left(\frac{A-B}{2}\right)$

But from example4ii we have that

$\sin\left(\frac{A+B}{2}\right) = \cos(\frac{C}{2})$

⇨ $\cos\left(\frac{C}{2}\right) + \sin\left(\frac{A-B}{2}\right) = \sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right)$

Using the factor formulae we have

$\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)$

Hence the proof

Example6: In any triangle ABC, show that

$i)$ $\sin A + \sin B + \sin C = 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

$ii)$ $\sin A + \sin B - \sin C = 4\sin(\frac{A}{2})\sin(\frac{B}{2})\cos(\frac{C}{2})$

Solution:

$i)$ $\sin A + \sin B + \sin C = (\sin A + \sin B) + \sin C$

$= 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2} + \frac{C}{2}\right)$

$= 2\cos(\frac{C}{2})\cos(\frac{A-B}{2}) + 2\sin(\frac{C}{2})\cos(\frac{C}{2})$

$= 2\cos(\frac{C}{2})\cos(\frac{A-B}{2}) + 2\cos(\frac{A+B}{2})\cos(\frac{C}{2})$

$= 2\cos\left(\frac{C}{2}\right)\left[\cos(\frac{A-B}{2}) + \cos(\frac{A+B}{2})\right]$

$= 2\cos\left(\frac{C}{2}\right)\left[2\cos(\frac{A}{2})\cos(\frac{B}{2})\right] = 4\cos(\frac{A}{2})\cos(\frac{B}{2})\cos(\frac{C}{2})$

Hence the proof

$ii)$Similar to $i)$ above

Inverse trigonometric functions:

The equation $\sin\theta = p$ define a unique value of p for each given angle θ, but when p is known, the equation may have many solutions or no solution. To express θ as a function of p we write

$\theta = arc\sin p$, $\theta = arc\cos p$ or $\theta = arc\tan p$

Example1

$\sin\theta = \frac{1}{2}$

⇨ $\theta = arc\sin(\frac{1}{2})$

⇨ $\theta = \ ...\ -690^o, -570^o, -330^o,\ -210^o,\ 30^o, 150^o,\ 390^o, 510^o, 750^o,. ..$

Example2

$\sin\theta = -0.515$

⇨ $\theta = arc\sin(-0.515)$

⇨ $\theta = \ ...,-149^o,\ -31^o,\ 30^o, 211^o,\ 329^o, 571^o\ ...$

Example3

$\cos\theta = \frac{1}{\sqrt{2}}$

⇨ $\theta = arc\sin(\frac{1}{\sqrt{2}})$

⇨ $\theta = \ ...\ -675^o, -405^o, -315^o,\ -45^o,\ 45^o, 315^o,\ 405^o, 675^o,\ ...$

Example4

$\cos\theta = \frac{3}{2}$

⇨ $\theta = arc\sin(\frac{3}{2})$ ⇨ $\theta = \infty$

⇨ there is no solution.

Example5

$\tan\theta = \frac{1}{\sqrt{3}}$

⇨ $\theta = arc\tan(\frac{1}{\sqrt{3}})$

⇨ $\theta = \ ...\ -510^o, -330^o, -150^o,\ -30^o,\ 210^o,\ 390^o, 570^o, 750^o,. ..$

Example6: $\tan\theta = -\sqrt{3}$ ⇨ $\theta = arc\tan(-\sqrt{3})$

⇨ $\theta = ...,-420^o, -240^o,\ -60^o,\ 120^o,,\ 300^o, 480^o,\ ...$

Definition: It is necessary to consider the inverse trigonometric function as a singled valued. To do these, it is generally agreed to select one out of many values corresponding to the given values of p. This selected value is called the principal value.

Note: When only the principal value is called for, we write $Arc\sin p$, $Arc\cos p$, $Arc\tan p$ respectively.

Selecting the principal value:

1. When p is positive or zero and the solution exist, the principal value is the value of θ which lies between $0\ and\ \frac{\pi}{2}$ inclusive.

   $0 \le Arc\sin p \le \frac{\pi}{2}$

   $0 \le Arc\cos p \le \frac{\pi}{2}$

   $0 \le Arc\tan p \le \frac{\pi}{2}$

2. When p is negative and the solution exist

   $-\frac{\pi}{2} \le Arc\sin p < 0$

   $\frac{\pi}{2} \le Arc\cos p \le \pi$

   $-\frac{\pi}{2} \le Arc\tan p < 0$

Example 1

$i)$ $\sin\theta = \frac{1}{2}$ ⇨ $\theta = Arc\sin\left(\frac{1}{2}\right) = 30^o$

$ii)$ $\cos\theta = \frac{1}{\sqrt{2}}$ ⇨ $\theta = Arc\cos(\frac{1}{\sqrt{2}}) = 45^o$

$iii)$ $\tan\theta = \frac{1}{\sqrt{3}}$ ⇨ $\theta = Arc\tan\left(\frac{1}{\sqrt{3}}\right) = 30^o$

$iv)$ $\sin\theta = -0.515$ ⇨ $\theta = Arc\sin(-0.515) = -31^o$

$v)$ $\cos\theta = -\frac{1}{\sqrt{3}}$ ⇨ $\theta = Arc\cos\left(-\frac{1}{\sqrt{3}}\right) = 125.26^o$

$vi)$ $\tan\theta = -\sqrt{3}$ ⇨ $\theta = Arc\tan(-\sqrt{3}) = -60^o$

General expression for angles with given trigonometric ratio:

The general expression for equations in which the trigonometric ratio of the unknown angles occurs is given as follows.

1. $\sin\theta = p \quad \Rightarrow \quad \theta = n\pi + (-1)^n\theta^I$ where $n\varepsilon Z, and\ \theta^I\ the\ principal\ value$
2. $\cos\theta = p \quad \Rightarrow \quad \theta = 2n\pi \pm \theta^I$ where $n\varepsilon Z, and\ \theta^I\ the\ principal\ value$
3. $\tan\theta = p \quad \Rightarrow \quad \theta = n\pi + \theta^I$ where $n\varepsilon Z, and\ \theta^I\ the\ principal\ value$

Example1: Given that $Arc\tan 3x + Arc\tan 2x = \frac{3\pi}{4}$, find the value of x

Solution:

$$Arc\tan 3x + Arc\tan 2x = \frac{3\pi}{4}$$

$\Rightarrow \quad \tan[Arc\tan 3x + Arc\tan 2x] = \tan(\frac{3\pi}{4})$

$\Rightarrow \quad \frac{3x+2x}{1-(3x)(2x)} = -1 \quad \Rightarrow \quad x = 1 \quad or \quad x = -\frac{1}{6}$

Example2: Show that $Arc\tan\frac{1}{4} + Arc\tan\frac{2}{9} = Arc\tan\frac{1}{2}$

Solution:

$$Arc\tan\frac{1}{4} + Arc\tan\frac{2}{9}$$

$\Rightarrow \quad \tan\left[Arc\tan\frac{1}{4} + Arc\tan\frac{2}{9}\right] = \frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\right)\left(\frac{2}{9}\right)} = \frac{1}{2}$

$\Rightarrow \quad Arc\tan\frac{1}{4} + Arc\tan\frac{2}{9} = Arc\tan\frac{1}{2}$

Hence the proof

Example3: Evaluate $Arc\sin\frac{1}{\sqrt{5}} + Arc\sin\frac{1}{\sqrt{10}}$

Solution:

Let $A = Arc\sin\frac{1}{\sqrt{5}}$ $\qquad B = Arc\sin\frac{1}{\sqrt{10}}$

$\Rightarrow \quad \sin A = \frac{1}{\sqrt{5}}$ $\qquad \sin B = \frac{1}{\sqrt{10}}$

$\Rightarrow \quad \sin^2 A = \frac{1}{5}$ $\qquad \sin^2 B = \frac{1}{10}$

$\Rightarrow \quad \cos^2 A = \frac{4}{5}$ $\qquad \cos^2 B = \frac{9}{10}$

$\Rightarrow \quad \cos A = \frac{2}{\sqrt{5}}$ $\qquad \cos B = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \sin(A+B) = \sin A\cos B + \cos A\sin B$

⇨ $\sin(A+B) = \left(\frac{1}{\sqrt{5}} * \frac{3}{\sqrt{10}}\right)\left(\frac{2}{\sqrt{5}} * \frac{1}{\sqrt{10}}\right) = \frac{1}{\sqrt{2}}$

⇨ $\sin(A+B) = \frac{1}{\sqrt{2}}$

⇨ $A + B = Arc\sin(\frac{1}{\sqrt{2}})$

$Arc\sin\frac{1}{\sqrt{5}} + Arc\sin\frac{1}{\sqrt{10}} = Arc\sin\frac{1}{\sqrt{2}}$

Hence the proof

Example1: Find the value of x given that $Arc\tan(2x+1) - Arc\tan(2x-1) = Arc\tan(\frac{1}{8})$

Solution:

Trigonometric Equations:

Here we shall consider and solve equations in which the trigonometric ratios of the unknown angles occur. The solution of such equations can be reduce to the solution of one or more equations of the form $\sin\theta = p,\ \cos\theta = p,\ \text{or}\tan\theta = p,$

Example1. Find the solution of $8\cos^2 x + 6\sin x - 9 = 0 \qquad 0 \le x \le 2\pi$

Solution:

$8\cos^2 x + 6\sin x - 9 = 0$

⇨ $8(1-\sin^2 x) + 6\sin x - 9 = 0$

⇨ $8\sin^2 x - 6\sin x + 1 = 0$

Let $\sin x = t$

⇨ $8t^2 - 6t + 1 = 0$

⇨ $(4t-1)(2t-1) = 0$

⇨ $t = \frac{1}{4}\ \ or\ \ t = \frac{1}{2}$

⇨ $\sin x = \frac{1}{4}\ \ or\ \ \sin x = \frac{1}{2}$

⇨ $x = Arc\sin\frac{1}{4} = 14.48^o\ \ or\ \ x = Arc\sin\frac{1}{2} = 30^o$

⇨ $x = n\pi + (-1)^n 14.48^o\ \ or\ \ x = n\pi + (-1)^n 30^o$

⇨ $x = 14.48^o, 165.52^o\ \ or\ \ x = 30^o, 150^o$

Hence the required solution is $x = 14.48^o, 30^o, 150^o 165.52^o$

Example2: Solve the equation $2\cos^2 x = 2 - \sin x \qquad 0 \le x \le 2\pi$

Solution:

$2\cos^2 x + \sin x - 2 = 0$

⇨ $2(1-\sin^2 x) + \sin x - 2 = 0$

⇨ $2\sin^2 x - \sin x = 0$

Let $\sin x = t$

⇨ $2t^2 - t = 0$

⇨ $t(2t - 1) = 0$

⇨ $t = 0 \ \ or \ \ t = \frac{1}{2}$

⇨ $\sin x = 0 \ \ or \ \ \sin x = \frac{1}{2}$

⇨ $x = Arc\sin 0 = 0^o \ \ or \ \ x = Arc\sin\frac{1}{2} = 30^o$

⇨ $x = n\pi + (-1)^n 0^o \ \ or \ \ x = n\pi + (-1)^n 30^o$

⇨ $x = 0^o, 180^o, 360^o \ \ or \ \ x = 30^o, 150^o$

Hence the required solutions are $x = 0^o, 30^o, 150^o, \ 180^o, 360^o$,

Exercises: Solve the following equations for $0 \le x \le 2\pi$

1. $3\sin^2 x + 2\cos x + 1 = 0$
2. $4\tan^2 x + 5\tan x + 1 = 0$
3. $3\sec^2 x = 2\tan x + 4 = 0$
4. $\sin 2x = \tan x$
5. $\cos 2x = \cos x$

Expression of the form $a\cos\theta + b\sin\theta$

Expression of the form $a\cos\theta + b\sin\theta$ may be written in the form involving either the sine or cosine of the other angle, i.e. we can either express $a\cos\theta + b\sin\theta$ in the form,

$R\sin(\theta \pm \alpha) \ \ or \ \ R\cos(\theta \pm \alpha)$ where R is a constant.

a) The form $R\sin(\theta + \alpha)$

If $a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$

⇨ $a\cos\theta + b\sin\theta = R[\sin\theta\cos\alpha + \cos\theta\sin\alpha]$

⇨ $a\cos\theta + b\sin\theta = R\cos\theta\sin\alpha + R\sin\theta\cos\alpha]$

Comparing the coefficients we have

$a = R\sin\alpha \qquad and \qquad b = R\cos\alpha$

⇨ $a^2 = R^2\sin^2\alpha \quad and \qquad b^2 = R^2\cos^2\alpha$

⇨ $a^2 + b^2 = R^2[\sin^2\alpha + \cos^2\alpha] = R^2$

⇨ $R = \sqrt{a^2 + b^2}$

$a = R\sin\alpha \qquad\qquad b = R\cos\alpha \qquad\qquad \tan\alpha = \frac{a}{b}$

$\Rightarrow \quad \sin\alpha = \frac{a}{R} \qquad \Rightarrow \quad \cos\alpha = \frac{b}{R} \qquad \tan\alpha = \frac{a}{b}$

$\Rightarrow \quad \alpha = arc\sin\left(\frac{a}{R}\right) \qquad \Rightarrow \quad \alpha = arc\cos\left(\frac{b}{R}\right) \qquad \alpha = arc\tan\left(\frac{a}{b}\right)$

Example1: Express the following in the form $R\sin(\theta + \alpha)$

$i)$ $3\cos\theta + \sin\theta$ $\qquad ii)$ $3\cos\theta - 2\sin\theta$

Solution:

$i)$ $3\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

$a = 3,\ b = 1 \quad \Rightarrow \quad R = \sqrt{10}$

$\Rightarrow \quad \sin\alpha = \frac{3}{\sqrt{10}} \qquad \Rightarrow \quad \alpha = ArcSin\left(\frac{3}{\sqrt{10}}\right) \approx 72^0,$

$\cos\alpha = \frac{1}{\sqrt{10}} \qquad \Rightarrow \quad \alpha = Arccos\left(\frac{1}{\sqrt{10}}\right) \approx 72^0$

$\tan\alpha = \frac{3}{1} \qquad \Rightarrow \quad \alpha = Arctan\left(\frac{3}{1}\right) \approx 72^0$

Hence $3\cos\theta + \sin\theta = R\sin(\theta + 72^0)$

$ii)$ $3\cos\theta - 2\sin\theta = R\sin(\theta + \alpha)$

$a = 3,\ b = -2 \quad \Rightarrow \quad R = \sqrt{13}$

$\Rightarrow \quad \sin\alpha = \frac{3}{\sqrt{13}} \qquad \Rightarrow \quad \alpha = ArcSin\left(\frac{3}{\sqrt{13}}\right) \approx 56^0,$

$\cos\alpha = \frac{-2}{\sqrt{13}} \qquad \Rightarrow \quad \alpha = Arccos\left(\frac{-2}{\sqrt{13}}\right) \approx 56^0$

$\tan\alpha = \frac{3}{-2} \qquad \Rightarrow \quad \alpha = Arctan\left(\frac{3}{-2}\right) \approx 56^0$

Hence $3\cos\theta + \sin\theta = R\sin(\theta + 56^0)$

b) *The form* $R\sin(\theta - \alpha)$

If $\quad a\cos\theta + b\sin\theta = R\sin(\theta - \alpha)$

$\Rightarrow \quad a\cos\theta + b\sin\theta = R[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$

$\Rightarrow \quad a\cos\theta + b\sin\theta = -R\sin\alpha\cos\theta + R\cos\alpha\sin\theta]$

Comparing the coefficients we have

$a = -R\sin\alpha \qquad b = R\cos\alpha$

$\Rightarrow \quad a^2 = R^2\sin^2\alpha \qquad \Rightarrow \quad b^2 = R^2\cos^2\alpha$

$\Rightarrow \quad a^2 + b^2 = R^2[\sin^2\alpha + \cos^2\alpha] = R^2$

$\Rightarrow \quad R = \sqrt{a^2 + b^2}$

$a = -R\sin\alpha \qquad b = R\cos\alpha \qquad \tan\alpha = -\frac{a}{b}$

$\Rightarrow \quad \sin\alpha = -\frac{a}{R} \qquad \Rightarrow \quad \cos\alpha = \frac{b}{R} \qquad \Rightarrow \quad \tan\alpha = -\frac{a}{b}$

⇨ $\alpha = arc\sin(-\frac{a}{R})$ ⇨ $\alpha = arc\cos(\frac{b}{R})$ ⇨ $\alpha = arc\tan(-\frac{a}{b})$

Example1: Express the following in the form $R\sin(\theta - \alpha)$

$i)$ $2\sin\theta - 3\cos\theta$ $ii)$ $\cos\theta - \sin\theta$

Solution:

$i)$ $2\sin\theta - 3\cos\theta = -3\cos\theta + 2\sin\theta = R\sin(\theta + \alpha)$

$a = -3,\ b = 2$ ⇨ $R = \sqrt{13}$

⇨ $\sin\alpha = \frac{3}{\sqrt{13}}$ $\cos\alpha = \frac{2}{\sqrt{13}}$ $\tan\alpha = \frac{3}{2}$

⇨ $\alpha = Arcsin\left(\frac{3}{\sqrt{13}}\right) \approx 56^0$, ⇨ $\alpha = Arccos\left(\frac{2}{\sqrt{13}}\right) \approx 56^0$ ⇨ $\alpha = Arctan\left(\frac{3}{2}\right) \approx 56^0$

Hence $3\cos\theta + \sin\theta = R\sin(\theta + 56^0)$

$ii)$ $\cos\theta - \sin\theta = R\sin(\theta - \alpha)$

$a = 1,\ b = -1$ ⇨ $R = \sqrt{2}$

⇨ $\sin\alpha = -\frac{1}{\sqrt{2}}$ $\cos\alpha = \frac{-1}{\sqrt{2}}$ $\tan\alpha = 1$

⇨ $\alpha = Arcsin\left(-\frac{1}{\sqrt{2}}\right) \approx -45^0$ ⇨ $\alpha = Arccos\left(\frac{-1}{\sqrt{2}}\right) \approx 135^0$ ⇨ $\alpha = Arctan(1) \approx 45^0$

