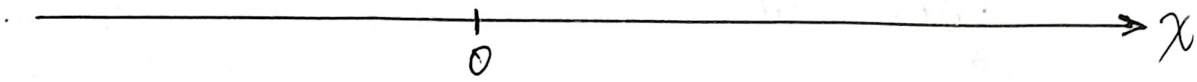
One Dimensional Random Walk



p 为当前步向右的概率. p 取 $\frac{1}{2}$
q 为 ------ 向左的概率 q 取 $\frac{1}{2}$ $> p+q=1$.

N 为总共经历的决策数. m 为其中决定向右的次数.

x 为位移. $x = L(m-(N-m)) = (2m-N)L$.

L 为每一次决策走出的长度. Δt 为对应时间. t 为总时间.

(1) 求 $E(x) = E((2m-N)L) = (2E(m)-N)L$ ~ 求 $E(m)$.

$$E(m) = \sum_{m=0}^{N}\left(C_N^m p^m q^{N-m}\right)\cdot m.$$

m^2	0	1	$\cdots$	k^2	$\cdots$	N^2
m	0	1	$\cdots$	k	$\cdots$	N
P	$C_N^0 p^0 q^N$	$C_N^1 p q^{N-1}$	$\cdots$	$C_N^k p^k q^{N-k}$	$\cdots$	$C_N^N p^N q^0$

又 ∵ $\frac{d}{dp}\left(C_N^m p^m q^{N-m}\right) = m\cdot C_N^m p^{m-1} q^{N-m}$

$$\therefore C_N^m p^m q^{N-m}\cdot m = p\left[m\cdot C_N^m \cdot p^{m-1} q^{N-m}\right] = p\cdot\frac{d}{dp}\left(C_N^m p^m q^{N-m}\right)$$

$$\therefore E(m) = \sum_{m=0}^{N}\left(C_N^m\cdot p^m\cdot q^{N-m}\right)\cdot m = \sum_{m=0}^{N} p\left[\frac{d}{dp}\left(C_N^m p^m q^{N-m}\right)\right]$$

$$= p\cdot\frac{d}{dp}\sum_{m=0}^{N} C_N^m p^m q^{N-m} = p\cdot\frac{d}{dp}\cdot(p+q)^N = p\cdot N\cdot(p+q)^{N-1}$$

$$= p\cdot N.$$

$\therefore E(x) = (2pN-N)L$. 取 $p=\frac{1}{2}$. $E(x) = 0$.

(2) 求 $E(x^2) = E\left(L^2(2m-N)^2\right) = E\left(L^2(4m^2-4mN+N^2)\right)$

$$= 4L^2 E(m^2) - 4NL^2 E(m) + N^2L^2.$$

~ 求 $E(m^2)$

$$E(m^2) = \sum_{m=0}^{N} m^2 \cdot C_N^m \cdot p^m \cdot q^{N-m}$$

又∵ $\frac{d^2}{(dp)^2}(C_N^m p^m \cdot q^{N-m}) = m(m-1) \cdot C_N^m p^{m-2} q^{N-m}$

且 $\frac{d}{dp}(C_N^m p^m \cdot q^{N-m}) = m \cdot C_N^m p^{m-1} q^{N-m}$

$$\therefore m^2 C_N^m p^m \cdot q^{N-m} = m(m-1) C_N^m \cdot p^m \cdot q^{N-m} + m C_N^m \cdot p^m \cdot q^{N-m}$$

$$= p^2 [m(m-1) C_N^m p^{m-2} q^{N-m}] + p [m C_N^m p^{m-1} q^{N-m}]$$

$$= p^2 \cdot \left[\frac{d^2}{(dp)^2} C_N^m p^m \cdot q^{N-m}\right] + p \left[\frac{d}{dp} C_N^m p^m q^{N-m}\right]$$

$$\therefore E(m^2) = \sum_{m=0}^{N} m^2 C_N^m p^m \cdot q^{N-m}$$

$$= \sum_{m=0}^{N} p^2 \cdot \left[\frac{d^2}{dp^2}(C_N^m p^m q^{N-m})\right] + \sum_{m=0}^{N} p \cdot \left[\frac{d}{dp} C_N^m p^m \cdot q^{N-m}\right]$$

$$= p^2 \cdot \frac{d^2}{dp^2} \sum_{m=0}^{N} C_N^m p^m \cdot q^{N-m} + p \frac{d}{dp} \sum_{m=0}^{N} C_N^m p^m q^{N-m}$$

$$= p^2 \cdot \left[\frac{d^2}{dp^2} \cdot (p+q)^N\right] + p\left[\frac{d}{dp}(p+q)^N\right]$$

$$= p^2 \cdot N \cdot (N-1)(p+q)^{N-2} + pN(p+q)^{N-1}$$

$$= p^2 (N)(N-1) + pN.$$

$$= p^2 N^2 - p^2 N + pN = pN(1-p) + p^2 N^2 = Npq + N^2 p^2.$$

$$\therefore E(x^2) = L^2 [4 p^2 N^2 - 4p^2 N + 4pN - 4pN^2 + N^2]$$

$$= L^2 [4p(p-1)N^2 + 4p(1-p)N + N^2] = L^2 [4pq(N-N^2) + N^2]$$

取 $p = \frac{1}{2}$

$E(x^2) = L^2 N$ 其中 $N = \frac{t}{\Delta t}$.

$$\therefore E(x^2) = L^2 \cdot \frac{t}{\Delta t} = 2\left[L^2 \times \frac{1}{2\Delta t}\right] t = 2\left(\frac{L^2}{2\Delta t}\right) t.$$

令 $D = \frac{L^2}{2\Delta t}$ 为扩散系数. $E(x^2) = 2Dt$.

(3) 另法.

认为 $m = m_1+m_2+\cdots+m_N$.

$m_i = 0$ 或 1. $i=1,2\cdots N$.

$E(m) = E(m_1+m_2+\cdots+m_N) = E(m_1)+E(m_2)+\cdots+E(m_N)$

其中 $E(m_i)=p$. ，两点分布. /0-1 分布.

∴ $E(m)=N\cdot p$.

独立事件.

$D(m) = D(m_1+m_2+\cdots+m_N) = D(m_1)+D(m_2)+\cdots+D(m_N)$.

其中 $D(m_i) = p\cdot(1-p) = p\cdot q$.

∴ $D(m) = Np\cdot q$.

又∵ $D(m) = E(m^2) - E^2(m)$

∴ $E^2(m) = D(m) + E^2(m) = Npq + N^2p^2$.

如此求得 $E(m)=Np$. $E^2(m) = Np\cdot q + N^2p^2$

与(1)(2)求得一致.

(4). 结论 ⟶ 推论.

$E(x^2) = 2Dt\times d$, $D = \frac{L^2}{2\Delta t}$.

一次碰撞后的位移.

碰撞间歇时长, d 维度.

$\langle x^2\rangle = 2Dt\times d$.

⇓

量纲: $[L]^2[T]^{-1}$. 单位 $\mu m^2/s$.

$x\to 0$. t 较小; $x\to+\infty$. t 较大.

$t = \frac{\langle x^2\rangle}{2D\times d}$. $\propto x^2$.

碰撞频次 $\frac{1}{\Delta t}$↑ D↑; L↑ D↑

vs.

$x\to 0$ t 较大; $x\to+\infty$. t 较小.

$t = \frac{x}{v}$. , Directed Motion. $\propto x$