ABUBAKAR TAFAWA BALEWA UNIVERSITY, BAUCHI
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICAL SCIENCES

MTH112

Calculus I

Lecture Note

Course outline:

i) **Real Numbers:** *Real numbers and their properties, the number line, intervals and their properties, absolute values and their properties, solving inequalities using sign-chart.*

ii) ***Functions from real number to real number:*** *Functions, domain and range of a function, monotonically increasing and monotonically decreasing functions, inverse function and inverse of a function, composition (or, product) of functions, even and odd functions, periodic functions*

iii) ***Sequence, Limit and Continuity:*** *Sequence, convergent of a sequence, limit of a function, left and right hand limit, evaluation of limits, continuity of a function.*

iv) ***Differentiation:*** *Differentiability at a point and on an interval, rules of differentiation, (sum, product, quotient, and chain rules.), differentiation of inverse trigonometric functions, implicit differentiation*

v) ***Integration:*** *Fundamental theorem of calculus, methods of integration (change of variables, integration by parts, trigonometric substitution), integration of rational algebraic functions using partial fraction decomposition method, numerical integration (mid-point rule, trapezoidal rule, Simpson rule)*

Reference books:

i) *Brief Calculus and its applications (eighth edition)*
- *Larry J. Goldstein. David C. lay. David I. Schneider*

ii) *Advance Engineering Mathematics (ninth edition)*
- *Erwin Kreyszig*

iii) *Calculus (second edition)*
- *Robert T. Smith. Ronald B. Minton*

iv) *Engineering Mathematics with addition (sixth edition)*
- *K. A. Stroud. Dexter J.*

v) *Advance Engineering Mathematics*
- *H. K. Dass*

vi) *Calculus with analytic geometry (5th edition)*
- *Robert Ellis and Denny Gulick*

Real Numbers:

Definition1: The Set of numbers denoted $N = \{1, 2, 3, 4, 5, 6 \dots\}$ is called natural numbers.

Note:

i) The subsets of natural numbers include, Even numbers, Odd numbers, Prime numbers and Composite numbers

ii) Natural numbers are the first set of numbers, and are used for counting.

- The set of natural numbers has defect, since not all equation of the form $a + b = n$ (where, $a, b, n \,\varepsilon\, N$) have a solution in N.

  e.g. $5 + x = 5$ has no solution in N

Definition2: The set formed by adding zero to the set of natural numbers denoted, $W = \{0, 1, 2, \dots\}$ is called whole numbers.

- Again it was discovered that the set of whole numbers has defect since not all equations of the form $a + b = n$ $(where\ a, b, n \,\varepsilon\, W)$ has a solution in W. *e.g.* $5 + x = 2$ *has no solution in* W.

To overcome this problem, the directed number system are invented such that for each $a \,\varepsilon\, N$, two directed numbers positive and negative($-a$ & a)where associated with the number zero at the Centre, the positive numbers are on the right of zero, and the negative numbers on the left of zero.

Definition3: The Set of directed numbers denoted $Z = \{0, \pm1, \pm2, \pm3 \dots\}$is called an integer.

- In an attempt to find solution to the equation of the form $ax = b$ where $a, b, \varepsilon\, Z$, it was also discovered that not all such equations has solution in Z.

e.g. $3x = 2$ has no solution in Z, these defect is remedied by adding to the set of integers numbers called common fractions.

Definition4: The set formed by adding common fractions to the set of integers is called rational numbers, denoted $Q = \{x : x = \frac{p}{q}\ q \neq 0, p, q \,\varepsilon\, Z\}$.

Note: Decimal numbers can also be used to express rational numbers; rational numbers can take any of the two forms of decimal numbers viz.

i) Terminating decimal e.g. 0.5, 0.13, 4.72, 3.825,

ii) Non-terminating but repeating in blocks decimal

e.g. 0.27272727 . . . . , 2.54545454 . . . . $-0.6666666 \dots\dots\dots$

- In an attempt to find solutions of equations such as $x^2 = 3$, it was discovered that there is no solution in the set of rational numbers, therefore the need arise to enlarge the set of rational numbers.

Definition5: The set of real numbers that are not rational are called Irrational denoted, Q^I .

Note: In the same way, irrational numbers can be expressed as a non-terminating and non-repeating in blocks decimal number.

e.g. i) 1.73205214. . . . *ii) 0.328471250765. . . .*

Properties of Irrational Numbers:

Let P and m be any positive prime numbers and $n \varepsilon Q$,

$i)$ All numbers of the form $n\sqrt{p}$, where $n \neq 0$ are irrational numbers.

$ii)$ All numbers of the form $(n + \sqrt{p}$), are irrational numbers

$iii)$ All numbers of the form, $\sqrt{m} + \sqrt{p}$, where $m \neq p$, are irrational numbers

$iv)$ Any number that is not a perfect square, the square root is an irrational number.

$v)$ The number " e " and " π " are also irrational

Definition6: The union of rational numbers and irrational numbers form a set called real numbers i.e. $R = Q \cup Q^I$

Example1: Show that $\sqrt{3}$ is not a rational number.

Proof:

Suppose the equation $x^2 = 3$ has solution in Q,

let, $x = \frac{p}{q}$, and that $\frac{p}{q}$ is reduced to its lowest terms,

then $x^2 = 3$ and $x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$ $\Rightarrow$ $\frac{p^2}{q^2} = 3$ and $p^2 = 3q^2$.

It follows that p^2 is a multiple of 3 which makes p also multiple of 3.

Since p is a multiple of we can then write

$p = 3m$ for some $m \,\varepsilon\, Z$, then

$\Rightarrow$ $p^2 = (3m)^2 = 9m^2$

$\Rightarrow$ $3q^2 = 9m^2$ $\Rightarrow$ $q^2 = 3m^2$

It follows that q^2 is a multiple of 3 which makes q also multiple of 3.

Now p and q have been found multiple of 3 which contradict our assumption that $\frac{p}{q}$ was expressed in its simplest form, Therefore $x^2 = 3$ cannot have solution in Q.

Hence $\sqrt{3}$ is not a rational number.

Exercise: Show that $\sqrt{5}$ is not a rational number

Some properties of real numbers:

$i)$ All real numbers can be represented on the real line.

$ii)$ All real numbers are closed under the four basic operations, except division by zero.

$iii)$ All real numbers are commutative under the operation of addition and multiplication.

$iv)$ All real numbers are associative under the operation of addition and multiplication.

$v)$ Real numbers obey the cancellation law, i.e. if $a + b = q$ and $a + c = q$ $\Rightarrow b = c$ and if $a * b = r$ and $a * c = r$ $\Rightarrow$ $b = c$ $\forall\, a, b, c \,\varepsilon\, R$.

$vi)$ There exist, $0 \,\varepsilon\, R$, such that $a + 0 = 0 + a = a$ $\forall\, a \,\varepsilon\, R$ (*existence of additive identity*)

$vii)$ There exist, $1 \,\varepsilon\, R$, such that $a * 1 = 1 * a = a$, $\forall a \,\varepsilon\, R$ (*existence of multiplicative identity*)

$viii)$ $\forall a \,\varepsilon\, R$ $a \neq 0$, $\ni a^{-1} \varepsilon\, R$, such that $a * a^{-1} = 1$ (*existence of multiplicative inverse*)

$ix)$ $\forall a \,\varepsilon\, R$, $a \neq 0$ $\exists (-a) \varepsilon\, R$, such that $a + (-a) = (-a) + a = 0$ (*additive inverse*)

$x)$ Multiplication is distributive over addition i.e. $a * (b + c) = (a * b) + (a * c)$.

Note: The multiplicative inverse element denoted a^{-1} *or* $\frac{1}{a}$ *is called the reciprocal of* a.

Real number line inequality:

An inequality is a statement that one expression is smaller than the other expression or greater than the other expression. Hence the following expression defines inequality. If a, b and x are real numbers, then

$i)$ $a < b$ ⇨ a is less than b, and $a > b$ ⇨ a is greater than b.

$ii)$ $a \leq b$ ⇨ a is less than or equal to b and $a \geq b$ ⇨ a is greater than or equal to b.

$iii)$ $a < x < b$ ⇨ x is greater than a but less than b.

$iv)$ $a \leq x \leq b$ ⇨ x is greater than or equal to a but less than or equal to b

Other properties of real numbers:

$i)$ $\forall\, a, b \,\varepsilon\, R$, One and only one of the following properties hold good.

Either $a < b$, *or* $a = b$, *or* $a > b$ (Tracheotomy law)

$ii)$ $\forall a, b \,\varepsilon\, R$, with $a < b$, then $\exists\, c \,\varepsilon\, R$ such that $a < c < b$ (density law)

Principles of inequality:

$i)$ The sense of inequality remains unchanged if each side is increase or decrease by the same real number.

$ii)$ The sense of inequality is unchanged if each side is multiplied by the same positive real number.

$iii)$ If $a > 0$, and $b > 0$, then $a + b > 0$.

$iv)$ If $a - b < 0$, then $a < b$, and if $a - b > 0$, then $a > b$.

$v)$ If $a < b$, and $b < a$ then $a = b$

Theorem (transitivity law): If $a > b$ and $b > c$, then $a > c$

Proof:

Since $\left.\begin{array}{l} a > b \Rightarrow a - b > 0 \\ b > c \Rightarrow b - c > 0 \end{array}\right\}$ ⇨ $(a - b) + (b - c) > 0$

⇨ $a - b + b - c > 0$

⇨ $a - c > 0$

⇨ $a > c$

Hence the proof

Theorem: The arithmetic mean of any two positive real numbers a and b is greater than or equal to their geometric mean ($AM \geq GM$)

Proof:

$AM = \frac{a+b}{2}$ $GM = \sqrt{a * b}$

If $AM \geq GM$, then $AM - GM \geq 0$

So we have to show that $AM - GM \geq 0$

$AM - GM = \frac{a+b}{2} - \sqrt{a * b}$

⇨ $2AM - 2GM = a + b - 2\sqrt{a * b} = a - 2\sqrt{a}\sqrt{b} + b$

⇨ $2AM - 2GM = (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 = \left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$

⇨ $AM - GM \geq 0$ Hence, $AM \geq GM$

Example: Show that if a, b, c are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$

Proof:

Since a, b, and c are positive real numbers

⇨ $\frac{a+b}{2} \geq \sqrt{ab}$, $\frac{b+c}{2} \geq \sqrt{bc}$, $\frac{c+a}{2} \geq \sqrt{ca}$

⇨ $\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) \geq (\sqrt{ab})(\sqrt{bc})(\sqrt{ca})$

⇨ $(a+b)(b+c)(c+a) \geq 8\sqrt{(ab)(bc)(ca)}$

⇨ $(a+b)(b+c)(c+a) \geq 8\sqrt{a^2b^2c^2}$

⇨ $(a+b)(b+c)(c+a) \geq 8\sqrt{(abc)^2}$

⇨ $(a+b)(b+c)(c+a) \geq 8abc$

Hence the proof

Theorem: The sum of any positive real number and its reciprocal is greater than or equal to two

Proof:

Let a be any positive real number

⇨ $a - 1 \geq 0$

Squaring both side, we have $(a-1)^2 \geq 0$

⇨ $a^2 - 2a + 1 \geq 0$

⇨ $a^2 + 1 \geq 2a$

⇨ $\frac{a^2+1}{a} \geq 2$ ⇨ $a + \frac{1}{a} \geq 2$

Hence the proof

Theorem: A linear factor $ax + b$ changes sign only at $x = -\frac{b}{a}$ where it takes on the value 0.

Solving inequalities using sign chart:

Example1: Find the range of values of x for which $i)$ $x^2 - 7x + 12 > 0$ $ii)$ $3x^2 - 3x \geq 6 - 10x$ $iii)$ $x^2 - 2x \leq 16 + 4x$ $iv)$ $2x^2 + 2x \leq 3x + 1$

Solution;

$i)$ $x^2 - 7x + 12 > 0$

⇨ $(x-3)(x-4) > 0$

Hence this is true if both the factors are positive or both the factors are negative. This can be clearly seen if we make the following table showing the sign of the factors.

----	$x < 3$	$3 < x < 4$	$x > 4$
$(x-3)$	$-$	$-$	$+$
$(x-4)$	$-$	$+$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if $x < 3$ or $x > 4$

$ii)$ $3x^2 + 2x \geq 6 - 10x$

⇨ $(3x-2)(x+3) \geq 0$

Hence this will be true if both the factors are either positive or both are negative, or at least one of the factors is zero. This can be clearly seen if we represent it on the sign chart

----	$x < -3$	$-3 < x < \frac{2}{3}$	$x > \frac{2}{3}$
$(3x-2)$	$-$	$-$	$+$
$(x+3)$	$-$	$+$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if

$x \leq -3$ or $x \geq \frac{2}{3}$

Note: The table showing the sign of the factors is called Sign Chart.

Example2: Find the range of values of x for which

i) $x^2 - 2x \leq 16 + 4x$ *ii)* $2x^2 + 2x < 3x + 1$

Solution;

i) $x^2 - 2x \leq 16 + 4x$

⇨ $(x+2)(x-8) \leq 0$

Hence this will be true only if one of the factors is positive and the other negative, or at-least one of the factors is zero. This can be clearly seen if we represent it on the sign chart

----	$x < -2$	$-2 < x < 8$	$x > 8$
$(x+2)$	$-$	$-$	$+$
$(x-8)$	$-$	$+$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if

$-2 < x < 8$

ii) $2x^2 + 2x < 3x + 1$

⇨ $(2x+1)(x-1) < 0$

Hence this will be true only if one of the factors is positive and the other negative. This can be clearly seen if we represent it on the sign chart

----	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 1$	$x > 1$
$(2x+1)$	$-$	$+$	$+$
$(x-1)$	$-$	$-$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if

$-\frac{1}{2} < x < 1$

Example3: Find the range of values of x for which

i) $\frac{2x+1}{x+2} > \frac{1}{2}$ *ii)* $\frac{2x-1}{x+3} < \frac{2}{3}$ *iii)* $\frac{x}{x+2} > \frac{3}{x-2}$ *iv)* $\frac{3+x}{(1-2x)(2+x)} < 0$

Solution:

i) $\frac{2x+1}{x+2} > \frac{1}{2}$

Multiply both side by $2(x+2)^2$ and simplify we have $(3x)(x+2) > 0$

Hence this will be true if both the factors are positive or both factors are negative, and this can be clearly seen on the sign chart below.

----	$x < -2$	$-2 < x < 0$	$x > 0$
$3x$	$-$	$-$	$+$
$(x+2)$	$-$	$+$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if

$x < -2$ or $x > 0$

ii) $\frac{2x-1}{x+3} < \frac{2}{3}$

Multiply both side by $3(x+3)^2$ and simplify we have $(x+3)(4x-9) < 0$

This will be true if one of the factor is positive and the other factor is negative, and this can be clearly seen on the sign chart below.

----	$x < -3$	$-3 < x < \frac{9}{4}$	$x > \frac{9}{4}$
$(x+3)$	$-$	$-$	$+$
$(4x-9)$	$-$	$+$	$+$
Prd	$+$	$-$	$+$

Thus the original inequality is true if

$-3 < x < \frac{9}{4}$

Example4: Find the range of values of x for which

i) $\frac{x}{x+2} > \frac{3}{x-2}$ *ii)* $\frac{3+x}{(1-2x)(2+x)} < 0$

Solution:

i) $\frac{x}{x+2} > \frac{3}{x-2}$

Multiply both side by $(x+2)^2(x-2)^2$ and simplify we have$(x-2)(x+2)(x-6)(x+1) > 0$

This will be true if all the factors are positive or all the factors are negative, or any of the two factors are both positive or negative, and this can be clearly seen on the sign chart below.

*	$x<-2$	$-2<x<-1$	$-1<x<2$	$2<x<6$	$x>6$
$(x-2)$	$-$	$-$	$-$	$+$	$+$
$(x+2)$	$-$	$+$	$+$	$+$	$+$
$(x-6)$	$-$	-	$-$	$-$	$+$
$(x+1)$	$-$	$-$	$+$	$+$	$+$
$product$	$+$	$-$	$+$	$-$	$+$

Thus the original inequality is true if $x<-2$, $-1<x<2$ and $x>6$

ii) $\frac{3+x}{(1-2x)(2+x)}<0$

Multiply both side by $(1+2x)^2(2+x)^2$ and simplify we have $(1-2x)(2+x)(3+x)<0$

This will be true if all the factors are positive or all the factors are negative, or any of the two factors are both positive or negative, and this can be clearly seen on the sign chart below.

*	$x<-3$	$-3<x<-2$	$-2<x<\frac{1}{2}$	$x>\frac{1}{2}$
$(1-2x)$	$+$	$+$	$+$	$-$
$(2+x)$	$-$	$-$	$+$	$+$
$(3+x)$	$-$	$+$	$+$	$+$
$product$	$+$	$-$	$+$	$-$

Thus the original inequality is true if $-3<x<-2$ and $x>\frac{1}{2}$

Exercises; Find the range of values of x for which

i) $\frac{1}{x+2}<\frac{3}{x-3}$ *ii)* $\frac{2}{x-1}<\frac{x}{x+2}$ *iii)* $\frac{x+1}{x-2}>\frac{3}{x-2}-\frac{1}{2}$ *iv)* $\frac{x^2+2x-19}{x-4}\geq 4$

Theorem: If $a_1, a_2, a_3, \ldots a_n$ and $b_1, b_2, b_3, \ldots b_n$ are real numbers, then

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$$

Proof: For all real number λ, we have

$$(a_1\lambda + b_1)^2 + (a_2\lambda + b_2)^2 + (a_3\lambda + b_3)^2 + \ldots + (a_n\lambda + b_n)^2 \geq 0$$

Expanding and collecting like terms we have

$$(a_1^2 + a_2^2 + \ldots + a_n^2)\lambda^2 + 2(a_1b_1 + a_2b_2 + \ldots + a_nb_n)\lambda + (b_1^2 + b_2^2 + \ldots + b_n^2) \geq 0$$

Now setting $A = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2$, $B = b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2$ and

$$C = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n \quad ⇨ \quad C^2 = (a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n)^2$$

We have $A\lambda^2 + 2C\lambda + B \geq 0$ . . . (1)

Dividing both sides of eqn. (1) by A, and completing the squares we have

$$\lambda^2 + \left(\frac{2C}{A}\right)\lambda + \left(\frac{B}{A}\right) \geq 0$$

$$⇨ \quad \lambda^2 + \left(\frac{2C}{A}\right)\lambda + \left(\frac{B}{A}\right) + \frac{C^2}{A^2} - \frac{C^2}{A^2} \geq 0$$

$$⇨ \quad \left(\lambda + \frac{C}{A}\right)^2 + \frac{B}{A} - \frac{C^2}{A^2} \geq 0 \quad \ldots (2)$$

Equation (2) is true for all real λ if and only if $\frac{B}{A} - \frac{C^2}{A^2} \geq 0$ ⇨ $\frac{C^2}{A^2} \leq \frac{B}{A}$

Multiplying through by A^2

$$⇨ \quad C^2 \leq AB \quad \ldots (3)$$

Now substituting the values of A, B and C^2 in eqn. (3) we have

$$(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2)$$

Hence the proof

Intervals:

Definition: An interval is collection of the points defined by an inequality operation on the set of real numbers.

Definition: An open interval is the collection of the points defined by an inequality operation defined on the set of real numbers, where by the end points are excluded

Definition: A closed interval is the collection of the points defined by an inequality operation defined on the set of real numbers, whereby the end points are Included.

Intervals in real numbers:

1. ϕ 2. $[a, a]$ 3. (a, b) 4. $[a, b]$ 5. $[a, b)$ 6. $(a, b]$ 7. $(-\infty, a]$
8. $[a, \infty)$ 9. $(-\infty, a)$ 10. (a, ∞) 11. $(-\infty, \infty)$

Note: We should note that ∞, and $-\infty$ are not numbers but only symbols to denote that x may take values as large as we pleased without any bound.

Properties of Intervals:

Let τ be the family of all intervals on the real line, we include in τ the null set and a singleton set i.e. ϕ and $[a, a]$, then the intervals satisfies the following properties.

1. The intersection of any two intervals is an interval. (i.e. If $A \varepsilon \tau, B \varepsilon \tau$,then $(A \cap B) \varepsilon \tau$).
2. The union of any two non-disjoint intervals is an interval (i.e. if $A \varepsilon \tau, B \varepsilon \tau$, and $A \cap B \neq \phi$, then $(A \cup B) \varepsilon \tau$)
3. The relative complement of any two non-comparable intervals is an interval. (i.e. if $A \varepsilon \tau, B \varepsilon \tau$, and $A \not\subset B$, $B \not\subset A$, then $(A \backslash B) \varepsilon \tau$)

Modulus (or absolute value)

Geometrically, the modulus of x,($x \varepsilon R$) is the distance between the point x on the real line and the origin. Modulus of x is denoted $|x|$ and is define as

$$|x| = \begin{cases} x & if \ \ x > 0 \\ 0 & if \ \ x = 0 \\ -x & if \ \ x < 0 \end{cases}$$

Note;

i) The modulus of any real number is always non-negative

ii) The distance between any two real numbers x and y is giving by $|x - y| = |y - x|$.

iii) We should also note that $||x|| = |x|$

iv) There is also a single defining equation for modulus of x which involve square root denoted

$|x| = \sqrt{x^2} \ \forall \ x \ \varepsilon R$ *(the principal square root).*

Properties of modulus:

i) $|x * y| = |x| * |y|$

ii) $|x + y| \leq |x| + |y|$

iii) $|x - y| \geq |x| - |y|$

iv) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, *provided that* $y \neq 0$

Theorem: Let $a, d, \varepsilon R$, and $d > 0$, then

i) $|x| < d \quad \Rightarrow \quad -d < x < d$

ii) $|x| > d \quad \Rightarrow \quad -d > x > d.$

iii) $|x - a| < d \quad \Rightarrow \quad -d < x - a < d$ so that $a - d < x < a + d.$

iv) $|x - a| > d \quad \Rightarrow \quad -d > x - a > d$ so that $a - d > x > a + d.$

Example1: Express the following in modulus for

i) $-2 < 3x < 4$ *ii)* $-6 \leq x \leq 0$

Solution:

i) $-2 < 3x < 4$

$\Rightarrow \quad -\frac{2}{3} < x < \frac{4}{3}$

$\Rightarrow \quad -\frac{2}{3} - \frac{1}{3} < x - \frac{1}{3} < \frac{4}{3} - \frac{1}{3}$

$\Rightarrow \quad -1 < x - \frac{1}{3} < 1$

$\Rightarrow \quad -3 < 3x - 1 < 3$

$\Rightarrow \quad |3x - 1| < 3$

ii) $-6 < x < 0$

$\Rightarrow \quad -6 + 3 < x + 3 < 0 + 3$

$\Rightarrow \quad -3 < x + 3 < 3$

$\Rightarrow \quad |x + 3| < 3$

Exercises1: Express the following in modulus for

i) $-3 < 2x < 5$ *ii)* $(x + 1)^2 > 9$

Example2: Find the range of values of x for which *i)* $\left|\frac{x-7}{1-x}\right| < 3$ *ii)* $\left|\frac{x+4}{x-3}\right| \leq 2$

Solution:

i) $\left|\frac{x-7}{1-x}\right| < 3 \quad \Rightarrow \quad -3 < \frac{x-7}{1-x} < 3 \quad \Rightarrow \quad -3 < \frac{x-7}{1-x}$ or $\frac{x-7}{1-x} < 3$

$\Rightarrow$ *a)* $\frac{x-7}{1-x} > -3$

Multiply both side by $(1 - x)^2$ and simplify we have $(x - 1)(x + 2) > 0$

This will be true if either both the factors are positive, both factors are negative, and this can be clearly seen on the sign chart below.

