振动：
- 线性
  - 简谐
  - 阻尼
  - 受迫
- 非线性
  - 自激振动
    - 软激励
    - 硬激励

(一) 线性振动 ⟹ $\boxed{\dfrac{d^2s}{dt^2}+B\dfrac{ds}{dt}+C\cdot S+D=0}$ 线性运动学微分方程.

1. 简谐：

$$\vec{F}_{合}=m\vec{a}=-u_0''\vec{x} \qquad \left(\frac{u_0''}{m}=\omega_0^2\right).$$

$$m\vec{a}+u_0''\vec{x}=0 \qquad \vec{a}+\frac{u_0''}{m}\vec{x}=0.$$

$$\vec{a}+\omega_0^2\vec{x}=0 \qquad \boxed{\frac{d^2s}{dt^2}+\omega_0^2 S=0}.$$

② 恒定力(大小)阻尼.

$$\vec{F}_{合}=m\vec{a}=(-u_0''\vec{x})+|\vec{f}|\left(-\frac{\vec{v}}{|\vec{v}|}\right).$$

$$m\vec{a}+u_0''\vec{x}+\frac{\vec{v}}{|\vec{v}|}\cdot|\vec{f}|=0 \qquad \vec{a}+\frac{u_0''}{m}\vec{x}+\frac{|\vec{f}|}{m}\frac{\vec{v}}{|\vec{v}|}=0.$$

$$\operatorname{sgn}\left(\frac{ds}{dt}\right)=\frac{\frac{ds}{dt}}{\left|\frac{ds}{dt}\right|}=\frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d^2s}{dt^2}+\frac{|\vec{f}|}{m\cdot|\vec{v}|}\frac{ds}{dt}+\frac{u_0''}{m}s=0.$$

$$\boxed{\frac{d^2s}{dt^2}+g(\vec{v})\cdot\frac{ds}{dt}+\omega_0^2 s=0}$$

非线性? ✓

2. 阻尼振动.

$$\vec{F}_{合}=m\vec{a}=(-u_0''\vec{x})+\underbrace{(-\gamma\vec{v})}_{阻力}$$

阻力系数（γ）

$$m\vec{a}+\gamma\vec{v}+u_0''\vec{x}=0.$$

$$\frac{u_0''}{m}=\omega_0^2 \qquad 令\ \beta=\frac{\gamma}{2m}$$

阻尼常量（β）

$$\boxed{\frac{d^2s}{dt^2}+2\beta\frac{ds}{dt}+\omega_0^2 s=0}$$

$$令\ \Lambda=\frac{\beta}{\omega_0}$$

阻尼度（Λ）.

3. 受迫振动.

$$\vec{F}_{合}=m\vec{a}=(-u_0''\vec{x})+(-\gamma\vec{v})+\underbrace{(F\cos\omega t)}_{驱动}.$$

$$\boxed{\frac{d^2s}{dt^2}+2\beta\frac{ds}{dt}+\omega_0^2 s=F\cos\omega t}.$$

(*) 阻尼相关参数

$\vec{f}_{阻} = -\gamma\vec{v}$ 阻力系数. $[\gamma] = \frac{N}{m/s} = kg\cdot s^{-1}$

针对振动
$$\begin{cases} \beta = \frac{\gamma}{2m} & \text{阻尼常量.} \quad [\beta] = \frac{kg\cdot s^{-1}}{kg} = s^{-1} \\ \Lambda = \frac{\beta}{\omega_0}. & \text{阻尼度.} \quad [\Lambda] = \frac{s^{-1}}{s^{-1}} = 1 \end{cases}$$

4. 运动学方程(S(t)的微分方程)的解.

(1) $\frac{d^2s}{dt^2} + \omega_0^2 s = 0$. $\quad s(t) = A\cos(\omega_0 t + \varphi_0)$.

(2) $\frac{d^2s}{dt^2} + 2\beta\frac{ds}{dt} + \omega_0^2 s = 0$

① $\Lambda < 1$. $\begin{cases} s(t) = A\cdot e^{-\beta t}\cos(\omega_r t + \varphi_0). \\ \omega_r = \sqrt{\omega_0^2 - \beta^2}. \end{cases}$

② $\Lambda > 1$ $\begin{cases} s(t) = \cancel{A}e^{-(\beta+\beta_r)t} + \cancel{A}e^{-(\beta-\beta_r)t}. \\ \beta_r = \sqrt{\beta^2 - \omega_0^2}. \end{cases}$

③ $\Lambda = 1$. $s(t) = Ae^{-\beta t}$

(3) $\frac{d^2s}{dt^2} + 2\beta\frac{ds}{dt} + \omega_0^2 s = F\cos\omega t$.

$$\left.\begin{cases} \text{定态}: A\cos(\omega t + \varphi_0) = S(t) \\ \text{暂态}: Ae^{-\beta t}\cos(\omega_r t + \varphi_0) = S(t). \end{cases}\right\}\text{叠加为实际振动.}$$

$$\begin{cases} A = \frac{F}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \\ \varphi = \arctan\frac{-2\beta\omega}{\omega_0^2 - \omega^2}. \end{cases}$$