Hence $3\cos\theta + \sin\theta = R\sin(\theta + 225^0)$

$c)$ *The form* $\boldsymbol{Rcos(\theta + \alpha)}$

If $a\cos\theta + b\sin\theta = R\cos(\theta + \alpha)$

⇨ $a\cos\theta + b\sin\theta = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$

⇨ $a\cos\theta + b\sin\theta = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta]$

Comparing the coefficients we have

$a = R\cos\alpha$ *and* $b = -R\sin\alpha$

⇨ $a^2 = R^2\cos^2\alpha$ *and* $b^2 = R^2\sin^2\alpha$

⇨ $a^2 + b^2 = R^2[\sin^2\alpha + \cos^2\alpha] = R^2$

⇨ $R = \sqrt{a^2 + b^2}$

$a = R\cos\alpha$ $b = -R\sin\alpha$ $\tan\alpha = -\frac{b}{a}$

⇨ $\cos\alpha = \frac{a}{R}$ ⇨ $\sin\alpha = -\frac{b}{R}$ $\tan\alpha = -\frac{b}{a}$

⇨ $\alpha = arc\cos(\frac{a}{R})$ $\alpha = arc\sin(-\frac{b}{R})$ $\alpha = arc\tan(-\frac{b}{a})$

Example1: Express the following in the form $R\sin(\theta - \alpha)$

$i)$ $3\cos\theta - 4\sin\theta$ $ii)$ $\sin\theta + 3\cos\theta$

Solution:

$i)$ $3\cos\theta - 4\sin\theta = R\sin(\theta + \alpha)$

$a = 3,\ b = -4 \quad ⇨\ R = \sqrt{25} = 5$

⇨ $\sin\alpha = \frac{4}{5}$ $\quad\quad$ $\cos\alpha = \frac{3}{5}$ $\quad\quad$ $\tan\alpha = \frac{4}{3}$

⇨ $\alpha = Arcsin\left(\frac{4}{5}\right) \approx 53^0$ $\quad$ ⇨ $\alpha = Arccos\left(\frac{3}{5}\right) \approx 53^0$ $\quad$ ⇨ $\alpha = Arctan\left(\frac{4}{3}\right) \approx 53^0$

Hence $3\cos\theta - 4\sin\theta = R\sin(\theta + 53^0)$

$ii)$ $\sin\theta + 3\cos\theta = 3\cos\theta + \sin\theta = R\sin(\theta - \alpha)$

$a = 3,\ b = 1 \quad ⇨\ R = \sqrt{10}$

⇨ $\sin\alpha = -\frac{1}{\sqrt{10}}$ $\quad\quad$ $\cos\alpha = \frac{3}{\sqrt{10}}$ $\quad\quad$ $\tan\alpha = -\frac{1}{3}$

⇨ $\alpha = Arcsin\left(-\frac{1}{\sqrt{10}}\right) \approx -18^0$

⇨ $\alpha = Arccos\left(\frac{3}{\sqrt{10}}\right) \approx 18^0$

⇨ $\alpha = Arctan\left(-\frac{1}{3}\right) \approx -18^0$

Hence $\sin\theta + 3\cos\theta = R\sin(\theta + 341^0)$

$\boldsymbol{d})$ *The form* $\boldsymbol{Rcos(\theta - \alpha)}$

If $\quad a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$

⇨ $\quad a\cos\theta + b\sin\theta = R[\cos\theta\cos\alpha + \sin\theta\sin\alpha]$

⇨ $\quad a\cos\theta + b\sin\theta = R\cos\alpha\cos\theta + R\sin\alpha\sin\theta]$

Comparing the coefficients we have

$a = R\cos\alpha \quad and\ b = R\sin\alpha$

⇨ $\quad a^2 = R^2\cos^2\alpha \quad and\ \ b^2 = R^2\sin^2\alpha$

⇨ $\quad a^2 + b^2 = R^2[\sin^2\alpha + \cos^2\alpha] = R^2$

⇨ $\quad R = \sqrt{a^2 + b^2}$

$a = R\cos\alpha \quad ⇨ \quad \cos\alpha = \frac{a}{R} \quad\quad b = R\sin\alpha \quad ⇨ \quad \sin\alpha = \frac{b}{R}$ and $\tan\alpha = \frac{b}{a}$

⇨ $\quad \alpha = arc\cos(\frac{a}{R}) \quad or \quad \alpha = arc\sin(\frac{b}{R}) \quad or \quad \alpha = arc\tan(\frac{b}{a})$

Example1: Express the following in the form $R\sin(\theta - \alpha)$

$i)$ $4\cos\theta - \sin\theta$ $\quad\quad$ $ii)$ $7\sin\theta + \cos\theta$

Solution:

$i)$ $4\cos\theta - \sin\theta = R\sin(\theta + \alpha)$

$a = 4,\ b = -1 \quad ⇨\ R = \sqrt{17}$

$\Rightarrow \quad \sin\alpha = \frac{-1}{\sqrt{17}} \quad \Rightarrow \quad \alpha = Arcin\left(\frac{-1}{\sqrt{17}}\right) \approx -14^0$

$\cos\alpha = \frac{4}{\sqrt{17}} \quad \Rightarrow \quad \alpha = Arccos\left(\frac{4}{\sqrt{17}}\right) \approx \ 14^0$

$\tan\alpha = \frac{-1}{4} \quad \Rightarrow \quad \alpha = Arctan\left(\frac{-1}{4}\right) \approx \ -14^0$

Hence $3\cos\theta - 4\sin\theta = R\sin(\theta + 346^0)$

$ii) \ \ 7\sin\theta + \cos\theta = \ \cos\theta + 7\sin\theta = R\sin(\theta - \alpha)$

$a = 1, \ b = 7 \quad \Rightarrow \quad R = \ \sqrt{50} = 5\sqrt{2}$

$\Rightarrow \quad \sin\alpha = \frac{7}{5\sqrt{2}} \quad \Rightarrow \quad \alpha = Arcsin\left(\frac{7}{5\sqrt{2}}\right) \approx \ 82^0$

$\cos\alpha = \frac{1}{5\sqrt{2}} \quad \Rightarrow \quad \alpha = Arccos\left(\frac{1}{5\sqrt{2}}\right) \approx \ 82^0$

$\tan\alpha = \frac{7}{1} \quad \Rightarrow \quad \alpha = Arctan(7) \approx \ 82^0$

Hence $7\sin\theta + \cos\theta = R\sin(\theta + 82^0)$

The equation $\boldsymbol{cos\,\theta \pm b\,sin\,\theta = c}$ ***:***

Equation of the type $a\,cos\,\theta \pm b\,sin\,\theta = c$, where a, b, c are constants, may be solved by solving the expression of the form $a\,cos\,\theta \pm b\,sin\,\theta$ in the form $R\cos(\theta \mp \beta)$ or $R\sin(\theta \pm \beta)$. If we always write the equation with a positive, we need only use the first form.

Example1: Find the value of θ between 0 and 360^o which satisfy the equations

$i) \ \ 3\cos\theta - 4\sin\theta = -2.5 \qquad\qquad ii) \ \ -7\sin\theta + 24\cos\theta = 12.5$

Solution:

$i) \ \ 3\cos\theta - 4\sin\theta = R\sin(\theta + \alpha) = -2.5$

$a = 3, \ b = -4 \quad \Rightarrow R = \ \sqrt{25} = 5$

$\Rightarrow \quad \sin\alpha = \frac{3}{5} \quad \Rightarrow \quad \alpha = Arcsin\left(\frac{3}{5}\right) \approx \ 37^0$

$\cos\alpha = \frac{-4}{5} \quad \Rightarrow \quad \alpha = Arccos\left(\frac{-4}{5}\right) \approx \ 143^0$

$\tan\alpha = \frac{3}{-4} \quad \Rightarrow \quad \alpha = Arctan\left(\frac{3}{-4}\right) \approx \ -37^0 = 143^0$

Hence $\ \ 3\cos\theta - 4\sin\theta = 5\sin(\theta + 143^0) = -2.5$

$\Rightarrow \quad 5\sin(\theta + 143^0) = -2.5$

$\Rightarrow \quad \sin(\theta + 143^0) = -0.5$

$\Rightarrow \quad \theta + 143^0 = Arcsin\left(-\frac{1}{2}\right) = -30^o$

$\Rightarrow \quad \theta + 143^0 = -30^o$

$\Rightarrow \quad \theta = -30^o - 143^0 = -173^o = 187^0$

$i) \quad -7\sin\theta + 24\cos\theta = \ \ 24\cos\theta - 7\sin\theta = R\sin(\theta+\alpha) = 12.5$

$\Rightarrow \quad 24\cos\theta - 7\sin\theta = R\sin(\theta+\alpha) = 12.5$

$a = 24,\ b = -7 \quad \Rightarrow \ R = \ \sqrt{625} = 25$

$\Rightarrow \quad \sin\alpha = \frac{24}{25} \qquad \Rightarrow \quad \alpha = Arcsin\left(\frac{24}{25}\right) \approx \ 74^0$

$\cos\alpha = \frac{-7}{25} \qquad \Rightarrow \quad \alpha = Arccos\left(\frac{-7}{25}\right) \approx \ 106^0$

$\tan\alpha = \frac{24}{-7} \qquad \Rightarrow \quad \alpha = Arctan\left(\frac{24}{-7}\right) \approx \ -74^0 = 106^0$

Hence $\ 24\cos\theta - 7\sin\theta = 25\sin(\theta + 106^0) = 12.5$

$\Rightarrow \quad 25\sin(\theta + 106^0) = 12.5$

$\Rightarrow \quad \sin(\theta + 106^0) = 0.5$

$\Rightarrow \quad \theta + 106^0 = Arcsin(0.5) = 30^o$

$\Rightarrow \quad \theta + 106^0 = 30^o$

$\Rightarrow \quad \theta = 30^o - 106^0 = -76^o = 104^0$

Section B

– ***Exponential functions and Logarithmic functions:***

– ***Algebraic functions and Rational functions***:

Course Outline:

Exponential functions: *Definition of a^x for any positive number a and any real number x, graphs of exponential functions, Laws of Exponents(indices), the number e, Natural exponential function.*

Logarithmic functions: *Definition of $log_a x$ for any positive number a and any positive real number x, graphs of logarithmic functions, Laws of Logarithms, the number e, Natural logarithmic function.*

Algebraic functions: *Polynomials, Division algorithms, Long division, Synthetic division, Factor theorem, remainder theorem.*

Rational functions: *Asymptotes, Partial fraction decomposition, Roots of a rational functions, finding the domain.*

a) Index functions:

Definition: Any function of the form

$$y = a^x \qquad . \ . \ . \ (1)$$

Is called an exponential function, where a is any positive real number called the base, and x is any real number called the index (or exponent or power).

Example: $i)\ y = 2^x \qquad ii)\ y = \left(\frac{2}{3}\right)^x \qquad iii)\ y = 2^{-(x+1)} \qquad iv)\ y = \left(\frac{5}{2}\right)^{-x}$

Laws of Indices;

For any positive real numbers a, b, and any real numbers x, and y, the following postulate hold

For any positive number a, b, and any real numbers x, and y, the following postulate hold good.

1) $a^x * a^y = a^{x+y}$

Proof:

$a^x = a * a * a * \ . \ . \ . * a\ (x\ factors) a^y = a * a * a * \ . \ . \ . * a\ (y\ factors)$

⇨ $(a^x) * (a^y) = [a * a * a * \ . \ . \ . * a\ (x\ factors)][a * a * a * \ . \ . \ . * a\ (y\ factors)$

$= a * a * a * \ . \ . \ . * a\ (x + y\ factors)$

$= a^{x+y}$

2) $\frac{a^x}{a^y} = a^{x-y}$

Proof:

$\frac{a^x}{a^y} = \frac{a*a*a* \ . \ . \ . \ *a\ (x\ factors)}{a*a*a* \ . \ . \ . *a\ \ (y\ factors)}$

$= a * a * a * \ . \ . \ .* a\ (x - y\ factors)$

$= a^{x-y}$

3) $(a^x)^y = (a^y)^x = a^{xy}$

Proof:

$a^x = a * a * a * \ . \ . \ . * a \quad (x\ factors)$

$(a^x)^y = \big(a * a * a * \ . \ . \ . * a \quad (x\ factors)\big)\big(a * a * a * \ . \ . \ . * a\ \ (x\ factors)\big) \ . \ . \ .$

$(a * a * a * \ . \ . \ . * a \quad (x\ factors))(y\ \ factors)$

$= (a * a * a * \ . \ . \ . * a \quad (xy\ factors) = a^{xy}$

4) $a^0 = 1$

Proof:

The proof follows from (2) when $x = y$

5) $a^{-x} = \frac{1}{a^x}$

Proof:

$a^x * a^{-x} = a^{x+(-x)}$ from (1)

⇨ $a^x * a^{-x} = a^{x+(-x)} = a^{x-x} = a^0 = 1$ . . . (a)

Also $a^x * \left(\frac{1}{a^x}\right) = \frac{a^x}{a^x} = 1$ . . . (b)

From (a) and (b) we have that $a^{-x} = \frac{1}{a^x}$

Hence the proof

6) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$ and $a^x * b^x = (a * b)^x$

7) If $a^x = a^y$ $then\ x = y$

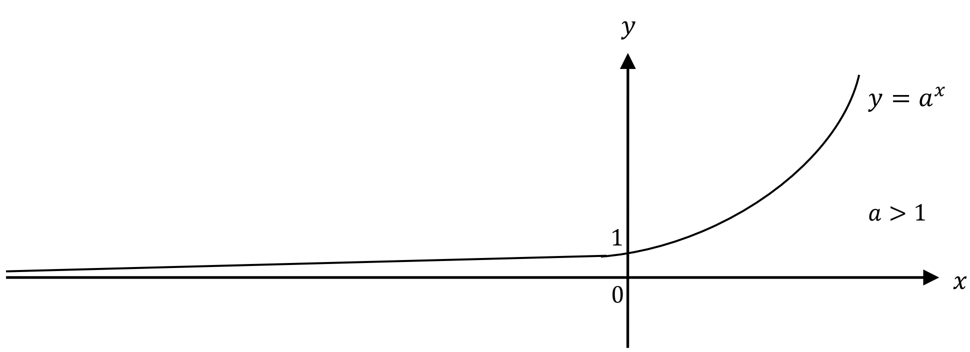
8) $a^{\frac{1}{n}} = \sqrt[n]{a}$ $and\ a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

The graph of exponential function:

1) This is a typical graph of Index function $y = a^x$ $for\ a > 1$.

If $a > 1$ and x_1 and x_2 are real numbers such that $x_1 < x_2$,then $a^{x_1} < a^{x_2}$ i.e. $f(x_1) < f(x_2)$

⇨ If $a > 1$, then $y = a^x$ is increasing throughout its domain.



Observation: If $a > 1$, we can observe that, as x decreases through negative values, the graph approaches the x-axis but never intersect it since $a^x > 0$ for all x. That means that the x-axis is a horizontal asymptote for the graph. As x increases through positive values, the graph rises very rapidly.

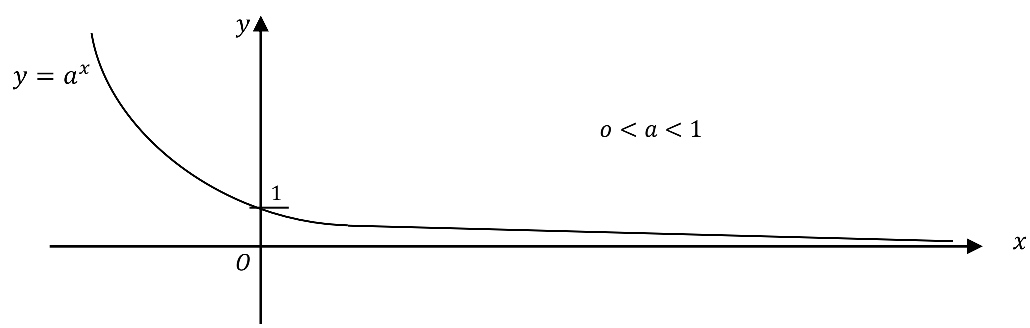
The y-intercept is always equal to 1, since $a^0 = 1$.

2) This is a typical graph of exponential function $y = a^x$ for $0 < a < 1$.

If $0 < a < 1.$ and x_1 and x_2 are real numbers such that $x_1 < x_2$, then $a^{x_1} > a^{x_2}$

i.e. $f(x_1) > f(x_2)$

⇨ If $0 < a < 1$, then $y = a^x$ is decreasing throughout its domain.



Observation: If$0 < a < 1$, we can observe that, as x increases through the positive values the graph approaches the x-axis but never intersect it. Here also the x-axis is the horizontal asymptote for the graph. As x decreases through the negative values the graph rises rapidly.

The y-intercept is always equal to 1, since $a^0 = 1$

The number e: The quantity e is defined as being the limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$, and its value is approximated to be 2.7182818281828 . . .

The Index function with base e is called natural Index.

e.g. $y = e^{3x+1}$

Normal and Index form of writing numbers

Normal form	Index form	Normal form	Index form	Normal form	Index form
2	2^1	3	3^1	5	5^1
4	2^2	9	3^2	25	5^2
8	2^3	27	3^3	125	5^3
16	2^4	81	3^4	625	5^4
32	2^5	243	3^5	3125	5^5
64	2^6	729	3^6	15625	5^6
.	.	.	.	.	.
.	.	.	.	.	.

Examples:

$\boldsymbol{a}$) *index expression:*

Simplify the following index expressions;

$i)\ (64)^{\frac{2}{3}} \times (9)^{\frac{1}{2}} \times (2)^{-3}$ $\quad ii)\ (125)^{-\frac{1}{3}} \times (49)^{-\frac{1}{2}} \times 10^0$

Solution:

$i)\ (64)^{\frac{2}{3}} \times (9)^{\frac{1}{2}} \times (2)^{-3}$

$= (2^6)^{\frac{2}{3}} \times (3^2)^{\frac{1}{2}} \times (2)^{-3}$

$= 2^4 \times 3 \times 2^{-3}$

$= 2^4 \times 2^{-3} \times 3$

$= 2 \times 3 = 6$

$ii)\ (125)^{-\frac{1}{3}} \times (49)^{-\frac{1}{2}} \times 10^0$

$= (5^3)^{-\frac{1}{3}} \times (7^2)^{-\frac{1}{2}}$

$= 5^{-1} \times 7^{-1}$

$= \frac{1}{5} \times \frac{1}{7}$

$= \frac{1}{35}$

Exercises:

Simplify the following index expressions .