*	$x < -2$	$-2 < x < 1$	$x > 1$
$(x - 1)$	$-$	$-$	$+$
$(x + 2)$	$-$	$+$	$+$

$\Rightarrow$ *b)* $\frac{x-7}{1-x} < 3$

Multiply both side by $(1 - x)^2$ and simplify we have $(x - 1)(2x - 5) > 0$

This will be true if either both the factors are positive, both factors are negative, and this can be clearly seen on the sign chart below

*	$x < 1$	$1 < x < \frac{5}{2}$	$x > \frac{5}{2}$
$(x - 1)$	$-$	$+$	$+$
$(2x - 5)$	$-$	$-$	$+$
$(x - 1)(2x - 5)$	$+$	$-$	$+$

Putting the two results together, we have that the original inequality is true if $x < \frac{2}{3}$ or $x > 10$

ii) $\left|\frac{x+4}{x-3}\right| \leq 2 \quad \Rightarrow \quad -2 \leq \frac{x+4}{x-3} \leq 2 \quad \Rightarrow \quad -2 \leq \frac{x+4}{x-3}$ or $\frac{x+4}{x-3} \leq 2$

$\Rightarrow$ a) $\frac{x+4}{x-3} \geq -2$

Multiply both side by $(x-3)^2$ and simplify we have $(x-3)(3x-2) \geq 0$

This will be true if either both the factors are positive, both factors are negative or one of the factors is zero, and this can be clearly seen on the sign chart below.

*	$x < \frac{2}{3}$	$\frac{2}{3} < x < 3$	$x > 3$
$(x-3)$	$-$	$-$	$+$
$(3x-2)$	$-$	$+$	$+$

$\Rightarrow$ b) $\frac{x+4}{x-3} \leq 2$

Multiply both side by $(x-3)^2$ and simplify we have $(x-3)(x-10) \leq 0$

This will be true if one of the factors is positive, and one of the factors is negative or one of the factors is zero, and this can be clearly seen on the sign chart below.

*	$x < 3$	$3 < x < 10$	$x > 10$
$(x-3)$	$-$	$+$	$+$
$(x+10)$	$-$	$-$	$+$

Putting the two results together, we have that the original inequality is true if $x < \frac{2}{3}$ or $x > 10$

Exercises: Find the range of values of x for which $i)$ $\left|\frac{x-3}{x+1}\right| < 2$ $ii)$ $\left|\frac{x+2}{x-1}\right| \leq 3$ $iii)$ $\left|\frac{x+1}{x}\right| > 2$

Example: Show that $|x-y| \geq ||x|-|y||$

Proof:

Let $x = (x-y) + y$

$\Rightarrow$ $|x| = |(x-y) + y|$

But $|(x-y) + y| \leq |x-y| + |y|$ (from property 2)

$\Rightarrow$ $|x| \leq |x-y| + |y|$

Rearranging we have

$$|x-y| \geq |x| - |y| \quad \ldots (1)$$

Taking modulus on both sides we have

$||x-y|| \geq ||x|-|y||$ $\Rightarrow$ $|x-y| \geq ||x|-|y||$

Hence the proof

Functions from real number to another real number $f: R \to R$

Introduction: The area A of a square depends on the length of its sides (x say), So the area is given by the formula $A = x^2$. Similarly the distance S (in feet) that a freely falling object drops in t seconds is describe by the formula$s = 16t^2$. These formulas illustrate the Mathematical notion of a function.

Definition: Let A and B be any two non-empty sets, if there is any rule which assigned to every element of set Aone and only one element of set B, then the rule is called a function or (mapping), denoted $f: A \to B$. And read f is a mapping from A to B.

Note:

i) Functions are also written as $y = f(x)$, and read y is a function of x.

ii) Arrow diagrams are also used to represent functions.

Definition: Let $f: A \to B$ be a function, if $a \varepsilon A$, then $f(a) \varepsilon B$, and $f(a)$ is called the image of a under f.

Definition: The set of all acceptable inputs of a function f is called the domain of f. In other words, if $f: A \to B$ is a function, then set A is called the domain of f, and set B is called the co domain of f.

Definition: The set of all outputs of a function f is called the range of f. In other words the set of those elements which appear as an image is called the range of f.

Note: The range of a function is always a subset of the Co-domain.

Definition: The range of values of x for which y is defined as a function of x is called the interval of definition (or the domain)

Types of functions:

Definition: Let A be any non-empty set, let $f: A \to A$ be a function defined by $f(a) = a$, then f is called an identity function.

Definition: Let A and B be any two non-empty sets, and let $f: A \to B$ be a function, then f is called a constant function if the same element $b \,\varepsilon\, B$ appear as an image of every element in A, denoted $f(a) = b$.

Definition: Let A and B be any two non-empty sets, and let $f: A \to B$ be a function and $g: A \to B$ be another function, then f and g are said to be equal if $f(a) = g(a) \quad \forall a \,\varepsilon\, A$.

e.g. $i) f(x) = x^2 - 2x,\qquad g(x) = x(1+x) - 3x$

$ii) \quad f(x) = x^2, \qquad g(x) = x(2+x) - 2x.$

Definition: Let A and B be any two non-empty sets, and let $f: A \to B$ be a function, if distinct elements of set A are mapped into distinct elements of set B, then f is called an injective or (one-one) function

i. e. if $a, b, \varepsilon\, A$ and $a \neq b$, then $f(a) \neq f(b)$ or if $a, b, \varepsilon\, A$, and $f(a) = f(b)$ then $a = b$

e.g. $f(x) = 2x + 7$

Definition: Let A and B be any two non-empty sets, and $f: A \to B$ be a function. If every element b ε B appear as the image of at least one element a ε A, then f is called a surjective (or onto) function.

Definition: A function $f: A \to B$ is called bijective if f is one - one and onto.

Definition(Inverse of a function):Let A and B be any two non-empty sets, $f: A \to B$ be a function, and let $b \,\varepsilon\, B$, then the inverse of b denoted $f^{-1}(b)$ consist of those elements in A which are mapped into b. i.e. those elements in A which have b as their image.

Definition (Inverse function): Let A and B be any two non-empty sets, if $f: A \to B$ is one-one and onto, then for each $b \,\varepsilon\, B$, the inverse $f^{-1}(b)$ will consist of a single element $a \,\varepsilon\, A$. Accordingly, f^{-1} is a function from set B to set A, denoted $f^{-1}: B \to A$.

Note:f^{-1} is a function if and only if f is bijective.

Definition: Let A, B, and C be any three non-empty sets, Also let $f: A \to B$ and $g: B \to C$ be two functions, then the function $gof: A \to C$ is called the composition of f and g.

Even and odd functions:

Definition: Let $y = f(x)$ be a function, then $y = f(x)$ is called an even function if $f(-x) = f(x)$.

Definition: Let $y = f(x)$ be a function, then $y = f(x)$ is called an odd function if $f(-x) = -f(x)$.

Properties of even and odd functions:

1. The product of two even functions is an even function.
2. The product of two odd functions is an even function.
3. The product of one even one odd functions is an odd function.
4. The sum of two even functions is an even function.
5. The sum of two odd functions is an odd function.
6. The sum of one even one odd functions is neither even nor odd.

Definition: Any function which repeats itself regularly over a given interval of space or time is called a periodic function. In other words, a function is called periodic if there exist a number τ such that $f(x + n\tau) = f(x)$, where $n \varepsilon Z$ and τ is called the period.

Example1: Find the domain and the range of the following functions.

$i)\quad y = \sqrt{4x - x^2}$ $\qquad ii)\quad y = \sqrt{24 + 10x - x^2}$

Solution:

$i)$ For y to be a real number $4x - x^2 \geq 0$ $\quad ⇨ \quad$ $x(4 - x) \geq 0$

Solving the inequality using sign chart we have that the range of values of x are

$x \geq 0$ and $x \leq 4$

Hence the domain $D = [0, 4]$and the range $R = [0, 2]$

$ii)$ For y to be a real number $24 + 10x - x^2 \geq 0$ $\quad ⇨ \quad$ $x^2 - 10x - 24 \leq 0$

⇨ $\quad (x - 12)(x + 2) \leq 0$

Solving the inequality using sign chart we have that the range of values of x are $-2 \leq x \leq 12$

Hence the domain $D = [-2,\ 12]$ and the Range $R = [0,\ \infty)$

Example2: Find the domain of the following functions.

$i)\quad y = \frac{1}{6\sqrt{x}}$ $\qquad ii)\quad y = \frac{2+x}{1-x^2}$

Solution:

$i)$ For y to be a real number $6\sqrt{x} > 0$ $\quad ⇨ \quad$ $\sqrt{x} > 0$ $\quad ⇨ \quad$ $x > 0$

Hence the domain $D = [1,\ \infty)$, and the range $R = [\frac{1}{6},\ \infty)$

$ii)$ For y to be a real number $(1 - x^2) \neq 0$

⇨ $\quad (1 - x)(1 + x) > 0$ $\quad$ or $\quad$ $(1 - x)(1 + x) < 0$

Solving these two inequalities using sign chart we have that y is a real number for all values of x except at point $x = \pm 1$ hence the domain $D = (-\infty,\ \infty)$ except at point $x = \pm 1$ and the corresponding values of y is the range.

Example3: Classify the following functions as either even, odd or neither.

$i)\quad f(x) = 3x^5 - 2x^3 + x \qquad ii)\quad f(x) = \sin 3x \qquad iii)\quad f(x) = \cos 3x$

$iv)\quad f(x) = e^x + e^{-x}v) \qquad v)\quad f(x) = 4x^3 + 3x \qquad vi)\quad f(x) = (\sin 2x)\cos x$

Solution:

$i)\quad f(x) = 3x^5 - 2x^3 + x$

$\Rightarrow\ -f(x) = -3x^5 + 2x^3 - x$, and

$\Rightarrow\ f(-x) = -3x^5 + 2x^3 - x$

$\Rightarrow\ f(-x) = -f(x)$

Hence the function $f(x) = 3x^5 - 2x^3 + x$ is an odd function.

$ii)\quad f(x) = \sin 3x$

$\Rightarrow\ -f(x) = -\sin 3x$ and

$\Rightarrow\ f(-x) = \sin 3(-x) = -\sin 3x$

$\Rightarrow\ f(-x) = -f(x)$

Hence the function $f(x) = \sin 3x$ is an odd function

Example4: Show that the composition of a function is associative.

Proof: Let $f: A \to B$, $g: B \to C$ and $h: C \to D$

Then we have to show that, $ho(gof) = (hog)of$

Let $a\ \varepsilon\ A$, then $f(a)\ \varepsilon\ B$, say $f(a) = b$ where $b\ \varepsilon\ B$, Now $b\ \varepsilon\ B$, $\Rightarrow$ $g(b) \varepsilon\ C$, say $g(b) = c$ where $c\ \varepsilon\ C$, also since $c\ \varepsilon\ C$, $h(c)\ \varepsilon\ D$, say $h(c) = d$ where $d\ \varepsilon\ D$ d ε D,

Now, $[ho(gof)]a = hog(f(a)) = hog(b) = h(g(b)) = h(c) = d$ . . . (1)

And $[(hog)of]a = (hog)f(a) = (hog)b = h(g(b)) = h(c) = d$ . . . (2)

From (1) and (2) we have that, $[ho(gof)]a = [(hog)of]a = d \quad \forall a\ \varepsilon\ A$

Hence the proof

Example5: Let $f: R \to R$ be defined by $f(x) = 3x + 1$. Show that f is a bijective function.

Proof:

Suppose $x, y, \varepsilon\ R$, such that $x \neq y$ $\Rightarrow$ $f(x) = 3x + 1$ and $f(y) = 3y + 1$

But since $x \neq y$ $\Rightarrow$ $3x \neq 3y$ $\Rightarrow$ $f(x) \neq f(y)$, therefore f is one-to-one.

Next we show that f is onto.

Let $n\ \varepsilon\ R$ contain in the co-domain of f such that $f(m) = n$

$\Rightarrow\quad 3m + 1 = n \quad \Rightarrow \quad m = \frac{n-1}{3} \quad \Rightarrow \quad f(m) = f\left(\frac{n-1}{3}\right) = 3\left(\frac{n-1}{3}\right) + 1 = n$

Therefore f is onto

Hence f is a bijective function.

Example6: Show that the product of

i) two even functions is even.

ii) two odd functions is even.

iii) one even function and one odd function is odd.

Proof:

Let $f(x) = f_1(x) * f_2(x)$

i) If $f_1(x)$ and $f_2(x)$ are both even functions, then

$f(-x) = f_1(-x) * f_2(-x) = f_1(x) * f_2(x) = f(x)$ Hence the proof

ii) If $f_1(x)$ and $f_2(x)$ are both odd functions, then
$f(-x) = f_1(-x) * f_2(-x) = -f_1(x) * -f_2(x) = f(x)$

iii) Without any loss of generality, let $f_1(x)$ be even and $f_2(x)$ be odd.
Then, $f(-x) = f_1(-x) * f_2(-x) = f_1(x) * -f_2(x) = -(f_1(x) * f_2(x)) = -f(x)$
Hence the proof

Example7: Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = 3x^2$ and $g(x) = 2x^3 + 11$.
Find the formula which defines the product function $fog: R \to R$.
Solution:
$fog(x) = f(g(x)) = f(2x^3 + 1) = 3(2x^3 + 1)^2 = 12x^6 + 12x^3 + 3$

Example8: let $f(x)$ be a function over the set of real numbers be defined by $f(x) = \frac{x}{2} - 3$,
Find the inverse $f^{-1}(x)$, if it exist
Solution:
Let $y = f(x)$, $\Rightarrow$ $y = \frac{x}{2} - 3$,
Making x the subject we have
$x = 2(y + 3)$, $\Rightarrow$ $x = f^{-1}(y) = 2(y + 3)$ $\Rightarrow$ $f^{-1}(y) = 2(y + 3)$
Now, replacing y by x we have
$f^{-1}(x) = 2(x + 3)$.

Example 9: Let $f: x \to \frac{x+1}{x-2}$, be a function defined on the set of real numbers excluding $x = 2$.
Find f^{-1}, and the largest domain of f^{-1},
Solution:
$f(x) = \frac{x+1}{x-2}$ $(x \neq 2)$ $\Rightarrow$ $y = \frac{x+1}{x-2}$
Making x the subject we have
$x = \frac{2y+1}{y-1}$ $\Rightarrow$ $f^{-1}(y) = \frac{2y+1}{y-1}$
Now, replacing y by x we have
$f^{-1}(x) = \frac{2x+1}{x-1}$
The largest domain of f^{-1} is D, where $D = R$ excluding $x = 1$.

Sequence, Limit and Continuity:

Sequence:

Introduction: In elementary algebra, we draw the graph of a straight lines and parabola. In drawing them, we made use of two sets of values of x and y, the curves in each case were smooth because of the definite relationship that exist between x and y. The numbers or terms are generated by definite rule that enable one to know what the next term is, in some cases, increase or decrease by a constant value or increase or decrease at a constant rate.

Definition1: A Sequence is a function whose domain is a set of positive integers greater than or equal to some positive integer m (usually 0 or 1).

Definition2: The numbers in the range of a sequence is called the terms of the sequence or members of the sequence.

Note:

$i)$ *Sequence are usually denoted by* a_n, b_n, c_n, u_n, *etc., or sometimes enclosed in a curly bracket* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{u_n\}$, *etc.* $n \varepsilon Z$.

$ii)$ *Sequence is usually specified in two ways viz.*

$a)$ *By displaying its first few terms or* — $b)$ *By writing the formula for the* n^{th} *term*

e.g. $a)$ $3,\ 5,\ 7,\ 9,\ \ldots$ — $e.g$ $a_n = 2n + 1$

Note:

$i)$ *If we know the* n^{th} *term of a sequence we can generate the first few terms. e.g. Given that* $a_n = 3n^2$, *then the first five terms are 3, 12, 27, 48, 75, . . .*

$ii)$ *If the first few terms of a sequence are known, the* n^{th} *term can be generated by studying the arrangement carefully and obtain a kind of order that will satisfy the arrangement.*

e.g. Given the first few terms as $5,\ 11,\ 17,\ 23,\ 29,\ \ldots$

Then we have

$$a_1 = 5 = (6 * 1) - 1$$
$$a_2 = 11 = (6 * 2) - 1$$
$$a_3 = 17 = (6 * 3) - 1$$
$$. \quad . \quad . \quad . \quad . \quad .$$
$$a_n = (6 * n) - 1$$

Definition3:

$i)$ A sequence having a fixed number of terms is called a finite sequence, while a sequence having an uncountable number of terms is called an infinite sequence.

$ii)$ A sequence in which all the terms are the same is called constant sequence.

$iii)$ A sequence in which the terms approaches zero as n approaches infinity is called a null set.

Definition4:

$i)$ A sequence $\{a_n\}$ is said to be bounded if there exist some numbers U and L such that

$L \leq a_n \leq U \qquad \forall n \geq 0$

$ii)$ A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n \quad \forall\, n \geq 0.$

$iii)$ A sequence $\{ a_n \}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \quad \forall n \geq 0$.

Convergence of a sequence:

For any sequence $\{ a_n \}$, we may be interested to know the behavior of a_n as n increases without bound, e.g. $a_n = 1 - \frac{1}{n}$ as n increase without bound, it appears that a_n approaches 1. However, if $a_n = \left(1 + \frac{1}{2n}\right)^n$, it might not be clear whether a_n approaches some specific number as n increases without bound

Definition5: Let $\{ a_n \}$ be a sequence, a number L is called the limit of a_n if for every $\epsilon > 0$, there exist $N > 0$ such that if $n \geq N$, then $|a_n - L| < \epsilon$, and is written as $\lim\limits_{n \to \infty} a_n = L$

$i)$ If such a number L exist, we say that the sequence $\{ a_n \}$ is convergent, and has sum L.

$ii)$ If such a number L does not exist, we say that the sequence $\{ a_n \}$ is divergent.

Exercise: Determine whether the following sequence is convergent or divergent; if it is convergent find the sum.

$i)$ $a_n = \left(1 + \frac{1}{n}\right)^n$ $\quad ii)$ $a_n = 3n^2$ $\quad iii)$ $a_n = \frac{n}{2n+1}$

Properties of convergent sequence:

Let $\{a_n\}$ and $\{b_n\}$ be any two convergent sequence and C any arbitrary constant, then

$i)$ The sum $\{a_n\} + \{b_n\}$ is also convergent. $\quad ii)$ The product $\{a_n\} * \{b_n\}$ is also convergent.

$iii)$ Any scalar multiple $C * \{a_n\}$ is also convergent.

$iv)$ The quotient $\{ \frac{a_n}{b_n} \}$ is convergent provided b_n is not a null sequence.

Limits:

Introduction: Defining a Limit has never been an easy one, the earliest attempts was made by the French Mathematician *Joseph Louis Lagrange (1736 – 1813).* However, his reasoning was shown to be faulty, further progress was made when the French Mathematician *Augustine Louis Cauchy* (1789 –1837) and the *Czech Priest Bern hard – Bolzano (1781- 1848)* independently gave definition of Limit and Continuity. Nevertheless, they still had a degree of vagueness which is no longer acceptable. The present day definitions were first published by the German Mathematician *Hein Rich Edward Hein in 1872.*

The meaning of limit informally:

When we say that "L is the Limit of a function $f(x)$ as x approaches some number a we mean roughly speaking that $f(x)$ get closer and closer to L as x get closer and closer to a. And we express this idea symbolically by the notation

$$\lim_{x \to a} f(x) = L$$

Example1: The following table gives some values of x and the corresponding values of $f(x)$ as $x \to 2$ where $f(x) = \frac{3x-5x-2}{x-2}$

Case 1

x	1	1.2500	1.5000	1.7500	1.9000	1.9500	1.9950	1.9990	. . . .
$f(x)$	4								

It can be seen that as x get closer to the value 2, $f(x)$ get closer to the value 7

Case 2

x	3	2.7500	2.5000	2.2500	2.1000	2.0100	2.0010	2.0001	. . . .
$f(x)$	10								

Here also we can see that as x get closer to the value 2, $f(x)$ get closer to the value 7.

Note:

i) In case1, we can see that x get closer to the value 2 from the left, and is therefore called left-hand limit denoted $\lim_{x \to 2^-} \frac{3x^2-5x-2}{x-2} = 7$

ii) In case2, we can see that x get closer and closer to the value 2 from the right, and is therefore, called right- hand limit denoted $\lim_{x \to 2^+} \frac{3x^2-5x-2}{x-2} = 7$

Example2: Now if $f(x) = \sqrt{9 - x^2}$, forming a similar table we have that.

Case1

x	2	2.7500	2.9000	2.9500	2.9900	2.9950	2.9990	2.9999	......
$f(x)$	2.2361	1.1990	0.7681	0.5454	0.2447	0.1732	0.0775	0.0245	......

We can observe that as x get closer to the value 3, $f(x)$ get closer to the value 0

Case 2

x	4	3.5000	3.1000	3.0500	3. 0100	3.0050	3.0010	3.0001	......
$f(x)$	Im	Im	Im	Im	Im	Im	Im	Im	......

Here we observe that as x get closer to 3 from the right, $f(x)$ is not a real number.

Note: $i)$ In case1 $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$. $ii)$ in case2 $\lim_{x \to 3^+} \sqrt{9 - x^2} = ?$.

Condition for existence of limit:

We say the Limit of a function $f(x)$ exist if the left hand limit is equal to the right hand Limit

i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Limit formally:

Definition: Let $f(x)$ be a function defined at each point in some open interval containing "a", then a number L is called the Limit of $f(x)$ as x approaches "a" if for every positive number $\epsilon > 0$ there exist a positive number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$

Example

*

*

Basic concepts of limits:

Given that $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} g(x) = B$, and C is any arbitrary constant, then the following postulates hold good.

1) $\lim_{x \to a} Cf(x) = C \lim_{x \to a} f(x) = C * A$

2) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B.$

3) $\lim_{x \to a} [f(x) * g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x) = A * B$

4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}$, provided that $\lim_{x \to a} g(x) \neq 0.$

5) $\lim_{x \to a} [\sqrt[n]{f(x)}] = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$ provided $\sqrt[n]{A}$ is a real number.