$V = \omega A$

$\varphi_0 = \frac{\pi}{2} + \varphi$.

$$\begin{cases} V = \frac{\omega F}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \\ \varphi_0 = \frac{\pi}{2} + \varphi = \frac{\pi}{2} - \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}. \end{cases}$$

5. 受迫振动的分析

$f = F\cos\omega t$

$\downarrow$

定态：
$$\begin{cases} A = \dfrac{F}{m\sqrt{(\omega^2-\omega_0^2)^2+4\beta\omega}} \\ \varphi = \arctan -\dfrac{2\beta\omega}{\omega_0^2-\omega^2} \end{cases} \Longrightarrow S(t) = A\cos(\omega t+\varphi)$$

$$\begin{cases} V = \omega A \\ \varphi_V = \varphi + \dfrac{\pi}{2} \end{cases} \Longrightarrow V(t) = \omega A\cos(\omega t+\varphi_V)$$

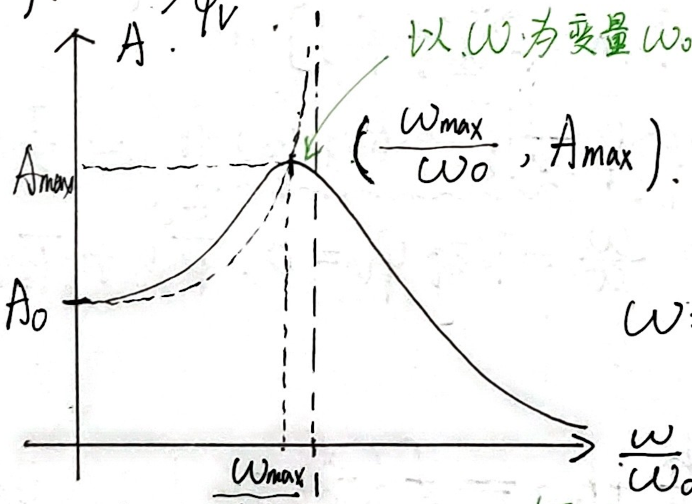
$S(t) = A\cos(\omega t+\varphi)$

(1) ω：由 $f = F\cos\omega t$ 驱动 f 频率决定

(2) A / V：由 $\underbrace{(F,\omega)}_{\text{驱动力}} + \underbrace{(\beta)}_{\text{阻尼}} + \underbrace{(m,\omega_0)}_{\text{自身}}$ 共同决定

(3) φ / φ_V：由 $(\omega) + (\beta) + (\omega_0)$ 共同决定 $\uparrow$ 与F无关

(2).

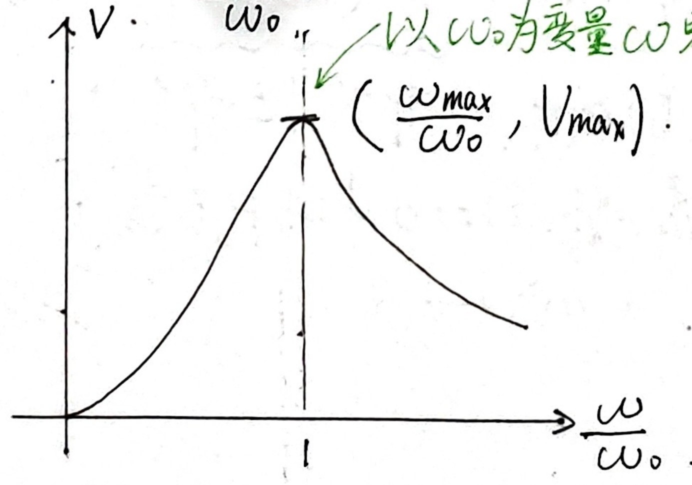


(a) 存在 $\left(\dfrac{\omega_{max}}{\omega_0}, A_{max}\right)$ 的前提

$\Lambda < \dfrac{1}{\sqrt{2}}$; $\beta^2 < \dfrac{\omega_0^2}{2}$

(b) $\omega = \omega_{max}$
$$\begin{cases} \dfrac{\omega_{max}}{\omega_0} \text{ 与 } \Lambda: \begin{cases} \Lambda = 0, \dfrac{\omega_{max}}{\omega_0} = \\ \Lambda\uparrow, \dfrac{\omega_{max}}{\omega_0}\downarrow \end{cases} \\ A_{max} \text{ 与 } \Lambda: \Lambda\downarrow\ A_{max}\uparrow\uparrow \end{cases}$$

(c)
$$\left.\begin{cases} \lim\limits_{\frac{\omega}{\omega_0}\to 0} A = A_0 \\ \lim\limits_{\frac{\omega}{\omega_0}\to +\infty} A = 0 \end{cases}\right\} \text{与}\beta\text{无关，与}\Lambda\text{无关}$$



(d)
$$\begin{cases} \dfrac{\omega_{max}}{\omega_0} = 1 \text{ 与 } \Lambda \text{ 无关} \\ V_{max} \text{ 与 } \Lambda: \Lambda\downarrow V_{max}\uparrow\uparrow \end{cases}$$