$i)\ (0.027)^{-\frac{1}{3}}$ $\quad ii)\ (125)^{-\frac{1}{3}} \times (64)^{\frac{1}{3}} \times 81^0$ $\quad iii)\ (625)^{\frac{3}{8}} \times (5)^{\frac{1}{2}} \div 25$

$iv)\ (27)^{-\frac{1}{3}} \times (64)^{-\frac{1}{2}} \times 4^{-\frac{1}{2}}$ $\quad v)\ \left(\frac{4}{25}\right)^{-\frac{1}{2}} \times (2)^4 \div \left(\frac{1}{8}\right)^{-2}$ $\quad vi)\ \frac{1}{3^{5n}} \times (9)^{n-1} \times (27)^{n+1}$

$vii)\ \frac{4^{\sqrt{3}} \times 16^{\sqrt{3}}}{4^{3\sqrt{3}-1}}$ $\quad viii)\ (8)^{\frac{1}{3}} \times (25)^{\frac{1}{2}} \times 8^0$ $\quad ix)\ \frac{(27)^{\frac{2}{3}} \times (32)^{-\frac{2}{5}}}{(64)^{-\frac{1}{2}}}$ $\quad x)\ (27)^{m+2} \times (4)^m \div 6^{2m}$

$xi)\ 2a^{-\frac{1}{2}} \times t^{\frac{3}{2}} \div 2a^{-\frac{3}{2}} \times t^{-\frac{1}{2}}$ $\quad xii)\ \frac{a^{m-\frac{n}{3}} \times b^{3m-n}}{(ab^3)^m \sqrt[3]{a^n}}$

b) *Index equations:*

Example1: Solve the following index equations.

$i)\ 3^x = 81$ $\quad ii)\ 4^{2x} = 2^{6x-2}$ $\quad iii)\ 2^x + 2^{x-1} = 48$

Solution:

$i)\ 3^x = 81$

⇨ $3^x = 3^4$

⇨ $x = 4$

$ii)\ 4^{2x} = 2^{6x-2}$

⇨ $(2^2)^{2x} = 2^{6x-2}$

⇨ $6x - 2 = 4x$

⇨ $x = 1$

(iii) $2^x + 2^{x-1} = 48$

⇨ $2^x + (2^x)(2^{-1}) = 48$

⇨ $2(2^x) + 2^x = 96$

⇨ $3(2^x) = 96$

⇨ $2^x = 32 = 2^5$ ⇨ $x = 5$

Exercises:

Solve the following index equations.

$i)\ 4^{2x-1} - \frac{1}{8} = 0$ $\quad ii)\ 4^x = 4\sqrt{2}$ $\quad iii)\ 4^x = 0.25$ $\quad iv)\ 9^x = \frac{1}{729}$

$v)\ 32^x = 0.5$ $\quad vi)\ 8^x = 0.25$ $\quad vii)\ 10^x = 0.0001$ $\quad viii)\ 3^{2x} = 243$

$ix)\ 4^x = 2^{\frac{1}{2}} \times 8$ $\quad x)\ 3^{2(x+1)} = 27^x$ $\quad xi)\ 5^{x-1} = 0.2$ $\quad xii)\ 2^{3x+1} = 1$

$xiii)\ 9^{2x-1} = \frac{81^{x-2}}{3^x}$ $\quad xiv)\ \frac{1}{81^{x-2}} = 27^{1-x}$ $\quad xv)\ \frac{9^{2x-3}}{3^{x+3}} = 1$ $\quad xvi)\ 8^{\frac{x}{2}} = 2^{\frac{3}{8}} \times 4^{\frac{3}{4}}$

$xvii)\ 9^{2x-1} \times 3^{3x+1} = 27^{x+3}$

c) *index equations reducible to quadratic equation:*

Example1: Solve the following index equations.

$i)\ 2(2^{2x}) - 5(2^x) + 2 = 0$ $\quad ii)\ 2^{2x} + 4(2^x) - 32 = 0$

Solution:

$i)\ 2(2^{2x}) - 5(2^x) + 2 = 0$

⇨ $2(2^x)^2 - 5(2^x) + 2 = 0$

Let $2^x = t$

⇨ $2t^2 - 5t + 2 = 0$

⇨ $(2t - 1)(t - 2) = 0$

⇨ $t = \frac{1}{2}$ or $t = 2$

⇨ $2^x = \frac{1}{2}$ or $2^x = 2$

⇨ $2^x = 2^{-1}$ or $2^x = 2^1$

⇨ $x = -1$ or $x = 1$

$ii)\ 2^{2x} + 4(2^x) - 32 = 0$

⇨ $(2^x)^2 + 4(2^x) - 32 = 0$

Let $2^x = t$

⇨ $t^2 + 4t - 32 = 0$

⇨ $(t - 4)(t + 8) = 0$

⇨ $t = 4$ or $t = -8$

⇨ $2^x = 4$

⇨ $2^x = 2^2$

⇨ $x = 2$

Exercises:

Solve the following equations:

$i)\ \ 2^x + 2^{-x} = 2$ $\quad ii)\ \ 2^{2(x+1)} + 8 = 33(2^x)$ $\quad iii)\ \ 2^{2x+1} - 15(2^x) = 8$

$iv)\ \ 3^{2(x-1)} - 8(3^{x-2}) = 1$ $\quad v)\ \ 4^x - 6(2^x) - 16 = 0$ $\quad vi)\ \ 3^{2x+1} + 26(3^x) - 9 = 0$

$vii)\ 2^{2x} - 5(2^x) + 4 = 0$ $\quad viii)\ \ 2^{x^2} - \frac{1}{4}(8^x) = 0$ $\quad ix)\ \ 3^{x^2} = 9^{x+4}$

$x)\ \ 2^{2x+1} - 9(2^x) + 4 = 0$ $\quad xi)\ \ 2^{x+3} - 15 = 2^{1-x}$ $\quad xii)\ 3^x + 3^{1-x} = 4$

$xiii)\ 4(4^x - 2^x + 1) = 2^x$ $\quad xiv)\ \ 3^{2x-3} - 4(3^{x-2}) + 1 = 0$ $\quad xv)\ \ 5^{2x} + 1 = 26(5^{x-1})$

$xvi)\ \ 9^x - 4(3^x) + 3 = 0$ $\quad xvii)\ \ 27^{x-3} = 3(9^{x-2})$ $\quad xviii)\ 6(3^x + 3^{-x}) = 20$

$xix)\ 3^{2x} - 3^{x+2} = 3^{x+1} - 27$ $\quad xx)\ 5^{2x} + 1 = 26(5^{x-1})$

$\boldsymbol{d})$ *Index equations reducible to simultaneous linear equations:*

Example1: Solve the following simultaneous index equations

$i)\ 2^{x+y} = 8$ $\qquad ii)\ 2^{x-y} = 8$

. $3^{2x-y} = 27$ $\qquad 2^{3x-y} = 128$

Solution:

$i)\ 2^{x+y} = 8$ $\qquad ii)\ 2^{x-y} = 8$

. $3^{2x-y} = 27$ $\qquad 2^{3x-y} = 128$

⇨ $2^{x+y} = 2^3$	⇨ $2^{x-y} = 2^3$
and $3^{2x-y} = 3^3$	and $2^{3x-y} = 2^7$
⇨ $x + y = 3$	⇨ $x - y = 3$
and $2x - y = 3$	and $3x - y = 7$
⇨ $x = 2\ and\ y = 1$	⇨ $x = 2\ and - 1$

Exercises:

$i)\ 8^{4-x} = 2^{1-y}$ $\qquad ii)\ 3^{x+y} = 9^{x+y}$ $\qquad iii)\ 3^x = 9^y$

. $3^{x+1} = 9^{-y}$ $\qquad 2^{x-y} = 8^{x-3}$ $\qquad 4^{xy} = 2^{x+y}$

b) Logarithmic functions:

Definition1: Any function of the form

$$y = \log_a x \qquad . . . (2)$$

is called a logarithmic function, where a, is any positive number called the base, and x is any positive real number.

Examples:

$i)\ y = \log_2 x \qquad ii)\ y = \log_4(5 + x) \qquad iii)\ y = \log_6(2x - 3)$

Definition2: if x is any positive real number, then the unique exponent y such that $a^y = x$ is called the logarithm of x with base a denoted $y = \log_a x$

Note: To every logarithmic equation there is a corresponding exponential equation.

⇨ if $y = \log_a x$, then $a^y = x$

e.g. $\log_2 8 = 3 \quad ⇨ \quad 2^3 = 8. \qquad \log_7 49 = 2 \quad ⇨ \quad 7^2 = 49.$

$\log_4 2 = \frac{1}{2} \quad ⇨ \quad 4^{\frac{1}{2}} = 2. \qquad \log_{10} 1000 = 3 \quad ⇨ \quad 10^3 = 1000.$

Law of logarithms:

If M and N are any positive real numbers, and a is a positive real number, then the following Postulates hold good.

1) $\log_a a = 1$ and $\log_a 1 = 0$.

Proof:

The proof follows from definition 2.

2) $\log_a MN = \log_a M + \log_a N$

Proof:

Let $\log_a M = p \quad ⇨ \quad a^p = M$ . . . (1)

And $\log_a N = q \quad ⇨ \quad a^q = N$ . . . (2)

⇨ $MN = a^p * a^q = a^{p+q}$ (from rules of exponent) . . . (3)

Taking log to base a on both side of equation (3)

⇨ $\log_a MN = \log_a a^{p+q}$.

⇨ $\log_a MN = p + q$ (from definition 2)

⇨ $\log_a MN = \log_a M + \log_a N$

3) $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$

Proof:

Let $\log_a M = p \quad \Rightarrow a^p = M$ . . . (1)

And $\log_a N = q \quad \Rightarrow \quad a^q = N$ . . . (2)

$\Rightarrow \quad \frac{M}{N} = \frac{a^p}{a^q} = a^{p-q}$ (from rules of exponent)

$\Rightarrow \quad \frac{M}{N} = a^{p-q}$ . . . (3)

Taking log to base a on both side of equation (3)

$\Rightarrow \quad \log_a \left(\frac{M}{N}\right) = \log_a a^{p-q} \quad \Rightarrow \log_a \left(\frac{M}{N}\right) = p - q$

$\Rightarrow \quad \log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$

4) $\log_a M^k = k \log_a M$ where k is any arbitrary constant.

Proof;

Let $\log_a M = p \quad \Rightarrow \quad a^p = M$

$\Rightarrow \quad M = a^p$

$\Rightarrow \quad M^k = a^{pk}$

Taking log to base a on both side

$\Rightarrow \quad \log_a M^k = \log_a a^{kp}$

$\Rightarrow \quad \log_a M^k = (\text{kp})\log_a a = kp \quad \Rightarrow \quad \log_a M^k = k \log_a M$

5) If $\log_a M = \log_a N$, then $M = N$

6) Change of base.

$\log_a M = \frac{\log_b M}{\log_b a}$

Proof:

Let $\log_a M = x \quad \Rightarrow \quad a^x = M$

Taking log to base b on both side

$\Rightarrow \quad \log_b a^x = \log_b M$

$\Rightarrow \quad x \log_b a = \log_b M$

$\Rightarrow \quad x = \frac{\log_b M}{\log_b a}$,

$\Rightarrow \quad \log_a M = \frac{\log_b M}{\log_b a}$,

Hence the proof

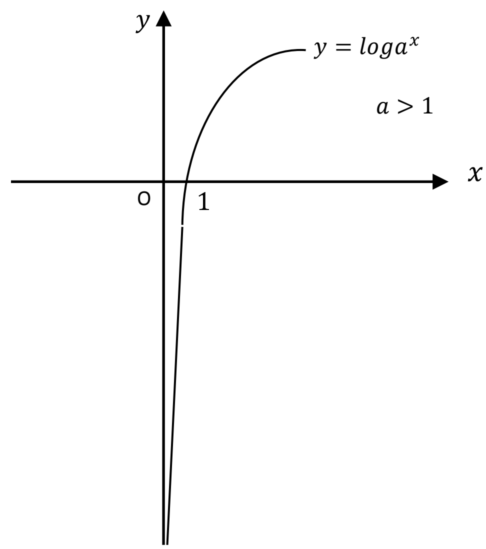
The nature of graph of logarithmic function:

1) This is a typical graph of logarithmic function $y = \log_a x$ or $x = a^y$ for $a > 1$.

If $a > 1$ and y_1 and y_2 are real numbers such that $y_1 < y_2$, then $a^{y_1} < a^{y_2}$ i.e.

$f(y_1) < f(y_2)$

⇨ If $a > 1$, then $x = a^y$ is increasing throughout its domain (i.e. as y increases, x increases).



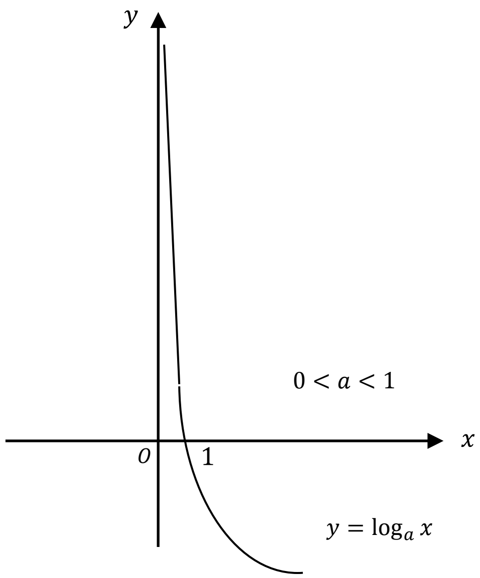
Observation: If $a > 1$, we can observe that, as y decreases through negative values, the graph approaches the y-axis but never intersect it since $a^y > 0$ for all y. That means that the y-axis is a vertical asymptote for the graph. Asy increases through positive values, the graph rises very rapidly.

The x-intercept is always equal to 1, since $a^0 = 1$.

2) This is a typical graph of Logarithmic function $y = \log_a x \quad or \quad x = a^y$, for $0 < a < 1$.

If $0 < a < 1$ and y_1 and y_2 are real numbers such that $y_1 > y_2$, then $a^{y_1} > a^{y_2}$ i.e. $f(y_1) > f(y_2)$

⇨ If $0 < a < 1$, then $x = a^y$ is decreasing throughout its domain (i.e. as y increases, x decreases)



Observations: If $0 < a < 1$, we can observe that, as x increases through the positive values, the graph approaches the y-axis but never intersect it. Here also the y-axis is the vertical asymptote for the graph. As y decreases through the negative values the graph rises rapidly The x-intercept is always equal to 1, since $a^0 = 1$

Definition: The logarithmic function with base e is called natural (or Napier an) logarithm usually denoted. $y = \ln x$

Examples:

a) *Simplifying logarithmic expressions:*

Simplify the following logarithmic expression

$i)$ $\log 64 + 2\log 5 - 2\log 40$

$ii)$ $\log_5(\frac{3}{5}) + 3\log_5(\frac{15}{2}) - \log_5(\frac{81}{8})$

Solution:

$i)$ $\log 64 + 2\log 5 - 2\log 40$

$= \log 64 + \log 5^2 - 2\log 40^2$

$= \log 64 + \log 25 - 2\log 1600$

$= \log\left(\frac{64\times25}{1600}\right) = \log\left(\frac{1600}{1600}\right) = \log 1 = 0$

$ii)$ $\log_5(\frac{3}{5}) + 3\log_5(\frac{15}{2}) - \log_5(\frac{81}{8})$

$= \log_5(\frac{3}{5}) + \log_5\left(\frac{15}{2}\right)^3 - \log_5(\frac{81}{8})$

$= \log_5(\frac{3}{5}) + \log_5(\frac{3375}{8}) - \log_5(\frac{81}{8})$

$= \log_5\left(\frac{\frac{3}{5}\times\frac{3375}{8}}{\frac{81}{8}}\right) = \log_5\left(\frac{3}{5}\times\frac{3375}{8}\times\frac{8}{81}\right)$

$= \log_5 25 = \log_5 5^2 = 2\log_5 5 = 2$

Exercises:

Simplify the following logarithmic expressions:

$i)$ $\log_9 27 - \log_3 81 - 3\log_{27} 3$

$ii)$ $\log_2(\frac{5}{3}) + \log_2(\frac{6}{7}) - \log_2(\frac{5}{28})$

$b)$ *Logarithmic equations:*

1) Solve the following equations.

$i)$ $\log_4 3^x - 4\log_4 3 = 0$ $\quad ii)$ $\log_3(4x + 1) - \log_3(3x - 5) = 2$

Solution:

$i)$ $\log_4 3^x - 4\log_4 3 = 0$

⇨ $\log_4 3^x - \log_4 3^4 = 0$

⇨ $\log_4 3^x = \log_4 3^4$

⇨ $3^x = 3^4$

⇨ $x = 4$

$ii)$ $\log_3(4x + 1) - \log_3(3x - 5) = 2$

⇨ $\log_3 (\frac{4x + 1}{3x - 5}) = 2$

⇨ $\frac{4x + 1}{3x - 5} = 3^2$

⇨ $9(3x - 5) = 4x + 1$

⇨ $x = 2$

Exercises:

Solve the following logarithmic equations

$i)$ $\log_3 x + \log_x 3 = \frac{10}{3}$ $\quad ii)$ $\log_x 9 + \log_{x^2} 3 = 2.5$ $\quad iii)$ $\log x + 2\log 5 = 2$

$iv)$ $\log(2x + 1) - \log(3x + 2) = 1$ $\quad v)$ $\log_5 0.04 = x$ $\quad vi)$ $\log_{27} x = \frac{2}{3}$

$vii)$ $\log_3(4x + 1) - \log_3(3x - 5) = 2$

$c)$ *Logarithmic equations reducible to quadratic equations:*

1) Solve the following logarithmic equations

$i)$ $2\log_5 x - \log_5(3 + 2x) = 0$ $\quad ii)$ $\log_6(x + 9) = 1 + \log_6(x + 3) - \log_6(x + 2)$.