6) $\lim_{x \to a} [\log_b f(x)] = \log_b [\lim_{x \to a} f(x)] = \log_b A.$

Special limits:

Theorem: Prove that: $i)$ $\lim_{x \to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$ $\quad ii)$ $\lim_{\theta \to 0} \frac{Sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{tan\theta}{\theta} = 1.$

$iii)$ $\lim_{n \to \infty} [1 + \frac{1}{n}]^n = e$ where $2 < e < 3$

Proof:

$$i) \lim_{x \to 1} \frac{x^n - 1^n}{x^m - 1^n} = \lim_{x \to 1} \frac{(x-1)(x^{n-1}1^0 + x^{n-2}1^1 + x^{n-3}1^2 + x^{n-4}1^3 + x^{n-5}1^4 + \ldots + x^{n-n}1^{n-1})}{(x-1)(x^{m-1}1^0 + x^{m-2}1^1 + x^{m-3}1^2 + x^{m-4}1^3 + x^{m-5}1^4 + \ldots + x^{m-m}1^{m-1})}$$

$$= \lim_{x \to 1} \frac{(x^{n-1} + x^{n-2} + x^{n-3} + x^{n-4} + x^{n-5} + \ldots + x^{n-n})}{(x^{m-1} + x^{m-2} + x^{m-3} + x^{m-4} + x^{m-5} + \ldots + x^{m-m})}$$

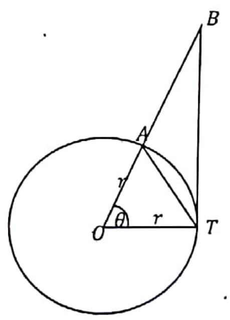
$$= \lim_{x \to 1} \frac{(x^{n-1} + x^{n-2} + x^{n-3} + x^{m-4} + x^{m-5} + \ldots + 1)}{(x^{m-1} + x^{m-2} + x^{m-3} + x^{m-4} + x^{m-5} + \ldots + 1)}$$

$$= \frac{1+1+1+\ldots+1 \ (n-factors)}{1+1+1+\ldots+1 \ (m-factors)} = \frac{n}{m}$$

$ii)$ $\lim_{\theta \to 0} \frac{Sin\theta}{\theta}$

Consider a circle of any radius

From the circle we observe that



$\tan\theta = \frac{BT}{OT} \quad \Rightarrow \quad BT = OT \tan\theta = r\tan\theta$, *and* $OT = OA = r$

Area of $\Delta OAT <$ *Area of sector* $\widehat{OAT} <$ *Area of* ΔOBT

$\Rightarrow \quad \frac{1}{2} r^2 \sin\theta < \frac{\theta}{360^o} \pi r^2 < \frac{1}{2} r(BT)$

$\Rightarrow \quad \frac{1}{2} r^2 \sin\theta < \frac{\theta}{360^o} 180^o r^2 < \frac{1}{2} r(r) \tan\theta$

$\Rightarrow \quad \frac{1}{2} r^2 \sin\theta < \frac{1}{2} \theta r^2 < \frac{1}{2} r^2 \tan\theta$ (Multiplying through by 2)

$\Rightarrow \quad r^2 \sin\theta < \theta r^2 < r^2 \tan\theta$ (Dividing through by r^2)

$\Rightarrow \quad \sin\theta < \theta < \tan\theta$ $\quad \ldots (1)$

$a)$ Dividing eqn. (1) through by $\sin\theta$ we have

$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta} \Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

Take the reciprocal the sense of the inequality changes

$\Rightarrow \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta$

Taking limits throughout we have

$\lim_{\theta\to 0} 1 > \lim_{\theta\to 0}\frac{Sin\theta}{\theta} > \lim_{\theta\to 0}\cos\theta \quad \Rightarrow \quad 1 > \lim_{\theta\to 0}\frac{Sin\theta}{\theta} > 1$

$\Rightarrow \quad \lim_{\theta\to 0}\frac{Sin\theta}{\theta} = 1$

$b)$ Dividing eqn. (1) through by $\tan\theta$ we have

$\frac{\sin\theta}{\tan\theta} < \frac{\theta}{\sin\theta} < 1 \quad \Rightarrow \quad \frac{1}{\cos\theta} < \frac{\theta}{\tan\theta} < 1$

Take the reciprocal the sense of the inequality changes

$\Rightarrow \quad \cos\theta > \frac{\text{tnn}\,\theta}{\theta} > 1$

Taking limits throughout we have

$\lim_{\theta\to 0}\cos\theta > \lim_{\theta\to 0}\frac{tan\theta}{\theta} > \lim_{\theta\to 0} 1 \quad \Rightarrow \quad 1 > \lim_{\theta\to 0}\frac{tan\theta}{\theta} > 1 \quad \Rightarrow \quad \lim_{\theta\to 0}\frac{tan\theta}{\theta} = 1$

Hence the proof

$iii)\quad \underset{x\to\infty}{Lim}\left(1+\frac{1}{n}\right)^n = \underset{x\to\infty}{Lim}\; 1+\frac{n}{1!}\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{1}{n}\right)^4+\ldots+\left(\frac{1}{n}\right)^n$

$= \underset{x\to\infty}{Lim}\; 1+\frac{1}{1!}+\frac{\frac{1}{n}(n-1)}{2!}+\frac{\frac{1}{n}(n-1)\frac{1}{n}(n-2)}{3!}+\frac{\frac{1}{n}(n-1)\frac{1}{n}(n-2)\frac{1}{n}(n-3)}{4!}+\frac{\frac{1}{n}(n-1)\frac{1}{n}(n-2)\frac{1}{n}(n-3)\frac{1}{n}(n-4)}{5!}+\ldots+\frac{1}{n^n}$

$= \underset{x\to\infty}{Lim}\; 1+\frac{1}{1!}+\frac{(1-\frac{1}{n})}{2!}+\frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!}+\frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})}{4!}+\frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})(1-\frac{4}{n})}{5!}+\ldots+\frac{1}{n^n}$

$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\ldots \quad = e$

Example1: Verify that $\lim_{n\to\infty}[1+\frac{x}{n}]^n = e^x$ where $2 < e < 3$

Solution:

$\underset{x\to\infty}{Lim}\left(1+\frac{x}{n}\right)^n = \underset{x\to\infty}{Lim}\; 1+\frac{n}{1!}\left(\frac{x}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{n}\right)^4+\ldots+\left(\frac{x}{n}\right)^n$

$= \underset{x\to\infty}{Lim}\; 1+\frac{x}{1!}+\frac{\frac{1}{n}(n-1)x^2}{2!}+\frac{\frac{1}{n}(n-1)\frac{1}{n}(n-2)x^3}{3!}+\frac{\frac{1}{n}(n-1)\frac{1}{n}(n-2)\frac{1}{n}(n-3)x^4}{4!}+\ldots+\left(\frac{x}{n}\right)^n$

$= \underset{x\to\infty}{Lim}\; 1+\frac{x}{1!}+\frac{(1-\frac{1}{n})x^2}{2!}+\frac{(1-\frac{1}{n})(1-\frac{2}{n})x^3}{3!}+\frac{(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})x^4}{4!}+\ldots+\left(\frac{x}{n}\right)^n$

$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\ldots \quad = e^x$

Example1: Evaluate the following limits if it exists

$i)\ \underset{x\to 1}{Lim}\frac{x^3-1}{x-1} \qquad ii)\ \underset{x\to 1}{Lim}\frac{x^2-1}{x^3-1} \qquad iii)\ \underset{x\to 1}{Lim}\frac{x^4-1}{x^3-1}$

Solution:

$i)\ \underset{x\to 1}{Lim}\frac{x^3-1}{x-1} = \underset{x\to 1}{Lim}\frac{(x-1)(x^2+x+1)}{x-1} = \underset{x\to 1}{Lim}(x^2+x+1) = 3$

$ii)\ \underset{x\to 1}{Lim}\frac{x^2-1}{x^3-1} = \underset{x\to 1}{Lim}\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = i)\ \underset{x\to 1}{Lim}\frac{(x+1)}{(x^2+x+1)} = \frac{2}{3}$

$iii)\ \underset{x\to 1}{Lim}\frac{x^4-1}{x^3-1} = \underset{x\to 1}{Lim}\frac{(x-1)(x^3+x^2+x+1)}{(x-1)(x^2+x+1)} = i)\ \underset{x\to 1}{Lim}\frac{(x^3+x^2+x+1)}{(x^2+x+1)} = \frac{4}{3} = 1\frac{1}{3}$

Example2: Evaluate the following limits if it exists

$i)\ \lim_{x\to 0} \frac{(3+x)^2-9}{x}$, $ii)\ \lim_{x\to 3} \frac{x^3-27}{x^2-9}$ $iii)\ \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

Solution;

$$i)\ \lim_{x\to 0} \frac{(3+x)^2-9}{x} = \lim_{x\to 0} \frac{(3+x)^2-3^2}{x} = \lim_{x\to 0} \frac{[(3+x)-3][(3+x)+3]}{x} = \lim_{x\to 0} \frac{x(x+6)}{x} = \lim_{x\to 0} (x+6) = 6$$

$$ii)\ \lim_{x\to 3} \frac{x^3-27}{x^2-9} = \lim_{x\to 3} \frac{x^3-3^3}{x^2-3^2} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{(x^2+3x+9)}{(x+3)} = \frac{27}{6} = \frac{9}{2} = 4\tfrac{1}{2}$$

$$iii)\ \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} * \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \lim_{x\to 0} \frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)}$$

$$= \lim_{x\to 0} \frac{(4+x)-4}{x(\sqrt{4+x}+2)} = \lim_{x\to 0} \frac{x}{x(\sqrt{4+x}+2)} = \lim_{x\to 0} \frac{1}{(\sqrt{4+x}+2)} = \frac{1}{4}$$

Example3: Evaluate the following limits if it exists

$i)\ \lim_{x\to 1} \frac{x^3-1}{x^2-3x+2}$ $ii)\ \lim_{x\to 5} \frac{x^2-7x+10}{x^2-25}$ $iii)\ \lim_{x\to 1} \frac{(x-1)\sqrt{2-x}}{x^2-1}$

Solution;

$$i)\ \lim_{x\to 1} \frac{x^3-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{(x^2+x+1)}{(x-2)} = \frac{3}{-1} = -3$$

$$ii)\ \lim_{x\to 5} \frac{x^2-7x+10}{x^2-25} = \lim_{x\to 5} \frac{(x-5)(x-2)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{(x-2)}{(x+5)} = \frac{3}{10}$$

$$iii)\ \lim_{x\to 1} \frac{(x-1)\sqrt{2-x}}{x^2-1} = \lim_{x\to 1} \frac{(x-1)\sqrt{2-x}}{(x-1)(x+1)} = \lim_{x\to 1} \frac{\sqrt{2-x}}{(x+1)} = \frac{1}{2}$$

Example4: Evaluate the following limits if it exists

$i)\ \lim_{x\to -2} \frac{2x^2+5x-7}{3x^2-x-2}$ $ii)\ \lim_{x\to \frac{1}{2}} \frac{8x^3-1}{6x^2-5x+1}$ $iii)\ \lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

Solution: Do

Example5: Evaluate the following limits if it exists

$i)\ \lim_{x\to 2} \frac{x^2-5x+6}{x^2-12x+20}$ $ii)\ \lim_{x\to 0} \frac{3x^3+2x^2-x}{5x}$ $iii)\ \lim_{x\to 1} \frac{x(x-1)}{2(x^2-1)}$

Solution:

$$i)\ \lim_{x\to 2} \frac{x^2-5x+6}{x^2-12x+20} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x-2)(x-10)} = \lim_{x\to 2} \frac{(x-3)}{(x-10)} = \frac{-1}{-8} = \frac{1}{8}$$

$$ii)\ \lim_{x\to 0} \frac{3x^3+2x^2-x}{5x} = \lim_{x\to 0} \frac{x(3x^2+2x-1)}{5x} = \lim_{x\to 0} \frac{(3x^2+2x-1)}{5} = \frac{-1}{5} = -\frac{1}{5}$$

$$iii)\ \lim_{x\to 1} \frac{x(x-1)}{2(x^2-1)} = \lim_{x\to 1} \frac{x(x-1)}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{x}{2(x+1)} = \frac{1}{4}$$

Example6: Evaluate the following limits if it exists

$i)\ \lim_{a\to 0} \frac{\sqrt{x+a}-\sqrt{x}}{a}$ $ii)\ \lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}$ $iii)\ \lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49}$ $iv)\ \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

Solution:

$$i)\ \lim_{a\to 0} \frac{\sqrt{x+a}-\sqrt{x}}{a} = \lim_{a\to 0} \frac{\sqrt{x+a}-\sqrt{x}}{a} * \frac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}+\sqrt{x}} = \lim_{a\to 0} \frac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{a(\sqrt{x+a}+\sqrt{x})}$$

$$= \lim_{a\to 0} \frac{[(x+a)-x]}{a(\sqrt{x+a}+\sqrt{x})} = \lim_{a\to 0} \frac{a}{a(\sqrt{x+a}+\sqrt{x})} = \lim_{a\to 0} \frac{1}{(\sqrt{x+a}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x^2} = \lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x^2} * \dfrac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} = \lim\limits_{x\to 0} \dfrac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{x^2(\sqrt{1+x^2}+1)}$

$= \lim\limits_{x\to 0} \dfrac{[(1+x^2)-1]}{x^2(\sqrt{1+x^2}+1)} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2(\sqrt{1+x^2}+1)} = \lim\limits_{x\to 0} \dfrac{1}{(\sqrt{1+x^2}+1)} = \dfrac{1}{2}$

iii) $\lim\limits_{x\to 7} \dfrac{2-\sqrt{x-3}}{x^2-49} = \lim\limits_{x\to 7} \dfrac{2-\sqrt{x-3}}{x^2-49} * \dfrac{2+\sqrt{x-3}}{2+\sqrt{x-3}} = \lim\limits_{x\to 7} \dfrac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{(x-7)(x+7)(2+\sqrt{x-3})}$

$= \lim\limits_{x\to 7} \dfrac{[4-(x-3)]}{(x-7)(x+7)(2+\sqrt{x-3})} = \lim\limits_{x\to 7} \dfrac{-(x-7)}{(x-7)(x+7)(2+\sqrt{x-3})} = \lim\limits_{x\to 7} \dfrac{-1}{(x+7)(2+\sqrt{x-3})} = \dfrac{-1}{56} = -\dfrac{1}{56}$

Example7: Evaluate the following limits if it exists

i) $\lim\limits_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1}$ ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ iii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-(x+1)}{\sqrt{x+1}-1}$

Solution:

i) $\lim\limits_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1} = \lim\limits_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1} * \dfrac{\sqrt{5-x}+2}{\sqrt{2-x}+1} * \dfrac{\sqrt{2-x}+1}{\sqrt{5-x}+2} = \lim\limits_{x\to 1} \dfrac{(\sqrt{5-x}-2)(\sqrt{5-x}+2)(\sqrt{2-x}+1)}{(\sqrt{2-x}-1)(\sqrt{2-x}+1)(\sqrt{5-x}+2)}$

$= \lim\limits_{x\to 1} \dfrac{[(5-x)-4](\sqrt{2-x}+1)}{[(2-x)-1](\sqrt{5-x}+2)} = \lim\limits_{x\to 1} \dfrac{(1-x)(\sqrt{2-x}+1)}{(1-x)(\sqrt{5-x}+2)} = \lim\limits_{x\to 1} \dfrac{(\sqrt{2-x}+1)}{(\sqrt{5-x}+2)} = \dfrac{1}{2}$

ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = \lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} * \dfrac{\sqrt{x^2+1}+1}{\sqrt{x^2+16}+4} * \dfrac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}$

$= \lim\limits_{x\to 0} \dfrac{(\sqrt{x^2+1}-1)(\sqrt{x^2+1}+1)(\sqrt{x^2+16}+4)}{(\sqrt{x^2+16}-4)(\sqrt{x^2+16}+4)(\sqrt{x^2+1}+1)} = \lim\limits_{x\to 0} \dfrac{[(x^2+1)-1](\sqrt{x^2+16}+4)}{[(x^2+16)-16](\sqrt{x^2+1}+1)}$

$= \lim\limits_{x\to 0} \dfrac{x^2(\sqrt{x^2+16}+4)}{x^2(\sqrt{x^2+1}+1)} = \lim\limits_{x\to 0} \dfrac{(\sqrt{x^2+16}+4)}{(\sqrt{x^2+1}+1)} = \dfrac{8}{2} = 4$

iii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-(x+1)}{\sqrt{x+1}-1} = \lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-(x+1)}{\sqrt{x+1}-1} * \dfrac{\sqrt{x+1}+(x+1)}{\sqrt{x+1}+1} * \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+(x+1)}$

$= \lim\limits_{x\to 0} \dfrac{(\sqrt{x+1}-(x+1))(\sqrt{x+1}+(x+1))(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)(\sqrt{x+1}+(x+1))} = \lim\limits_{x\to 0} \dfrac{[(x+1)-(x+1)^2](\sqrt{x+1}+1)}{[(x+1)-1](\sqrt{x+1}+(x+1))}$

$= \lim\limits_{x\to 0} \dfrac{-x(x+1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+(x+1))} = -\lim\limits_{x\to 0} \dfrac{(x+1)(\sqrt{x+1}+1)}{(\sqrt{x+1}+(x+1))} = -\dfrac{1(2)}{1+1} = -\dfrac{2}{2} = -1$

Example8: Use special limits above to verify the following limits

i) $\lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{\theta} = p$ ii) $\lim\limits_{\theta\to 0} \dfrac{tanq\theta}{\theta} = q$ iii) $\lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{Sinq\theta} = \dfrac{p}{q}$

Solution:

i) $\lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{\theta} = \lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{\theta} * \dfrac{p}{p} = \lim\limits_{\theta\to 0} \dfrac{pSinp\theta}{p\theta} = p \lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{p\theta}$

Now we set $p\theta = t$ $\Rightarrow$ as $\theta \to 0, t \to 0$

Therefore $p \lim\limits_{\theta\to 0} \dfrac{Sinp\theta}{p\theta} = p \lim\limits_{t\to 0} \dfrac{Sint}{t} = p$ (Since $\lim\limits_{t\to 0} \dfrac{Sint}{t} = 1$)

ii) $\lim\limits_{\theta\to 0} \dfrac{tanq\theta}{\theta} = \lim\limits_{\theta\to 0} \dfrac{tanq\theta}{\theta} * \dfrac{q}{q} = \lim\limits_{\theta\to 0} \dfrac{qtanq\theta}{q\theta} = q \lim\limits_{\theta\to 0} \dfrac{tanq\theta}{q\theta}$

Now we set $q\theta = y$ $\Rightarrow$ as $\theta \to 0, y \to 0$

Therefore $q \lim\limits_{\theta\to 0} \dfrac{tanq\theta}{q\theta} = q \lim\limits_{y\to 0} \dfrac{tany}{y} = q$ (Since $\lim\limits_{y\to 0} \dfrac{tany}{y} = 1$)

iii) $\lim\limits_{\theta\to 0}\frac{Sinp\theta}{Sinq\theta}=\lim\limits_{\theta\to 0}\frac{Sinp\theta}{Sinq\theta}*\frac{\theta}{\theta}=\lim\limits_{\theta\to 0}\left[\frac{Sinp\theta}{\theta}*\frac{\theta}{Sinq\theta}\right]$

$=\lim\limits_{\theta\to 0}\frac{Sinp\theta}{\theta}*\lim\limits_{\theta\to 0}\frac{\theta}{Sinq\theta}=p\left(\frac{1}{q}\right)=\frac{p}{q}$

Example9: Evaluate the following limits if it exists

i) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$ ii) $\lim\limits_{x\to 0}\frac{\sin 3x}{\tan x}$, iii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$

Solution:

i) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\lim\limits_{x\to 0}\frac{\sin 5x}{x}*\frac{5}{5}=5\lim\limits_{x\to 0}\frac{\sin 5x}{5x}$

Let $t=5x \Rightarrow$ as $x\to 0,\ t\to 0$

$\Rightarrow 5\lim\limits_{x\to 0}\frac{\sin 5x}{5x}=5\lim\limits_{t\to 0}\frac{\sin t}{t}=5$

ii) $\lim\limits_{x\to 0}\frac{\sin 3x}{\tan x}=\lim\limits_{x\to 0}\left[\frac{\sin 3x}{1}*\frac{x}{x}*\frac{1}{\tan x}\right]=\lim\limits_{x\to 0}\frac{\sin 3x}{x}*\lim\limits_{x\to 0}\frac{x}{\tan x}=3*1=3$

iii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}*\frac{1+\cos x}{1+\cos x}=\lim\limits_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=\lim\limits_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$=\lim\limits_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}=\lim\limits_{x\to 0}\frac{\sin^2 x}{x^2}*\frac{1}{(1+\cos x)}=\lim\limits_{x\to 0}\frac{\sin x}{x}*\lim\limits_{x\to 0}\frac{\sin x}{x}*\lim\limits_{x\to 0}\frac{1}{(1+\cos x)}=\frac{1}{2}$

Example10: Evaluate the following limits if it exists

i) $\lim\limits_{x\to\frac{1}{2}}\cdot\sqrt{\frac{4x^2-4x+1}{4x^2-1}}$, ii) $\lim\limits_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}$ iii) $\lim\limits_{x\to -3}\frac{(x+3)(3x-1)}{x^2+x-6}$

Solution:

i) $\lim\limits_{x\to\frac{1}{2}}\sqrt{\frac{4x^2-4x+1}{4x^2-1}}=\sqrt{\lim\limits_{x\to\frac{1}{2}}\frac{4x^2-4x+1}{4x^2-1}}=\sqrt{\lim\limits_{x\to\frac{1}{2}}\frac{(2x-1)(2x-1)}{(2x-1)(2x+1)}}=\sqrt{\lim\limits_{x\to\frac{1}{2}}\frac{(2x-1)}{(2x+1)}}=0$

ii) $\lim\limits_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}=\lim\limits_{x\to 8}\frac{x-8}{x^{\frac{1}{3}}-2}$

Let $y^3=x \Rightarrow$ as $x\to 8,\ y\to 2$

$\Rightarrow \lim\limits_{x\to 8}\frac{x-8}{x^{\frac{1}{3}}-2}=\lim\limits_{y\to 2}\frac{y^3-8}{y-2}=\lim\limits_{y\to 2}\frac{y^3-2^3}{y-2}=\lim\limits_{y\to 2}\frac{(y-2)(y^2+2y+4)}{y-2}=\lim\limits_{y\to 2}(y^2+2y+4)=12$

iii) $\lim\limits_{x\to -3}\frac{(x+3)(3x-1)}{x^2+x-6}=\lim\limits_{x\to -3}\frac{(x+3)(3x-1)}{(x+3)(x-2)}=\lim\limits_{x\to -3}\frac{(3x-1)}{(x-2)}=\frac{-7}{-5}=\frac{7}{5}=1\frac{2}{5}$

Example11: Evaluate the following limits if it exists

i) $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ ii) $\lim\limits_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$ iii) $\lim\limits_{x\to 64}\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$

Solution:

i) $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}*\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim\limits_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim\limits_{x\to 4}\frac{(x-4)}{(x-4)(\sqrt{x}+2)}=\lim\limits_{x\to 4}\frac{1}{(\sqrt{x}+2)}=\frac{1}{4}$

iii) $\lim\limits_{x\to 64}\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}=\lim\limits_{x\to 64}\frac{x^{\frac{1}{2}}-8}{x^{\frac{1}{3}}-4}$

Let $y^6=x \Rightarrow$ as $x\to 64,\ y\to 2$

$\Rightarrow \lim\limits_{x\to 64}\frac{x^{\frac{1}{2}}-8}{x^{\frac{1}{3}}-4}=\lim\limits_{y\to 2}\frac{y^3-8}{y^2-4}=\lim\limits_{y\to 2}\frac{(y-2)(y^2+2y+4)}{(y-2)(y+2)}\lim\limits_{y\to 2}\frac{(y^2+2y+4)}{(y+2)}=\frac{12}{4}=3$

Exercises: Evaluate the following limits if it exists

$i)\ \lim_{a\to0}\frac{(x+a)^3-x^3}{a}$ $\quad ii)\ \lim_{a\to0}\frac{\sin(x+a)\ -\sin x}{a}$ $\quad iii)\ \lim_{x\to0}\frac{\sin^2 4x}{x^2},$ $\quad iv)\ \lim_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$

$v)\ \lim_{x\to a}\frac{\sin x-\sin a}{x-a}$ $\quad vi)\ \lim_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$ $\quad vii)\ \lim_{x\to4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$ $\quad viii)\ \lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

$ix)\ \lim_{x\to0} x\sin\frac{1}{x}$ $\quad x)\ \lim_{x\to0}\frac{\sin\pi x}{\sin 3\pi x}$ $\quad xi)\ \lim_{x\to0}\frac{\sin\left(\frac{x}{4}\right)}{x}$ $\quad xii)\ \lim_{x\to0}\frac{\cos mx-\cos nx}{x}$

$xiii)\ \lim_{x\to a}\frac{\cos x-\cos a}{x-a}$ $\quad xiv)\ \lim_{x\to1}\frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ $\quad xv)\ \lim_{x\to0}\frac{\sin^2\left(\frac{x}{4}\right)}{x^2}$ $\quad xvi)\ \lim_{x\to0}\frac{1-\cos^2 x}{x\sin^2 x}$

$xvii)\ \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin a}{\cos 2x}$ $\quad xviii)\ \lim_{x\to0}\frac{2ax\sin x}{3x}$ $\quad ixx)\ \lim_{x\to0}\frac{\cos 3x-\cos x}{\cos^2 x-1}$

Infinite limit and Limit at infinity:

We shall now look at the limit of the function of the form $\frac{x}{a}$, and $\frac{a}{x}$, when x approaches zero, and when x approaches infinity. Where a is any arbitrary constant.

