Oriented Models Based On Ordinary Differential Equations

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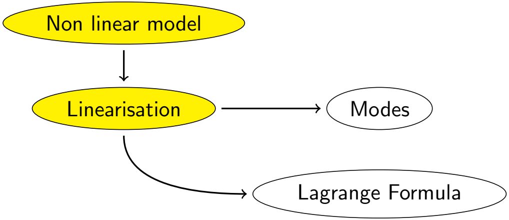
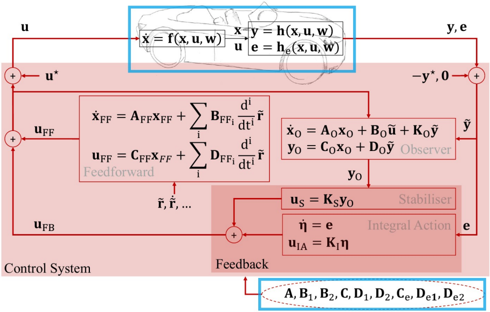
Motivations and Goals

- This course deals with the so-called **model-based** controls.
- Therefore, we need (oriented) models!
- Specifically, we focus on models written as sets of 1^{st}-order Ordinary Differential Equations (ODEs)
- The goals is the formal definition of on oriented model of this kind

Where are we?

Regarding the course contents ...





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Basic Mathematics Notions

We use the following mathematical symbols throughout this presentation:

- The colon symbol $:$ and the arrow $\to$ are read as **such that** and **to**
- The symbol $:=$ reads as **defined as**
- We use $\exists$ for **exists**
- We read the symbols $=$ and $>$ as **equal to** and **greater than**
- We read $\implies$, $\not\Longrightarrow$, and $\iff$ as **implies**, **does not imply**, and **is equivalent to**
- The symbol $\in$ reads as **belongs to**

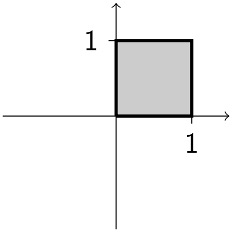
Basic Mathematics Notions

Definition - Set

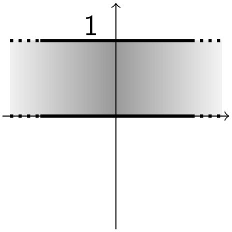
A **set** is a collection of objects, which are called **elements** of the set. Sets are usually denoted with calligraphic letters, *e.g.*, $\mathcal{X}$. However, we refer to the sets of real and natural numbers with $\mathbb{R}$ and $\mathbb{N}$

Examples

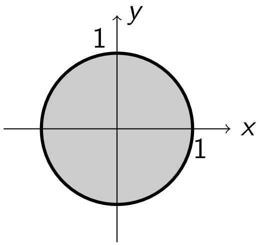
$\mathcal{X} := [0, 1] \times [0, 1]$



$\mathcal{X} := \mathbb{R} \times [0, 1]$



$\mathcal{X} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$



Basic Mathematics Notions

Definition - Matrices, Vectors

Let $a_{ij} \in \mathbb{R}$, with $i = 1, \dots, n$, $j = 1, \dots, m$, and $n, m \in \mathbb{N}$, then

$$\mathbf{A} := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

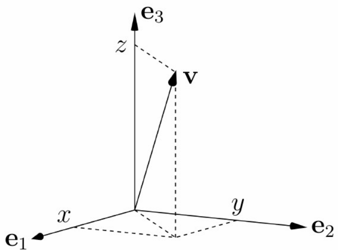
is called **matrix**. A matrix of m rows and n columns, whose entries belong to $\mathbb{R}$, represents an element of the space $\mathbb{R}^{m \times n}$, *i.e.*, $\mathbf{A} \in \mathbb{R}^{m \times n}$. When $n = 1$, a matrix is said to be a **vector**, and it is denoted with $\mathbf{v} \in \mathbb{R}^m$.