Solution:

$i)$ $2\log_5 x - \log_5(3 + 2x) = 0$

⇨ $\log_5 x^2 - \log_5(3 + 2x) = 0$

⇨ $\log_5 x^2 = \log_5(3 + 2x)$

⇨ $x^2 = 3 + 2x$

⇨ $x^2 - 2x - 3 = 0$

⇨ $(x - 3)(x + 1) = 0$

⇨ $x = 3$ or $x = -1$

$ii)$ $\log_6(x + 9) = 1 + \log_6(x + 3) - \log_6(x + 2)$.

⇨ $\log_6(x + 9) - \log_6(x + 3) + \log_6(x + 2) = 1$

⇨ $\log_6 \frac{(x+9)(x+2)}{(x+3)} = 1$

⇨ $\frac{x^2 + 11x + 18}{x + 3} = 6$

⇨ $x^2 + 5x = 0$

$x(x + 5) = 0$

⇨ $x = 0$ or $x = -5$

Exercises:

1) Solve the following equations.

$i)$ $\log_{10}(x^2 + 9) - 2\log_{10} x = 1$

$ii)$ $\log_{10}(x^2 + 24) - \log_{10} x = \log_{10} 10$

$iii)$ $\log_4 x + \log_x 4 = 2.5$

$iv)$ $\log_3 x + \log_3(2x + 3) = 3$

$v)$ $\log_{10}(x - 8) = 1 - \log_{10}(x + 1)$

$vi)$ $\log_3(x^2 - x - 2) = 2\log_3(x + 1)$

$vii)$ $\log_{10}(x + 9) = 1 + \log_{10}(x + 1) - \log_{10}(x - 2)$

$viii)$ $\log_3(x - 7) = 2 - \log_3(x + 1)$

$xi)$ $\log_6(x + 9) = 1 + \log_6(x + 3) - \log_6(x + 2)$

$x)$ $\log_3(x - 7) = 2 - \log_3(x + 1)$

$xi)$ $\log_{10}(x^2 + 24) - \log_{10} x = \log_{10} 10$

$xii)$ $\log_3 x + \log_x 3 = \frac{10}{3}$

$xiii)$ $\log_x 9 + \log_{x^2} 3 = 2.5$

$xiv)$ $\log_{10}(x^2 + 9) - 2\log_{10} x = 1$

$d)$ *Logarithmic equations reducible to simultaneous linear equations:*

Given that $\log_4(x - 1) + \log_4(\frac{y}{2}) = 1$

$\log_2(x + 1) + \log_2(y) = 2$

Solve for x and y

Solution:

$$\log_4(x - 1) + \log_4(\tfrac{y}{2}) = 1 \qquad . \ . \ . \ (1)$$

$$\log_2(x + 1) + \log_2(y) = 2 \qquad . \ . \ . \ (2)$$

From equation (1) we have

$$y(x - 1) = 8 \qquad . \ . \ . \ (3)$$

From equation (2) we have

$$y(x + 1) = 4 \qquad . \ . \ . \ (4)$$

Solving (3) and (4) simultaneously we have

$$x = -3, \ y = 4$$

General exercises:

1) Show that $\log_a(a^2 - x^2) = 2 + \log_a(1 - \frac{x^2}{a^2})$

2) Given that, $p = \log_a(1 + \frac{1}{8})$, $q = \log_a(1 + \frac{1}{15})$, and $r = \log_a(1 + \frac{1}{24})$,

Show that, $p + q + r = \log_a(1 + \frac{1}{80})$,

3) Given that, $l = \log_a(1 - \frac{1}{8})$, $m = \log_a(1 - \frac{1}{15})$, and $n = \log_a(1 - \frac{1}{24})$,

Show that, $l - m - n = \log_a(1 - \frac{1}{46})$,

4) Given that $\log_3(x - 6) = 2y$, and $\log_3(x - 7) = 3y$, Show that $x^2 - 13x + 42 = 72^y$. If $y = 1$ find the possible values of x.

5) Show that $2\log(a + b) = 2\log a + \log\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$

6) Given that $\log_4 a = 1 + \log_2 b$,. Find a relation between a and b without involving logarithms

7) Given that $1 + \log_3 p = \log_{27} q$,. Find a relation between p and q without involving logarithms

8) Show that $\log_b a * \log_a b = 1$

9) Show that $\log_c a * \log_b c * \log_a b = 1$

10) Given that $p = \log_a bc$, $q = \log_b ca$, $r = \log_c ab$,

Show that $p * q * r = p + q + r + 2$

11) Given that $x = \log_a n$, $y = \log_c n$, $n > 1$,

Show that $\frac{x+y}{x-y} = \frac{\log_b c + \log_b a}{\log_b c - \log_b a}$ $\quad Hint: find\ (x + y), then\ find\ (x - y), then\ divide$

12) If a, b, and c, are real numbers, Show that

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$$

13) Suppose $\log_{27} m = \frac{t}{2}$, and $\log_3 3m = l$, where $l - t = 4$

Show that $m = 27^3$

Algebraic functions:

Polynomials:

Definition: Any function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ . \ . \ . \ + a_0 x^0 \ = \ \sum_{i=0}^{n} a_i x^i$$

Is called a polynomial function, where a_i are real numbers called the coefficients, and n is a non-negative integer.

Example:

$i)$ $P(x) = 3x + 1,$ $ii)$ $P(x) = 2x^2 - 5x + 3$ $iii)$ $P(x) = x^3 - 2x^2 + 3x - 1$

$iv)$ $P(x) = 12 + 2x^2 - x^4$

Definition: Let $P(x) = \sum_{i=0}^{n} a_i x^i$ where $a_i \ \varepsilon \ R$

$i)$ A polynomial which has only one term is called a *monomial***.**

e.g. $P(x) = 2x^3$

$ii)$ The highest power of x in a polynomial is called the *degree* of the polynomial.

e.g. $P(x) = x^3 - 2x^2 + 3x - 1$ is a polynomial of degree 3

$iii)$ The coefficient of the term with the highest power of x is called the *leading coefficient.*

e.g. If $P(x) = 12 + 2x^2 - x^4$, then the leading coefficient is -1

$iv)$ A polynomial whose leading coefficient is equal to one is called *monic polynomial.*

e.g. $P(x) = x^3 - 2x^2 + 3x - 1$is a monic polynomial.

$v)$ If all the coefficients of a polynomial are zero, the polynomial is called zero Polynomial.

e.g. $P(x) = 0$

$vi)$ A polynomial of degree one is called linear, of degree two is called quadratic, of degree three is called cubic, of degree four is called quartic.

Definition: Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$and $Q_n(x) = \sum_{i=0}^{n} b_i x^i$, Then $P_n(x) = Q_n(x)$ iff $a_i = b_i$

Algebraic operations:

– *Addition:*

Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$ and $Q_n(x) = \sum_{i=0}^{n} b_i x^i$ Let m $\leq$ n.

Then $P_n(x) + Q_n(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n}(a_i + b_i)x^i + \sum_{i=0}^{n} a_i x^i$

Example: Given that $P(x) = 2x^4 + 3x^3 - 11x^2 + x - 6$ and $Q(x) = 2x^3 + 20x^2 - 3x + 8$
Then.

$$
\begin{array}{rl}
P(x) + Q(x) = & 2x^4 + 3x^3 - 11x^2 + x - 6 \\
. & \underline{\quad\quad 2x^3 + 20x^2 - 3x + 8} \\
 & 2x^4 + 5x^3 + 9x^2 - 2x + 2
\end{array}
$$

Example: Consider the following polynomials

$P(x) = x^5 + 3x^4 + 2x^3$ and $Q(x) = 2x^5 - 4x^4 - 3x^3 + 2x^2 - 7.$

Find $P(x) + Q(x)$

Solution: Do.

– *Subtraction:*

Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$ and $Q_m(x) = \sum_{i=0}^{m} b_i x^i$ Let m $\leq$ n.

Then $P_n(x) - Q_m(x) = \sum_{i=0}^{n} a_i x^i - \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{n}(a_i - b_i)x^i + \sum_{i=n+1}^{m} a_i x^i$

Example: Given that $P(x) = 2x^4 + 3x^3 - 11x^2 + x - 6$ and $Q(x) = 2x^3 + 20x^2 - 3x + 8$

Then,

$$
\begin{array}{rl}
P(x) - Q(x) = & 2x^4 + 3x^3 - 11x^2 + x - 6 \\
 & \underline{0x^4 + 2x^3 + 20x^2 - 3x + 8} \\
 & 2x^4 + x^3 - 31x^2 + 4x - 14
\end{array}
$$

Example: Consider the following polynomials

$P(x) = 10x^4 + 5x^3 + 3x^2 + 7x$ and $Q(x) = 2x^5 - 4x^4 - 2x^3 + 5x^2 + 10x - 5.$

Find $P(x) - Q(x)$

Solution: Do.

− Multiplication:

Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$ and $Q_m(x) = \sum_{i=0}^{m} b_i x^i$ m ≤ n

Then $P_n(x) * Q_m(x) = \left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{i=0}^{n} (a_i * b_i) x^{i+j}$

Note: If $P_n(x)$ has degree n and $Q_m(x)$ has degree m, then their product $P_n(x) * Q_m(x)$ has degree $(m + n)$.

Example: Consider the following polynomials. $P(x) = 6x^2 + 2x - 3$ and $Q(x) = x + 3$

Then $P(x) * \mathrm{Q}(x) = (6x^2 + 2x - 3)(x + 3) = x(6x^2 + 2x - 3) + 3(6x^2 + 2x - 3)$

$$= 6x^3 + 2x^2 - 3x + 18x^2 + 6x - 9 = 6x^3 + 20x^2 + 3x - 9$$

Example: Given that $P(x) = x^3 + 4x^2 + 5x$ and $Q(x) = 2x^3 - 1$. Find $P(x) * \mathrm{Q}(x)$

Solution: Do.

− *Division:*

In the process of dividing the polynomials $P(x)$ and $Q(x)$ we usually arrive at a result of the form $\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$ $\quad Q(x) \neq 0$

Where $P(x)$ is called the dividend, $Q(x)$ the divisor, $q(x)$ the quotient, and $r(x)$ the Remainder, and that the degree of $Q(x)$ is less than or equal to the degree of $P(x)$.

Definition: The process of carrying out the division (or the steps in carrying out the division) is called $division\ algorithm.$

Theorem: Let $P(x)$ and $Q(x)$ be two polynomials, and $Q(x) \neq 0$, then there exist a unique Polynomials$q(x)$ and $r(x)$ such that, $P(x) = \mathrm{Q(x)}q(x) + r(x)$ Where, either $r(x) = 0$, or the degree of $r(x)$ is less than the degree of $Q(x)$.

$a)$ *Long division:*

Steps in the process of long division:

$i)$ Arrange the terms of the polynomials $P(x)$ and $Q(x)$ in descending powers of x, and if a power is missing write the term with zero coefficient.

$ii)$ Divide the first term of the dividend by the first term of the divisor, and write the answer directly above the first term of the dividend.

$iii)$ Multiply the divisor by the quotient and subtract the product from the dividend.

$iv)$ Repeat steps 2 and step 3 until the degree of the remainder is less than the degree of the divisor.

$v)$ Write the answer in the form $\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$.

Example: Divide the polynomial $P(x) = 3x^3 - 2x^2 + 5x - 7$ and $Q(x) = x - 2$.
using the process of long division.

Solution:

$$\begin{array}{r}
3x^2 + 4x + 13 \\
(x-2)\overline{\big)\,3x^3 - 2x^2 + 5x - 7} \\
\underline{3x^3 - 6x^2} \qquad\qquad \\
4x^2 + 5x - 7 \\
\underline{4x^2 - 8x} \qquad \\
13x - 7 \\
\underline{13x - 26} \\
19.
\end{array}$$

$\Rightarrow \quad \frac{3x^3-2x^2+5x-7}{x-2} = (3x^2 + 4x + 13) + \frac{19}{x-2}$.

Example: Divide the polynomial $P(x) = x^5 + 4x^4 - 3x + 5$ by $Q(x) = x^3 - 2x + 1$ using the process of long division.

Solution:

$$\begin{array}{r}
x^2 + 4x + 2 \qquad\qquad \\
(x^3 + 0x^2 - 2x + 1)\overline{\big)\,x^5 + 4x^4 + 0x^3 + 0x^2 - 3x + 5} \\
\underline{x^5 + 0x^4 - 2x^3 + x^2} \qquad\qquad\quad \\
4x^4 + 2x^3 - x^2 - 3x + 5 \\
\underline{4x^4 + 0x^3 - 8x^2 + 4x} \\
2x^3 + 7x^2 - 7x + 5 \\
\underline{2x^3 + 0x^2 - 4x + 2} \\
7x^2 - 3x + 3
\end{array}$$

$\Rightarrow \quad \frac{x^5 + 4x^4 - 3x + 5}{x^3 - 2x + 1} = (x^2 + 4x + 2) + \frac{7x^2 - 3x + 3}{x^3 - 2x + 1}$

Exercises:

$i)$ Divide the polynomial $P(x) = x^5 + 4x^4 + 3x + 5$ by $Q(x) = x^3 - 2x + 1$
sing the process of long division.

$ii)$ Divide the polynomial $5x^4 - 4x^3 + 3x^2 + 22x + 55$ by $Q(x) = 5x^2 + 11x + 11$
using the process of long division.

iii) Find $r(x)$ such that $5x^4 + 3x^2 + 2x + 1 = (x + 2)(x^2 + 3x + 1) + r(x)$

iv) Find $q(x)$ such that $3x^3 - 5x^2 - x + 4 = (3x + 1)q(x) + (10x + 7)$

Synthetic division:

In this method, the divisor $Q(x)$ must be a linear factor of the form $(x - c)$, and $r(x) = 0$ or the degree of $r(x)$ is equals zero. In the method we usually arrive at a result of the form.

$$\frac{P(x)}{x-c} = q(x) + \frac{r}{x-c}$$

Steps in the process of Synthetic division:

i) Since the divisor is of the form $x - c$, we write c in the box, and arrange on the same Row1 the coefficients of the dividend in descending powers of x, putting zero coefficient for every missing power.

ii) Copy the leading coefficient of the dividend on the third row.

iii) Multiply the leading coefficient you copied by the number c in the box and write the result in the second row under the coefficient next to the leading coefficient and add the numbers in that column.

iv) Multiply the sum in iii) by the number c in the box and write the result in the second row next to the one obtained in step 3 and add the numbers in that column.

v) Continue as in step 3 and step 4 until there is an entry on the third row for each entry in the first row.

vi) The last number in the third row is the remainder r, and the other numbers before the last are the coefficient of the quotient.

Example: Use synthetic division to obtain the quotient and the remainder when

$P(x) = 2x^4 - 10x^2 - 23x + 6$ is divided by $x - 3$.

Solution:

$$\begin{array}{r|rrrrr} 3 & 2 & 0 & -10 & -23 & 6 \\ & & 6 & 18 & 24 & 3 \\ \hline & 2 & 6 & 8 & 1 & \underline{9} \end{array}$$

$\Rightarrow \quad \frac{2x^4 - 10x^2 - 23x + 6}{x-3} = (2x^3 + 6x^2 + 8x + 1) + \frac{9}{x-3}$

Example: Use synthetic division to obtain the quotient and the remainder when
$P(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$ is divided by $(x + 2)$.

Solution:

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \underline{0} \end{array}$$

⇨ $$\frac{3x^4 + 8x^3 - 2x^2 - 10x + 4}{x+2} = (3x^3 + 2x^2 - 6x + 2) + \frac{0}{x+2}$$

Exercises:

1. If $P(x) = x^4 - 4x^3 + x^2 - 3x - 5$, Use synthetic division to find $P(2)$ and $p(-2)$.
2. Find the quotient and the remainder when $3x^4 - 8x^3 + 9x + 54$ is divided by $x - 2$ using synthetic division.
3. Use synthetic division to find the value of$k$ if $x^3 + (k-1)x^2 + x + (5-k)$ is divided by $(x + i)$, and leave remainder zero.

Remainder theorem:
Whenever we divide a polynomial $P(x)$ by $(x - c)$ we always get some quotient $q(x)$ and a remainder r such that

$$P(x) = (x - c)q(x) + r \qquad . . . (1)$$

Setting $x = c$ in equation (1) we have,

$$P(c) = r \qquad . . . (2)$$

From equation (2), we have that whenever$P(x)$ is divided by $(x - c)$ the remainder is $P(c)$.

Theorem: When a polynomial $P(x)$ is divided by $(x - c)$, then the remainder is $P(c)$.

Example1: Find the value of k if the remainder when $P(x) = 2x^4 + kx^3 - 11x^2 + 4x + 12$ Is divided by $(x - 3)$ is 60.