Recall:

$i)$ Zero divided by any number is equals zero, $\frac{0}{a}=0$, and any number divided by zero is equals infinity, $\frac{a}{0}=\infty$

$ii)$ Infinity divided by any number is equals Infinity, $\frac{\infty}{a}=\infty$, and any number divided by Infinity is equals zero, $\frac{a}{\infty}=0$

$i)\ \lim_{x\to0}\frac{x}{a}=0,$ $\quad ii)\ \lim_{x\to0}\frac{a}{x}=\infty,$ $\quad iii)\ \lim_{x\to\infty}\frac{x}{a}=\infty,$ $\quad i)\ \lim_{x\to0}\frac{a}{x}=0,$

Definition: Any Limit of the form $\lim_{x\to\infty}\frac{f(x)}{g(x)}=a$ is called limit at Infinity. On the other hand any Limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}=\pm\infty$ is called Infinite limit.

Example1: Evaluate the following limits if it exists

$i)\ \lim_{x\to\infty}\frac{4x-3}{x^2+5},$ $\quad ii)\ \lim_{x\to\infty}\frac{(1+x)(2+x)}{x^2+1},$ $\quad iii)\ \lim_{x\to\infty}\frac{(x+1)^2}{x^2+1},$ $\quad iv)\ \lim_{x\to\infty}\left[3x+\frac{1}{x^2}\right],$

Solution:

$$i)\ \lim_{x\to\infty}\frac{4x-3}{x^2+5}=\lim_{x\to\infty}\frac{x^2\left(\frac{4}{x}-\frac{3}{x^2}\right)}{x^2\left(1-\frac{5}{x^2}\right)}=\lim_{x\to\infty}\frac{\left(\frac{4}{x}-\frac{3}{x^2}\right)}{\left(1-\frac{5}{x^2}\right)}=\frac{0}{1}=0$$

$$ii)\ \lim_{x\to\infty}\frac{(1+x)(2+x)}{x^2+1}=\lim_{x\to\infty}\frac{x^2+3x+2}{x^2+1}=\lim_{x\to\infty}\frac{x^2\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{\left(1+\frac{3}{x}-\frac{2}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}=\frac{1}{1}=1$$

Exercise: Evaluate the following limits if it exists

$i)\ \lim_{x\to\infty}\left[\frac{x^3}{x^2+1}-x\right],$ $\quad ii)\ \lim_{x\to\infty}\frac{(x-1)^3}{2x^3+3x-1},$ $\quad i)\ \lim_{x\to\infty}\frac{x^2-2x}{x^2-4x+4},$ $\quad ii)\ \lim_{x\to\infty}\left[3x-\frac{1}{x^2}\right]$

Continuity:

Here we are going to learn about continuous functions and how to find the point of discontinuity of functions without drawing the graph.

Definition: A function $f(x)$ is said to be continuous at point $x = a$ if the following conditions are hold good.

1. $f(a)$ be defined 2. $\lim_{x \to a} f(x)$ exist 3. $\lim_{x \to a} f(x) = f(a)$.

A function will therefore said to be discontinuous at point $x = a$ if one or more of these conditions fail at point $x = a$.

Example1: Examine the following functions for continuity

$i)$ $f(x) = x - 2$ $at\ x = 4$ $ii)$ $f(x) = 2x^2 - 3x + 1$ $at\ x = 2$ $iii)$ $f(x) = \sin x$ $at\ x = \frac{\pi}{2}$

Solution:

$i)$ $f(x) = x - 2$ $at\ x = 4$

1. $f(4) = 2 \Rightarrow f(4)$ is define
2. $\lim_{x \to 4^-} (x - 2) = 2$ and $\lim_{x \to 4^+} (x - 2) = 2 \Rightarrow$ *Limit of f(x) exist*
3. $\lim_{x \to 4} (x - 2) = f(4) = 2$

Hence $f(x) = x + 2$ is continuous at point $x = 4$

$ii)$ $f(x) = 2x^2 - 3x + 1$ $at\ x = 2$

1. $f(2) = 3 \Rightarrow f(2)$ is define
2. $\lim_{x \to 2^-} (2x^2 - 3x + 1) = 3$ and $\lim_{x \to 2^+} (2x^2 - 3x + 1) = 3 \Rightarrow$ *Limit of f(x) exist*
3. $\lim_{x \to 2} (2x^2 - 3x + 1) = f(2) = 3$

Hence $f(x) = 2x^2 - 3x + 1$ is continuous at point $x = 2$

$iii)$ $f(x) = \sin x$ $at\ x = \frac{\pi}{2}$

1. $f\left(\frac{\pi}{2}\right) = 1 \Rightarrow f(\frac{\pi}{2})$ is define
2. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \sin x = 1$ and $\lim_{x \to \left(\frac{\pi}{2}\right)^+} \sin x = 1 \Rightarrow$ Limit of $f(x)$ exist
3. $\lim_{x \to \frac{\pi}{2}} \sin x = f\left(\frac{\pi}{2}\right) = 1$

Hence $f(x) = \sin x$ is continuous at point $x = \frac{\pi}{2}$

Example2: Examine the following functions for continuity

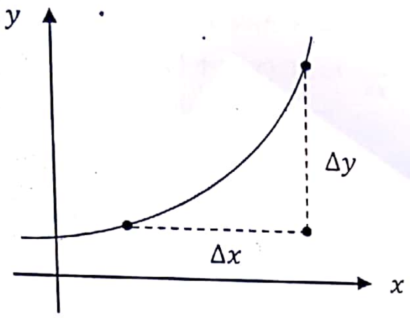
$i)$ $f(x) = \begin{cases} \frac{\sin x}{x} & x = 0 \\ 0 & x \neq 0 \end{cases}$ $ii)$ $f(x) = \sin x$ $at\ x = \frac{\pi}{2}$

Solution:

1. $f(0) = \infty \Rightarrow f(0)$ is not define
2. $\lim_{x \to 0^-} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \Rightarrow$ *Limit of f(x) exist*
3. $\lim_{x \to 0} \frac{\sin x}{x} \neq f(0)$

Hence $f(x) = \frac{\sin x}{x}$ is not continuous at point $x = 0$

Concept of derivatives from first principle:



Suppose y is a single-valued function of defined by the equation

$y = f(x)$ . . . (1)

An increase Δx in x will produce an increase Δx in y. Adding Δx to x and Δy to y

$y + \Delta y = f(x + \Delta x)$

Making Δy the subject we have

$\Delta y = f(x + \Delta x) - f(x)$ . . . (2)

Dividing both sides by Δx (where $\Delta x \neq 0$) we have

$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$\frac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to x in the interval Δx and is called difference quotient.

Taking limit as $\Delta x \to 0$ on both sides we have

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

As $\Delta x \to 0$ $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ tends to a finite limit if $f(x)$ is continuous. This limit is interpreted as the rate of change of y with respect to x for the initial value of x, and it is called the differential coefficient or the derivative of y with respect to x, and is written as $\frac{dy}{dx}$.

Therefore $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Hence $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ . . . (3)

Basic concept of Differentiation:

1. *Derivative of a constant function:*

Let $y = c$, where c is any arbitrary constant

$\Rightarrow \quad y + \Delta y = c$

$\Rightarrow \quad \Delta y = c - c = 0$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = 0 \qquad \Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$

$\Rightarrow$ *Derivatives of any constant is zero*

2. *Derivative of the sum of functions:* The derivative of a function which contains two or more terms is the sum of the individual terms. i.e. if $y = u(x) + v(x)$, then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Proof:

Let $y = u + v$

$\Rightarrow \quad y + \Delta y = (u + \Delta u) + (v + \Delta v)$

$\Rightarrow \quad \Delta y = [(u + \Delta u) + (v + \Delta v)] - [u + v]$

$\Rightarrow \quad \Delta y = \Delta u + \Delta v$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u + \Delta v}{\Delta x}$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Hence the proof

3. *Derivative of the difference of functions:* The derivative of a difference of two or more terms is the difference of the individual terms. i.e. If $y = u(x) - v(x)$, then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$.

Proof:

Let $y = u - v$

$\Rightarrow \quad y + \Delta y = (u + \Delta u) - (v + \Delta v)$

$\Rightarrow \quad \Delta y = [(u + \Delta u) - (v + \Delta v)] - [u - v]$

$\Rightarrow \quad \Delta y = \Delta u - \Delta v$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u - \Delta v}{\Delta x}$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} - \frac{\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} - \frac{\Delta v}{\Delta x}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

Hence the proof

4. *Derivative of the Product functions:*

If $y = u(x) * v(x)$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

Proof:

Let $y = u * v$

$\Rightarrow \quad y + \Delta y = (u + \Delta u) * (v + \Delta v)$

$\Rightarrow \quad y + \Delta y = uv + v\Delta u + u\Delta v + \Delta u\Delta v$

$\Rightarrow \quad \Delta y = (uv + v\Delta u + u\Delta v + \Delta u\Delta v) - uv$

$\Rightarrow \quad \Delta y = v\Delta u + u\Delta v + \Delta u\Delta v$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{v\Delta u + u\Delta v + \Delta u\Delta v}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta u\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta u\Delta v}{\Delta x}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = v\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u\Delta v}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$. (Neglecting $\lim_{\Delta x \to 0} \frac{\Delta u\Delta v}{\Delta x}$)

Hence the proof

5. *Derivative of the Quotient functions:*

If $y = \frac{u}{v}$ where u and v are functions of x, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Proof:

Let $\quad y = \frac{u}{v}$

$\Rightarrow \quad y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$

$\Rightarrow \quad \Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$

$\Rightarrow \quad \Delta y = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)} = \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v} * \frac{1}{\Delta x} = \frac{\frac{v\Delta u}{\Delta x} - \frac{u\Delta v}{\Delta x}}{v^2 + v\Delta v} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(Neglecting $v\Delta v$)

6. *Derivative of the composite functions:*

If $y = f(g(x))$ then, $\frac{dy}{dx} = f'(g(x)) * g'(x)$

Proof;

Let $y = f(g(x))$ . . . (1)

Let $u = g(x)$

$\Rightarrow \quad y = f(u)$

$\Rightarrow \quad y + \Delta y = f(u + \Delta u)$

$\Rightarrow \quad \Delta y = f(u + \Delta u) - f(u)$

Dividing both sides by Δu (where $\Delta u \neq 0$) we have

$$\frac{\Delta y}{\Delta u} = \frac{f(u+\Delta u)-f(u)}{\Delta u}$$

$$\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u)-f(u)}{\Delta u}$$

Hence $\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u)-f(u)}{\Delta u}$ . . . (2)

And from eqn. (1) we have

$$u = g(x)$$

$$u + \Delta u = g(x + \Delta x)$$

$$\Delta u = g(x + \Delta x) - g(x)$$

Dividing both sides by Δx (where $\Delta x \neq 0$) we have

$$\frac{\Delta u}{\Delta x} = \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-f(x)}{\Delta x}$$

Hence, $\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-f(x)}{\Delta x}$ . . . (3)

From (2) and (3) we have

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} * \lim_{\Delta u \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u)-f(u)}{\Delta u} * \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-f(x)}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = f'(g(x)) * g'(x)$$

Differentiation of polynomial algebraic functions x^n:

$$y = x^n$$

$$\Rightarrow \quad y + \Delta y = (x + \Delta x)^n$$

$$\Rightarrow \quad \Delta y = (x + \Delta x)^n - x^n$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(x^n + \frac{n}{1!}x^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\Delta x)^3 + \ldots + \Delta x^n\right) - x^n}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{x^n + \frac{n}{1!}x^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\Delta x)^3 + \ldots + \Delta x^n - x^n}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{n}{1!}x^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\Delta x)^3 + \ldots + \Delta x^n}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta x\left(\frac{n}{1!}x^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\Delta x)^2 + \ldots + \Delta x^{n-1}\right)}{\Delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{n}{1!}x^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\Delta x)^2 + \ldots + \Delta x^{n-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

Example1: Differentiate the following functions using first principle

$i)\ y = 2x^2 + 3x - 4,\qquad ii)\ y = \sqrt{x}\qquad iii)\ y = x^3 - 2x^2 + 2x - 1$

Solution;

$i)\qquad y = 2x^2 + 3x - 4,$

$\Rightarrow\qquad y + \Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) - 4,$

$\Rightarrow\qquad \Delta y = [2(x + \Delta x)^2 + 3(x + \Delta x) - 4] - [2x^2 + 3x - 4]$

$\Rightarrow\qquad \Delta y = [2x^2 + 4x\Delta x + 2(\Delta x)^2 + 3x + 3\Delta x - 4] - [2x^2 + 3x - 4]$

$\Rightarrow\qquad \Delta y = 2x^2 + 4x\Delta x + 2(\Delta x)^2 + 3x + 3\Delta x - 4 - 2x^2 - 3x + 4$

$\Rightarrow\qquad \Delta y = 4x\Delta x + 2(\Delta x)^2 + 3\Delta x = \Delta x(4x + 2\Delta x + 3)$

$\Rightarrow\qquad \frac{\Delta y}{\Delta x} = \frac{\Delta x(4x+2\Delta x+3)}{\Delta x} = 4x + 2\Delta x + 3$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 4x + 2\Delta x + 3 = 4x + 3$

$ii)\qquad y = \sqrt{x}$

$\Rightarrow\qquad y + \Delta y = \sqrt{x + \Delta x}$

$\Rightarrow\qquad \Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$\Rightarrow\qquad \frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \frac{1}{2\sqrt{x}}$

$iii)\quad y = x^3 - 2x^2 + 2x - 1$

Solution: Similar to $i)$

Differentiation of basic trigonometric functions: (sine and cosine)

Let $y = \sin x$ where x is in radian

$\Rightarrow\qquad y + \Delta y = \sin(x + \Delta x)$

$\Rightarrow\qquad \Delta y = \sin(x + \Delta x) - \sin x$

$\Rightarrow\qquad \frac{\Delta y}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} * \frac{\frac{1}{2}}{\frac{1}{2}} = \lim_{\Delta x \to 0} \frac{\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} = \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) * \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) * \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$

Let $\frac{\Delta x}{2} = t \qquad \Rightarrow \qquad$ as $\Delta x \to 0 t \to 0$

$\Rightarrow\qquad \frac{dy}{dx} = \lim_{t \to 0} \cos(x + t) * \lim_{t \to 0} \frac{\sin t}{t} = \cos x$

$\Rightarrow\qquad \frac{dy}{dx} = \cos x$

ii) Let $y = \cos x$ where x is in radian

$\Rightarrow \quad y + \Delta y = \cos(x + \Delta x)$

$\Rightarrow \quad \Delta y = \cos(x + \Delta x) - \cos x$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\cos(x+\Delta x)-\cos x}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x)-\text{cosin}\, x}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0} \frac{-2\sin\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{-2\sin\left(\frac{2x+\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} * \frac{\frac{1}{2}}{\frac{1}{2}} = \lim_{\Delta x\to 0} \frac{-\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} = -\lim_{\Delta x\to 0} \sin\left(x + \frac{\Delta x}{2}\right) * \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$

$\Rightarrow \quad \frac{dy}{dx} = -\lim_{\Delta x\to 0} \sin\left(x + \frac{\Delta x}{2}\right) * \lim_{\Delta x\to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$

Let $\frac{\Delta x}{2} = t \quad \Rightarrow \quad$ as $\Delta x \to 0 t \to 0$

$\Rightarrow \quad \frac{dy}{dx} = -\lim_{t\to 0} \sin(x + t) * \lim_{t\to 0} \frac{\sin t}{t} = \sin x$

$\Rightarrow \quad \frac{dy}{dx} = -\sin x$

Differentiation of natural exponential functions:

Let $\quad y = e^x$

$\Rightarrow \quad y + \Delta y = e^{(x+\Delta x)}$

$\Rightarrow \quad \Delta y = e^{(x+\Delta x)} - e^x = e^x e^{\Delta x} - e^x = e^x(e^{\Delta x} - 1)$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{e^x(e^{\Delta x}-1)}{\Delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{e^x(e^{\Delta x}-1)}{\Delta x} = \lim_{\Delta x\to 0} e^x * \frac{(e^{\Delta x}-1)}{\Delta x} = \lim_{\Delta x\to 0} e^x * \lim_{\Delta x\to 0} \frac{(e^{\Delta x}-1)}{\Delta x}$

$\quad = e^x * \lim_{\Delta x\to 0} \frac{(e^{\Delta x}-1)}{\Delta x}$

But from example 1, under limits we have that

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots$

$\Rightarrow \quad e^{\Delta x} = 1 + \frac{\Delta x}{1!} + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \frac{(\Delta x)^4}{4!} + \frac{(\Delta x)^5}{5!} \ldots$

$\Rightarrow \quad e^{\Delta x} - 1 = \frac{\Delta x}{1!} + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \frac{(\Delta x)^4}{4!} + \frac{(\Delta x)^5}{5!} \ldots$

$\Rightarrow \quad \frac{e^{\Delta x}-1}{\Delta x} = \frac{1}{1!} + \frac{\Delta x}{2!} + \frac{(\Delta x)^2}{3!} + \frac{(\Delta x)^3}{4!} + \frac{(\Delta x)^4}{5!} + \frac{(\Delta x)^5}{6} \ldots$

$\Rightarrow \quad \lim_{\Delta x\to 0} \frac{(e^{\Delta x}-1)}{\Delta x} = \lim_{\Delta x\to 0} \left[\frac{1}{1!} + \frac{\Delta x}{2!} + \frac{(\Delta x)^2}{3!} + \frac{(\Delta x)^3}{4!} + \frac{(\Delta x)^4}{5!} + \frac{(\Delta x)^5}{6} \ldots\right] = 1$

$\Rightarrow \quad \frac{dy}{dx} = e^x * \lim_{\Delta x\to 0} \frac{(e^{\Delta x}-1)}{\Delta x} = e^x * 1 = e^x$

Differentiation of logarithmic functions:

Let $y = \log_a x$

$\Rightarrow \quad y + \Delta y = \log_a(x + \Delta x)$

$\Rightarrow \quad \Delta y = \log_a(x + \Delta x) - \log_a x$

$\Rightarrow \quad \Delta y = \log_a \left[\frac{x+\Delta x}{x}\right] = \log_a \left[1 + \frac{\Delta x}{x}\right]$

$\Rightarrow \quad \Delta y = \log_a \left[1 + \frac{\Delta x}{x}\right]$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{x}{x} * \frac{1}{\Delta x} \log_a \left[1 + \frac{\Delta x}{x}\right]$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{1}{x} * \frac{x}{\Delta x} \log_a \left[1 + \frac{\Delta x}{x}\right]$

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{1}{x} * \log_a \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}$

$\Rightarrow \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lim} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lim} \left[\frac{1}{x} * \log_a \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}\right] = \underset{\Delta x \to 0}{Lim} \frac{1}{x} * \underset{\Delta x \to 0}{Lim} \left[\log_a \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lim} \frac{1}{x} * \log_a \left[\underset{\Delta x \to 0}{Lim} \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} * \log_a \left[\underset{\Delta x \to 0}{Lim} \left[1 + \frac{\Delta x}{x}\right]^{\frac{x}{\Delta x}}\right]$

Now let $\frac{x}{\Delta x} = n, \quad \Rightarrow \quad \frac{\Delta x}{x} = \frac{1}{n}$, as $\Delta x \to 0,\ n \to \infty$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} * \log_a \left[\underset{n \to \infty}{Lim} \left[1 + \frac{1}{n}\right]^n\right] \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} * \log_a e$

Example1: Given that, $f(x) = 4 - 5x + 2x^3 - 2x^5$, Show that $f'(2) = f'(-2)$

Solution:

$f(x) = 4 - 5x + 2x^3 - 2x^5$

$\Rightarrow \quad f'(x) = -5 + 6x^2 - 10x^4$

$\Rightarrow \quad f'(2) = -5 + 6(2)^2 - 10(2)^4 = -141$

$f'(-2) = -5 + 6(-2)^2 - 10(-2)^4 = -141$

Hence $f'(2) = f'(-2)$

Example2: Find the derivative of the following functions

i) $y = x^4 e^x$ $\qquad$ *ii)* $y = (x^2 - 2x + 1)2^x$

Solution:

i) $y = x^4 e^x$

$\Rightarrow \quad \frac{dy}{dx} = 4x^3 e^x + x^4 e^x$

$\Rightarrow \quad \frac{dy}{dx} = x^3(4 + x)e^x$

ii) $y = (x^2 - 2x + 1)2^x$

$\Rightarrow \quad \frac{dy}{dx} = 2(x-1)2^x + 2^x ln2(x^2 - 2x + 1)$

$\Rightarrow \quad \frac{dy}{dx} = \left(2(x-1) + ln2(x^2 - 2x + 1)\right)2^x$

Example3: Find the derivative of the following functions

i) $y = x^3 \log_5 x$ ii) $y = x^3 \tan x$

i) $y = x^3 \log_5 x$

$\Rightarrow \frac{dy}{dx} = 3x^2 \log_5 x + x^3 \frac{1}{x} \log_5 e$

$\Rightarrow \frac{dy}{dx} = 3x^2 \log_5 x + x^2 \log_5 e$

$\Rightarrow \frac{dy}{dx} = x^2(3 \log_5 x + \log_5 e)$

ii) $y = x^3 \tan x$

$\Rightarrow \frac{dy}{dx} = 3x^2 \tan x + x^3 sec^2 x$

$\Rightarrow \frac{dy}{dx} = x^2(3 \tan x + x sec^2 x)$

Example4: Find the derivative of the following functions

i) $y = e^x \sin x$ ii) $y = 2^x \ln x$

Solution:

i) $y = e^x \sin x$

$\Rightarrow \frac{dy}{dx} = e^x \sin x + e^x \cos x$

$\Rightarrow \frac{dy}{dx} = (\sin x + \cos x)e^x$

ii) $y = 2^x \ln x$

$\Rightarrow \frac{dy}{dx} = 2^x \ln 2 \ln x + \frac{1}{x} 2^x$

$\Rightarrow \frac{dy}{dx} = \left(\ln 2 \ln x + \frac{1}{x}\right) 2^x$

Exercise: Find the derivative of the following functions

i) $y = 7^x(x^5 - 4x^3 + 2x - 3)e^x$ ii) $y = 2^x \csc x$ iii) $y = x \sin x \log_5 x$

Example4: Find the derivative of the following functions

i) $y = \frac{e^x}{x^3}$ ii) $y = \frac{2^x}{\ln x}$ iii) $y = \frac{\ln x}{1+x^2}$

Solution:

i) $y = \frac{e^x}{x^3}$

$\Rightarrow \frac{dy}{dx} = \frac{x^3 e^x - 3x^2 e^x}{(x^3)^2} = \frac{x^2(x-3)e^x}{x^6} = \frac{(x-3)e^x}{x^4}$

ii) $y = \frac{2^x}{\ln x}$

$\Rightarrow \frac{dy}{dx} = \frac{2^x \ln 2 \ln x - \frac{2^x}{x}}{\ln^2 x} = \frac{2^x(x \ln 2 \ln x - 1)}{x \ln^2 x}$

iii) $y = \frac{\ln x}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x}(1+x^2) - 2x \ln x}{(1+x^3)^2}$.