Examples

$$\mathbf{A} = \begin{bmatrix} -0.2176 & 0.0513 & 0.4669 \\ -0.3031 & 0.8261 & -0.2097 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -0.2741 \\ 1.5301 \\ -0.2490 \end{bmatrix}$$

Basic Mathematics Notions

Commonly, we graphically represent vectors as arrows in the $\mathbb{R}^n$ space. For example, let $\mathbf{e}_1 := \text{col}(1, 0, 0)$, $\mathbf{e}_2 := \text{col}(0, 1, 0)$, and $\mathbf{e}_3 := \text{col}(0, 0, 1)$ be three orthonormal vectors belonging to $\mathbb{R}^3$. Moreover, let $x, y, z \in \mathbb{R}$ be three real numbers collected to form the vector $\mathbf{v} := \text{col}(x, y, z)$. Then, we depict $\mathbf{v}$ as in the following figure by highlighting that x, y, and z represent the projections of $\mathbf{v}$ on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively.



Basic Mathematics Notions

Definition - Dot Product

The operator "$\cdot$" defines the **dot matrix product**, representative of the well-known rule **row-by-column**. In details, let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$, and $\mathbf{C} \in \mathbb{R}^{m\times p}$, with $m, n, p \in \mathbb{N}$. Let a_{ik}, b_{kj}, and c_{ij} be the elements of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, with $i = 1, \ldots, m$, $k = 1, \ldots, n$, and $j = 1, \ldots, p$. Then, $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ if

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Remark

It is worth noting that the matrix product is well-posed if the number of columns of **A** is equal to the number of rows of **B**.

Example

$$\mathbf{A} = \begin{bmatrix} 0.2176 & -0.0513 \\ -0.3031 & 0.8261 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -0.2741 \\ 1.5301 \end{bmatrix}, \mathbf{A} \cdot \mathbf{v} = \begin{bmatrix} -0.2176 \times 0.2741 - 0.0513 \times 1.5301 \\ 0.3031 \times 0.2741 + 0.8261 \times 1.5301 \end{bmatrix}$$

Basic Mathematics Notions

Definition - Transpose

Let $\mathbf{A}$ be a matrix with elements a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $n, m \in \mathbb{N}$. Then, its **transpose**, namely $\mathbf{A}^\top$, is defined as

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \ldots & a_{nm} \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} -0.2176 & 0.0513 & 0.4669 \\ -0.3031 & 0.8261 & -0.2097 \end{bmatrix}, \qquad \mathbf{A}^\top = \begin{bmatrix} -0.2176 & -0.3031 \\ 0.0513 & 0.8261 \\ 0.4669 & -0.2097 \end{bmatrix}$$

Basic Mathematics Notions

Definition - Function

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets. Then, a **function f** from $\mathcal{X}$ to $\mathcal{Y}$ is a relation that assigns to each element of $\mathcal{X}$ exactly one element of $\mathcal{Y}$. The set $\mathcal{X}$ is called the **domain** of the function and the set $\mathcal{Y}$ is called the **codomain** of the function. We encapsulate these concepts in the symbol $\mathbf{f} : \mathcal{X} \to \mathcal{Y}$. Moreover, $\mathbf{f}(\mathbf{x}) \in \mathcal{Y}$ for any $\mathbf{x} \in \mathcal{X}$.

Definition - Norm

Let $\mathcal{V}$ be a set. Then, the **norm** is a real-valued function denoted with $\| \cdot \|$ and such that $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathcal{V}$ and $\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$. Let $\mathbf{v} \in \mathbb{R}^n$ be a vector, with $n \in \mathbb{N}$. Then, we define $\|\mathbf{v}\| := \sqrt{\mathbf{v}^\top \cdot \mathbf{v}}$ as the Euclidean norm.

Definition - Continuous Function

Let $\mathbf{f} : \mathcal{X} \to \mathbb{R}^n$ be a function defined on $\mathcal{X} \subseteq \mathbb{R}^m$, with $n, m \in \mathbb{N}$. Then, we say that $\mathbf{f}$ is **continuous** on $\mathcal{X}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that all $\mathbf{s} \in \mathcal{X}$ satisfying $\| \mathbf{s} - \mathbf{x} \| < \delta$ will also satisfy $\|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{x})\| < \varepsilon$ for all $\mathbf{x} \in \mathcal{X}$.