Solution:

$P(x) = 2x^4 + kx^3 - 11x^2 + 4x + 12$

$P(3) = 60 \qquad . . . (1)$

$P(3) = 2(3)^4 + k(3)^3 - 11(3)^2 + 4(3) + 12 = 87 + 27k \qquad . . . (2)$

⇨ $87 + 27k = 60$ ⇨ $k = -1$

Example2: Find the value of k for which a polynomial $P(x) = x^5 + kx^4 - 2x + 1$ leaves a remainder 5 when divided by $x + 2$

Solution:

$P(x) = x^5 + kx^4 - 2x + 1$

$P(-2) = 5$ . . . (1)

$P(-2) = (-2)^5 + k(-2)^4 - 2(-2) + 1 = 16k - 27$ . . . (2)

⇨ $16k - 27 = 5$ ⇨ $k = 2$

Example3: A polynomial $P(x) = ax^3 + bx^2 - x + 3$ has factor $x + 1$ and when divided by $x - 4$ leaves a remainder 15. Find a and b and factorize $p(x)$ completely and also obtain its zero's

Solution:

$P(x) = ax^3 + bx^2 - x + 3$

$P(-1) = 0$

$P(-1) = a(-1)^3 + b(-1)^2 - (-1) + 3 = -a + b + 4$

⇨ $-a + b = -4$ . . . (1)

$P(4) = 15$

$P(4) = a(4)^3 + b(4)^2 - (4) + 3 = 64a + 16b - 1 = 15$

⇨ $4a + b = 1$ . . . (2)

Solving equations (1) and (2) simultaneously we have

$a = 1$ and $b = -3$

Hence $P(x) = ax^3 + bx^2 - x + 3 = x^3 - 3x^2 - x + 3 = 0$

⇨ $(x^3 - 3x^2) - (x - 3) = x^2(x - 3) - (x - 3) = (x^2 - 1)(x - 3) = 0$

⇨ $(x - 1)(x + 1)(x - 3) = 0$ ⇨ $x = 1, x = -1, x = 3$

Example4: Find the value of a and b for which a polynomial $P(x) = ax^3 + bx^2 - 3x - 4$ leaves a remainder $35x - 32$ when divided by $x^2 - 3x + 2$

Solution:

$$\frac{ax^3 + bx^2 - 3x - 4}{x^2 - 3x + 2} = q(x) + \frac{35x - 32}{x^2 - 3x + 2}$$

⇨ $ax^3 + bx^2 - 3x - 4 = (x^2 - 3x + 2)q(x) + 35x - 32$

⇨ $ax^3 + bx^2 - 3x - 4 = (x - 2)(x - 1)q(x) + 35x - 32$

Now we set $x = 1$

⇨ $a(1)^3 + b(1)^2 - 3(1) - 4 = 35(1) - 32$

⇨ $a + b = 10$ $\quad . . . (1)$

Again we set $x = 2$

⇨ $a(2)^3 + b(2)^2 - 3(2) - 4 = 35(2) - 32$

⇨ $2a + b = 12$ $\quad . . . (2)$

Solving (1) & (2) simultaneously we have

$a = 2$ *and* $b = 8$

Example5: A polynomial $P(x) = x^3 + ax^2 + bx + 6$ has remainder $4 - 4x$ when divided by $x^2 - 3x + 2$. Find the values of a and b

Solution:

$$\begin{array}{r l}
 & x + (a + 3) \\
(x^2 - 3x + 2) & \overline{\big) x^3 + ax^2 + bx + 6} \\
 & \underline{x^3 - 3x^2 + 2x} \\
 & \quad (a + 3)x^2 + (b - 2)x + 6 \\
 & \quad \underline{(a + 3)x^2 - 3(a + 3)x + 2(a + 3)} \\
 & \qquad\qquad (3a + b + 7)x - 2a
\end{array}$$

⇨ $(3a + b + 7)x - 2a = -4x + 4$

Comparing the coefficients

⇨ $3a + b + 7 = -4$ and $-2a = 4$

⇨ $a = -2$ and $b = -5$

Example6: The remainder when a polynomial $P(x) = 4x^3 + ax^2 + bx + 8$ is divided by $2x^2 + x - 1$ is $8x + 5$, find the values of a and b

Solution:

$$\frac{4x^3 + ax^2 + bx + 8}{2x^2 + x - 1} = q(x) + \frac{8x + 5}{2x^2 + x - 1}$$

⇨ $4x^3 + ax^2 + bx + 8 = (2x^2 + x - 1)q(x) + 8x + 5$

⇨ $4x^3 + ax^2 + bx + 8 = (2x - 1)(x + 1)q(x) + 8x + 5$

Now we set $x = \frac{1}{2}$

⇨ $4(\frac{1}{2})^3 + a(\frac{1}{2})^2 + b\left(\frac{1}{2}\right) + 8 = 8\left(\frac{1}{2}\right) + 5$

⇨ $a + 2b = 2$ $\quad . . . (1)$

Again we set $x = -1$

⇨ $4(-1)^3 + a(-1)^2 + b(-1) + 8 = 8(-1) + 5$

⇨ $a - b = -7$ . . . (2)

Solving (1) & (2) simultaneously we have

$a = -4$ *and* $b = 3$

Example7: The remainder when $x^3 + ax^2 + bx + 1$ is divided by $(x - 1)$ is 7, when divided by $(x - 2)$ the remainder is 39. Find a and b.

Solution:

$P(x) = x^3 + ax^2 + bx + 1$

$P(1) = 1^3 + a(1)^2 + b(1) + 1 = 7$

⇨ $a + b = 5$ . . . (1)

$P(2) = 2^3 + a(2)^2 + b(2) + 1 = 39$

⇨ $2a + b = 15$ . . . (2)

Solving eqn. (1) and (2) simultaneously, we have

$a = 10 \quad and \quad b = -5$

Example8: When $5x^3 + ax^2 + x - b$ is divided by $(x - 2)$ the remainder is 3, given that $(x - 1)$ is also a factor. Find a and b.

Solution:

$P(x) = 5x^3 + ax^2 + x - b$

$\left.\begin{array}{l} P(2) = 3 \\ P(2) = 42 - 4a - b \end{array}\right\}$ ⇨ $42 - 4a - b = 3$ . . . (1)

$\left.\begin{array}{l} P(1) = 0 \\ P(1) = 6 - a - b \end{array}\right\}$ ⇨ $6 - a - b = 0$ . . . (2)

Solving eqn. (1) and (2) simultaneously we have

$a = 11$ and $b = -5$

Example9: The expression $ax^3 + bx^2 + cx + 6$ is divisible by $(x - 1)$, it has remainders -4 and 18 when divided by $(x - 2)$ and $(x - 4)$ respectively. Find the values of $a, b, and\ c$

Solution:

$P(x) = ax^3 + bx^2 + cx + 6$

$P(1) = 0$

$P(1) = a + b + c + 6$

⇨ $a + b + c + 6 = 0$ . . . (1)

$P(2) = -4$

$P(2) = 8a + 5b + 2c$

⇨ $\quad 8a + 4b + 2c + 6 = -4$ . . . (2)

$P(4) = 18$

$P(4) = 64a + 16b + 4c + 6$

⇨ $\quad 64a + 16b + 4c + 6 = 18$ . . . (3)

Solving eqn. (1), (2) and (3) we have

$a = 1, b = -2, \ and \ c = -5$

Exercises:

1. Find the remainder when $P(x) = 3x^{100} + 5x^{85} - 4x^{38} + 2x^{17} - 6$ is divided by $x + 1$
2. Find the value of k for which $x^5 + kx^4 - x^2 - 3x + 1$, leaves a remainder 3 when divided by $(x + 2)$

Factor theorem:

Whenever we divide a polynomial $P(x)$ by $(x - c)$ we always get some quotient $q(x)$ and a remainder r such that

$$P(x) = (x - c)q(x) + r \qquad . . . (1)$$

Now when the remainder $r = 0$, equation (1) becomes,

$$P(x) = (x - c)q(x) \qquad . . . (2)$$

Equation (2), implies that$(x - c)$ is a factor of $P(x)$. And if $x = c$, equation (2) reduces to.

$$P(c) = 0 \qquad . . . (3)$$

Theorem: A polynomial $P(x)$ has a factor $(x - c)$ if and only if $P(c) = 0$.

Definition: If $P(c) = 0$, c is called the zero (or the root) of the polynomial $P(x)$.

Example1: For what value of a is $(x + 1)$ a factor of $P(x) = 2x^4 + 2x^3 + ax^2 - 3x + 2$.

Solution:

$P(x) = 2x^4 + 2x^3 + ax^2 - 3x + 2.$

$P(-1) = 0.$ . . . (1)

$P(-1) = 2(-1)^4 + 2(-1)^3 + a(-1)^2 - 3(-1) + 2 = a + 5$ . . . (2)

⇨ $\quad a + 5 = 0.$

⇨ $\quad a = -5$

Example2: Find the value of a which makes $(2x-1)$ a factor of $P(x) = 2x^3 + ax^2 - 13x + 6$

Solution:

$P(x) = 2x^3 + ax^2 - 13x + 6$

$P(\frac{1}{2}) = 0.$ . . . (1)

$P(\frac{1}{2}) = 2(\frac{1}{2})^3 + a(\frac{1}{2})^2 - 13(\frac{1}{2}) + 6 = \frac{1}{4}a - \frac{1}{4}$ . . . (2)

⇨ $\frac{1}{4}a - \frac{1}{4} = 0.$

⇨ $a - 1 = 0$

⇨ $a = 1$

Example3: When is $(x+4)$ a factor of $P(x) = x^3 + x^2 + ax + 8$.

Solution:

$P(x) = x^3 + x^2 + ax + 8.$

$P(-4) = 0.$ . . . (1)

$P(-4) = 2(-4)^3 + (-4)^2 + a(-4) + 8 = -4a - 40$ . . . (2)

⇨ $-4a - 40 = 0.$

⇨ $a = -10$

Exercises:

1. Determine the value of k so that $P(x) = x^3 + kx^2 + kx + 3$ is divisible by $(x+3)$.
2. Show that $(x+2)$ is a factor of $x^{12} - 4096$
3. For what value of m is $P(x) = x^3 + 3x^2 - x + m$ divisible by $(x+8)$.

Rational functions

Definition: A rational function is a quotient of two polynomial functions, Thus, $f(x)$ is a rational function if $\forall x$ in its domain.$f(x) = \frac{g(x)}{h(x)}$.where $g(x)$ and $h(x)$ are polynomials, and that $h(x) \neq 0$.

Example: $i)$ $f(x) = \frac{x-1}{x^2-x-6}$. $ii)$ $f(x) = \frac{x^2-x}{16-x^2}$ $iii)$ $f(x) = \frac{5x}{4-x^2}$

Note: The numerator and the denominator will not have the same zero.

Example: if one root of the equation $ax^2 + bx + c = 0$ is twice the other, Show that $2b^2 = 9ac$, hence if $a = 3$ and $b = 2$, find the value of c.

***Algebraic operations*:**

− *Addition and Subtraction:*

We are familiar with the technique of forming the sum or difference of two or more than two algebraic fractions.

Example:

$i)\ \frac{2}{x+1}+\frac{1}{x+3}=\frac{3x+7}{(x+1)(x+3)}$ $\qquad ii)\ \frac{4}{x-3}-\frac{2}{x+5}=\frac{2x+26}{(x-3)(x+5)}$

$iii)\ \frac{x+1}{x-4}+\frac{5}{x+2}=\frac{-4x^2+3x+22}{(x-2)(x+2)}$, $\qquad iv)\ \frac{1}{x-1}-\frac{2}{x+5}+\frac{x+1}{x+1}=\frac{4x^3+2x+3}{(x-1)(x+1)(x+1)}$

Partial Fraction decomposition:

From the examples above, we notice that the addition or subtraction of algebraic fraction results into an algebraic fraction. The reversed operation is known as partial fraction decomposition.

Definition: The process of decomposing a single fraction into the sum or difference of two or more than two fractions is called partial fraction decomposition.

Guidelines in partial fraction decomposition:

G_1: If the degree of the numerator is greater than or equal to the degree of the denominator, carry out long division to obtain a quotient together with a fraction whose degree of the numerator is less than the degree of the denominator.

Example:

$i)\ \frac{x^3+x^2+4x}{x^2+x-2}=x+\frac{6x}{x^2+x-2}$ $\qquad ii)\ \frac{x^3+2x^2-11x-11}{x^2-7x+12}=(x+9)+\frac{40x-119}{x^2-7x+12}$

$iii)\ \frac{x^3+x^2+5}{x^2+2x-3}=(x-1)+\frac{5x+2}{x^2+2x-3}$ $\qquad iv)\ \frac{2x^2+2x-1}{x^2+x-2}=2+\frac{3}{x^2+x-2}$

$v)\ \frac{2x^3-4x^2-6x+1}{x^2-4x+3}=(2x+4)+\frac{4x-11}{x^2-4x+3}$ $\qquad vi)\ \frac{3x^2+21x+32}{x^2+7x+10}=3+\frac{2}{x^2+7x+10}$

Definition: Linear factors are factors of the form $(ax+b)$ where a and b are real constants.

G_2: If the denominator can be factored into linear factors of which none is repeated, then to each linear factor in the denominator, there corresponds a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be determine.

Note: To determine the values of A, we employ any of the following methods.

i) Cover-up rule. ii) Elimination method. iii) Method of undetermined coefficients.

Example: Decompose into partial fraction $\frac{6x}{x^2+x-2}$

Solution:

$$\frac{6x}{x^2+x-2} = \frac{6x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

− Using cover-up rule: To obtain A we set $x+2=0$ so that $x=-2$, then in the expression $\frac{6x}{(x+2)(x-1)}$, we cover $x+2$ and substitute -2 for x in the remaining,we have A = 4. And to obtain B we set $x-1=0$ so that $x=1$, then in the expression $\frac{6x}{(x+2)(x-1)}$, we cover $x-1$ and substitute 1 for$x$ in the remaining,wehave $B=2$.

− *Using elimination method:* In elimination method we clear fraction by multiplying both sides by $(x+2)(x-1)$, so that.

$$6x = A(x-1) + B(x+2) \qquad \text{. . . (1)}$$

To obtain A we set $(x+2)=0$, so that$x=-2$. Substituting $x=-2$ in equation (1) we have $A=4$. And to obtain B we set $(x-1)=0$, so that $x=1$. Substituting $x=1$ in equation (1) we have $B=2$.

− *Using the method of undetermined coefficients:* Here also we clear fraction by multiplying both sides by $(x+2)(x-1)$, so that.

$$6x = A(x-1) + B(x+2) \qquad \text{. . . (1)}$$

Expanding the RHS of eqn. (1) and collecting like terms we have

$$6x = (A+B)x + (-A+2B).$$

Comparing the coefficient we have

$$A+B=6 \qquad \text{. . . (2)}$$

$$-A+2B=0 \qquad \text{. . . (3)}$$

Solving equations (2) and (3) simultaneously we have $A=4$, and $B=2$

⇨ $$\frac{6x}{x^2+x-2} = \frac{6x}{(x+2)(x-1)} = \frac{4}{(x+2)} + \frac{2}{(x-1)}$$

Exercises:

Decompose the following into partial fraction

$i)\ \frac{6x-9}{x^2-1}$ $\quad ii)\ \frac{x+2}{x^2-x-2}$ $\quad iii)\ \frac{3x^2-4x+5}{(x+1)(x-3)(2x-1)}$ $\quad iv)\ \frac{7x-4}{x^2-7x+12}$

$v)\ \frac{5-x}{x^2-2x-8}$ $\quad vi)\ \frac{5x+2}{x^2-x-2}$ $\quad vii)\ \frac{3x+7}{(x+1)(x+3)}$ $\quad viii)\ \frac{2+x}{(x-5)(x+3)}$

$ix)\ \frac{2x+5}{(x-2)(x+1)}$ $\quad x)\ \frac{2x+26}{(x-3)(x+5)}$ $\quad xi)\ \frac{1}{(x-1)(x-2)}$ $\quad xii)\ \frac{4x-1}{(x-1)(x+3)}$

G_3 **:** If the denominator can be factored into linear factors with some being repeated, then to each linear factor of the form $(ax+b)^n$ in the denominator, there corresponds a partial fraction of the form.

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ \ldots\ +\frac{A_n}{(ax+b)^n}$$

Where $A_1,\ A_2,\ A_3\ .\ .\ .\ A_n$ are constants to be determine.

Note: To determine the value of $A_1,\ A_2,\ A_3\ .\ .\ .\ A_n$, we employ cover-up rule method, elimination method and method of undetermined coefficient combine.

Example: Decompose into partial fraction $\frac{x}{(x+1)^2(x+2)}$

Solution:

$$\frac{x}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$$

We obtain the value of $C=-2$ using cover-up rule. $\Rightarrow C=-2$

$$\Rightarrow \quad \frac{x}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{-2}{x+2}$$

Next clear the fraction by multiplying both sides by $(x+1)^2(x+2)$, so that.

$$x=A(x+1)(x+2)+B(x+2)-2(x+1)^2$$

Setting $x=-1$, we have $B=-1$. (Using elimination method)

$$\Rightarrow \quad x=A(x+1)(x+2)-(x+2)-2(x+1)^2$$

Expanding the RHS and collecting like terms we have.

$$x=(A-2)x^2+(3A-5)x+(2A-4)$$

Comparing the coefficients we have that, $A-2=0,\ \ 3A-5=1\ and\ \ 2A-4=0 \quad \Rightarrow\ A=2$

$$\Rightarrow \quad \frac{x}{(x+1)^2(x+2)}=\frac{2}{x+1}-\frac{1}{(x+1)^2}-\frac{2}{(x+2)}$$

Example: Decompose into partial fraction $\frac{x^2+10x-36}{x(x-3)^2}$

Solution:

$$\frac{x^2+10x-36}{x(x-3)^2} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$$

Using cover-up rule we obtain the value of $A = -4$.

⇨ $$\frac{x^2+10x-36}{x(x-3)^2} = \frac{-4}{x} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$$

Next clear the fraction by multiplying both sides by $x(x-3)^2$, so that.

$$x^2 + 10x - 36 = -4(x-3)^2 + B(x-3)x + Cx.$$

Setting $x = 3$, we have $C = 1$. (Using elimination method)

⇨ $$x^2 + 10x - 36 = -4(x-3)^2 + B(x-3)x + x,$$

Expanding the RHS and collecting like terms we have.

$$x^2 + 10x - 36 = (-4+B)x^2 + (25-3B)x - 36$$

Comparing the coefficients we have that, $B - 4 = 1$ *and* $25 - 3B = 10$, ⇨ $B = 5$

⇨ $$\frac{x^2+10x-36}{x(x-3)^2} = \frac{-4}{x} + \frac{5}{(x-3)} + \frac{1}{(x-3)^2}$$

Exercises:

Decompose the following into partial fraction

$i)$ $\frac{x^2-6}{(2x-1)(x+2)^2}$ $ii)$ $\frac{2}{(x+2)(x+3)^3}$ $iii)$ $\frac{3}{(x-2)(x-3)^2}$ $iv)$ $\frac{2x}{(x-1)^3(x+2)}$

$v)$ $\frac{1}{(x+1)(x-5)^2}$ $vi)$ $\frac{3}{(x-3)(x+2)^2}$ $vii)$ $\frac{1}{(x+1)(x+2)^2(x+3)}$ $viii)$ $\frac{1}{(x-1)^2(x+2)}$

G_4: If the denominator contains irreducible quadratic factor(s) $ax^2 + bx + c$, none of which is repeating, then to each quadratic factor in the denominator there corresponds a partial fraction of the form. $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determine.