Exercise: Find the derivative of the following functions

i) $y = \frac{x-1}{\log_2 x}$ ii) $y = \frac{1-e^x}{1+e^x}$ iii) $y = \frac{1-10^x}{1+10^x}$

Example5: Find the derivative of the following functions

i) $y = (x^2 + 4)^5$ ii) $y = \left(\frac{x}{1-x}\right)^3$ iii) $y = \sqrt{1 - x^2}$

Solution:

i) $y = (x^2 + 4)^5$ Let $t = x^2 + 4$

$\Rightarrow y = (t)^5$ $\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow \frac{dy}{dt} = 5t^4 = 5(x^2 + 4)^4$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = 5(x^2 + 4)^4 * 2x$

$\Rightarrow \frac{dy}{dx} = 10x(x^2 + 4)^4$

ii) $y = \left(\frac{x}{1-x}\right)^3$ Let $t = \frac{x}{1-x}$

$\Rightarrow$ $y = (t)^3$ $\Rightarrow$ $\frac{dt}{dx} = \frac{(1-x)+x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$\Rightarrow$ $\frac{dy}{dt} = 3t^2 = 3\left(\frac{x}{1-x}\right)^2$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = 3\left(\frac{x}{1-x}\right)^2 * \frac{1}{(1-x)^2}$

$\Rightarrow$ $\frac{dy}{dx} = 3\left(\frac{x}{1-x}\right)^2 * \frac{1}{(1-x)^2}$

iii) $y = \sqrt{1-x^2}$

$\Rightarrow$ $y = (1-x^2)^{\frac{1}{2}}$ Let $t = 1 - x^2$

$\Rightarrow$ $y = (t)^{\frac{1}{2}}$ $\Rightarrow$ $\frac{dt}{dx} = -2x$

$\Rightarrow$ $\frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2t^{\frac{1}{2}}} = \frac{1}{2\sqrt{t}} = \frac{1}{2\sqrt{1-x^2}}$ $\frac{dt}{dx} = -2x$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{1}{2\sqrt{1-x^2}} * (-2x)$

Example6: Find the derivative of the following functions

i) $y = 5\left(\frac{x}{x^2+2}\right)^4$ ii) $y = \left(x^2 + \frac{1}{x} - 5x\right)^{\frac{1}{3}}$ iii) $y = \sqrt[5]{(3x^2+4x-5)^3}$

Solution:

i) $y = 5\left(\frac{x}{x^2+2}\right)^4$ Let $t = \frac{x}{x^2+2}$

$\Rightarrow$ $y = 5(t)^4$ $\Rightarrow$ $\frac{dt}{dx} = \frac{(x^2+2)-2x^2}{(x^2+2)^2} = \frac{x^2+2-2x^2}{(x^2+2)^2}$

$\Rightarrow$ $\frac{dy}{dt} = 20t^3 = 20\left(\frac{x}{x^2+2}\right)^3$ $\Rightarrow$ $\frac{dt}{dx} = \frac{(2-x^2)}{(x^2+2)^2}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = 20\left(\frac{x}{x^2+2}\right)^3 * \frac{(2-x^2)}{(x^2+2)^2}$

$\Rightarrow$ $\frac{dy}{dx} = 20\left(\frac{x}{x^2+2}\right)^3 * \frac{(2-x^2)}{(x^2+2)^2}$

ii) $y = \left(x^2 + \frac{1}{x} - 5x\right)^{\frac{1}{3}}$ Let $t = x^2 + \frac{1}{x} - 5x$

$\Rightarrow$ $y = (t)^{\frac{1}{3}}$ $\Rightarrow$ $\frac{dt}{dx} = 2x - \frac{1}{x^2} - 5$

$\Rightarrow$ $\frac{dy}{dt} = \frac{1}{3}(t)^{-\frac{2}{3}} = \frac{1}{3}\left(x^2 + \frac{1}{x} - 5x\right)^{-\frac{2}{3}}$ $\Rightarrow$ $\frac{dt}{dx} = 2x - \frac{1}{x^2} - 5$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{1}{3}\left(x^2 + \frac{1}{x} - 5x\right)^{-\frac{2}{3}} * \left(2x - \frac{1}{x^2} - 5\right)$

$\Rightarrow$ $\frac{dy}{dx} = \frac{1}{3}\left(x^2 + \frac{1}{x} - 5x\right)^{-\frac{2}{3}} * \left(2x - \frac{1}{x^2} - 5\right)$

iii) $y = \sqrt[5]{(3x^2+4x-5)^3} = (3x^2+4x-5)^{\frac{3}{5}}$ Let $t = 3x^2 + 4x - 5$

$\Rightarrow$ $y = (t)^{\frac{3}{5}}$ $\Rightarrow$ $\frac{dt}{dx} = 6x + 4$

$\Rightarrow$ $\frac{dy}{dt} = \frac{3}{5}(t)^{-\frac{2}{5}} = \frac{3}{5}(3x^2+4x-5)^{-\frac{2}{5}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{3}{5}(3x^2 + 4x - 5)^{-\frac{2}{5}} * (6x + 4)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{5}(3x^2 + 4x - 5)^{-\frac{2}{5}} * (6x + 4)$

Example7: Find the derivative of the following functions

$i)\ y = \ln(\sin x) \qquad ii)\ y = \sin(2x^3) \qquad iii)\ y = \ln\left(\frac{1}{x^2}\right)$

Solution:

$i) \quad y = \ln(\sin x)$ Let $t = \sin x$

$\Rightarrow \quad y = \ln t$ $\Rightarrow \frac{dt}{dx} = \cos x$

$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{t} = \frac{1}{\sin x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{1}{\sin x} * \cos x = \frac{\cos x}{\sin x} = \cot x$

$\Rightarrow \quad \frac{dy}{dx} = \cot x$

$ii) \quad y = \sin(2x^3)$ Let $t = 2x^3$

$\Rightarrow \quad y = \sin t$ $\Rightarrow \frac{dt}{dx} = 6x^2$

$\Rightarrow \quad \frac{dy}{dt} = \cos t = \cos(2x^3)$ $\Rightarrow \frac{dt}{dx} = 6x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \cos(2x^3) * 6x^2$

$iii) \quad y = \ln\left(\frac{1}{x^2}\right)$ Let $t = \frac{1}{x^2}$

$\Rightarrow \quad y = \ln t$ $\Rightarrow \frac{dt}{dx} = -\frac{2}{x^3}$

$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{t} = x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = x^2 * \left(-\frac{2}{x^3}\right) = -\frac{2}{x}$

Example8: Find the derivative of the following functions

$i)\ y = \sqrt[3]{\frac{1}{1+x^2}} \qquad ii)\ y = \sqrt{\frac{1-x}{1+x}} \qquad iii)\ y = sec^2(4x^3 - 1)$

Solution:

$i) \quad y = \sqrt[3]{\frac{1}{1+x^2}} = \left(\frac{1}{1+x^2}\right)^{\frac{1}{3}}$ Let $t = \frac{1}{1+x^2}$

$\Rightarrow \quad y = t^{\frac{1}{3}}$ $\Rightarrow \frac{dt}{dx} = -\frac{2x}{(1+x^2)^2}$

$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{3}t^{-\frac{2}{3}} = \frac{1}{3}\left(\frac{1}{1+x^2}\right)^{-\frac{2}{3}}$ $\Rightarrow \frac{dt}{dx} = -\frac{2x}{(1+x^2)^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{1}{3}\left(\frac{1}{1+x^2}\right)^{-\frac{2}{3}} * \left(-\frac{2x}{(1+x^2)^2}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{1+x^2}\right)^{-\frac{2}{3}} * \left(-\frac{2x}{(1+x^2)^2}\right)$

ii) $y = \sqrt{\frac{1-x}{1+x}} = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$ Let $t = \frac{1-x}{1+x}$

$\Rightarrow \quad y = t^{\frac{1}{2}}$ $\Rightarrow \quad \frac{dt}{dx} = -\frac{2}{(1+x)^2}$

$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} * \left(-\frac{2}{(1+x)^2}\right) = \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} * \left(-\frac{1}{(1+x)^2}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} * \left(-\frac{1}{(1+x)^2}\right)$

iii) $y = sec^2(4x^3-1) = (\sec(4x^3-1))^2$ Let $t = \sec(4x^3-1)$ and $u = 4x^3 - 1$

$\Rightarrow \quad y = t^2 \quad \Rightarrow \quad t = \sec u \quad \Rightarrow \quad u = 4x^3 - 1$

$\Rightarrow \quad \frac{dy}{dt} = 2t = 2(\sec u) = 2\sec(4x^3-1)$ $\Rightarrow \quad \frac{dt}{du} = \sec u . \tan u$ $\Rightarrow \quad \frac{du}{dx} = 12x^2$

$\Rightarrow \quad \frac{dy}{dt} = 2\sec(4x^3-1)$ $\Rightarrow \quad \frac{dt}{du} = \sec(4x^3-1)\tan(4x^3-1)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{du} * \frac{du}{dx} = 2\sec(4x^3-1) * \sec(4x^3-1)\tan(4x^3-1) * 12x^2$

$= 24x^2\sec(4x^3-1) * \sec(4x^3-1)\tan(4x^3-1)$

Exercises: Find the derivative of the following functions

i) $y = e^{-\cos 3x}$ ii) $y = \ln(\sqrt{1-x^2})$ iii) $y = \sqrt{1 + sin^2 xiv)}$

iv) $y = \frac{1}{1+\cos 4x}$ v) $y = \ln^4(\sin x)$ vi) $y = (1 + \ln(\sin x))^5$

vii) $y = \cos(\ln 2x)$ viii) $y = e^{\sqrt{\frac{1-x}{1+x}}}$

Differentiation of Implicit functions:

Often a function $f(x,y)$ is specified by an equation in x and y, and that y is not the subject of the equation. If such situation exists we say that y is an implicit function; by using the rules of differentiation the differential coefficients $\frac{dy}{dx}$ may be found without making y the subject of the equation. Differentiate both sides with respect to x and make $\frac{dy}{dx}$ the subject of the equation.

Example1: Find the derivative of the following functions

i) $x^3 + y^3 = 3xy$ ii) $x^2 + y^2 = x + y + 8$ iii) $x^3 + y^3 = x^3y^2 - 1$

Solution:

i) $x^3 + y^3 = 3xy$

Differentiating with respect to x we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = \frac{3(y-x^2)}{3(y^2-x)} = \frac{(y-x^2)}{3(y^2-x)}$$

ii) $x^2 + y^2 = x + y + 8$

Differentiating with respect to x we have

$$2x + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = \frac{(1-2x)}{(2y-1)}$$

iii) $x^3 + y^3 = x^3y^2 - 1$

Differentiating with respect to x, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3x^2y^2 + x^3 2y\frac{dy}{dx}$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = \frac{3x^2y^2 - 3x^2}{3y^2 - 2x^3y} = \frac{3x^2(y^2-1)}{y(3y-2x^3)}$$

Example2: Find the derivative of the following functions

i) $2^x + 2^y = 2^{x+y}$ *ii)* $x^y = y^x$ *iii)* $y = x^x$

Solution:

i) $2^x + 2^y = 2^{x+y} \Rightarrow 2^x + 2^y = 2^x 2^y$

Differentiating with respect to x we have

$$2^x \ln 2 + \left(2^y \ln 2\frac{dy}{dx}\right) = (2^x \ln 2)2^y + 2^x\left(2^y \ln 2\frac{dy}{dx}\right)$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = \frac{2^x \ln 2(2^y - 1)}{2^y \ln 2(1 - 2^x)} = \frac{2^x(2^y-1)}{2^y(1-2^x)}$$

ii) $x^y = y^x$

Taking natural logarithm on both sides we have

$$\ln x^y = \ln y^x$$

$$\Rightarrow \quad y \ln x = x \ln y$$

Differentiating with respect to x we have

$$(\ln x)\frac{dy}{dx} + \frac{y}{x} = \ln y + \left(\frac{x}{y}\right)\frac{dy}{dx}$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = \frac{\left(\ln y - \frac{y}{x}\right)}{\left(\ln x - \frac{x}{y}\right)}$$

iii) $y = x^x$

Taking natural logarithm on both sides we have

$$\ln y = \ln x^x = x \ln x$$

Differentiating with respect to x we have

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \ln x) \quad \Rightarrow \quad \frac{dy}{dx} = x^x(1 + \ln x)$$

Example3: Find the derivative of the following functions

i) $y = 3^{\sin x}$ *ii)* $y = (\sin x)^{\cos x}$ *iii)* $y = x^{\ln x}$

Solution:

i) $y = 3^{\sin x}$

Taking natural logarithm on both sides we have

$$\ln y = \ln 3^{\sin x} = \sin x \ln 3$$

Differentiating with respect to x we have

$$\frac{1}{y} * \frac{dy}{dx} = \cos x \ln 3$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = y(\cos x \ln 3)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3^{\sin x}(\cos x \ln 3)$$

ii) $y = (\sin x)^{\cos x}$

Taking natural logarithm on both sides we have *ii)* $y = (\sin x)^{\cos x}$

Taking natural logarithm on both sides we have

$$\ln y = \ln(\sin x)^{\cos x} = \cos x \ln(\sin x)$$

Differentiating with respect to x we have

$$\frac{1}{y} * \frac{dy}{dx} = -\sin x \ln(\sin x) + \cos x\left(\frac{\cos x}{\sin x}\right)$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = y\left(-\sin x \ln(\sin x) + \cos x\left(\frac{\cos x}{\sin x}\right)\right) = y(-\sin x \ln(\sin x) + \cos x \cot x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x}(-\sin x \ln(\sin x) + \cos x \cot x)$$

iii) $y = x^{\ln x}$

Taking natural logarithm on both sides we have

$$\ln y = \ln x^{\ln x} = \ln x\,(\ln x) = \ln^2 x$$

Differentiating with respect to x we have

$$\frac{1}{y} * \frac{dy}{dx} = \frac{2}{x}\ln x$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = y\left(\frac{2}{x}\ln x\right) \quad \Rightarrow \quad \frac{dy}{dx} = x^{\ln x}\left(\frac{2}{x}\ln x\right)$$

Example4: Find the derivative of the following functions

i) $y = (x+1)^{\frac{2}{x}}$ *ii)* $y = (\ln x)^x$ *iii)* $y = x^{x^2}$

Solution:

i) $y = (x+1)^{\frac{2}{x}}$

Taking natural logarithm on both sides we have

$$\ln y = \ln(x+1)^{\frac{2}{x}} = \frac{2}{x}\ln(x+1)$$

Differentiating with respect to x we have

$$\frac{1}{y} * \frac{dy}{dx} = -\frac{2}{x^2}\ln(x+1) + \frac{2}{x}\left(\frac{1}{x+1}\right)$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = y\left(-\frac{2}{x^2}\ln(x+1) + \frac{2}{x}\left(\frac{1}{x+1}\right)\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = (x+1)^{\frac{2}{x}}\left(-\frac{2}{x^2}\ln(x+1) + \frac{2}{x}\left(\frac{1}{x+1}\right)\right)$$

ii) $y = (\ln x)^x$

Taking natural logarithm on both sides we have

$$\ln y = \ln(\ln x)^x = x\ln(\ln x)$$

Differentiating with respect to x we have

$$\frac{1}{y} * \frac{dy}{dx} = \ln(\ln x) + x\left(\frac{1}{x\ln x}\right) = \ln(\ln x) + \frac{1}{\ln x}$$

Making $\frac{dy}{dx}$ the subject we have

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\ln(\ln x) + \frac{1}{\ln x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\ln x)^x\left(\ln(\ln x) + \frac{1}{\ln x}\right)$$

iii) $y = x^{x^2}$

Taking natural logarithm on both sides we have

$$\ln y = \ln x^{x^2} = x^2 \ln x$$

Differentiating with respect to x we have

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x = x(2\ln x + 1)$$

Making $\frac{dy}{dx}$ the subject we have

$$\frac{dy}{dx} = y(2x\ln x + x) \qquad \Rightarrow \qquad \frac{dy}{dx} = x^{x^2}(2x\ln x + x)$$

Exercises:

1. Find the derivative of the following functions

i) $y = (x^2+1)^{\sin x}$ *ii)* $y = a^{\sin 3x}$ *iii)* $y = x^{x^x}$

iv) $y = e^{x^n}$ *v)* $y = \frac{1}{3^x}$ *vi)* $y = 10^{1-\sin^2 3x}$

vii) $y = (x)^{\frac{1}{x}}$ *viii)* $y = 5e^{-x^2}$ *ix)* $y = x^2 2^x$

Differentiation of inverse trigonometric functions:

i) Given that

$$y = arc\sin x$$

$$\Rightarrow \quad \sin y = x \qquad\qquad \Rightarrow \quad sin^2 y = x^2$$

Differentiating with respect to x we have

$$\Rightarrow \quad \cos y\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-sin^2 y}} \qquad \text{since } sin^2 y + cos^2 y = 1 \Rightarrow \quad cos y = \sqrt{1 - sin^2 y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

ii) Given that

$$y = arc\cos x$$

$$\Rightarrow \quad \cos y = x \qquad\qquad \Rightarrow \quad cos^2 y = x^2$$

Differentiating with respect to x we have

$\Rightarrow \quad -\sin y \frac{dy}{dx} = 1$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin y} = \frac{1}{\sqrt{1-cos^2y}}$ since $sin^2y + cos^2y = 1 \Rightarrow \quad siny = \sqrt{1-cos^2y}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$iii)$ Given that

$y = arc\tan x$

$\Rightarrow \quad \tan y = x \qquad \Rightarrow \quad tan^2y = x^2$

Differentiating with respect to x we have

$\Rightarrow \quad sec^2y\frac{dy}{dx} = 1$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y}$ since, $1 + tan^2y = sec^2y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$

Example1: Find the derivative of the following inverse trigonometric functions

$i)\ y = arcsin\left(\frac{x}{2}\right)$ $\quad ii)\ y = arc\cos\left(\frac{1}{x}\right)$ $\quad iii)\ y = arc\tan(\ln x)$

Solution:

$i)\quad y = arc\sin\left(\frac{x}{2}\right)$,

$\Rightarrow \quad \sin y = \frac{x}{2} \qquad \Rightarrow \quad sin^2y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

Differentiating with respect to x we have

$\Rightarrow \quad \cos y\frac{dy}{dx} = \frac{1}{2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{2\sqrt{1-sin^2y}}$ since $sin^2y + cos^2y = 1 \quad \Rightarrow \quad cosy = \sqrt{1-sin^2y}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{1-\frac{x^2}{4}}} = \frac{1}{2\sqrt{\frac{4-x^2}{4}}} = \frac{1}{2\frac{\sqrt{4-x^2}}{2}} = \frac{1}{\sqrt{4-x^2}}$

$ii)\quad y = arc\cos\left(\frac{1}{x}\right)$

$\Rightarrow \quad \cos y = \frac{1}{x} \qquad \Rightarrow \quad cos^2y = \frac{1}{x^2}$

Differentiating with respect to x we have

$\Rightarrow \quad -\sin y\frac{dy}{dx} = -\frac{1}{x^2}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2\sin y} = -\frac{1}{x^2\sqrt{1-cos^2y}}$ since $sin^2y + cos^2y = 1 \Rightarrow \quad siny = \sqrt{1-cos^2y}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = -\frac{1}{x^2\frac{\sqrt{x^2-1}}{x}} = -\frac{1}{x\sqrt{x^2-1}}$

iii) $y = \arctan(\ln x)$

$\Rightarrow \quad \tan y = \ln x \qquad\qquad \Rightarrow \quad \tan^2 y = \ln^2 x$

Differentiating with respect to x we have

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x\sec^2 y} = \frac{1}{x(1+\tan^2 y)} \qquad \text{since } \sec^2 y = 1+\tan^2 y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(1+\ln^2 x)}$$

Example2: Find the derivative of the following inverse trigonometric functions

i) $y = \arcsin\left(\frac{x^2-1}{x^2}\right)$ ii) $y = \arccos\left(\frac{x-1}{x+1}\right)$ iii) $y = \arctan\left(\frac{2x}{1-x^2}\right)$

