曲线积分与曲面积分比较

曲线积分（二维平面）

定义

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(1) 对弧长 $\int_L f(x,y)\,ds$

路线（弧段／闭曲线）

(2) 对坐标（位移矢量）（线段方向向量）

$$\int_L \vec F\cdot d\vec s;\quad \begin{cases}\vec F = X(x,y)\,\hat i + Y(x,y)\,\hat j\\ d\vec s = dx\cdot\hat i + dy\cdot\hat j\end{cases}$$

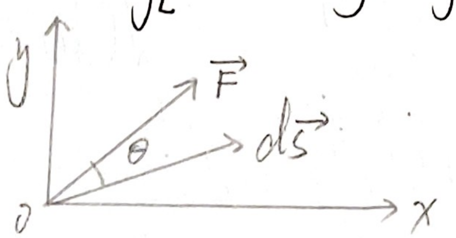
$$\int_L \vec F\cdot d\vec s = \int_L X(x,y)\,dx + Y(x,y)\,dy$$

(3) 关联

含+/-

$$\int_L \vec F\cdot d\vec s = \int_L X\cdot dx + Y dy = \int_L |\vec F|\cos\theta\cdot ds$$

恒正



$$\vec F = X\hat i + Y\hat j$$

$$d\vec s = \underbrace{(ds\cos\alpha)}_{dx}\,\hat i + \underbrace{(ds\cos\beta)}_{dy}\,\hat j$$

$$\vec F\cdot d\vec s = X\cos\alpha\,ds + Y\cos\beta\,ds$$

$$\boxed{= (X\cos\alpha + Y\cos\beta)\,ds}$$

$$\int_L \vec F\cdot d\vec s = \int_L X dx + Y dy \quad \text{（对坐标）}$$

$$\|$$

$$\int_L |\vec F|\cdot\cos\theta\,ds = \int_L (X\cos\alpha + Y\cos\beta)\,ds \quad \text{（对弧长）}$$

$\theta = \langle \vec F, d\vec s\rangle$　　$\cancel{\vec F}$ 与 x 轴

$$\begin{cases} ds\cos\alpha = dx\\ ds\cos\beta = dy\end{cases}$$

曲面积分（三维曲面）

面积元素恒正

(1) 对面积，$\iint_S f(x,y,z)\cdot dS$

曲面

(2) 对坐标（平面法向量）

$$\iint_S \vec A\cdot d\vec S;\quad \begin{cases}\vec A = X\hat i + Y\hat j + Z\hat k\\ d\vec S = dydz\,\hat i + dzdx\,\hat j + dxdy\,\hat k\end{cases}$$

$$\iint_S \vec A\cdot d\vec S = \iint_S X dydz + Y dzdx + Z dxdy$$

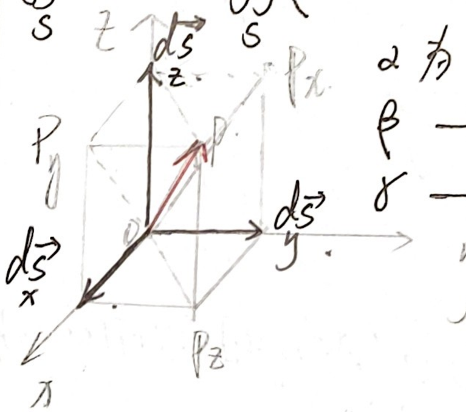
(3) 关联 $= \iint_S |\vec A|\cdot\cos\theta\cdot dS$

$$\iint_S \vec A\cdot d\vec S = \iint_S (X\cos\alpha + Y\cos\beta + Z\cos\gamma)\,dS.$$

α 为 $\vec{OP}$ 与 x 轴正向夹角

β ⟶ y 轴

γ ⟶ z 轴.



$$d\vec S = (dS\cos\alpha)\,\hat i + (dS\cos\beta)\,\hat j + (dS\cos\gamma)\,\hat k$$

$$\iint_S \vec A\cdot d\vec S = \iint_S X\,dydz + \iint_S Y\,dzdx + \iint_S Z\,dxdy$$

$$\|$$

$$\iint_S |\vec A|\cdot\cos\theta\,dS = \iint_S (X\cos\alpha + Y\cos\beta + Z\cos\gamma)\,dS$$

$\vec A$ 与 x 轴夹角

$$\begin{cases} dS\cos\alpha = dydz\\ dS\cos\beta = dzdx\\ dS\cos\gamma = dxdy\end{cases}$$

x 轴法向量对应 yz 面

计算方法.

1. $\int_L f(x,y)\,ds$

(1) 参数方程 $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

相类似

$z = z(x,y)$

$\begin{cases} dx = x'(t)\,dt \\ dy = y'(t)\,dt \end{cases}$ $t\in[\alpha,\beta]$

$y = y(x)$

$$\int_L f(x,y)\,ds = \int_\alpha^\beta f[x(t),y(t)]\sqrt{x'^2(t)+y'^2(t)}\,dt$$

(2) $y = y(x) \Leftrightarrow \begin{cases} x = x(t) = t \\ y = y(x) = y(t) \end{cases}$

$x'(t) = \frac{dx}{dx} = 1$

$$\boxed{\int f(x,y)\,ds = \int_\alpha^\beta f(x,y(x))\sqrt{1+y'^2(x)}\,dx}$$

化为对一元函数的积分

2. $\int_L X\,dx + Y\,dy$

(1) 参数方程

$$\int_L X\,dx + Y\,dy = \int_\alpha^\beta \{X[x(t),y(t)]\,x'(t) + Y[x(t),y(t)]\,y'(t)\}\,dt$$

(2) $y = y(x)$

$$\sim = \int_a^b \{X[x,y(x)] + Y[x,y(x)]\,y'\}\,dx$$

关于 x 的单变量积分

定理

格林定理

$$\boxed{\iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)dx\,dy = \oint_L X\,dx + Y\,dy}$$

1. $\iint_S f(x,y,z)\,ds$

$$ds = \sqrt{1+z_x'^2+z_y'^2}\;d\sigma.$$

$$\boxed{\iint_S f(x,y,z)\,ds = \iint_D f(x,y,z(x,y))\sqrt{1+z_x'^2+z_y'^2}\,d\sigma}$$

化为对二元函数的重积分

$z = f(x,y) \Leftrightarrow F(x,y,z) = f(x,y) - z = 0$

$\vec{n} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$

$\frac{\partial F}{\partial x} = \frac{\partial z}{\partial x}$ $\frac{\partial F}{\partial y} = \frac{\partial z}{\partial y}$ $\frac{\partial F}{\partial z} = -1$

$$\cos\gamma = \frac{\vec{n}\cdot(0,0,1)}{|\vec{n}|} = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}}$$

$$\cos\gamma = \frac{-1}{\sqrt{1+\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2}}$$

$$ds\cos\gamma = d\sigma$$

2. $\iint_S X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy.$

以 $\iint_S Z\,dx\,dy$ 为例.

$$= \iint_{D_{xy}} Z(x,y,z(x,y))\,dx\,dy.$$ 为二重积分.

奥氏定理

$$\boxed{\iiint_V \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)\underbrace{dx\,dy\,dz}_{}^{dV} = \oiint_S X\,dy\,dz + Y\,dz\,dx + Z\,dx\,dy}$$