Note: To determine the values of A and B, we employ elimination method and the method of undetermined coefficients combine.

Example: Decompose the following into partial fractions. $\frac{x^2-x-21}{(x^2+4)(2x-1)}$

Solution:

$$\frac{x^2-x-21}{(x^2+4)(2x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}$$

Clear the fraction by multiplying both sides by $(x^2+4)(2x-1)$, so that.

$x^2 - x - 21 = (Ax + B)(2x - 1) + C(x^2 + 4).$

Setting $x = \frac{1}{2}$, we have $C = -5$

⇨ $x^2 - x - 21 = (Ax + B)(2x - 1) - 5(x^2 + 4).$

Expanding the RHS and collecting like terms we have.

$$x^2 - x - 21 = (2A - 5)x^2 + (2B - A)x - (20 + B).$$

Comparing the coefficients we have

$2A - 5 = 1$ ⇨ $A = 3$ and

$-20 - B = -21$ ⇨ $B = 1$

⇨ $\frac{x - x - 21}{(x^2 + 4)(2x - 1)} = \frac{3x + 1}{x^2 + 4} - \frac{5}{2x - 1}$

Example: Decompose the following into partial fractions. $\frac{9x^2 - 3x + 8}{x^3 + 2x}$

Solution:

$$\frac{9x^2 - 3x + 8}{x^3 + 2x} = \frac{9x^2 - 3x + 8}{x(x^2 + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2}$$

Using cover-up rule we have that $A = 4$

Clear the fraction by multiplying both sides by $x^3 + 2x$ so that.

$$9x^2 - 3x + 8 = 4(x^2 + 2) + (Bx + C)x.$$

Expanding the RHS and collecting like terms we have

$$9x^2 - 3x + 8 = (4 + B)x^2 + Cx + 8$$

Comparing the coefficients we have,

$4 - B = 9$ ⇨ $B = 5$ and $C = -3$

⇨ $\frac{9x^2 - 3x + 8}{x^3 + 2x} = \frac{9x^2 - 3x + 8}{x(x^2 + 2)} = \frac{4}{x} + \frac{5x - 3}{x^2 + 2}$

Exercises:

Decompose the following into partial fractions.

$i)$ $\frac{1}{(x+1)(x^2+1)}$ $ii)$ $\frac{3}{(x+2)(x^2+2x+3)}$ $iii)$ $\frac{2x^4 + 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1}$ $iv)$ $\frac{1}{(x-3)(x^2-5x+3)}$

$v)$ $\frac{2x-1}{(x-1)(x^2+3x+1)}$

G₅: If the denominator contains irreducible repeated quadratic factor(s) $(ax^2 + bx + c)^{\boldsymbol{n}}$, then to each repeated quadratic factor in the denominator there corresponds a partial fraction of the form.

$$\frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \frac{A_3x + B_3}{(ax^2+bx+c)^3} + \ . \ . \ . \ + \frac{A_nx + B_n}{(ax^2+bx+c)^n}$$

Where $A_1, B_1, A_2, B_2, A_3, B_3, \ ... A_n, B_n$, are constants to be determine.

Note: To determine the values of $A_1, B_1, A_2, B_2, A_3, B_3, \ ... A_n, B_n$, we employ the method of undetermined coefficients.

Example: Decompose the following into partial fractions. $\frac{x^3 + x^2 + 2x}{(x^2+2x+2)^2}$

Solution:

$$\frac{x^3 + x^2 + 2x}{(x^2+2x+2)^2} = \frac{Ax+B}{(x^2+2x+2)} + \frac{Cx+D}{(x^2+2x+2)^2}$$

Clear the fraction by multiplying both sides by $(x^2 + 2x + 2)^2$, so that.

$$x^3 + x^2 + 2x = (x^2+2x+2)(Ax+B) + (Cx+D)$$

Expanding the RHS and collecting like terms we have.

$$x^3 + x^2 + 2x = Ax^3 + (2A+B)x^2 + (2A+2B+C)x + (D+2B)$$

Comparing the coefficients we have

$A = 1,$

$2A + B = 1 \quad \Rightarrow \quad B = -1,$

$2A + 2B + C = 2 \quad \Rightarrow \quad C = 2, \quad and$

$D + 2B = 0 \quad \Rightarrow \quad D = 2$

$$\Rightarrow \quad \frac{x^3 + x^2 + 2x}{(x^2+2x+2)^2} = \frac{x-1}{(x^2+2x+2)} + \frac{2x+2}{(x^2+2x+2)^2}$$

Example: Decompose the following into partial fractions. $\frac{4x^3 - x^2 + 4x + 2}{(x^2+1)^2}$

Solution:

$$\frac{4x^3 - x^2 + 4x + 2}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$$

Clear the fraction by multiplying both sides by $(x^2 + 1)^2$, so that.

$$4x^3 - x^2 + 4x + 2 = (x^2+1)(Ax+B) + (Cx+D)$$

Expanding the RHS and collecting like terms we have.

$$4x^3 - x^2 + 4x + 2 = Ax^3 + Bx^2 + (A+C)x + (D+B)$$

Comparing the coefficients we have

$A = 4, \quad B = -1,$

$A + C = 4 \quad \Rightarrow C = 0, and$

$B + D = 2 \quad \Rightarrow D = 3$

⇨ $$\frac{4x^3 - x^2 + 4x + 2}{(x^2+1)^2} = \frac{4x-1}{(x^2+1)} + \frac{3}{(x^2+1)^2}$$

Exercises:

Decompose the following into partial fractions.

$i)$ $\frac{3x^3 + 13x - 1}{(x^2+4)^2}$ $\quad$ $ii)$ $\frac{x^3}{(x^2+1)^2}$

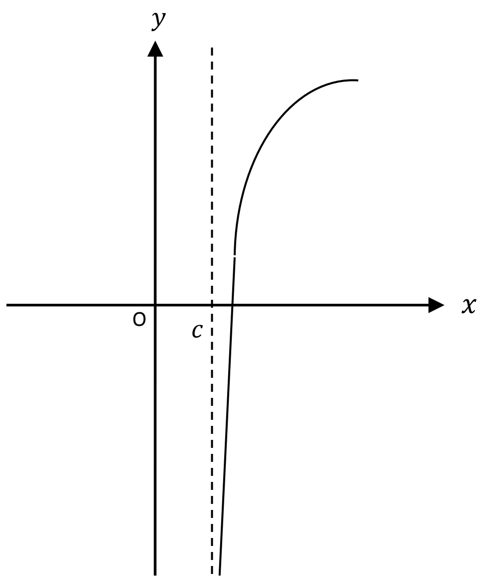
− *Multiplication and division:*

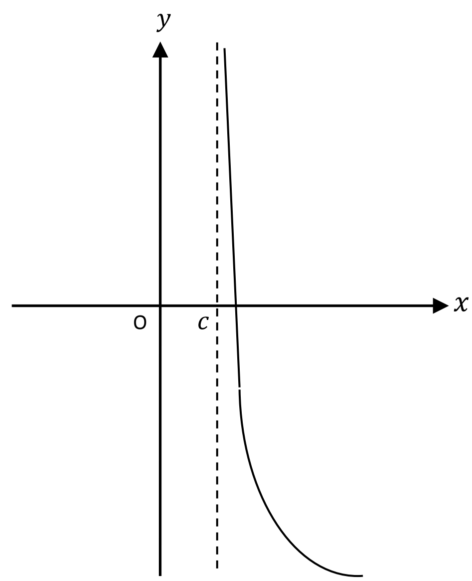
We multiply (divide) algebraic fractions in the same way we multiply (divide) fractional numbers.

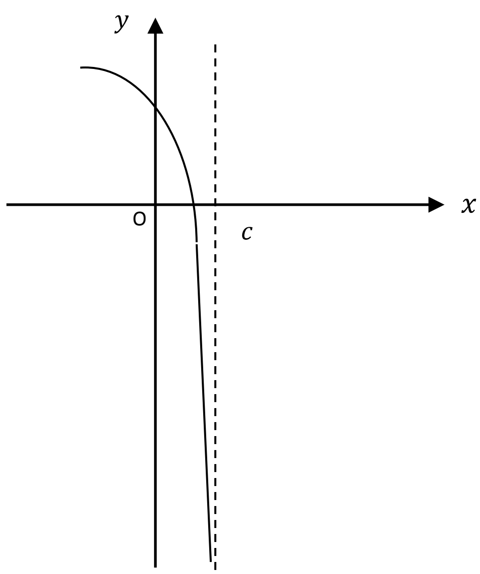
Note: Of major importance are the zero's (roots) of the numerator and the denominator. $f(c) = 0$ if and only if $g(c) = 0$, and hence the zero's of the numerator$g(x)$ are the zero's of $f(x)$. However, if c is the zero of the denominator $h(x)$, then $f(c) = \infty$, and the behavior of $f(x)$ when x is near c requires special attention.

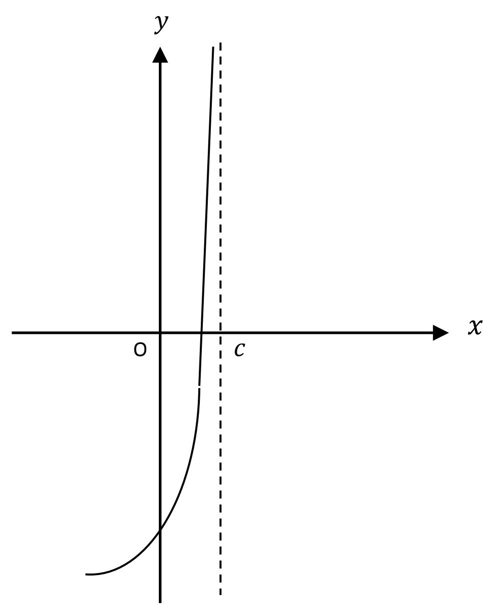
– *Graph of rational functions:*

Definition: The line $x = c$ is a vertical asymptote for the graph of a function f, if $f(x) \to \pm\infty$ as $x \to c$ either from the right from the left.









Guidelines for Sketching the graph of rational functions:

G_1: Find the real zero's of the numerator, and use them to plot the points corresponding to the x-intercept.

G_2: Find the real zero's of the denominator; for each zero c, the line $x = c$ is a vertical asymptote. (represent the line $x = c$ with dotted lines).

G_3: Find the sign of $f(x)$ in each of the intervals determine by the zero's of the numerator and denominator, use these signs to determine whether the graph lies above or below the x-axis in each interval.

G_4: If $x = c$ is a vertical asymptote, use the information in G_3 to determine whether $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to c^-$ or $x \to c^+$.

G_5: Use the information in G_3 to determine the manner in which the graph intersects the x-axis.

G_6: Determine the behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$, if $f(x) \to b$, then the line $y = b$ is a horizontal asymptote. (if $b \neq 0$, represent line $y = b$ by dotted lines)

G_7: Sketch the graph, plotting points whenever necessary.

Example: Sketch the graph of f if $f(x) = \frac{x-1}{(x^2-x-6)}$.

Solution:

$$f(x) = \frac{(x-1)}{(x^2-x-6)} = \frac{(x-1)}{(x+2)(x-3)}$$

G_1: $(x-1) = 0$ ⇨ $x = 1$. Hence the numerator has zero $c = 1$.

G_2: $(x+2)(x-3) = 0$ ⇨ $x = -2$ or $x = 3$. Hence the denominator has zero's $c = -2$ and $c = 3$.

G_3: The zero's -2, 1 and 3 determine the following intervals $(-\infty\ \ -2), (-2\ \ 1), (1\ 3)$ and $(3\ \ \infty)$, Since $f(x)$ is a quotient of two polynomials, it follows that $f(x)$ always positive or always negative throughout each of these intervals. Using test values to determine the sign of $f(x)$, we arrives at the following table.

Interval	*Test Values*	*Sign*	*Position of the graph*
$(-\infty - 2)$	$f(-3) = -\frac{2}{3}$	$-ve$	Below x-axis
$(-2 \;\; 1)$	$f(0) = \frac{1}{6}$	$+ve$	Above x-axis
$(1 \quad 3)$	$f(2) = -\frac{1}{4}$	$-ve$	Below x-axis
$(3 \quad \infty)$	$f(4) = \frac{1}{2}$	$+ve$	Above x-axis

G_4: We shall use the fourth column of the table to investigate the behavior of $f(x)$ near each vertical asymptote.

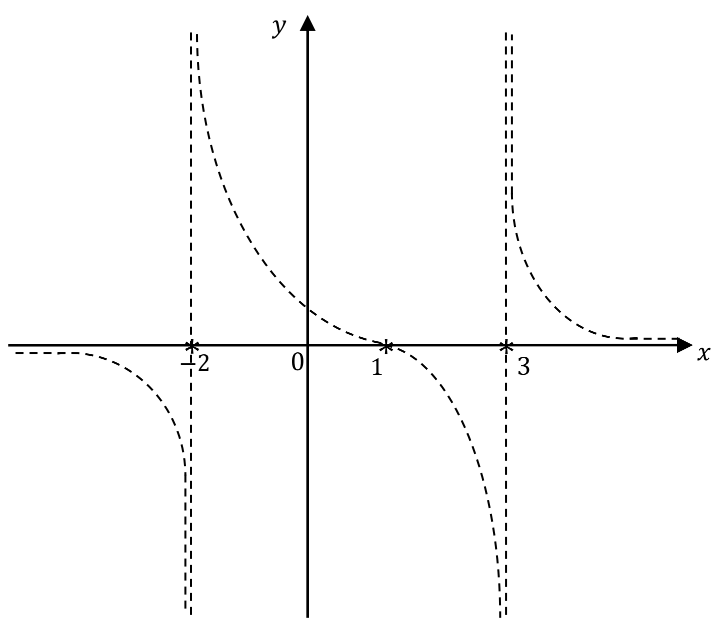
$a)$ Consider the vertical asymptote $x = -2$, since the graph lie below the x-axis throughout the interval$(-\infty - 2)$, it follows that $f(x) \to -\infty \; as \; x \to -2^-$. And since the graph lie above the x-axis throughout the interval $(-2 \;\; 1)$, it follows that$f(x) \to \infty \; as \; x \to -2^+$.

$b)$ Consider the vertical asymptote $x = 3$, since the graph lie below the x-axis throughout the interval $(1 \;\; 3)$, it follows that$f(x) \to -\infty \; as \; x \to 3^-$. And since the graph lie above the$x$-axis throughout the interval $(3 \;\; \infty)$ it follows that $f(x) \to \infty \; as \; x \to 3^+$.

G_5 From the fourth column of the table, we see that the graph crosses the x-axis at $x = 1$.

G_6 To determine what is true if $x \to \infty$or $\to -\infty$, we divide both the numerator and the denominator by x^2 obtaining $f(x) \to 0$. Thus the line $y = 0$ (i.e. the x-axis) is the horizontal asymptote for the graph.

G_7 Using the information in $G_1 \; to \; G_6$ and plotting several points gives us the sketch



Example: Sketch the graph of f if $f(x) = \frac{x^2 - x}{(16 - x^2)}$.

Solution:

$$f(x) = \frac{x^2-x}{(16-x)} = \frac{x(x-1)}{(4-x)(4+x)}$$

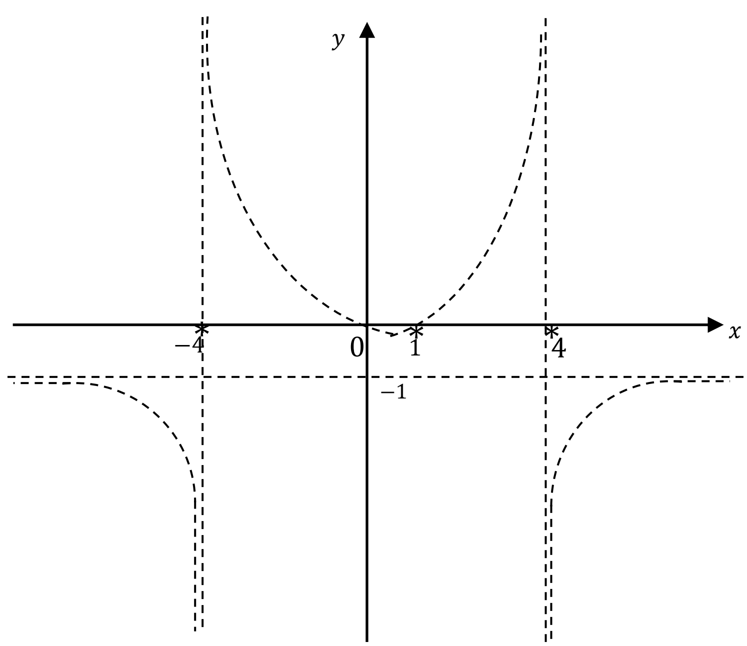
G_1: $x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$. Hence the numerator has zero's $c = 0$ and $c = 1$.

G_2: $(4-x)(4+x) = 0 \Rightarrow x = -4$ or $x = 4$. Hence the denominator has zero's -4 and 4.

G_3 The zero's $-4,\ 0,\ 1$ and 4 determine the following intervals $(-\infty\ \ -4), (-4\ \ 0), (0\ \ 1)$, $(1\ \ 4)$, and $(4\ \ \infty)$. Since $f(x)$ is a quotient of two polynomials, it follows that $f(x)$ is always positive or always negative throughout each of these intervals.

Using test values to determine the sign of $f(x)$, we arrives at the following table.