Solution:

i) $y = \arcsin\left(\frac{x^2-1}{x^2}\right)$

$\Rightarrow \quad \sin y = \frac{x^2-1}{x^2} \qquad\qquad \Rightarrow \quad \sin^2 y = \left(\frac{x^2-1}{x^2}\right)^2 = \frac{x^4-2x^2+1}{x^4}$

Differentiating with respect to x we have

$$\Rightarrow \quad \cos y \frac{dy}{dx} = \frac{2}{x^3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{x^3\cos y} = \frac{2}{x^3\sqrt{1-\sin^2 y}} \qquad \text{since } \sin^2 y + \cos^2 y = 1 \Rightarrow \quad \cos y = \sqrt{1-\sin^2 y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{x^3\sqrt{1-\frac{x^4-2x^2+1}{x^4}}} = \frac{2}{x^3\sqrt{\frac{2x^2-1}{x^4}}} = \frac{2}{x^3\frac{\sqrt{2x^2-1}}{x^2}} = \frac{2}{x\sqrt{2x^2-1}}$$

ii) $y = \arccos\left(\frac{x-1}{x+1}\right)$

$\Rightarrow \quad \cos y = \frac{x-1}{x+1} \qquad\qquad \Rightarrow \quad \cos^2 y = \frac{(x-1)^2}{(x+1)^2}$

Differentiating with respect to x we have

$$-\sin y \frac{dy}{dx} = \frac{2}{(x+1)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{(x+1)^2} * \frac{1}{\sin y} = -\frac{2}{(x+1)^2} * \frac{1}{\sqrt{1-\cos^2 y}}, \qquad \text{since} \quad \sin y = \sqrt{1-\cos^2 y}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{(x+1)^2} * \frac{1}{\sqrt{1-\frac{(x-1)^2}{(x+1)^2}}} = -\frac{2}{(x+1)^2} * \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}} = -\frac{2}{(x+1)^2} * \frac{(x+1)}{2\sqrt{x}} = -\frac{1}{(x+1)\sqrt{x}}$$

iii) $y = \arctan\left(\frac{2x}{1-x^2}\right)$

$\Rightarrow \quad \tan y = \frac{2x}{1-x^2} \qquad\qquad \Rightarrow \quad \tan^2 y = \frac{4x^2}{(1-x^2)^2}$

Differentiating with respect to x we have

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = \frac{2(1+x^2)}{(1-x^2)^2} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{2(1+x^2)}{(1-x^2)^2} * \frac{1}{\sec^2 y} = \frac{2(1+x^2)}{(1-x^2)^2} * \frac{1}{(1+\tan^2 y)}$$

since $\sec^2 y = 1+\tan^2 y$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1+x^2)}{(1-x^2)^2} \cdot \frac{1}{\left(1+\frac{4x^2}{(1-x^2)^2}\right)} = \frac{2(1+x^2)}{(1-x^2)^2} * \frac{(1-x^2)^2}{(1-x^2)^2+4x^2} = \frac{2(1+x^2)}{(1-x^2)^2+4x^2} = \frac{2(1+x^2)}{1+2x^2+x^4}$$

$$= \frac{2(1+x^2)}{(1+x^2)(1+x^2)} = \frac{2}{(1+x^2)}$$

Example3: Find the derivative of the following inverse trigonometric functions

i) $y = arcsin\left(\frac{x}{\sqrt{1-x^2}}\right)$, ii) $y = arc\cos(2x^2-1)$ iii) $y = arc\tan\left(\frac{x-1}{x}\right)$

Solution:

i) $y = arc\sin\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\Rightarrow \quad \sin y = \frac{x}{\sqrt{1-x^2}}$ $\Rightarrow \quad sin^2y = \frac{x^2}{1-x^2}$

Differentiating with respect to x we have

$$\Rightarrow \quad \cos y\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}\ \cos y} = -\frac{x}{\sqrt{1-x^2}\sqrt{1-sin^2y}} \qquad \text{since} \quad cosy = \sqrt{1-sin^2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}\sqrt{1-\frac{x^2}{1-x^2}}} = -\frac{x}{\sqrt{1-x^2}\sqrt{\frac{1-2x^2}{1-x^2}}} = -\frac{x}{\sqrt{(1-x^2)\frac{1-2x^2}{1-x^2}}} = -\frac{x}{\sqrt{1-2x^2}}$$

ii) $y = arc\cos(2x^2-1)$

$\Rightarrow \quad \cos y = 2x^2-1$ $\Rightarrow \quad cos^2y = (2x^2-1)^2$

Differentiating with respect to x we have

$$\Rightarrow \quad -\sin y\frac{dy}{dx} = 4x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4x}{\sin y} = -\frac{4x}{\sqrt{1-cos^2y}}, \qquad \text{since} \quad siny = \sqrt{1-cos^2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4x}{\sqrt{1-(2x^2-1)^2}} = -\frac{4x}{\sqrt{4x^2(x^2+1)}} = -\frac{4x}{2x\sqrt{x^2+1}} = -\frac{2}{\sqrt{x^2+1}}$$

iii) $y = arc\tan\left(\frac{x-1}{x}\right)$

$\Rightarrow \quad \tan y = \frac{x-1}{x}$ $\Rightarrow \quad tan^2y = \frac{(x-1)^2}{x^2}$

Differentiating with respect to x we have

$$\Rightarrow \quad sec^2y\frac{dy}{dx} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^2} * \frac{1}{sec^2y} = \frac{1}{x^2} * \frac{1}{(1+tan^2y)} \qquad \text{since } sec^2y = 1+tan^2y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^2} * \frac{1}{\left(1+\frac{(x-1)^2}{x^2}\right)} = \frac{1}{x^2} * \frac{x^2}{2x^2-2x+1} = \frac{1}{2x^2-2x+1}$$

Exercises:

1) Find the derivative of the following inverse trigonometric functions

i) $y = \frac{arcsin(4x)}{1-4x}$ ii) $y = arc\cos\left(\frac{1}{x+1}\right)^3$ iii) $y = \frac{1+arc\tan x}{\sqrt{1+x^2}}$

Integration as inverse of differentiation:

Introduction: If the derivative of a function is given, can we found the function having this derivative? Most of the time, the answer is yes. We know that the derivative of $3x$ is 3. If we wish to find the function whose derivative is 3, $3x$ is a correct answer. But we immediately realize that 3 is also the derivative of $3x + 2$, $3x + 5$, and in fact $3x + C$ where C is any arbitrary constant. It is clear then that if we wish to find the function whose derivative is 3, $3x + C$ is the most general answer, where C will be determine, if we are given further information.

The process of obtaining a function whose derivative is a given expression is called integration. Thus, integration is the reverse of differentiation.

Example

1. Find a function $F(x)$, such that its derivative is:

i) $2x$ ii) $5x^4$ iii) $3x^2$

Solution:

i) $F(x) = x^2 + C$, where C is a constant of integration

ii) $F(x) = x^5 + C$, where C is a constant of Integration

iii) $F(x) = x^3 + C$, were C is a constant of Integration

Primitive functions:

When we studied how to differentiate, you used symbols such as $\frac{dy}{dx}$ or $y'(x)$, to represent the derivative of y with respect to x. In integration, we shall use $\int y dx$ to indicate the integral of y with respect to x. Using this sign of integration, we observe that we have accepted that $\int 3dx = 3x + c$, that is the function whose derivative with respect to x equals 3 is $3x + c$ were c is any constant.

In general $\int f(x)dx = F(x) + c$ means that $F(x) + c$ is a function whose derivation with respect to x is $f(x)$. For any particular value of that c, c_1 for instance, $F(x) + c_1$ is called a primitive function for $f(x)$ and c is called constants of integration. The symbol is called the integral sign or sign of integration. $f(x)$ is called the integrand and the process of finding $F(x)$ is called the integration.

The symbols $\int f(x)dx$ is the integral of $f(x)$ with respect to x

Example: In each of the following write the integrand and primitive function for the integrand

i) $\int (3x^2 + 2)dx$ ii) $\int (4x + x^2)dx$ iii) $\int (10 + x)dx$

iv) $\int 4x^4 dx$ v) $\int \left(\frac{1}{2}x^{\frac{1}{2}}\right) dx$ vi) $\int 10x^{-2}dx$

Solution:

i) The integrand is $3x^2 + 2$ and the primitive function for the integral is $x^3 + 2x + c$

ii) The integrand is $4x + x^2$ and the primitive function for the integral is $2x^2 + \frac{x^3}{3} + c$

iii) The integrand is and the primitive function for the integral is

Note: From the above examples we observed two things; first, integration involves establishing the function whose derivative is given. Secondly, the results can be checked by differentiation.

Theorem: If two functions of x had the same derivative in an interval $a < x < b$, then their difference is a constant in that interval. Conversely, if the difference of two functions is constant, then they have the same derivative provided that the derivative exist.

Proof:

Suppose $f(x)$ and $g(x)$ are two functions having the same derivative in the internal $a \leq x \leq b$,

Then $\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x)]$ *or* $f'(x) = g'(x)$

Let $h(x) = f(x) - g(x)$

From differentiation we have

$$h'(x) = f'(x) - g'(x) \quad But \quad f'(x) = g'(x)$$

$\Rightarrow \quad h'(x) = f'(x) = g'(x) = 0$

$\Rightarrow \quad h'(x) = 0$

$\Rightarrow \quad h(x) = const.$

Since the derivative of $h(x)$ with respect to x is zero, it follows that $h(x)$ is a constant.

The Indefinite integrals:

We have mentioned that integration is the reverse of differentiation. In other words, given

$$\frac{dy}{dx} = f'(x) \qquad \ldots (1)$$

We are to find the function $F(x)$ such that when we differentiate $f(x)$, we obtain $f(x)$. If we can find such $F(x)$, then it is called an indefinite integral of $f(x)$. $F(x)$ is also called an anti-derivative of $f(x)$. because it is obtained by reversing the process of differentiation.

We observe that eqn.(1) can be used to obtain y.

$$\int \frac{dy}{dx} dx = \int f'(x)dx \quad \Rightarrow \quad y = \int f'(x)dx = F(x) + c \qquad \ldots (2)$$

Observe that in eqn.(2) we have added a constant of integration as there are many functions whose derivative equals $f(x)$. Again $F(x) + c$ is called indefinite integral of $f(x)$. The word indefinite is used because c can take any value.

Properties of indefinite integrals

Theorem1: Suppose $f(x)$ and $g(x)$ have anti-derivatives. Then for any constant a and b

$\int [af(x) + bg(x)]dx = a \int f(x)dx + b \int g(x)dx$

Proof:

We have that $\frac{d}{dx}\int f(x)dx = f(x)$ and $\frac{d}{dx}\int g(x)\,dx = g(x)$ it then follows that

$\frac{d}{dx}[a \int f(x)dx + b \int g(x)dx] = af(x) + bg(x)$

Note: *These results are usually applied when integrating a function by breaking the function up into the sum of parts whose integral can easily be obtained. By reversing any derivative formulae, we get corresponding integration formulae. The following table contains a number of important formulas. The proofs of these are left as straight forward, yet important exercise.*

A brief Table of Integration

S/No	$f(x)$	$\int f(x)dx$	S/No	$f(x)$	$\int f(x)dx$
1	$x^n \quad n \neq -1$	$\frac{x^{n+1}}{n+1} + c$	12	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
2	$\frac{1}{x}$	$\ln x + c$	13	$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x + c$
3	e^x	$e^x + c$	14	$\frac{1}{1+x^2}$	$\tan^{-1} x + c$
4	e^{-x}	$-e^{-x} + c$	15	$\ln x$	$x \ln x - x + c$
5	$\sin x$	$-\cos x + c$	16	$\sin^2 x$	$\frac{1}{2}x + \frac{1}{2}\sin x \cos x + c$
6	$\cos x$	$\sin x + c$	17	$\cos^2 x$	$\frac{1}{2}x + \frac{1}{2}\sin x \cos x + c$
7	$\tan x$	$-\ln\|\cos x\| + c$	18	$\tan^2 x$	$\tan x - x + c$
8	$\csc x$	$\ln\|\csc x - \cot x\| + c$	19	$\csc^2 x$	$-\cot x + c$
9	$\sec x$	$\ln\|\sec x + \tan x\| + c$	20	$\sec^2 x$	$\tan x + c$
10	$\cot x$	$-\ln\|\sin x\| + c$	21	$\cot^2 x$	
11	$\csc x \cot x$	$-\csc x + c$	22	$\csc x \cot x$	$\sec x + c$

Techniques of integration:

a) *Direct integration:*

Example1: Evaluate the following integrals

i) $\int x^5\, dx$ *ii)* $\int \frac{1}{x^3} dx$ *iii)* $\int \sqrt{x}\, dx$ *iv)* $\int \frac{1}{\sqrt[3]{x}} dx$

Solution;

i) $\int x^5\, dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c = \frac{1}{6}x^6 + c$

ii) $\int \frac{1}{x^3} dx = \int x^{-3}\, dx = \frac{x^{-3+1}}{-3+1} + c = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

iii) $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

iv) $\int \frac{1}{\sqrt[3]{x}} dx = \int x^{-1/3}\, dx = \frac{x^{-1/3+1}}{-1/3+1} + c = \frac{x^{2/3}}{2/3} + c = \frac{3}{2}x^{2/3} + c$

Example2: Evaluate the following integrals

i) $\int (3\cos x + 4x^5)\, dx$ *ii)* $\int (3e^x - 2\sec^2 x)\, dx$ *iii)* $\int (4x - 3e^x)\, dx$

Solution:

i) $\int (3\cos x + 4x^5)\, dx = \int 3\cos x\, dx + \int 4x^5 dx$

$= 3\int \cos x\, dx + 4\int x^5 dx$

$= (3\sin x + \frac{2}{3}x^6) + c$

ii) $\int (3e^x - 2\sec^2 x)\,dx = \int 3e^x dx - \int 2\sec^2 x\,dx$

$= 3\int e^x dx - 2\int \sec^2 x\,dx$

$= (3e^x - \tan x) + c$

iii) $\int (4x - 3e^x)\,dx = \int 4x dx - \int 3e^x dx$

$= 4\int x dx - 3\int e^x dx = (2x^2 + 3e^x) + c$

b) *Simple Substitution:*

Theorem2: If, $\int f(x)dx = F(x) + c$, then for any constant $a \neq 0$ $\int f(ax)dx = \frac{1}{a}F(ax) + c$

Proof:

Since, $\int f(x)dx = F(x) + c$, it follows that $F'(x) = f(x)$. By chain rule, we have that

$\frac{d}{dx}\left[\frac{1}{a}F(ax)\right] = \frac{1}{a}\frac{d}{dx}[F(ax)] = \frac{1}{a}aF'(ax) = F'(ax) = f(ax)$

Hence the proof:

Example3: Evaluate the following integrals

i) $\int \sin 3x\,dx$ ii) $\int 5e^{4x}\,dx$ iii) $\int 8\sec^2 5x\,dx$

Solution:

i) $\int \sin 3x\,dx$ Let $t = 3x$

$\Rightarrow \frac{dt}{dx} = 3 \Rightarrow dt = 3dx \Rightarrow \frac{1}{3}dt = dx$

$\Rightarrow \int \sin 3x\,dx = \int \sin t \frac{1}{3}dt = \frac{1}{3}\int \sin t\,dt$

$= \left(-\frac{1}{3}\cos t\right) + c = \left(-\frac{1}{3}\cos 3x\right) + c$

ii) $\int 5e^{4x}\,dx = 5\int e^{4x}\,dx$ Let $t = 4x$

$\Rightarrow \frac{dt}{dx} = 4 \Rightarrow dt = 4dx \Rightarrow \frac{1}{4}dt = dx$

$\Rightarrow 5\int e^{4x}\,dx = 5\int e^t \frac{1}{4}dt = \frac{5}{4}\int e^t dt = \frac{5}{4}[e^t] + c = \frac{5}{4}[e^{4x}] + c$

iii) $\int 8\sec^2 5x\,dx = 8\int \sec^2 5x\,dx$ Let $t = 5x$

$\Rightarrow \frac{dt}{dx} = 5 \Rightarrow dt = 5dx \Rightarrow \frac{1}{5}dt = dx$

$\Rightarrow 8\int \sec^2 5x\,dx = 8\int \sec^2 t \frac{1}{5}dt = \frac{8}{5}\int \sec^2 dt$

$= \frac{8}{5}[\tan t] + c = \frac{8}{5}[\tan 5x] + c$

Example4: Evaluate the following integrals

i) $\int \frac{4x}{x^2+4}\,dx$ ii) $\int \left(2\cos x - \sqrt{e^x}\right)dx$ iii) $\int 5\sin 2x\,dx$

Solution:

i) $\int \frac{4x}{x^2+4}dx = 4\int \frac{x}{x^2+4}dx$ Let $t = x^2+4$

$\Rightarrow \frac{dt}{dx} = 2x \Rightarrow dt = 2xdx \Rightarrow \frac{1}{2}dt = xdx$

$\Rightarrow \quad 4\int \frac{x}{x^2+4}dx = 4\int \frac{1}{t}\cdot\frac{1}{2}dt = 2\int \frac{1}{t}dt = 2\ln|t| + c = 2\ln|x^2+4| + c$

ii) $\int\left(2\cos x - \sqrt{e^x}\right)dx = \int 2\cos x\,dx - \int \sqrt{e^x}\,dx = 2\int \cos x\,dx - \int e^{x/2}\,dx$ . . . (1)

$2\int \cos x\,dx = 2\sin x$

$\int e^{x/2}\,dx$ Let $t = \frac{x}{2}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2} \Rightarrow 2dt = dx$ . . . (2)

$\Rightarrow \quad \int e^{x/2}\,dx = \int e^t\,2dt = 2\int e^t\,dt = 2e^t = 2e^{x/2}$

From eqn. (1) & (2) we have

$2\int \cos x\,dx - \int e^{x/2}\,dx = 2\left(\sin x + e^{x/2}\right) + c$

iii) $\int 5\sin 2x\,dx = 5\int \sin 2x\,dx$ Let $t = 2x$

$\Rightarrow \frac{dt}{dx} = 2 \Rightarrow dt = 2dx \Rightarrow \frac{1}{2}dt = dx$

$\Rightarrow \quad 5\int \sin 2x\,dx = 5\int \sin t\frac{1}{2}dt = \frac{5}{2}\int \sin t\,dt$

$= -\frac{5}{2}\cos t = \left(-\frac{5}{2}\cos 2x\right) + c$

Example5: Evaluate the following integrals

i) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ ii) $\int \frac{x+1}{(x^2+2x-1)^2}dx$ iii) $\int \frac{\sin x}{\sqrt{\cos x}}dx$ vi) $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx$

Solution:

i) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ Let $t = \sqrt{x}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2dt = \frac{dx}{\sqrt{x}}$

$\Rightarrow \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^t\,2dt$

$= 2\int e^t\,dt = 2e^t + c$

$= 2\,e^{\sqrt{x}} + c$

ii) $\int \frac{x+1}{(x^2+2x-1)^2}dx$ Let $t = x^2+2x-1$

$\Rightarrow \frac{dt}{dx} = 2x+2$

$\Rightarrow \frac{1}{2}dt = (x+1)dx$

$\Rightarrow \int \frac{x+1}{(x^2+2x-1)^2}dx = \frac{1}{2}\int \frac{1}{t^2}2dt = \frac{1}{2}\int t^{-2}\,dt$

$= \frac{1}{2}\left(\frac{t^{-1}}{-1}\right) + c = -\frac{1}{2}(x^2+2x-1)^{-1} + c$

iii) $\int \frac{\sin x}{\sqrt{\cos x}}dx$ Let $t = \cos x$

$\Rightarrow \frac{dt}{dx} = -\sin x$

$\Rightarrow -dt = \sin x\, dx$

$\Rightarrow \int \frac{\sin x}{\sqrt{\cos x}}dx = \int \frac{1}{\sqrt{t}}(-dt)$

$= -\int t^{-\frac{1}{2}}dt = -2\sqrt{t} + c$

$= -2\sqrt{\cos x} + c$

vi) $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx$ Let $t = \sqrt{x} + 2$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$

$\Rightarrow 2dt = \frac{dx}{\sqrt{x}}$

$\Rightarrow \int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx = \int t^3\, 2dt$

$= 2\int t^3\, dt$

$= 2\left(\frac{t^4}{4}\right) + c = \frac{1}{2}(\sqrt{x} + 2)^4 + c$

Example6: Evaluate the following integrals

i) $\int x\,(x^2 - 3)^4 dx$ ii) $\int x^3\sqrt{x^4 + 3}\, dx$ iii) $\int \cos x\sqrt{\sin x + 1}dx$ vi) $\int 2x\, e^{x^2}dx$

Solution:

i) $\int x\,(x^2 - 3)^4 dx$ Let $t = x^2 - 3$

$\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow \frac{1}{2}dt = xdx$

$\Rightarrow \int x\,(x^2 - 3)^4 dx = \frac{1}{2}\int t^4\, dt$

$= \frac{1}{2}\left(\frac{t^5}{5}\right) + c$

$= \frac{1}{10}(x^2 - 3)^5 + c$

ii) $\int x^3\sqrt{x^4 + 3}\, dx$ Let $t = x^3 + 3$

$\Rightarrow \frac{dt}{dx} = 3x^2$

$\Rightarrow \frac{1}{3}dt = x^2 dx$

$\Rightarrow \int x^3\sqrt{x^4 + 3}\, dx = \frac{1}{2}\int \sqrt{t}\, dt$

$= \frac{1}{2}\int t^{\frac{1}{2}}\, dt$

$= \frac{1}{3}(x^3 + 3)^{\frac{3}{2}} + c$

iii) $\int \cos x\sqrt{\sin x + 1}\, dx$ Let $t = \sin x + 1$

$\Rightarrow \frac{dt}{dx} = \cos x$

$\Rightarrow dt = \cos x\, dx$

$\Rightarrow \int \cos x\sqrt{\sin x + 1}\, dx = \int \sqrt{t}\, dt$

$= \int t^{\frac{1}{2}}\, dt = \frac{2}{3}t^{\frac{3}{2}} + c$

$= \frac{2}{3}(\sin x + 1)^{\frac{3}{2}} + c$

vi) $\int 2x\, e^{x^2}dx$ Let $t = x^2$

$\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow dt = 2xdx$

$\Rightarrow \int 2x\, e^{x^2}dx = \int e^t\, dt$

$= e^t + c = e^{x^2} + c$

Exercise: Evaluate the following integrals

i) $\int \frac{e^x}{e^x+3}dx$ ii) $\int \frac{e^x+3}{e^x}dx$ iii) $\int 3\, sec^2 3xdx$ iv) $\int x^3\sqrt{x^4 + 3}\, dx$

v) $\int (2x + 1)(x^2 + x)^3\, dx$ vi) $\int \cos x\, e^{\sin x}dx$ vii) $\int \tan x\; dx$ ix) $\int x\cos x^2\, dx$

viii) $\int (3\sin x + 4)^5 \cos x\, dx$ xi) $\int \frac{4x+8}{[illegible]}dx$ xii) $\int x^2(x^3 - 2)^{14}\, dx$

c) *Integration by parts:*

Theorem2: if $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Proof:

