标准正态分布的双边 Laplace 变换

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y\in R$$

$$\mathcal{L}\circ p_Y(y) \overset{def}{=} \int_{\boxed{-\infty}}^{+\infty} e^{-py}\, dy \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right)$$

$p_Y(y)$ 使得负实轴也收敛 可以定义双侧 $\mathcal{L}$.

$$\mathcal{L}\circ p_Y(y) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2+2py}{2}}\, dy$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{y^2+2py+p^2}{2}\right)+\frac{p^2}{2}}\, dy$$

$$= e^{\frac{p^2}{2}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+p)^2}{2}}\, dy = e^{\frac{p^2}{2}}$$

即
$$\boxed{\mathcal{L}\circ \underbrace{p_Y(y)}_{Y\sim N(0,1)} = \underbrace{e^{\frac{p^2}{2}}}_{N(0,1)\text{的双侧}\mathcal{L}}}$$

以下记 $Y\sim N(0,1)$ 为 $p_Y(y)$

$X\sim$ 任意分布 为 $p_X(x)$

22例 $\mathcal{L}$ 变换与原点矩的关系

$$\lim_{p\to 0}\circ\mathcal{L}\circ p_X(x) = \lim_{p\to 0}\circ\int_{-\infty}^{+\infty}dx\circ e^{-px}\circ p_x(x)$$

$$= \int_{-\infty}^{+\infty}dx\circ\left[\lim_{p\to 0}\circ e^{-px}\right]\circ p_x(x)$$

$$= \int_{-\infty}^{+\infty}dx\circ 1\circ p_x(x) = \int_{-\infty}^{+\infty}p_x(x)\,dx = 1$$

$$\therefore\ \boxed{\lim_{p\to 0}\circ\mathcal{L}\circ p_x(x) = 1 = E(x^0)}$$

（概率的规范性）

$$\lim_{p\to 0}\circ\left(-\frac{d}{dp}\right)\circ\mathcal{L}\circ p_X(x) = \lim_{p\to 0}\circ\left(\frac{-d}{dp}\right)\circ\int_{-\infty}^{+\infty}dx\circ e^{-px}\circ p_X(x)$$

$$= \lim_{p\to 0}\circ\int_{-\infty}^{+\infty}dx\circ\boxed{\frac{-d}{dp}\circ e^{-px}}\circ p_X(x)$$

$$= \lim_{p\to 0}\int_{-\infty}^{+\infty}dx\circ\boxed{e^{-px}\circ x}\,p_x(x)$$

$$= \int_{-\infty}^{+\infty}dx\circ\underbrace{\left[\lim_{p\to 0}\circ e^{-px}\right]}_{=1}\circ x\,p(x)$$

$$= \int_{-\infty}^{+\infty}x\,p(x)\,dx = E(x)$$

即 $$\boxed{\lim_{p\to 0}\circ\left(-\frac{d}{dp}\right)\circ\mathcal{L}\circ p_x(x) = E(x)}$$

可进一步推广有

$$\lim_{p\to 0} \circ (-1)^n \circ \left(\frac{d^n}{dp^n}\right) \circ \mathcal{L} \circ p_x(x) = E(x^n)$$

可以简化写成

$$\boxed{\left[(-1)^n \mathcal{L}^{(n)}\{p_x(x)\}\right]\Big|_{p=0} = E(x^n)}$$

或 $F_x(p) \overset{def}{=} \mathcal{L}\{p_x(x)\}$

$$\boxed{(-1)^n F_x^{(n)}(0) = E(x^n)}$$

即. 对随机变量 x

① 先求 $\mathcal{L}$ ② 再求 n 阶导 ③ 再乘 $(-1)^n$ ④ 再令 $p\to 0$

$\Updownarrow$

求 x 的 n 阶原点矩.

$\Updownarrow$

① 先取 n 次幂 ② 再求期望.