Basic Mathematics Notions

Definition - Linear Function

Let $\mathcal{X}$ and $\mathcal{Y}$ be to sets, then a function $\mathbf{f} : \mathcal{X} \to \mathcal{Y}$ is said to be **linear** if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and for any $\alpha \in \mathbb{R}$ the following relations hold:

$$\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2), \qquad \mathbf{f}(\alpha\, \mathbf{x}_1) = \alpha \mathbf{f}(\mathbf{x}_1).$$

Definition - Differentiable Function

Let $\mathbf{f} : \mathcal{X} \to \mathbb{R}^n$ be a function, with $n \in \mathbb{N}$. It is said to be **differentiable** at point $\mathbf{x}_0 \in \mathcal{X}$ if there exists a linear function $\mathbf{J} : \mathcal{X} \to \mathbb{R}^n$ such that

$$\lim_{\mathbf{h} \to 0} \frac{\| \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{J}(\mathbf{h}) \|}{\| \mathbf{h} \|} = \mathbf{0}\,.$$

It is worth noting that differentiability implies continuity.

Definition - class $\mathcal{C}^1$

The class $\mathcal{C}^1$ consists of all differentiable functions whose derivatives are continuous; such functions are called **continuously differentiable**.

Basic Mathematics Notions

Definition - Time Derivative

Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$, with $n \in \mathbb{N}$, $\mathbf{f} \in \mathcal{C}^1$ and let $t \in \mathbb{R}$ be the time. Then, we define the *time derivative* of $\mathbf{f}$ as

$$\dot{\mathbf{f}} := \frac{d}{dt}\mathbf{f}(t).$$

The following figure reports the graphical representation of the derivative $\dot{\mathbf{f}}$

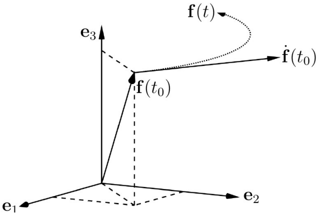


Figure: The dotted line graphically represents the function $\mathbf{f}$. The arrow at the end of the dotted line denotes the time evolution direction. The evaluations $\mathbf{f}(t)$ and $\dot{\mathbf{f}}(t)$, at time $t = t_0$, are represented as vectors. Note that $\dot{\mathbf{f}}(t)$ is tangent to $\mathbf{f}(t)$ for all t.

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Simulation Model

Let us give a name to each variable appearing in the oriented model

What are we talinkg about?

system

in British English

('sɪstəm)

NOUN

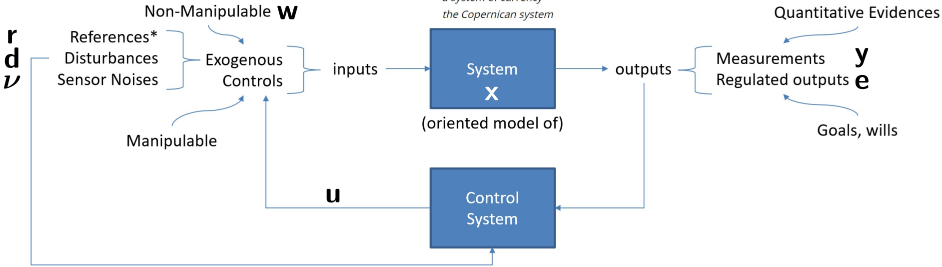
Word Frequency

1. a group or combination of interrelated, interdependent, or interacting elements forming a collective entity; a methodical or coordinated assemblage of parts, facts, concepts, etc

   a system of currency

   the Copernican system

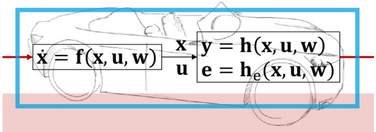
More formally



Simulation Model

Non linear model

- $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$, with $n \in \mathbb{N}$, called **state**
- $\mathbf{u} : \mathbb{R} \to \mathbb{R}^p$, with $p \in \mathbb{N}$, called **control inputs**
- $\mathbf{w} : \mathbb{R} \to \mathbb{R}^r$, with $r \in \mathbb{N}$, $\mathbf{w} := \text{col}(\mathbf{d}, \boldsymbol{\nu}, \mathbf{r})$, called **exogenous**
- $\mathbf{y} : \mathbb{R} \to \mathbb{R}^q$, with $q \in \mathbb{N}$, called **measurements / outputs**
- $\mathbf{e} : \mathbb{R} \to \mathbb{R}^m$, with $m \in \mathbb{N}$, called **regulated outputs / errors**
- $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^n$, called **process model**
- $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^q$, called **output function**
- $\mathbf{h}_e : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^m$, called **error function**