(e) $\lim\limits_{\omega\to 0} V = \lim\limits_{\omega\to +\infty} V = 0$ 与$\beta(\Lambda)$无关

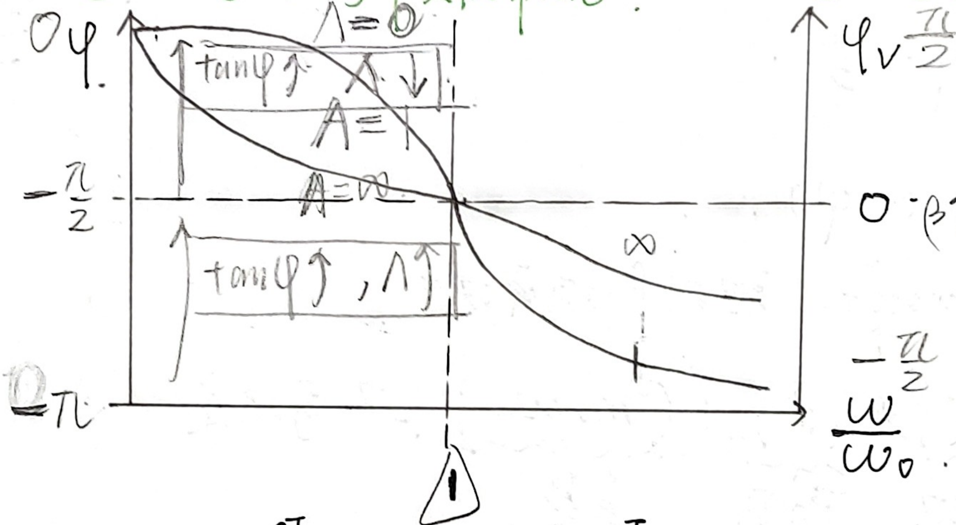
* 补充说明. F, m, β 取定.

A_{max} : $A = A(\omega_0, \omega)$. 令 $\begin{cases} ① \ \dfrac{\partial A}{\partial \omega} = 0 \Rightarrow \omega \neq \omega_0 \quad \boxed{\omega = \sqrt{\omega_0^2 - 2\beta^2}} \\ ② \ \dfrac{\partial A}{\partial \omega_0} = 0 \Rightarrow \omega = \omega_0 \end{cases}$

V_{max} : $V = V(\omega_0, \omega)$. 令 $\begin{cases} ① \ \dfrac{\partial V}{\partial \omega} = 0 \Rightarrow \omega = \omega_0 \\ ② \ \boxed{\dfrac{\partial V}{\partial \omega_0} = 0 \Rightarrow \omega = \omega_0} \end{cases}$

① : 力学中实际状况.

(3). ② : 电磁学中实际状况.



$\tan\varphi = \dfrac{2\beta}{\omega\left(1 - \left(\frac{\omega_0}{\omega}\right)^2\right)}$

β↑ Λ↑ tanφ↓

$\frac{\omega}{\omega_0} < 1$ $\quad \frac{\omega_0}{\omega} > 1$ $\quad \tan\varphi < 0$

$\frac{\omega}{\omega_0} > 1$ $\quad \frac{\omega_0}{\omega} < 1$ $\quad \tan\varphi > 0$

β↑ Λ↑ tanφ↑

$\overline{P} = \frac{1}{T}\int_0^T f v \cdot dt = \frac{1}{T}\int_0^T F\cos\omega t \, V\cos(\omega t + \varphi_v)\, dt = \frac{FV}{T}\int_0^T \cos\omega t \cos(\omega t + \varphi_v)\, dt$

$\int_0^T \cos\omega t \cdot \cos(\omega t + \varphi_v)\, dt = \frac{1}{2}\int_0^T [\cos(\varphi_v) + \cos(2\omega t + \varphi_v)]\, dt$

$\int_0^T \cos\omega t \cos(\omega t + \varphi_0)\, dt = \frac{T}{2}\cos\varphi_v + \frac{1}{2}\underbrace{\int_0^T \cos(2\omega t + \varphi_0)\, dt}_{=0} = \frac{T}{2}\cos\varphi_v$

$\therefore \overline{P} = \frac{1}{2} FV \cdot \cos\varphi_v \Rightarrow \frac{\omega}{\omega_0} = 1$ 时 $\varphi_v = 0$, $\overline{P}$ 达到 $\overline{P}_{max} = FV$.

(二) 非线性振动.

运动学方程: $m\frac{d^2 s}{dt^2} + \gamma\frac{ds}{dt} + ks = \underbrace{f\left(\frac{ds}{dt} - v_0\right)}$ "软"

"线性": 是 $\left(\frac{d^2 s}{dt^2}, \frac{ds}{dt}, s\right)$ 的线性 非线性部分

函数. 对于 $F\left(\frac{d^2 s}{dt^2}, \frac{ds}{dt}, s\right) = 0$ 中的 F 而言.

又如: $\frac{d^2\theta}{dt^2} + \frac{f_0 \, \mathrm{sgn}\left(\frac{d\theta}{dt}\right)}{m} + \omega_0^2\theta = 0$ "硬".