卷积与随机变量的和

$$z \overset{def}{=\!=} x_1 + x_2$$

$$p_{x_1+x_2}(x_1+x_2) = p_z(z) = \iint\limits_{\{(x_1,x_2)\mid x_1+x_2=z\}} p_x(x_1)\cdot p_x(x_2)\,dx_1\,dx_2$$

$$p_{x_1+x_2}(x_1+x_2) = p_z(z) = \int_{-\infty}^{+\infty} p_x(x_1)(z-x_1)\,dx_1$$

$$\therefore\ p_{x_1+x_2}(x_1+x_2) = p_x(x_1) * p_x(x_2)$$

卷积(据定义)

即 和的密度 = 密度的卷积

可以推广到多个(连续和～连续卷积)

用算子表示

$$\boxed{p\circ\sum_{i=1}^{n}}\circ x_i = \boxed{\left(\mathop{*}_{i=1}^{n}\right)\circ p}\circ x_i$$

先求和.再做概率测度映射

$\Updownarrow$

先做概率测度映射.再卷积

} 作用于随机变量 x_i

$$\sum_{i=1}^{n} = p^{-1}\circ\left(\mathop{*}_{i=1}^{n}\right)\circ p$$

即卷积与求和作为算符 通过$\underbrace{\text{概率测度映射}}_{p}$ 是相似的 $\sum_{i=1}^{n} \sim \mathop{*}_{i=1}^{n}$

乘积与随机变量的和.

$$\sum_{i=1}^{n} \sim \mathop{(\ast)}_{i=1}^{n} : \boxed{P \circ \sum_{i=1}^{n}} \circ x_i = \boxed{\mathop{(\ast)}_{i=1}^{n} \circ P} \circ x_i$$

$$\sum_{i=1}^{n} = P^{-1} \circ \mathop{(\ast)}_{i=1}^{n} \circ P$$

$$\prod_{i=1}^{n} \sim \mathop{(\ast)}_{i=1}^{n} : \boxed{\prod_{i=1}^{n} \circ \mathcal{L}} \circ f(x) = \boxed{\mathcal{L} \circ \mathop{(\ast)}_{i=1}^{n}} \circ f(x)$$

$$\mathop{(\ast)}_{i=1}^{n} = \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L}$$

$$\sum_{i=1}^{n} = P^{-1} \circ \mathop{(\ast)}_{i=1}^{n} \circ P$$

&

$$\mathop{(\ast)}_{i=1}^{n} = \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L}$$

$$\Downarrow$$

$$\sum_{i=1}^{n} = P^{-1} \circ \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L} \circ P$$

即

$$P\left(\sum_{i=1}^{n} x_i\right) = \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L} \{ P(x_i) \}$$

$$\begin{tikzcd}
\text{随机变量空间 } x_i \arrow[d, shift right, "P"'] \\
\text{实变函数空间 } f(x) \arrow[u, shift right, "P^{-1}"'] \arrow[d, shift right, "\mathcal{L}"'] \\
\text{复变函数空间 } F(p) \arrow[u, shift right, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

(概率密度函数)

(矩母生成函数)

$\sum_{i=1}^{n}$ 相似 $\underset{i=1}{\overset{n}{(*)}}$ 相似 $\prod_{i=1}^{n}$

x_i 独立同分布，$i=1,2,\cdots,n$

共有 n 个 $E(x_i)=\mu$ $D(x_i)=\sigma^2$

$Y_i = \dfrac{x_i-\mu}{\sigma}$，各自归一化

$E(Y_i)=0$ $D(Y_i)=1$

$\overline{x_i} = \dfrac{1}{n}\sum x_i$，平均 $E(\overline{x_i})=\mu$ $D(\overline{x_i})=\dfrac{\sigma^2}{n}$

$T_n = \dfrac{\overline{x_i}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$ 平均后归一化 $E(T_n)=0$ $D(T_n)=1$

$$T_n = \frac{\left(\frac{1}{n}\sum \sigma Y_i+\mu\right)-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sum Y_i}{\sqrt{n}}$$

$$\boxed{T_n = \frac{\sum Y_i}{\sqrt{n}}}$$

$$T_n = \frac{\sum Y_i}{d_n}$$

$$P \circ T_n = P \circ \sum_{i=1}^{n} \circ \left(\frac{Y_i}{d_n}\right)$$

$$= \left(\mathop{\ast}_{i=1}^{n}\right) \circ P \circ \left(\frac{Y_i}{d_n}\right)$$

$$= \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L} \circ P \circ \left(\frac{Y_i}{d_n}\right)$$

即 $$P(T_n) = \mathcal{L}^{-1} \circ \left(\underbrace{\mathcal{L}\left\{P\left(\frac{Y_i}{d_n}\right)\right\}}\right)^n$$

$$F(s) \overset{def}{=} \mathcal{L}\left\{P\left(\frac{Y_i}{d_n}\right)\right\}$$