Interval	*Test Values*	*Sign*	*Position of the graph*
$(-\infty\ \ -4)$	$f(-5) = -\frac{10}{3}$	$-ve$	*Below x-axis*
$(-4\ \ 0)$	$f(-3) = \frac{12}{7}$	$+ve$	*Above x-axis*
$(0\ \ 1)$	$f\left(\frac{1}{2}\right) = -\frac{1}{63}$	$-ve$	*Below x-axis*
$(1\ \ 4)$	$f(3) = \frac{6}{7}$	$+ve$	*Above x-axis*
$(4\ \ \infty)$	$f(5) = -\frac{20}{9}$	$-ve$	*Below x-axis*



Exercises:

Sketch the graph of f if

$i)\quad f(x) = \frac{5x}{4-x^2} \qquad ii)\quad f(x) = \frac{x^2}{x^2 - x - 2} \qquad iii)\quad f(x) = \frac{2}{x-5}$

$iv)\ f(x) = \frac{x}{x-5} \qquad v)\quad f(x) = \frac{2x^2}{x^2+1} \qquad vi)\quad f(x) = \frac{-3}{x-2}$

Note: Vertical asymptotes are common characteristics of graphs of rational functions. Indeed, if c is a zero of the denominator $h(x)$, then the graph of $f(x)$ has vertical asymptote at $x = c$. Moreover, we can make $f(x)$ as close to zero as we pleased by choosing$x$ sufficiently large. $f(x) \to 0\ \ as\ \ x \to \pm\infty$

Definition: The line $y = b$ is a horizontal asymptote for the graph of a function f, if $f(x) \to 0\ \ as\ \ x \to \pm\infty$

Note: The graphs of a rational function often have a horizontal asymptote. Some typical cases for $x \to \infty$ are illustrated below, the manner in which the graph approach the line $y = b$ may vary depending on the function. Similar sketches may be made for the case $x \to -\infty$

Section C

Complex numbers

Course Outline:

Complex numbers: Representation in the plane, Sum, Difference, Product and Quotient of complex numbers, Modulus and Argument of complex numbers, Complex conjugate and its properties, Polar representation of complex numbers, Unit circle, n^{th} *root, De Moivre's Theorem, Zero of polynomials, Quadratic formula.*

Complex Numbers:

Introduction: The concept of imaginary numbers has had its origin in the fact that the solution of the quadratic equation $ax^2 + bx + c = 0$, leads to the expression $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, which is not meaningful if $b^2 - 4ac < 0$, in that square root of number which had hitherto fore been developed could be negative . In this context, Euler (1707 – 1783) was the first Mathematician who introduce the symbol i for $\sqrt{-1}$ with the property $i^2 = -1$ and accordingly, a root of the equation, $x^2 + 1 = 0$. He also called the symbol i imaginary number

Definition1: Any number of the form $x + iy$ where $x \, and \, y$ are real numbers is called a complex numbers denoted, $z = x + iy$.

Note: The number x is called the real part, and the number y is called the imaginary part.

Example: $z = 3 + 2i$ *is a complex number having real part* $x = 3$ *and imaginary part* $y = 2$

Definition2: In any complex number, $z = x + iy$, then if

$i)$ $y = 0$, the number z is purely real, denoted $z = x + 0i$ or $z = x$

$ii)$ $x = 0$, the number z is purely imaginary, denoted $z = 0 + iy$ or $z = iy$

$iii)$ $x = y = 0$ the number z is called zero complex number denoted $z = 0 + 0i$ or $z = 0$

$iv)$ $x < 0$, $y < 0$, the number z is called negative complex number.

Note: The set of all complex numbers is denoted ₵

$$i.e. \text{ ₵} = \{z_1, z_2, z_3, . . . \; z_n\}$$

Properties of complex numbers:

If $z_1, z_2, z_3, ...z_n$ are complex numbers, and ₵ $= \{z_1, z_2, z_3, . . . \; z_n\}$ is the set of complex numbers the following properties hold good

1. $z_i + z_j = z_k$ ⇨ Closure
2. $z_i * z_j = z_r$ ⇨ Closure
3. $z_i - z_j = z_r$ ⇨ Closure
4. $\frac{z_i}{z_j} = z_r$ $\quad z_j \neq 0$ ⇨ Closure
5. $z_i + z_j = z_j + z_i$ ⇨ Commutative laws hold good
6. $z_i * z_j = z_j * z_i$ ⇨ Commutative laws hold good.

7. $z_i + (z_j + z_k) = (z_i + z_j) + z_k,$ ⇨ Associative laws hold good.

8. $z_i * (z_j * z_k) = (z_i * z_j) * z_k$ ⇨ Associative laws hold good.

9. $z_i * (z_j + z_k) = (z_i * z_j) + (z_i * z_k)$ ⇨ Distributive laws hold good.

10 $z_i + 0 = 0 + z_i = z_i,$ ⇨ Existence of additive identity.

11 $z_i * 1 = 1 * z_i = z_i$ ⇨ Existence of multiplicative identity.

12 If $z_i \neq 0$, there exist z_j in ₵ such that $z_i + z_j = 0,$ ⇨ Existence of additive inverse

13 $z_i * z_j = z_j * z_i = 1$ ⇨ Existence of multiplicative inverse

Definition; Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, we say that $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2,$

Example: Let $z_1 = x + 4i$ and $z_2 = 3 + iy$, Then, $z_1 = z_2$ if and only if Let $x = 3$ and $y = 4$

The number i

The symbol i was introduce to represent $\sqrt{-1}$ with the property that

$i^n,\ n\,\varepsilon\, Z^+$always assumed any of the following numbers $1,\ -1,\ i,\ -i$

$i.e.\quad i^n = 1,\ -1,\ i,\ or\ -i$

Example: Simplify the following $a)\ i^8$ $b)\ i^{10}$ $c)\ i^{30}$ $d)\ i^{19},$ $e)\ i^{91}$

Solution:

$a)\quad i^8 = (i^2)^4 = (-1)^4 = 1$ $\qquad b)\quad i^{10} = (i^2)^5 = (-1)^5 = -1$

$c)\quad i^{13} = (i^2)^6\, i = (-1)^6\, i = i$ $\qquad d)\quad i^{19} = (i^2)^9\, i = (-1)^9\, i = -i$

Exercises: Simplify the following $a)\ i^{12}$ $b)\ i^{27}$ $c)\ i^{30}$ $d)\ i^{120}$

Fundamental operations with Complex Numbers:

In performing operations with complex numbers we can proceed as in the algebra of real numbers replacing i^2 by -1 when it occurs.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

1. *Addition:* $z_1 + z_1 = (x_1 + x_2) + (y_1 + y_2)i$
2. *Subtraction:* $z_1 - z_1 = (x_1 - x_2) + i(y_1 - y_2)$
3. *Multiplication:* $z_1 * z_1 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$
4. *Division:* $\frac{|z_1|}{|z_2|} = \frac{x_1+iy_1}{x_2+iy_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$

Examples 1: Given that $z_1 = 3 + 4i \; and \; z_2 = 2 + 5i$

Find $a)\; z_1 + z_2 \quad b)\; z_1 - z_2 \quad c)\; z_1 * z_2$ and $d)\; \frac{z_1}{z_2}$

Solution:

$a)\; z_1 + z_2 = (3 + 4i) + (2 + 5i) = 5 + 9i$

$b)\; z_1 - z_2 = (3 + 4i) - (2 + 5i) = 1 - i$

$c)\; z_1 * z_1 = (3 + 4i) * (2 + 5i) = (3 * 2 - 4 * 5) + i(3 * 5 + 4 * 2) = -14 + 23i$

$d)\; \frac{z_1}{z_2} = \frac{3+4i}{2+5i} = \frac{(3+4i)(2-5i)}{(2+5i)(2-5i)} = \frac{26-3i}{29} = \frac{26}{29} - \frac{3}{29}i$

Exercises:

1. Given that, $z_1 = 7 - 5i \; and \; z_2 = 2 - 3i$.

   Find $a)\; z_1 + z_2 \quad b)\; z_1 - z_2 \quad c)\; z_1 * z_2 \quad d)\; \frac{z_1}{z_2}$

2. Write each of the following in the form $x + iy$

   $a)\; -(-3 + 7i) + (-6 + 6i)$

   $b)\; (16 + 10i) - (9 - 15i)$

   $c)\; (-7 + i)(3 - i)$

3. Express each of the following in the form $x + iy$

   $a)\; (1 + i)^4 \qquad b)\; \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 \qquad c)\; (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3$

4. Solve for $x \; and \; y$ in each of the following if $x \; and \; y$ are real numbers.

   $a)\; 3x + 6i = -9 + (2y)i$

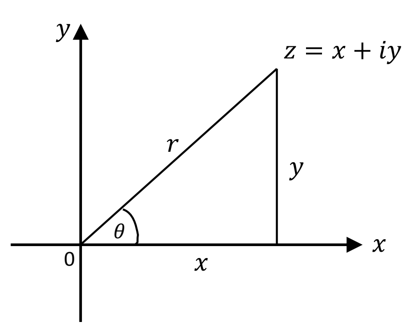
   $b)\; 9 - (2y)i = 6x + 8i$

   $c)\; i(2x + 4y) = (2x - 4) + (3y)i$

   $d)\; (2x + y) + (3x - 4y)i = (x - 2) + (4y - 5)i$

Geometrical representation of complex numbers (Argand Diagram):

If in the fig. below the point z represents the complex number $z = x + iy$ and if the length oz is denoted by r and the angle aoz by θ we have



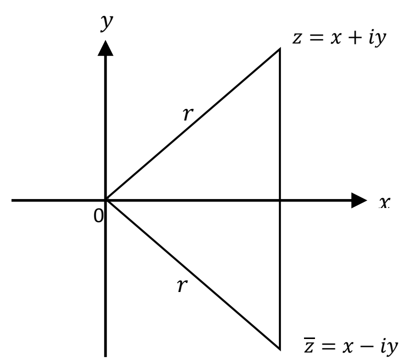
Definition3: The absolute value or Modulus of a complex number $z = x + iy$ is defined as $|z| = |x + iy| = \sqrt{x^2 + y^2}$

Properties of absolute values:

If $z_1,\ z_2,\ z_3, . . .\ z_n$ are complex numbers, the following properties hold good

1. $|z_1 z_2| = |z_1||z_2|$
2. $|z_1 + z_2| \leq |z_1| + |z_2|$
3. $|z_1 - z_2| \geq |z_1| - |z_2|$
4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ Provided $z_2 \neq 0$
5. $|z|^2 = z\bar{z}$

Definition 4: If $z = x + iy$. Is a complex number, then the number $\bar{z} = x - iy$. Is called the complex conjugate of $z = x + iy$.



Properties of complex conjugate:

If z_1 and z_2 are complex numbers, then

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1 z_2}$
3. $\overline{z^n} = \bar{z}^n$
4. $\bar{\bar{z}} = z$
5. $z\bar{z} = a^2 + b^2$
6. $z + \bar{z} = 2a$
7. $z - \bar{z} = 2ib$

$Exercise$:

$a)$ Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$ Showing the details of your work, Write the following complex numbers in the form $x + iy$

$i)$ $(5z_1 + 3z_2)^2$ $ii)$ $\frac{z_1 + z_2}{z_1 - z_2}$ $iii)$ $Re(z_2^2) * (Re(z_2))^2$ $iv)$ $\frac{\overline{z_1}}{\overline{z_2}}$ $v)$ $\overline{z_1/z_2}$

$b)$ Given that $z = x + iy$, find the values of x and y if $\frac{1}{z} + \frac{2}{\bar{z}} = 1 + i$ Ans: $x = \frac{3}{10}$ and $y = \frac{9}{10}$

$b)$ Given that $\sqrt{x + iy} = a + ib$, show that $x = a^2 + b^2$, $y = 2ab$ and find a and b if $x = 3, y = 4$

Definition 5: The positive angle between the complex number $z = x + iy$ and the positive real axis is called the amplitude (or argument) of the complex number $z = x + iy$ denoted $\arg(z) = \theta$ and define by $\theta = \tan^{-1}(\frac{y}{x})$.

Note: Every non-zero complex number admits of an infinite number of arguments and of all these arguments there is one and only one argument θ such that $0 < \theta \leq 2\pi$. This argument is called the principal argument.

Properties of amplitude (or argument):

If $z_1, z_2, z_3, ... z_n$ are complex numbers, the following properties hold good

1. $\arg(z_i * z_j) = \arg(z_i) + \arg(z_j)$
2. $\arg\left(\frac{z_i}{z_j}\right) = \arg(z_i) - \arg(z_j)$

Example 1:

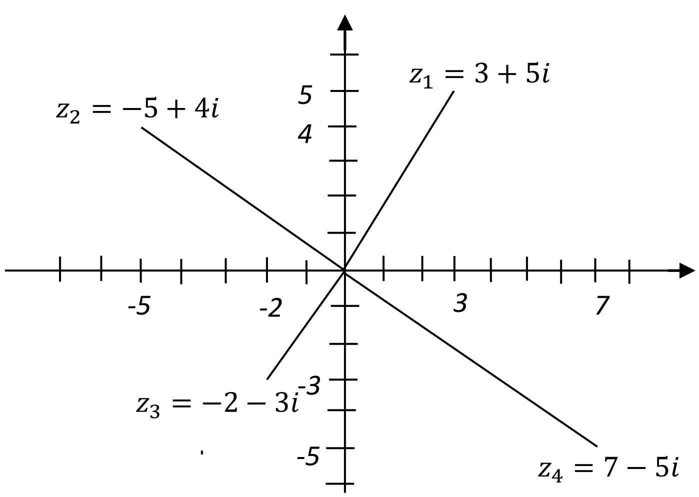
Given that, $z_1 = 3 + 5i, \quad z_2 = -5 + 4i, \quad z_3 = -2 - 3i \text{ and } z_4 = 7 - 5i$

$a)$ Represent the complex numbers on the Argand diagram and

$b)$ Find the modulus and the argument of, $z_1, \; z_2, \; z_3 \text{ and } z_4$.

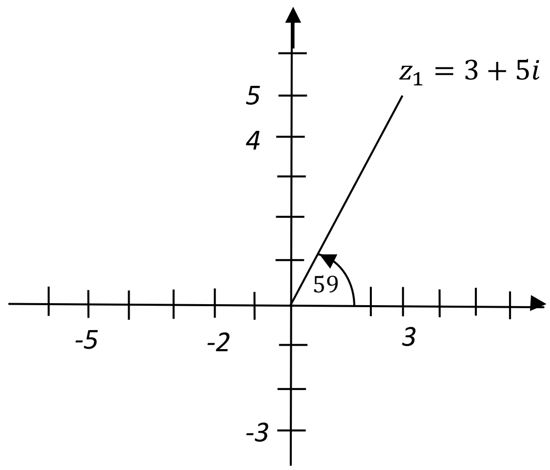
Solution:

$a) \; z_1 = 3 + 5i, \quad z_2 = -5 + 4i, \quad z_3 = -2 - 3i \text{ and } z_4 = 7 - 5i$



$bi)\quad z_1 = 3 + 5i,$

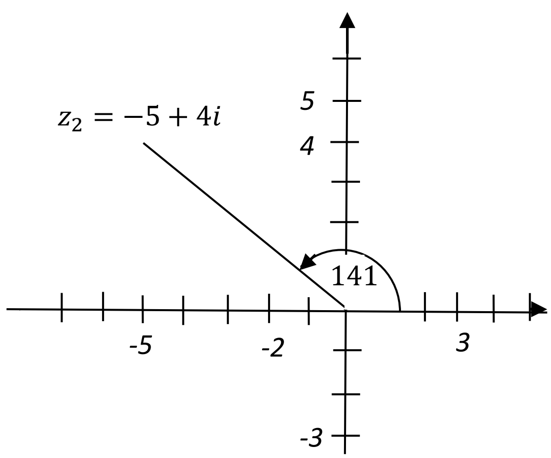
⇨ $\quad |z_1| = |3 + 5i| = \sqrt{3^2 + 5^2} = \sqrt{34}$ and $\arg(z_1) = \tan^{-1}\left(\frac{5}{3}\right) = 59.04$



$ii)\quad z_2 = -5 + 4i,$

⇨ $\quad |z_2| = |-5 + 4i| = \sqrt{(-5)^2 + 4^2} = \sqrt{41}$ and

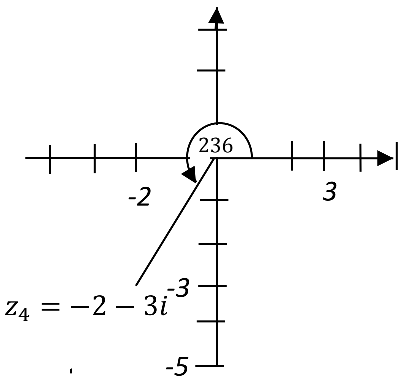
$\arg(z_2) = \tan^{-1}\left(\frac{4}{-5}\right) = (-38.65^0 + 180^0) = 141.34^0$



$iii)\quad z_3 = -2 - 3i,$

⇨ $\quad |z_3| = |-2 - 3i| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$ and

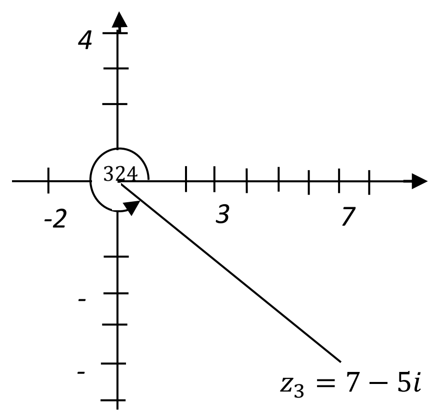
$\arg(z_4) = \tan^{-1}\left(\frac{-3}{-2}\right) = (56.31^0 + 180^0) = 236.31^0$



$iii)\quad z_3 = 7 - 5i,$

⇨ $\quad |z_4| = |7 - 5i| = \sqrt{7^2 + (-5)^2} = \sqrt{74}$ and

$\arg(z_4) = \arg(z_3) = \tan^{-1}\left(\frac{-5}{7}\right) = (-35.54^0 + 360^0) = 324.46^0$