Recall: The product rule of differentiation

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \quad \ldots (1)$$

If we integrate both sides of eqn. (1) we have :

$$\int \left[\frac{d}{dx}[f(x)g(x)]\right] dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

Of course, the integral on the left hand side is reduced to simply $f(x)g(x)$

$$\Rightarrow \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \quad \ldots (2)$$

Note: It is usually convenient to write eqn.(2) using the notation $u = f(x)$ *and* $v = g(x)$, *so that*

$du = f'(x)dx$ *and* $dv = g'(x)dx$

And so equation (2) reduces to

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du \quad \ldots (3)$$

Example: Evaluate the following integrals

i) $\int x \sin x\,dx$ *ii)* $\int \ln x\,dx$ *iii)* $\int e^x \sin 4x\,dx$

Solution:

i) $\int x\sin x\,dx$ Let $u = x$ and $dv = \sin x\,dx$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow \int dv = \int \sin x\,dx$

$\Rightarrow du = dx,$ $\Rightarrow v = -\cos x$

$$\Rightarrow \quad \int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$$
$$= -x\cos x + \int \cos x\,dx.$$
$$= -x\cos x + \sin x = (\sin x - x\cos x) + c$$

ii) $\int \ln x\,dx$ Let $u = \ln x$ and $dv = dx$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow \int dv = \int dx$

$\Rightarrow du = \frac{1}{x}dx,$ $\Rightarrow v = x$

$$\Rightarrow \quad \int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx$$
$$= x\ln x - \int dx = x\ln x - x = x(\ln x - 1) + c$$

Scanned by CamScanner

iii) $\int e^x \sin 4x\,dx$ Let $u = \sin 4x$ and $dv = e^x dx$

$\Rightarrow \frac{du}{dx} = 4\cos 4x$ $\Rightarrow \int dv = \int e^x dx$

$\Rightarrow du = 4\cos 4x\,dx,$ $\Rightarrow v = e^x$

$\Rightarrow \int e^x \sin 4x\,dx = e^x \sin 4x - 4\int e^x \cos 4x\,dx$

Again Let $u = \cos 4x$ and $dv = e^x dx$

$\Rightarrow \frac{du}{dx} = -4\sin 4x$ $\Rightarrow \int dv = \int e^x dx$

$\Rightarrow du = -4\sin 4x\,dx,$ $\Rightarrow v = e^x$

$\Rightarrow \int e^x \sin 4x\,dx = e^x \sin 4x - 4[e^x \cos 4x - \int -4e^x \sin 4x\,dx]$

$\Rightarrow \int e^x \sin 4x\,dx = e^x \sin 4x - 4[e^x \cos 4x + 4\int e^x \sin 4x\,dx]$

$\Rightarrow \int e^x \sin 4x\,dx = e^x \sin 4x - 4e^x \cos 4x - 16\int e^x \sin 4x\,dx$

$\Rightarrow 17\int e^x \sin 4x\,dx = (\sin 4x - 4\cos 4x)e^x$

$\Rightarrow \int e^x \sin 4x\,dx = \frac{1}{17}(\sin 4x - 4\cos 4x)e^x + c$

Note: *When using integration by parts, keep in mind that you are splitting up the integrand into two pieces. One of these pieces, corresponding to u, will be differentiated and the other, corresponding to dv, will be integrated. Since we can differentiate virtually every function we run across, we should think in terms of a dv for which we know an anti-derivative, as well as a choice of both that will result in an easier integral. Unfortunately, it's not always so easy to see the problem through from beginning to end. You will learn what works best by working through lots of problems. Even if we don't see how the problem is going to end up, try something (at least you'll learn what doesn't work).*

Example5: Evaluate the following integrals

i) $\int \frac{\ln x}{x}dx$ ii) $\int x^2 \sin x\,dx$ iii) $\int e^{2x} \sin x\,dx$

Solution:

i) $\int \frac{\ln x}{x}dx$ Let $u = \ln x$ and $dv = \frac{1}{x}dx$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow \int dv = \int \frac{1}{x}dx$

$\Rightarrow du = \frac{1}{x}dx,$ $\Rightarrow v = \ln x$

$\Rightarrow \int \frac{\ln x}{x}dx = ln^2 x - \int \frac{\ln x}{x}dx$

$\Rightarrow 2\int \frac{\ln x}{x}dx = ln^2 x$

$\Rightarrow \int \frac{\ln x}{x}dx = \frac{1}{2}(ln^2 x) + c$

$ii)\quad \int x^2 \sin x\, dx$ Let $u = x^2$ $dv = \sin x\, dx$

$\Rightarrow \frac{du}{dx} = 2x$ $\Rightarrow \int dv = \int \sin x\, dx$

$\Rightarrow du = 2x dx,$ $\Rightarrow v = -\cos x$

$\Rightarrow \quad \int x^2 \sin x\, dx = -x^2 \cos x - \int -\cos x * 2x\, dx$

$\Rightarrow \quad \int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$

Again let $u = x$ and $dv = \cos x\, dx$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow \int dv = \int \cos x\, dx$

$\Rightarrow du = dx,$ $\Rightarrow v = \sin x$

$\Rightarrow \quad \int x^2 \sin x\, dx = -x^2 \cos x + 2[x \sin x - \int \sin x\, dx]$

$\Rightarrow \quad \int x^2 \sin x\, dx = (-x^2 \cos x + 2x \sin x + 2\cos x) + c$

$iii)\quad \int e^{2x} \sin x\, dx$ Let $u = e^{2x}$ and $dv = \sin x\, dx$

$\Rightarrow \frac{du}{dx} = 2e^{2x}$ $\Rightarrow \int dv = \int \sin x\, dx$

$\Rightarrow du = 2e^{2x} dx,$ $\Rightarrow v = -\cos x$

$\Rightarrow \quad \int e^{2x} \sin x\, dx = -e^{2x} \cos x - \int -2e^{2x} \cos x\, dx = -e^{2x} \cos x + 2\int e^{2x} \cos x\, dx$

$\Rightarrow \quad \int e^{2x} \sin x\, dx = -e^{2x} \cos x + 2\int e^{2x} \cos x\, dx$

Again let $u = e^{2x}$ and $dv = \cos x\, dx$

$\Rightarrow \frac{du}{dx} = 2e^{2x}$ $\Rightarrow \int dv = \int \cos x\, dx$

$\Rightarrow du = 2e^{2x} dx$, and $\Rightarrow v = \sin x$

$\Rightarrow \quad \int e^{2x} \sin x\, dx = -e^{2x} \cos x + 2[e^{2x} \sin x - \int 2e^{2x} \sin x\, dx]$

$\Rightarrow \quad \int e^{2x} \sin x\, dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x\, dx$

$\Rightarrow \quad 5\int e^{2x} \sin x\, dx = (-\cos x + 2\sin x)e^{2x}$

$\Rightarrow \quad \int e^{2x} \sin x\, dx = \frac{1}{5}(2\sin x - \cos x)e^{2x} + c$

Exercises:

1.Evaluate the following integrals

$i)\ \int x \ln x\, dx$	$ii)\ \int x^4 e^x dx$	$iii)\ \int e^{2x} \cos x\, dx$
$iv)\ \int x^2 e^{x^3} dx$	$v)\ \int e^x \sin 4x\, dx$	$vi)\ \int x^2 \sin 3x\, dx$
$vii)\ \int \cos x \cos 2x\, dx$	$viii)\ \int \cos x \ln(\sin x)\, dx$	$ix)\ \int \ln^2 x\, dx$
$x)\ \int x^2 \ln x\, dx$	$xi)\ \int x^2 e^{-3x} dx$	$xii)\ \int x^4 e^x dx$

d) Trigonometric Substitution:

If an integral contains a term of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$, for some $a>0$, we can often evaluate the integral by making a substitution involving a trigonometric function (hence, the name trigonometric substitution)

→ First, suppose that an integrand contains a term of the form $\sqrt{a^2-x^2}$, for some $a>0$,

If we let $x = a\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then we can eliminate the square root as follows.

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta$$

Since for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$

Note: We should always first consider whether an integral can be done directly with a simple substitution or by parts. If none of these methods are helpful, we consider trigonometric substitution. Always keep in mind that the immediate objective here is to eliminate the square root term, so we make a substitution that will accomplish this.

Example1: Evaluate the following integrals

i) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ ii) $\int \frac{1}{x^2\sqrt{9-x^2}}dx$

Solution:

i) $\int \frac{1}{x^2\sqrt{4-x^2}}dx = \int \frac{1}{x^2\sqrt{2^2-x^2}}dx \quad \Rightarrow \quad a=2$

Let $x = 2\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \frac{dx}{d\theta} = 2\cos\theta$

$$\Rightarrow \int \frac{1}{x^2\sqrt{4-x^2}}dx = \int \frac{2\cos\theta}{4\sin^2\theta\sqrt{4-(2\sin\theta)^2}}d\theta = \int \frac{\cos\theta}{2\sin^2\theta\sqrt{4-4\sin^2\theta}}d\theta = \int \frac{\cos\theta}{4\sin^2\theta\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \frac{\cos\theta}{4\sin^2\theta\sqrt{\cos^2\theta}}d\theta = \int \frac{\cos\theta}{4\sin^2\theta\cos\theta}d\theta = \frac{1}{4}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + c$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{4-x^2}}dx = -\frac{1}{4}\cot\theta + c = -\frac{1}{4}\left(\frac{\cos\theta}{\sin\theta}\right) + c \quad \ldots (1)$$

The next thing is to write the anti-derivative back to the original variable x. Since $2\sin\theta = x$

$\Rightarrow \sin\theta = \frac{x}{2} \quad \Rightarrow \quad \sin^2\theta = \left(\frac{x}{2}\right)^2 \quad \Rightarrow \quad \cos^2\theta = 1-\left(\frac{x}{2}\right)^2 = \frac{4-x^2}{4} \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{4-x^2}}{2}$

Substituting for $\sin\theta$ and $\cos\theta$ in eqn. (1) we have

$$\Rightarrow \int \frac{1}{x^2\sqrt{4-x^2}}dx = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right) + c$$

ii) $\int \frac{1}{x^2\sqrt{9-x^2}}dx = \int \frac{1}{x^2\sqrt{3^2-x^2}}dx \quad \Rightarrow \quad a=3$

Let $x = 3\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \frac{dx}{d\theta} = 3\cos\theta$

$$\Rightarrow \int \frac{1}{x^2\sqrt{9-x^2}}dx = \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-(3\sin\theta)^2}}d\theta = \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}d\theta = \int \frac{3\cos\theta}{9\sin^2\theta 3\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \frac{\cos\theta}{9\sin^2\theta\sqrt{\cos^2\theta}}d\theta = \int \frac{\cos\theta}{9\sin^2\theta\cos\theta}d\theta = \int \frac{1}{9\sin^2\theta}d\theta = \frac{1}{9}\int \csc^2\theta\, d\theta = -\frac{1}{9}\cot\theta + c$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{9-x^2}}dx = -\frac{1}{9}\cot\theta + c = -\frac{1}{9}\left(\frac{\cos\theta}{\sin\theta}\right) + c \quad \ldots (2)$$

$$\Rightarrow \int \frac{1}{\sqrt{4+x^2}}dx = \int \frac{2\tan\theta}{\sqrt{4+4tan^2\theta}} 2sec^2\theta\, d\theta = \int \frac{4\tan\theta sec^2\theta}{2\sqrt{1+tan^2\theta}} d\theta = 2\int \frac{\tan\theta sec^2\theta}{\sqrt{sec^2\theta}} d\theta = 2\int \frac{\tan\theta sec^2\theta}{\sec\theta} d\theta$$

$$\Rightarrow \int \frac{1}{\sqrt{4+x^2}}dx = 2\int \tan\theta \sec\theta\, d\theta = 2\sec\theta + c \qquad \ldots (3)$$

The next thing is to write the anti-derivative back to the original variable x.Since $2\,\text{tnn}\,\theta = x$

$$\Rightarrow \tan\theta = \frac{x}{2} \quad \Rightarrow \quad tan^2\theta = \frac{x^2}{4} \quad \Rightarrow \quad sec^2\theta = 1 + \frac{x^2}{4} = \frac{\sqrt{4-x^2}}{2} \quad \Rightarrow \quad \sec\theta = \frac{\sqrt{4+x^2}}{2}$$

Substituting for $\sec\theta$ and $\tan\theta$ in eqn. (3) we have

$$\Rightarrow \int \frac{1}{\sqrt{4+x^2}}dx = \frac{\sqrt{4+x^2}}{2} + c$$

Exercise: Evaluate the following integrals

i) $\int \sqrt{x^2+16}\,dx$ ii) $\int \frac{x^2dx}{\sqrt{1+4x^2}}$ iii) $\int \frac{x^2dx}{\sqrt{1+16x^2}}$

→ Finally, suppose that an integrand contains a term of the form $\sqrt{x^2-a^2}$, for some $a > 0$, if we let $x = a\sec\theta$ where $\theta\ \varepsilon\ [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$, then we can eliminate the square root as follows.

$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2-a^2} = \sqrt{a^2sec^2\theta - a^2} = a\sqrt{sec^2\theta-1} = a\sqrt{tan^2\theta} = a|\tan\theta|$$

Notice that the absolute value is needed, as $\tan\theta$ can be both positive and negative on $[0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$.

Example1: Evaluate the following integrals

i) $\int \frac{\sqrt{x^2-25}}{x}dx,\ \ for\ x > 5$ ii) $\int \frac{2dx}{\sqrt{x^2-4}}$

Solution:

i) $\int \frac{\sqrt{x^2-25}}{x}dx,\ \ for\ x > 5$

Let $x = 5\sec\theta$, for $\theta\ \varepsilon\ \left[0,\frac{\pi}{2}\right)$. Notice that we choose the first half of the domain $[0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$, so that we would have $x = 5\sec\theta > 5$. (if we had $x < -5$, we would have chosen $\theta\ \varepsilon\ (\frac{\pi}{2},\pi]$).

Let $x = 5\sec\theta$, $x^2 = 25sec^2\theta$.

$$\Rightarrow \frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \Rightarrow \quad dx = 5\sec\theta\tan\theta\, d\theta$$

$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{25sec^2\theta-25}}{5\sec\theta} 5\sec\theta\tan\theta\, d\theta = \int \frac{\sqrt{25sec^2\theta-25}}{1}\tan\theta\, d\theta$$

$$= \int \sqrt{25(sec^2\theta-1)}\tan\theta\, d\theta = \int \sqrt{25tan^2\theta}\tan\theta\, d\theta$$

$$= 5\int tan^2\theta\, d\theta = 5\int (sec^2\theta - 1)\, d\theta = 5(\tan\theta - \theta) + c$$

$$\int \left(\frac{\sqrt{x^2-25}}{x}\right)dx = (\tan\theta - \theta) + c \qquad \ldots (4)$$

Since $x = 5\sec\theta$, for $\theta\ \varepsilon\ \left[0,\frac{\pi}{2}\right)$, $\Rightarrow \sec\theta = \frac{x}{5}$, $\Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$

$$\Rightarrow sec^2\theta = \frac{x^2}{25} \quad \Rightarrow \quad tan^2\theta = \frac{x^2}{25} - 1 = \frac{x^2-25}{25} \quad \Rightarrow \quad \tan\theta = \frac{\sqrt{x^2-25}}{5}$$

Substituting for $\tan\theta$ in eqn. (4) we have $\int \frac{\sqrt{x^2-25}}{x}dx = 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right) + c$

The next thing is to write the anti-derivative back to the original variable x

Since $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3} \Rightarrow \sin^2\theta = \left(\frac{x}{3}\right)^2 \Rightarrow \cos^2\theta = 1 - \left(\frac{x}{3}\right)^2 = \frac{9-x^2}{9}$

$\Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$

Substituting for $\sin\theta$ and $\cos\theta$ in eqn. (2) we have

$$\Rightarrow \int \frac{1}{x^2\sqrt{9-x^2}}\,dx = -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + c$$

Exercise: Evaluate the following integrals

i) $\int \frac{\sqrt{16-x^2}}{x^2}\,dx$ ii) $\int \frac{dx}{x^2\sqrt{16-x^2}}$ iii) $\int \frac{x\,dx}{\sqrt{4-x^2}}$

→ Next suppose that an integrand contains a term of the form $\sqrt{a^2+x^2}$, for some $a > 0$, if we let $x = a\tan\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then we can eliminate the square root as follows.

$\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta$

Since for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\sec\theta \ge 0$

Example1: Evaluate the following integrals

i) $\int \frac{1}{\sqrt{9+x^2}}\,dx$ ii) $\int \frac{x\,dx}{\sqrt{4+x^2}}$

Solution:

i) $\int \frac{1}{\sqrt{9+x^2}}\,dx = \int \frac{1}{\sqrt{3^2+x^2}}\,dx \quad \Rightarrow a = 3$

Let $x = 3\tan\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Rightarrow \frac{dx}{d\theta} = 3\sec^2\theta \Rightarrow dx = 3\sec^2\theta\,d\theta$

$\Rightarrow \int \frac{1}{\sqrt{9+x^2}}\,dx = \int \frac{3\sec^2\theta}{\sqrt{9+(3\tan\theta)^2}}\,d\theta = \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta = \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,d\theta$

$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + c$

$\Rightarrow \int \frac{1}{\sqrt{9+x^2}}\,dx = \ln|\sec\theta + \tan\theta| + c$ . . . (3)

The next thing is to write the anti-derivative back to the original variable x

Since $x = 3\,\text{tnn}\,\theta \Rightarrow \tan\theta = \frac{x}{3} \Rightarrow \tan^2\theta = \frac{x^2}{9} \Rightarrow \sec^2\theta = 1 + \frac{x^2}{9}$

$\Rightarrow \sec\theta = \frac{\sqrt{9+x^2}}{3}$

Substituting for $\sec\theta$ and $\tan\theta$ in eqn. (3) we have

$$\Rightarrow \int \frac{1}{\sqrt{9+x^2}}\,dx = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c = \ln\left|\frac{\sqrt{9+x^2}+x}{3}\right| + c$$

ii) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \int \frac{1}{\sqrt{2^2+x^2}}\,dx \quad \Rightarrow a = 2$

Let $x = 2\tan\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Rightarrow x^2 = 4\tan^2\theta$

$\Rightarrow \frac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

ii) $\int \frac{2dx}{\sqrt{x^2-4}}$

Let $x = 2\sec\theta,\ x^2 = 4sec^2\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = 2\sec\theta\tan\theta \quad \Rightarrow \quad dx = 2\sec\theta\tan\theta\, d\theta$

$\int \frac{2dx}{\sqrt{x^2-4}} = \int \frac{4\sec\theta\tan\theta d\theta}{\sqrt{4sec^2\theta-4}} = \int \frac{4\sec\theta\tan\theta d\theta}{2\sqrt{sec^2\theta-1}} = 2\int \frac{\sec\theta\tan\theta d\theta}{\sqrt{tan^2\theta}} = 2\int \frac{\sec\theta\tan\theta d\theta}{\tan\theta}$

$= \ 2\int \sec\theta\, d\theta = \ln|\sec x + \tan x| + c$

$\Rightarrow \quad \int \frac{2dx}{\sqrt{x^2-4}} = \ln|\sec x + \tan x| + c$

$x = 2\sec\theta$, for $\theta\ \varepsilon\ \left[0, \frac{\pi}{2}\right)$, $\Rightarrow\ \sec\theta = \frac{x}{2}$, $\quad \Rightarrow\ sec^2\theta = \frac{x^2}{4}$, $\quad \Rightarrow\ tan^2\theta = \frac{x^2}{4} - 1 = \frac{x^2-4}{4}$,

$\Rightarrow \quad \tan\theta = \frac{\sqrt{x^2-4}}{2}$,

$\Rightarrow \quad \int \frac{2dx}{\sqrt{x^2-4}} = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + c = \ln|$[illegible]$| + c$

Exercise: Evaluate the following integrals

i) $\int \quad dx$ $\qquad$ ii) $\int \quad dx$ $\qquad$ iii) $\int \quad dx$

d) *Integration of rational functions(Partial fraction decomposition Technique):*

Recall: Partial fraction decomposition

1. G_1 *When the degree of the dividend is higher than or equal to the degree of divisor*

E xample1: Evaluate the following

i) $\int \frac{x^3+x+2}{x^2+2x-8} dx$ $\qquad$ ii) $\int \frac{x^2+1}{x^2-5x-6} dx$

Solution:

Here we employ the process of long division as follows:

i) $\int \frac{x^3+x+2}{x^2+2x-8} dx$

$$\begin{array}{r l}
 & x - 2 \\
(x^2 + 2x - 8) & \overline{\big)\, x^3 + 0x^2 + \ x - 8} \\
 & \underline{x^3 + 2x^2 - 8x} \\
 & \quad - 2x^2 + 9x + 2 \\
 & \quad \underline{- 2x^2 - 4x + 16} \\
 & \qquad 13x - 14
\end{array}$$

$\Rightarrow \quad \frac{x^3+x+2}{x^2+2x-8} = (x - 2) + \frac{13x-14}{x^2+2x-8}$

Therefore

$\frac{x^3+x+2}{x^2+2x-8} = (x - 2) + \frac{13x-14}{(x+4)(x-2)}$, but $\frac{13x-14}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2} = \frac{11}{x+4} + \frac{2}{x-2}$

$\Rightarrow \quad \frac{x^3+x+2}{x^2+2x-8} = (x - 2) + \frac{11}{x+4} + \frac{2}{x-2}$

$$\int \frac{x^3+x+2}{x^2+2x-8}\,dx = \int (x-2)\,dx + \int \left(\frac{11}{x+4} + \frac{2}{x-2}\right) dx$$

$$= \int (x-2)\,dx + \int \frac{11}{(x+4)}\,dx + \int \frac{2}{(x-2)}\,dx$$

$$= \left[\frac{x^2}{2} - 2x + 11\ln|x+4| + 2\ln|x-2|\right] + c$$

ii) $\int \frac{3x^2-6}{x^2-x-2}\,dx$

$$\begin{array}{r} 3 \\ (x^2-x-2)\overline{\smash{)}\,3x^2+0x-6} \\ \underline{3x^2-3x-6} \\ 3x \end{array}$$

⇨ $\frac{3x^2-6}{x^2-x-2} = 3 + \frac{3x}{x^2-x-2}$.