先分析 $F(s)$

将 $F(s)$ 做 Taylor 展开 ($s=0$ 处)

$$F(s) = F(0) + F'(0)s + \frac{1}{2}F''(0)s^2 + o(s^2)$$

$$F(0) = E\left(\left(\frac{Y_i}{d_n}\right)^0\right) = 1$$

Y_i 归一化

$$F'(0) = -E\left(\left(\frac{Y_i}{d_n}\right)^1\right) = -\frac{1}{d_n}E(Y_i) = 0$$

$$F''(0) = E\left(\left(\frac{Y_i}{d_n}\right)^2\right) = \frac{1}{n}E(Y_i^2) = \frac{1}{n}$$

$$\therefore F(s) = 1 + \frac{1}{2n}s^2 + o(s^2)$$

所有铺垫都已完成

梳理一下思路

$$p \circ T_n = p \circ \sum_{i=1}^{n} \circ x_i = \left(\mathop{\ast}_{i=1}^{n}\right) \circ p \circ x_i$$

$$= \mathcal{L}^{-1} \circ \prod_{i=1}^{n} \circ \mathcal{L} \circ p \circ x_i$$

$$= \mathcal{L}^{-1} \circ \exp \circ \boxed{\ln \circ \prod_{i=1}^{n} \circ \underbrace{\boxed{\mathcal{L} \circ p \circ x_i}}_{F(s) = 1 + \frac{1}{2n} s^2 + o(s^2)}}$$

接下来看

$$\ln\left(1 + \frac{1}{2n} s^2 + o(s^2)\right)^n$$

$$= n \ln\left(1 + \frac{1}{2n} s^2 + o(s^2)\right)$$

$$\lim_{n \to \infty} n \ln\left(1 + \frac{1}{2n} s^2\right)$$

$$= n \times \frac{s^2}{2n} = \frac{s^2}{2}$$

$$\therefore \lim_{n \to \infty} p \circ T_n = \mathcal{L}^{-1}\left\{ e^{\frac{s^2}{2}} \right\} = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

一开始证了

$$\mathcal{L}\{P_Y(y)\} = e^{\frac{s^2}{2}}$$

$$\therefore \lim_{n\to\infty} P\left(a < \frac{\overline{x_i}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} < b\right)$$

$$\|$$

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Central Limit Theorem!

铺垫

① $\mathcal{L}\{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\} = e^{\frac{s^2}{2}}$

② $\lim_{s\to 0} (-1)^n \frac{d^n}{ds^n} \circ \mathcal{L} \circ P \circ z_i = E(z_i^n)$

s是$\mathcal{L}$后的变量

③ 令 $Y_i = \frac{x_i-\mu}{\sigma}$ x_i的归一化

$z_i = \frac{Y_i}{\sqrt{n}}$ T_n展开各项

$T_n = \frac{\overline{x_i}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \sum_{i=1}^{n} z_i$

$\overline{x_i}$的归化

总体思路

$$\lim_{n\to\infty}\circ P\circ T_n=\lim_{n\to\infty}\circ P\circ\sum_{i=1}^{n}\circ Z_i$$

$$=\lim_{n\to\infty}\circ\left(\mathop{*}_{i=1}^{n}\right)\circ P\circ Z_i$$

$$=\lim_{n\to\infty}\circ\mathcal{L}^{-1}\circ\prod_{i=1}^{n}\circ\mathcal{L}\circ P\circ Z_i$$

$$=\lim_{n\to\infty}\circ\mathcal{L}^{-1}\circ\exp\circ\ln\circ\prod_{i=1}^{n}\circ\mathcal{L}\circ P\circ Z_i$$

$$=\mathcal{L}^{-1}\circ\exp\circ\lim_{n\to\infty}\circ\ln\circ\prod_{i=1}^{n}\circ\mathcal{L}\circ P\circ Z_i$$

对这一部分 Taylor(在 $S=0$ 邻域)

展开到2阶并转化为原点矩计算

① 首先证明

$$\mathcal{L}\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\right\}=e^{\frac{s^2}{2}}$$

为后续 $\mathcal{L}^{-1}\left\{e^{\frac{s^2}{2}}\right\}$ 准备

② 再证明 $\lim_{s\to 0}(-1)^n\circ\frac{d^n}{ds^n}\circ\mathcal{L}\circ P\circ Z_i=E(Z_i^{\,n})$

为后续对 $\mathcal{L}\{P(Z_i)\}$ Taylor 展开

做准备

③ 随机变量归一化 $\begin{cases}T_i=\frac{X_i-\mu}{\sigma}\\ Z_i=\frac{T_i}{\sqrt{n}}\end{cases}$

为了 $\begin{cases}\text{CLT 定理叙述漂亮}\\ \text{Taylor 系数简洁}\end{cases}$