Exercise1:

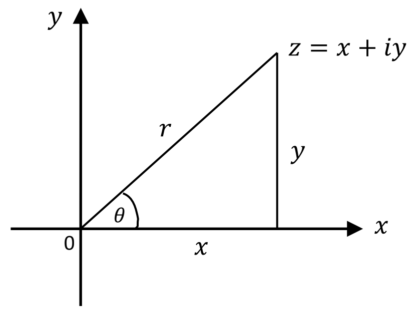
Given that, $z_1 = 2 + 3i,\ and\ z_2 = 4 - 5i,$

Find the modulus and the argument of, $\quad i)\ (5z_1 + 3z_2)^2 \qquad ii)\ \frac{z_1 + z_2}{z_1 - z_2}$

Polar Representation of Complex numbers:

The complex number $z = x + iy$ can be expressed in terms of its modulus r and argument θ by the relation $z = r(\cos\theta + i\sin\theta)$ as follows:

In the fig. below let the point z represents the complex number $z = x + iy$ and if the length oz is denoted by r and the angle aoz by θ we have



$$\cos\theta = \frac{x}{r} \quad \Rightarrow \quad x = r\cos\theta, \ \ and \quad \sin\theta = \frac{y}{r} \quad \Rightarrow \quad y = r\sin\theta,$$

$\Rightarrow \qquad z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$\Rightarrow \qquad z = r(\cos\theta + i\sin\theta) \qquad \qquad . . . (1)$

And is called the polar form of complex numbers

Example 1: Given that $z_1 = 2 + 3i, \ and \ \ z_2 = -3 + 4i$. Find the polar representation of $z_1 \ and \ z_2$

Solution:

$i) \ \ z_1 = 2 + 3i, \quad \Rightarrow \quad r = \sqrt{2^2 + 3^2} = \sqrt{13} \ \ and \ \ \theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.31$

$\Rightarrow \qquad z_1 = \sqrt{13}(\cos 56.31 + i\sin 56.31)$

$ii) \ \ z_2 = -3 + 4i, \quad \Rightarrow \quad r = \sqrt{(-3)^2 + 4^2} = 5 \ \ and \quad \theta = \tan^{-1}\left(\frac{4}{-3}\right) = -53.13 + 180$

$\Rightarrow \qquad z_2 = 5(\cos 126.87 + i\sin 126.87)$

Exercises: Represent the following complex numbers in the form $r(\cos\theta + i\sin\theta)$

$i) \ \frac{1}{2} + \frac{\pi}{4}i \qquad ii) \ \frac{-6 + 5i}{3i} \qquad iii) \ \frac{2 + 3i}{5 + 4i} \qquad iv) \ \frac{3\sqrt{2} + 2i}{-\sqrt{2} - \left(\frac{2}{3}\right)i}$

Note: For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leq \theta < 2\pi$, however, any interval of length 2π for example $-\pi < \theta \leq \pi$, can be used. Any particular choice, decided upon in advance, is called the $principal\ range$ and the value of θ is called its $principal\ value$.

Theorem: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then,

1. $z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$ . . . (a)
2. $\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$ . . . (b)

Proof:

1. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1 * z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) * r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \end{aligned}$$

Hence the proof:

2. $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\left(\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} * \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}\right)$

$= \frac{r_1}{r_2} * \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{sin^2\theta_2 + cos^2\theta_2}$

$= \frac{r_1}{r_2}[\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Hence the proof:

Example1:

Given that, $z_1 = 2\sqrt{3} - 2i$, $z_2 = -1 + \sqrt{3}i$, Obtain the polar representation of z_1 and z_2, and find $z_1 * z_2$, and $\frac{z_1}{z_2}$ in polar form

Solution:

$z_1 = 4\left[\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6})\right]$ and $z_2 = 2\left[\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})\right]$

⇨ $z_1 * z_2 = 4 * 2\left\{\cos(-\frac{\pi}{6} + \frac{2\pi}{3}) + i\sin(-\frac{\pi}{6} + \frac{2\pi}{3})\right\}$

$= 8\left\{\cos(\frac{5\pi}{2}) + i\sin(\frac{5\pi}{2})\right\} = 8\left\{\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right\} = 8i$

⇨ $\frac{z_1}{z_2} = \frac{4}{2}\left\{\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})\right\} = 2\left(-\frac{\sqrt{3}}{2} + i(-\frac{1}{2})\right) = -\sqrt{3} - i$

Example2:

Given that $z_1 = 1 - i$, $z_2 = -2 - 2i$. Find the polar representation of z_1 and z_2, and Find $z_1 * z_2$ and $\frac{z_1}{z_2}$, in polar form

Solution: Do

In the same way, if

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ and $z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$ then

$z_1 * z_2 * z_3 = r_1 * r_2\{\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)\}$

Now if $z = r(\cos\theta + i\sin\theta)$, then

$$z^2 = z * z = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = z^2 * z = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = z^3 * z = r^4(\cos 4\theta + i\sin 4\theta)$$

And in general

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Theorem (De Moivre's): Given that $z = r(\cos\theta + i\sin\theta)$is a non-zero complex number, then

$z^n = r^n(\cos n\theta + i\sin n\theta)$

Example 1:

Given that $z = 1 + i$, Find z^{20}.

Solution:

$$z = 1 + i, = \sqrt{2}\left\{\cos\left(\frac{\pi}{4}\right) + i\sin(\frac{\pi}{4})\right\}$$

$$z^{20} = (2^{\frac{1}{2}})^{20}\left[\cos 20\left(\frac{\pi}{4}\right) + i\cos 20\left(\frac{\pi}{4}\right)\right] = 2^{10}[\cos 5\pi + i\sin 5\pi]$$

Example 2:

Given that $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, Find z^{20}.

Solution:

Do

Example 3: Use *De' Moivre's* theorem to show that

$$\frac{\cos 3\theta + i\sin 3\theta}{\cos 5\theta - i\sin 5\theta} = \cos 8\theta + i\sin 8\theta$$

Solution:

$$\frac{\cos 3\theta + i\sin 3\theta}{\cos 5\theta - i\sin 5\theta} = \frac{\cos 3\theta + i\sin 3\theta}{\cos 5\theta - i\sin 5\theta} * \frac{\cos 5\theta + i\sin 5\theta}{\cos 5\theta + i\sin 5\theta} = \frac{\cos(3\theta+5\theta) + i\sin(3\theta+5\theta)}{cos^2 5\theta + sin^2 5\theta} = \cos 8\theta + i\sin 8\theta$$

Roots of Complex Numbers:

Let z be a non-zero complex number ($z \neq 0$), a number w is called an n^{th} $root$ of a complex number z if $w^n = z$ and we write $w = z^{\frac{1}{n}}$.

Let $\quad w = s(\cos\alpha + i\sin\alpha)$ . . . (1)

⇨ $\quad w^n = s^n(\cos n\alpha + i\sin n\alpha)$

and

$z = r(\cos\theta + i\sin\theta)$, then,

$s^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta)$ . . . (2)

⇨ $\quad s^n = r$

And since $s \geq 0,\ and\ r \geq 0$ we have $s = \sqrt[n]{r}$

Now substituting $s = \sqrt[n]{r}$ in equation (2) we have

$r(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta)$

⇨ $\quad (\cos n\alpha + i\sin n\alpha) = (\cos\theta + i\sin\theta)$

⇨ $\quad \cos n\alpha = \cos\theta$ and $\sin n\alpha = \sin\theta$ . . . (3)

Since both the two functions in two have period 2π, the two equations are true if and only if $n\alpha\ and\ \theta$ differ by a multiple of 2π. Thus for any integer $, n\alpha = \theta + 2\pi k$.

⇨ $\quad \alpha = \frac{\theta + 2\pi k}{n} \quad k = 0, 1, 2,\ .\ .\ .\ (n-1)$ . . . (4)

Substituting (4) into (1) we have

⇨ $\quad w = r^{\frac{1}{n}}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\} \ k = 0, 1, 2, 3, .\ .\ .\ (n-1)$ . . . (5)

Which follows that, there are n different values of $z^{\frac{1}{n}}$, i.e. n different n^{th} roots of z

Theorem: if $z = r(\cos\theta + i\sin\theta)$ is any non-zero complex number and if,n is any $+ve$ integer, then $z$has precisely $n$distinct n^{th} root. Moreover, the roots are given by.

$w_k = r^{\frac{1}{n}}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}, where\ \ k = 0, 1, 2, 3, .\ .\ .\ (n-1)$ . . . (6)

Note:

1. The n^{th} root of z all have modulus $\sqrt[n]{r}$, and hence they lie on the circle of radius $\sqrt[n]{r}$ with centre at o. Moreover, they are equally spaced on this circle since the difference in the arguments of successive n^{th}roots is $\frac{2\pi}{n}$.

2. It is sometimes convenient to use degree measure for 2π, so that equation (4) becomes

$$w = r^{\frac{1}{n}}\left\{\cos\left(\frac{\theta+k*360^o}{n}\right) + i\sin\left(\frac{\theta+k*360^o}{n}\right)\right\}, where\ \ k = 0, 1, 2, 3, . . . (n-1) \qquad . . . (7)$$

Example1: Find the four fourth root of the complex number $z = -8(1 + i\sqrt{3})$

Solution:

$$r = \sqrt{(-8)^2 + \left(-8\sqrt{3}\right)^2} \ = \ \sqrt{256} = 16 \text{and, } \theta = \tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right) = \tan^{-1}\left(\sqrt{3}\right)$$

$= (60^o + 180^o) = 240^o$ since the complex number lie in the third quadrant.

⇨ $z = 16(\cos 240^o + i\sin 240^o)$

⇨ $w_k = (16)^{\frac{1}{4}}\left\{\cos\left(\frac{240^o+k*360^o}{4}\right) + i\sin\left(\frac{240^o+k*360^o}{4}\right)\right\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

$= (2^4)^{\frac{1}{4}}\left\{\cos\left(\frac{240^o+k*360^o}{4}\right) + i\sin\left(\frac{240^o+k*360^o}{4}\right)\right\}, where\ \ k = 0,1, 2, 3, . . . (n-1)$

$= 2\{\cos(60^o + k*90^o) + i\sin(60^o + k*90^o)\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

For $k = 0, 1, 2, 3$. We have the four fourth roots as follows.

$w_0 = 2\{\cos 60^o + i\sin 60^o\} = 1 + \sqrt{3}i$ $\qquad$ $w_1 = 2\{\cos 150^o + i\sin 150^o\} = -\sqrt{3} + i$

$w_2 = 2\{\cos 240^o + i\sin 240^o\} = -1 - \sqrt{3}i$ $\qquad$ $w_3 = 2\{\cos 330^o + i\sin 330^o\} = \sqrt{3} - i$

Example2: Find the n n^{th} root of the following complex number

$i)\ \sqrt{-i}$ $\quad$ $ii)\ \sqrt[8]{1}$ $\quad$ $iii)\ \sqrt[4]{-1}$ $\quad$ $iv)\ \sqrt[3]{3+4i}$

Solution:

Example2: Find the six sixth root of the complex number $z = -1 + 0i$

Solution:

$r = \sqrt{(-1)^2 + (0)^2} \ = \ \sqrt{1} = 1$and, $\theta = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1}(0) = (0^o + 180^o) = 180^o$ since the complex number lie in the on the negative x-axis.

⇨ $z = 1(\cos 180^o + i\sin 180^o)$

⇨ $w_k = (1)^{\frac{1}{6}}\left\{\cos\left(\frac{180^o+k*360^o}{6}\right) + i\sin\left(\frac{180^o+k*360^o}{6}\right)\right\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

$= \{\cos(30^o + k*60^o) + i\sin(30^o + k*60^o)\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

For $k = 0, 1, 2, 3, 4, 5$. We have the six sixth roots as follows.

$w_0 = \{\cos 30^o + i\sin 30^o\} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\qquad$ $w_1 = \{\cos 90^o + i\sin 90^o\} = 0 + i$

$w_2 = \{\cos 150^o + i\sin 150^o\} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\qquad$ $w_3 = \{\cos 210^o + i\sin 210^o\} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$w_4 = \{\cos 270^o + i\sin 270^o\} = 0 - i$ $\qquad$ $w_5 = \{\cos 330^o + i\sin 330^o\} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

The n^{th} root of Unity:

The special case in which $z = 1$ is of particular interest that leads to the n distinctn^{th} root of 1 are called the n^{th} root of unity and are given by

$$z = \cos(0^o) + i\sin(0^o),$$
$$w_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right),$$

Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the $unit\ circle$.

Example1: Find the three third root of unity.

Solution:

$r = |z| = \sqrt{(1)^2 + (0)^2} = \sqrt{1} = 1$ and,

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1}(0) = 0^o$ since the complex number lie on the $+vex$-axis.

⇨ $\quad z = 1(\cos 0^o + i\sin 0^o)$

⇨ $\quad w = (1)^{\frac{1}{3}}\left\{\cos\left(\frac{k*360^o}{3}\right) + i\sin\left(\frac{k*360^o}{3}\right)\right\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

$\quad = \{\cos(k*120^o) + i\sin(k*120^o)\}, where\ \ k = 0, 1, 2, 3, . . . (n-1)$

For $k = 0, 1, 2, .$ We have the six sixth roots as follows.

$w_0 = \{\cos 0^o + i\sin 0^o\} = 1 + 0i$ $\qquad w_1 = \{\cos 120^o + i\sin 120^o\} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

$w_2 = \{\cos 240^o + i\sin 240^o\} = -\frac{1}{2} - \frac{\sqrt{3}}{2}$

Euler's formula:

By assuming that the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + . . . \qquad . . . (1)$$

Holds good, substituting $x = i\theta$ in (1) we arrive at a result

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and is called $Euler's\ formula$

⇨ $\quad re^{i\theta} = r(\cos\theta + i\sin\theta)$

⇨ $\quad z = re^{i\theta}$

⇨ $\quad z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Example: Given that $z = 3 + 5i$, find z^6 in exponential form

Solution:

$z = 3 + 5i$

⇨ $|z| = \sqrt{3^2 + 5^2} = \sqrt{34} = r$ and $\theta = \tan^{-1}\left(\frac{5}{3}\right) \approx 59^0$

$z^n = r^n e^{in\theta}$

⇨ $z^6 = r^6 e^{i6\theta} = \left(\sqrt{34}\right)^6 e^{6(59)i} = (34)^3 e^{6(59)i}$

Polynomial Equations:

Often in practice we require solutions of polynomial equations having the form

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ . \ . \ . + a_{n-1} z + a_n = 0$$

Where $a_0 \neq 0, a_1, a_2, . \ . \ . \ a_n$ are given real coefficient and n is a positive integer called the degree of the equation. Such solutions are also called $zeros$ of the polynomial, or $roots$ of the equation.

Theorem: Every polynomial equation

$P_n(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ . \ . \ . + a_{n-1} z + a_n = 0$

$a_n \neq 0$ of degree $n \geq 1$ has at least one solution in the field of complex numbers

Exercise:

Show that every real is a complex number, but not every complex number is a real number.

Proof:

Each complex number z may be express in the form $z = x + iy$, where x and y are real numbers.

So when $y = 0$, $z = x + 0i$ ⇨ $z = x$

⇨ z is purely a real number

But when $y \neq 0$, then z cannot be entirely real.

Proposition: In the field of complex numbers, the only polynomials which cannot be factorized further are of the form $a_1 z + a_0$ (i.e. the polynomial of degree one)

Note: Given a polynomial equation

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ . \ . \ . + a_{n-1} z + a_n = 0$$

If $x = c$ is a root (or zero, or solution) then, $P_n(c) = 0$, and $z = c$ is a factor.

Exercise: Show that the polynomial

$$f(z) = z^2 + \tfrac{3}{4}z + 7$$

Is factorizable over the complex field but not factorizable over the real field

Proposition: If $x + iy$ is a root of

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ . \ . \ . + a_{n-1} z + a_n = 0$$

Where $a_0 \neq 0, a_1, a_2, a_3, . \ . \ .$ are real number, then $x - iy$ is also a root.

Example: show that the polynomial $P(z) = 2z^2 + 2z + 1$

Has its two roots $z = -1 + i$ and $z = -1 - i$

Proposition:

At least one of the roots of

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ . \ . \ . + a_{n-1} z + a_n = 0$$

With real coefficients and n odd must be real.

Example; $P_5(z) = z^5 + 5z^4 + 10z^3 + \ 10z^2 + 9z + 5 = 0$

Proposition:

A polynomial of degree $n \geq 1$ has at most n distinct roots

Example; $P_5(z) = z^5 + 5z^4 + 10z^3 + \ 10z^2 + 9z + 5 = 0$

Has at most 5 distinct roots

Example; $P_3(z) = z^3 - 5z^2 + 16z - 5 = 0$, has at most 3 distinct roots

Complex Numbers as Ordered pairs:

From a strictly logical point of view it is desirable to define a complex number as an ordered pair (x, y) of real numbers x and y subject to certain operational definitions which turn out to be equivalent to those above.

Let $z_1 = (x_1, y_1)$and $z_2 = (x_2, \ y_2)$, be any two complex numbers then

1. *Addition:* $z_1 + z_1 = (x_1 + x_2) + i(y_1 + y_2)$
2. *Subtraction*: $z_1 - z_1 = (x_1 - x_2) + i(y_1 - y_2)$
3. *Multiplication*: $z_1 z_1 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$
4. *Division:* $\frac{|z_1|}{|z_2|} = \frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right)$

Exercises:

1. Given that $z_1 = 7 + 5i \; and \; z_2 = 2 - 3i$.

Write the following complex numbers in ordered pairs,

$i)$ $(z_1 + z_2)^9$

$ii)$ $\overline{z_1} - \overline{z_2}$

$iii)$ $\frac{z_1}{2} * z_2$ $\quad$ $iv)$ $\frac{2}{z_2}$

2. Write each of the following in the form $(x, \; y)$

$i)$ $-(-3 + 7i) + (-6 + 6i)$

$ii)$ $(16 + 10i) - (9 - 15i$

$iii)$ $(-7 + i)(3 - i)$