Therefore

$\frac{3x^2-6}{x^2-x-2} = 3 + \frac{3x}{(x+1)(x-2)}$, but $\frac{3x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{1}{x+1} + \frac{2}{x-2}$

⇨ $\frac{3x^2-6}{x^2-x-2} = 3 + \frac{1}{x+1} + \frac{2}{x-2}$

$\int \frac{3x^2-6}{x^2-x-2} = \int 3dx + \int \frac{1}{(x+1)}dx + \int \frac{2}{(x-2)}dx = [3x + \ln|x+1| + 2\ln|x-2|] + c$

Exercises: Evaluate the following

i) $\int \frac{2x^2+2x-1}{x^2+x-2}\,dx$ ii) $\int \frac{x^3+x^2+4x}{x^2+x-2}\,dx$

G_2 *When the divisor can be factored in linear factors all different*

Example1: Evaluate the following

i) $\int \frac{x-19}{x^2-3x-10}\,dx$ ii) $\int \frac{6x}{(x+2)(x-1)}\,dx$ iii) $\int \frac{3x^2-7x-2}{x^3-x}\,dx$

Solution:

i) Decomposing the function into partial fraction we have

$\frac{x-19}{x^2-3x-10} = \frac{x-19}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5} = \frac{3}{x+2} + \frac{2}{x-5}$

⇨ $\frac{x-19}{x^2-3x-10} = \frac{3}{x+2} + \frac{2}{x-5}$

⇨ $\int \frac{x-19}{x^2-3x-10}\,dx = \int \left(\frac{3}{x+2} + \frac{2}{x-5}\right) dx = 3\int \frac{1}{x+2}dx + 2\int \frac{1}{x-5}dx = 3\ln|x+2| + 2\ln|x-5| + c$

ii) Decomposing the function into partial fraction we have

$\frac{6x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{4}{x+2} + \frac{2}{x-1}$

⇨ $\frac{6x}{(x+2)(x-1)} = \frac{4}{x+2} + \frac{2}{x-1}$

⇨ $\int \frac{6x}{(x+2)(x-1)}\,dx = \int \left(\frac{4}{x+2} + \frac{2}{x-1}\right) dx = \int \frac{4}{x+2}dx + \int \frac{2}{x-1}dx = 4\ln|x+2| + 2\ln|x-1| + c$

iii) Decomposing the function into partial fraction we have

$$\frac{3x^2-7x-2}{x^3-x} = \frac{3x^2-7x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}$$

$$\Rightarrow \quad \frac{3x^2-7x-2}{x^3-x} = \frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}$$

$$\Rightarrow \quad \int \frac{3x^2-7x-2}{x^3-x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}\right)dx = 2\int \frac{1}{x}dx - 3\int \frac{1}{x-1}dx + 4\int \frac{1}{x+1}dx$$

$$= 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + c$$

G_3 When the divisor can be factored in linear factors with some repeating

Example1: Evaluate the following

i) $\int \frac{5x^2+20x+6}{x(x+1)^2}dx$ ii) $\int \frac{x-1}{x^3+4x^2+4x}dx$ iii) $\int \frac{2x}{x^2-6x+9}dx$

Solution:

i) Decomposing the function into partial fraction we have

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\Rightarrow \quad \frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\Rightarrow \quad \int \frac{5x^2+20x+6}{x(x+1)^2}dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right)dx = \int \frac{6}{x}dx - \int \frac{1}{x+1}dx + \int \frac{9}{(x+1)^2}dx$$

$$= 6\int \frac{1}{x}dx - \int \frac{1}{x+1}dx + 9\int \frac{1}{(x+1)^2}dx$$

$$= 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + c$$

ii) Decomposing the function into partial fraction we have

$$\frac{x-1}{x^3+4x^2+4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} = -\frac{1}{4x} + \frac{1}{4(x+2)} + \frac{3}{2(x+2)^2}$$

$$\Rightarrow \quad \frac{x-1}{x^3+4x^2+4x} = -\frac{1}{4x} + \frac{1}{4(x+2)} + \frac{3}{2(x+2)^2}$$

$$\Rightarrow \quad \int \frac{x-1}{x^3+4x^2+4x}dx = \int \left(-\frac{1}{4x} + \frac{1}{4(x+2)} + \frac{3}{2(x+2)^2}\right)dx$$

$$= -\frac{1}{4}\int \frac{1}{x}dx + \frac{1}{4}\int \frac{1}{4(x+2)}dx + \frac{3}{2}\int \frac{1}{(x+2)^2}dx$$

$$= -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2| - \frac{3}{2(x+2)} + c$$

iii) Decomposing the function into partial fraction we have

$$\frac{2x}{x^2-6x+9} = \frac{2x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{2}{x-3} + \frac{6}{(x-3)^2}$$

$$\Rightarrow \quad \frac{2x}{x^2-6x+9} = \frac{2}{x-3} + \frac{6}{(x-3)^2}$$

$$\Rightarrow \quad \int \frac{2x}{x^2-6x+9}dx = \int \left(\frac{2}{(x-3)} + \frac{6}{(x-3)^2}\right)dx = 2\int \frac{1}{(x-3)}dx + 6\int \frac{1}{(x-3)^2}dx$$

$$= 2\ln|x-3| - \frac{6}{(x-3)} + c$$

G₄ When the divisor can be factored in linear and non-reducible quadratic factors all different

Example1: Evaluate the following

i) $\int \frac{1}{(x+1)(x^2+1)}dx$ ii) $\int \frac{x-1}{(x^2+2)(x+1)}dx$

Solution:

i) Decomposing the function into partial fraction we have

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)} = \frac{1}{2(x+1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}$$

$$\Rightarrow \quad \frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}$$

$$\Rightarrow \quad \int \frac{1}{(x+1)(x^2+1)}dx = \int\left(\frac{1}{2(x+1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}\right)dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{x}{(x^2+1)}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + c$$

ii) Decomposing the function into partial fraction we have

$$\frac{x-1}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} = \frac{2x-1}{3(x^2+1)} - \frac{2}{3(x+1)} = \frac{2x}{3(x^2+2)} - \frac{1}{3(x^2+2)} - \frac{2}{3(x+1)}$$

$$\Rightarrow \quad \frac{x-1}{(x^2+2)(x+1)} = \frac{2x}{3(x^2+2)} - \frac{1}{3(x^2+2)} - \frac{2}{3(x+1)}$$

$$\Rightarrow \quad \int \frac{x-1}{(x^2+2)(x+1)}dx = \int\left(\frac{2x}{3(x^2+2)} - \frac{1}{3(x^2+2)} - \frac{2}{3(x+1)}\right)dx$$

$$= \frac{2}{3}\int \frac{x}{x^2+2}dx - \frac{1}{3}\int \frac{1}{(x^2+2)}dx - \frac{2}{3}\int \frac{1}{x+1}dx$$

$$= \frac{1}{3}\ln|x^2+1| - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) - \frac{2}{3}\ln|x+1| + c$$

Numerical Integration

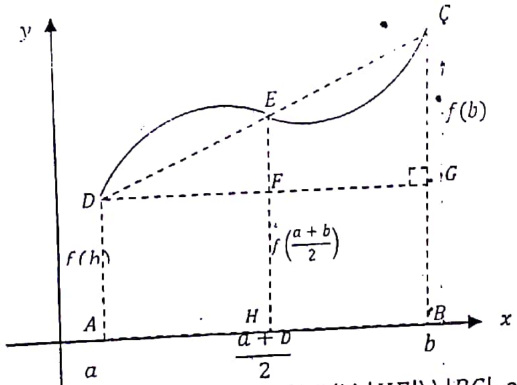
Introduction: We studied integration of various types of functions using various methods and techniques. You would discover that not all integrations can be performed exactly. One way of dealing with such integrations is by numerical integration which is a form of approximation for the integral of the form

$$\int_a^b f(x)dx = \sum_{i=0}^{n} C_i f(x_i)$$

Where C_i is a constant and x_i are points within the interval $(a \ b)$ of integration

Note: *There are various types of numerical integration formulae, three of which we are going to discuss*

The midpoint rule (one point formula): We would recall that the integral $\int_a^b f(x)dx$ represent the area bounded by the function $f(x)$ the ordinate $x = a$, $x = b$, and the x-axis.



From the figure, we observe that $|AD|\backslash\backslash|HE|\backslash\backslash|BC|$, also H is the midpoint of AB, and hence F is the midpoint of $|DG|$. Now since $|HE|\backslash\backslash|BC|$ ⇨ $|EF|\backslash\backslash|GC|$, It then follows from midpoint theorem that E is the midpoint of DC. Hence $|FE| = \frac{1}{2}|GC|$. Therefore, the area of $ABCD$ = area of $ABGD$ + area of $\Delta CDG = |AB| * |BG| + \frac{1}{2}|CG| * |DG|$

$$= |AB| * |HF| + |FE| * |AB|$$

$$= |AB|[|HF| + |FE|].$$

$$= |AB| * |HE| = (b-a)f(\tfrac{a+b}{2})$$

Since a numerical integration rule is an approximation, we can approximate the area under the curve $y = f(x)$ from point $x = a$, to $x = b$, and the x-axis by the area of the trapezium ABCD.

$$\int_a^b f(x)dx = (b-a)f(\tfrac{a+b}{2}) \qquad \ldots (1)$$

Example: Use midpoint rule to evaluate the integral $\int_1^3 \frac{1}{\sqrt{x^2+3}}dx$

Solution:

$a = 1\,, b = 3$ ⇨ $(b-a) = 2$, and $\frac{a+b}{2} = 2$

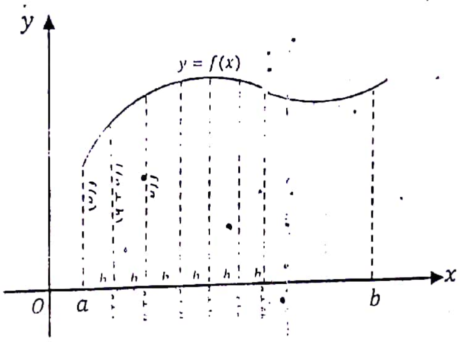
⇨ $f\left(\frac{a+b}{2}\right)\frac{a+b}{2} = f(2) = \frac{1}{\sqrt{7}} \int_1^3 \frac{1}{\sqrt{x^2+3}}dx \cong (3-1)f\left(\frac{1+3}{2}\right) = 2 * \frac{1}{\sqrt{7}} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7} =$ 0.7554 square unit.

Example: Evaluate approximately using midpoint rule the following integrals.

(i) $\int_2^5 \sqrt{x^2+1}\,dx$ (ii) $\int_0^3 \frac{dx}{x^2+3}$

The midpoint rule (n-point formula):

Suppose we divide the area under the curve $y = f(x)$ into n small strips (trapezia) each of width $h = \frac{b-a}{n}$, then we apply the midpoint rule to each trapezium then add the areas to give the required area under the curve,



The total area of the n strips is

$$\text{Area} = hf\left(\frac{a+(a+h)}{2}\right) + hf\left(\frac{(a+h)+(a+2h)}{2}\right) + hf\left(\frac{(a+2h)+(a+3h)}{2}\right) + \ldots + hf\left(\frac{(a+(n-1)h)+(a+nh)}{2}\right)$$

$$= hf\left(\frac{2a+h}{2}\right) + hf\left(\frac{2a+3h}{2}\right) + hf\left(\frac{2a+5h}{2}\right) + hf\left(\frac{2a+7h}{2}\right) + \ldots + hf\left(\frac{2a+(2n-1)h}{2}\right)$$

$$= h\sum_{i=1}^{n} f\left(\frac{2a+(2i-1)h}{2}\right)$$

$$\Rightarrow \int_a^b f(x)dx \cong h\sum_{i=1}^{n} f\left(\frac{2a+(2i-1)h}{2}\right) \qquad \ldots (2)$$

This method gives a better and better approximation as n increases in value.

Example: Evaluate approximately the integral $\int_0^3 \frac{dx}{1+x}$ with $n = 6$

Solution: $a = 0,\ b = 3,\ n = 6 \Rightarrow h = \frac{1}{2}$ or 0.5 and $f(x) = \frac{1}{1+x}$

$$\Rightarrow \text{Area} \cong hf\left(\frac{2a+h}{2}\right) + hf\left(\frac{2a+3h}{2}\right) + hf\left(\frac{2a+5h}{2}\right) + hf\left(\frac{2a+7h}{2}\right) + hf\left(\frac{2a+9h}{2}\right) + hf\left(\frac{2a+11h}{2}\right)$$

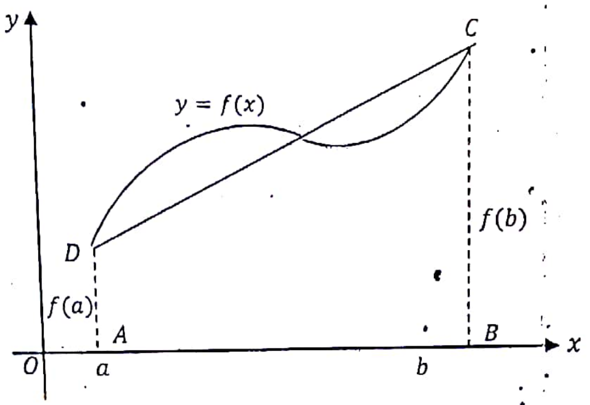
$$= 0.5\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right]$$

$$= 0.5[0.8000 + 0.5714 + 0.4444 + 0.3636 + 0.3077 + 0.2667]$$

$$= 0.5[2.7538] = 1.3769 \textit{ Square Units}$$

Trapezoidal Rule (one-point formula):

The midpoint rule was obtained by finding the area of the trapezium ABCD, however we did not use the usual formula for the area of a trapezium. Here we are going to find the area of the trapezium ABCD using the usual formula for the area of a trapezium.



Area of trapezium

$$ABCD = \frac{1}{2}(|AD| + |BC|) * |AB|$$

$$= \frac{1}{2}(f(a) + f(a))(b - a)$$

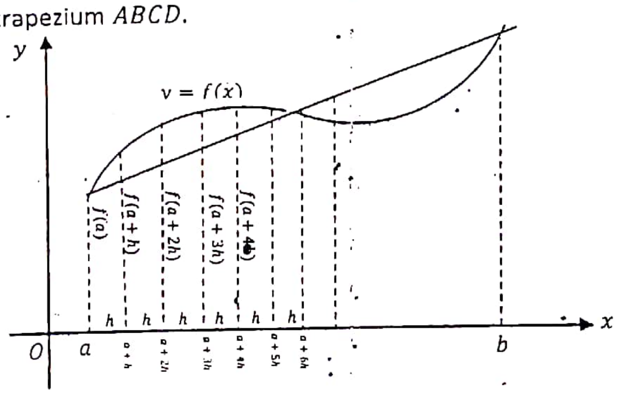
Hence $\int_a^b f(x)dx = \left(\frac{b-a}{2}\right)[f(a) + f(b)]$ . . . (3)

Example: Evaluate approximately the integral

Solution:

Trapezoidal rule (n-Point Formula):

Here also we divide the area under the curve $y = f(x)$ into n small strips (trapezia) each of width $h = \frac{b-a}{n}$, we then apply the one point trapezoidal rule to each trapezium and add their areas together to give the area of the trapezium $ABCD$.



The parallel sides of the trapezium are $f(a)$ and $f(a+h)$, $f(a+h)$ and $f(a+2h)$; $f(a+2h)$ and $f(a+3h)$, . . . $f(a+(n-1)h)$ and $f(a+nh)$,respectively.

Thus the total area of the n strips of trapezium is

$$A = \frac{h}{2}[f(a) + f(a+h)] + \frac{h}{2}[f(a+h) + f(a+2h)] + \ldots + \frac{h}{2}[f(a+(n-1)) + f(a+nh)]$$

$$= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + \ldots + 2f(a+(n-1)h) + f(b)]$$

$$\Rightarrow \quad \int_a^b f(x)dx \cong \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \ldots + 2f(a+(n-1)h) + f(b)]$$

$$\Rightarrow \quad \int_a^b f(x)dx \cong \frac{h}{2}[(f(a) + f(b)) + 2(\textstyle\sum Rem)]$$

Scanned by CamScanner

Example: Evaluate approximately the following integral $\int_0^1 \sqrt{1+x^2}\,dx \qquad n = 10$

Solution: $a = 0,\ b = 1,\ and\ n = 10 \Rightarrow h = 0.10$

x	$\sqrt{1+x^2}$	F & L	REM
0.000	1.000	1.0000	
0.100	1.0050		1.0050
0.200	1.0198		1.0198
0.300	1.0440		1.0440
0.400	1.0770		1.0770
0.500	1.1180		1.1180
0.600	1.1662		1.1662
0.700	1.2207		1.2207
0.800	1.2806		1.2806
0.900	1.3454		1.3454
1.000	1.4142	1.4142	
****	Total	2.4142	10.2767

$\Rightarrow \int_0^1 \sqrt{1+x^2}\,dx$

$\cong \frac{0.1}{2}\left[\left(f(0)+f(1)\right)+2\left(\sum Rem\right)\right]$

$\cong 0.05[(2.4142)+2(10.2767)]$

$\cong 1.1484\ Square\ unit$

OR

x	$f(x)$	C	$Cf(x)$
0.000	1.0000	1	1.0000
0.100	1.0050	2	2.0100
0.200	1.0198	2	2.0396
0.300	1.0440	2	2.0881
0.400	1.0770	2	2.1540
0.500	1.1180	2	2.2360
0.600	1.1662	2	2.3324
0.700	1.2207	2	2.4414
0.800	1.2806	2	2.5612
0.900	1.3454	2	2.6908
1.000	1.4142	1	1.4142
****	*****	*	22.9676

$\Rightarrow \int_0^1 \sqrt{1+x^2}\,dx \cong \frac{0.1}{2}\left[\sum Cf(x)\right]$

$\cong 0.05[22.9676]$

$\cong 1.1484\ Square\ unit$

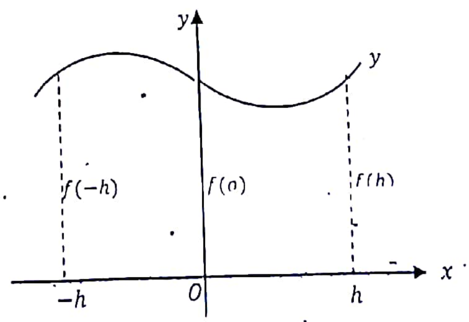
Simpson's rule (2-Point Formula):

Simpson's rule is obtained through an attempt to approximate the curve $y = f(x)$ with a parabola, in that case, we need at least three points to be able to determine the equation of the parabola exactly, we can consider a small portion of the curve $y = f(x)$, and try to approximate it with a parabolic curve i.e. a quadratic function of the form $ax^2 + bx + c$, where $a, b, and\ c$ are constant to be determine.

Then we can write

$$y = f(x) = ax^2 + bx + c$$

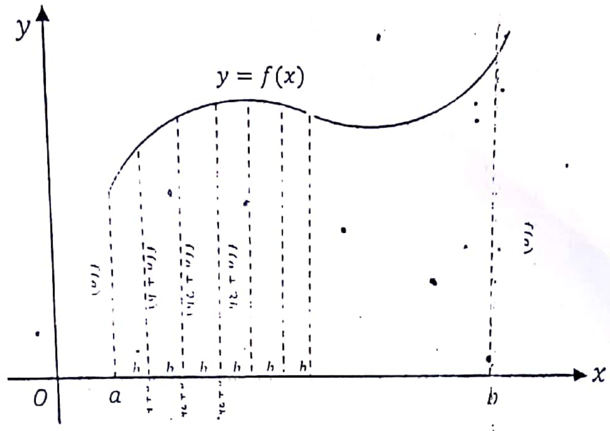
Now we let the small portion be from $x = -a, to\ x = a$



Scanned by CamScanner

Simpson's Rule(2n-Point Formula):

To obtain a more accurate approximation, we divide the area under the curve $y = f(x)$ into $2n$-Strips each of width $h = \frac{b-a}{2n}$



Note: *The number of division must be even because the formula covers two strips at a time.*

In the figure, the points of division are $a,\ a+h,\ a+2h,\ a+3h,\ a+4h,\ \dots\ a+(2n-1)h,\ b$ *and the values of* $f(x)$ *at this points are* $f(a),\ f(a+h),\ f(a+2h),\ f(a+3h),\ \dots\ f[a+(2n-1)h],$ $f[a+(2n-2)h]\ f(b)$respectively

Now, we can apply 2-point Simpson's rule taking two strips at a time.

Thus

$$\int_a^b f(x)dx = \int_a^{a+2h} f(x)dx + \int_{a+2h}^{a+4h} f(x)dx + \int_{a+4h}^{a+6h} f(x)dx + \dots + \int_{a+(2n-2)}^{b} f(x)dx$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+a+2h}{2}\right) + f(a+2h)\right] + \frac{h}{3}\left[f(a+2h) + 4f\left(\frac{(a+2h)+(a+4h)}{2}\right) + f(a+4h)\right] +$$

$$\dots + \frac{h}{3}\left[f(a+(2n-2)h) + 4f\left(\frac{(a+(2n-2)h+h)}{2}\right) + f(b)\right]$$

$$\Rightarrow \quad \int_a^b f(x)dx = \frac{h}{3}\left[\begin{array}{l} f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + 2f(a+4h) + \\ \quad \dots 4f[a+(2n-1)h] + 2f(a+(2n-2)h) + f(b) \end{array}\right]$$

$$\Rightarrow \quad \int_a^b f(x)dx = \frac{h}{3}\left[f(a) + f(b) + 2\sum even + 4\sum odd\right]$$

Example: Evaluate approximately the following integrals

i) $\int_1^4 x \ln x\, dx$ using twenty one ordinates   ii) $\int_{-1}^{2} \frac{dx}{3+x}$ using eleven ordinates

Solution;

i) $a = 1,\ b = 4,\ and \quad n = 10 \quad \Rightarrow \quad 2n = 20 \ and \ h = 0.15$

x	$x \ln x$	F &L	Rem	Odd	Even
1.000	0.0000	0.0000	----	---	---
1.150	0.1607		0.1607	0.1607	---
1.300	0.3411		0.3411		0.3411
1.450	0.5388		0.5388	0.5388	
1.600	0.7520		0.7520		0.7520
1.750	0.9793		0.9793	0.9793	
1.900	1.2195		1.2195		1.2195
2.050	1.4716		1.4716	1.4716	
2.200	1.7346		1.7346		1.7346
2.350	2.0079		2.0079	2.0079	
2.500	2.2907		2.2907		2.2907
2.650	2.5826		2.5826	2.5826	
2.800	2.8829		2.8829		2.8829
2.950	3.1913		3.1913	3.1913	
3.100	3.5073		3.5073		3.5073
3.250	3.8306		3.8306	3.8306	
3.400	4.1608		4.1608		4.1608
3.550	4.4977		4.4977	4.4977	
3.700	4.8408		4.8408		4.8408
3.850	5.1901		5.1901	5.1901	
4.000	5.5452	5.5452			
****	Total	5.5452		24.4506	21.7297

$\Rightarrow \quad \int_1^4 x \ln x \, dx$

$= 0.05[(5.5452) + 4(24.4506) + 2(21.7297)]$

$= 0.05[5.5452 + 97.8024 + 43.4594]$

$= 0.05[146.807] = 7.34035$ *square unit*