积分在几何问题上的应用.

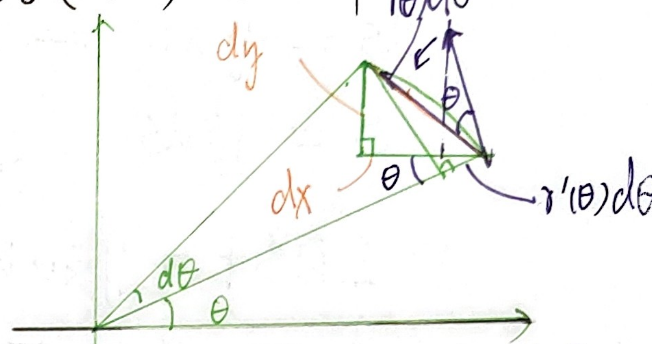
(一) 曲线的三种表示方式.

$$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$$

1. $y = f(x)$. 2. $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ 3. $r = r(\theta)$.

直角坐标系 $\xleftarrow{\text{转化}}$ 极坐标系 $\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r'd\theta \\ rd\theta \end{pmatrix}$

用 (x, y) 确定一个点, 用 (r, θ) 确定一个点



$dx^2 + dy^2 = (r(\theta)d\theta)^2 + (r'(\theta)d\theta)^2$.

$dx^2 + dy^2 = (r(\theta)^2 + r'(\theta)^2)\, d\theta^2$.

(二) 几何上四类问题

面积 $\begin{cases} \text{曲线围成的 ①} \\ \text{旋转体侧面积, ②} \end{cases}$

体积 —— 旋转体体积, ③

弧长 ④

(四) 公式.

1. 弧长

$$dl = \sqrt{dx^2+dy^2} = \begin{cases} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \sqrt{1+f'(x)}\, dx. \\ \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt = \sqrt{(x'(t))^2+(y'(t))^2}\, dt. \\ \quad\| \\ \sqrt{r^2(\theta)+(r'(\theta))^2}\, d\theta \end{cases}$$

先化为参数方程可得

$$L = \int_a^b dl$$

$$\begin{cases} x(\theta) = r(\theta)\cos\theta \\ y(\theta) = r(\theta)\sin\theta \end{cases} \Rightarrow \begin{cases} \frac{dx}{d\theta} = r'(\theta)\cos\theta + r(\theta)(-\sin\theta). \\ \frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta. \end{cases}$$

$$\Updownarrow$$

$$\begin{pmatrix} \frac{dx}{d\theta} \\ \frac{dy}{d\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r'(\theta) \\ r(\theta) \end{pmatrix} \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2} = \sqrt{r'^2(\theta)+r^2(\theta)}\cdots$$

2. 体积.

$$dV = \pi y^2 \cdot dx = \begin{cases} \pi f^2(x)\, dx. \\ \pi y^2(t)\; x'(t)\, dt. \\ \pi\,[r(\theta)\sin\theta]^2 \cdot [r'(\theta)\cos\theta + r(\theta)(-\sin\theta)]\, d\theta \end{cases}$$

$$V = \int_a^b dV$$

"高"

3. (1) 侧面积

弧长的微分

$$ds = 2\pi y \cdot dl = \begin{cases} 2\pi f(x) \cdot \sqrt{1+f'(x)^2} \cdot dx \\ 2\pi y(t) \cdot \sqrt{x'(t)^2+y'(t)^2}\, dt \\ 2\pi [r(\theta)\sin\theta] \cdot \sqrt{r(\theta)^2+r'(\theta)^2} \cdot d\theta \end{cases}$$

$$S = \int_a^b ds$$

"斜高"

(2) 围成面积

① 直角系下与x轴的面积

$$ds = y \cdot dx = \begin{cases} f(x)\, dx \\ y(t)\, x'(t)\, dt \\ r(\theta)\sin\theta\, [r'(\theta)\cos\theta - r(\theta)\sin\theta]\, d\theta \end{cases}$$

$$S = \int_a^b ds$$

② 直角系下与y轴的面积 ($f^{-1}(y)$ 存在)

$$ds = x \cdot dy = f^{-1}(y)\, dy$$

$$S = \int_a^b ds$$

$$\begin{cases} ds = 2\pi y \cdot dl & \text{"斜高"} \\ dV = \pi y^2 \cdot dx & \text{"高"} \end{cases}$$

③ 极坐标系下

$$ds = \frac{1}{2} r^2(\theta)\, d\theta$$

$$S = \int_\alpha^\beta ds$$