define a 1st-order ODEs-based oriented model if they satisfy

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\
\mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \qquad (1) \\
\mathbf{e} &= \mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w})
\end{aligned}
$$

We introduce $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ are called **initial state** and **initial time**, for the evaluation of (1). We refer to (1) as the **simulation model**

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Linearisation

Definition - Partial Derivatives

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function, with $m \in \mathbb{N}$. Take $\mathbf{x} \in \mathbb{R}^m$ whose elements are $x_i \in \mathbb{R}$, with $i = 1, \ldots, m$. Let $f(x_1, \ldots, x_m)$ denote the evaluation of f at $\mathbf{x}$. Then, we define the **partial derivatives** of f at $\mathbf{x}$ as

$$\frac{\partial f}{\partial x_i} := \left. \frac{df(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_m)}{ds} \right|_{s=x_i}$$

Definition - Jacobian

Let $f_i : \mathbb{R}^m \to \mathbb{R}$, with $i = 1, \ldots, n$, be $\mathcal{C}^1$-class functions, with $n, m \in \mathbb{N}$. Define $\mathbf{f} := [f_1, \ldots, f_n]^\top$. Let $\mathbf{x} \in \mathbb{R}^m$, with elements $x_i \in \mathbb{R}$, $i = 1, \ldots, m$. Then, we define the **Jacobian** as $\partial \mathbf{f} / \partial \mathbf{x} : \mathbb{R}^m \to \mathbb{R}^{n \times m}$ such that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

Linearisation

Definition - (Big) "O" Limiting Behaviour

Let $f, g : \mathbb{R}^m \to \mathbb{R}^n$ be two functions, with $m, n \in \mathbb{N}$. Take $\mathbf{x}^\star \in \mathbb{R}^m$. Then we say $f(\mathbf{x}) = O(g(\mathbf{x}))$ for $\mathbf{x} \to \mathbf{x}^\star$ if and only if there exist $\rho, c > 0$ such that

$$\|f(\mathbf{x})\| \leq c\|g(\mathbf{x})\| \quad \forall\, \mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{x}^\star\| \leq \rho$$

Taylor's Theorem

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function, with $f \in \mathcal{C}^2$. Then, for any $x^\star \in \mathbb{R}^m$, the difference

$$f(\mathbf{x}) - f(\mathbf{x}^\star) - \left[\left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^\star} \right] \cdot (\mathbf{x} - \mathbf{x}^\star) = O(\|\mathbf{x} - \mathbf{x}^\star\|^2)$$

for $\mathbf{x} \to \mathbf{x}^\star$.

Linearisation

Definition - Equilibrium triplet, output, and error

Consider the oriented model (1), hereafter recalled

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}), \quad \mathbf{e} = \mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w})$$

Then, we say $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ is an **equilibrium triplet** if $\mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) = \mathbf{0}$. Moreover, we call $\mathbf{y}^\star := \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ and $\mathbf{e}^\star := \mathbf{h}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ the **equilibrium output** and **equilibrium error**.

Remark

Usually, the equilibrium triplet is designed such that $\mathbf{e}^\star = \mathbf{0}$.

Definition - Variations

Consider the oriented model (1) and let $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$, $\mathbf{y}^\star$, and $\mathbf{e}^\star$ be the equilibrium triplet, output, and error. Then, we define the variations as

$$\tilde{\mathbf{x}}_{\text{NL}} := \mathbf{x} - \mathbf{x}^\star, \quad \tilde{\mathbf{u}} := \mathbf{u} - \mathbf{u}^\star, \quad \tilde{\mathbf{w}} := \mathbf{w} - \mathbf{w}^\star, \quad \tilde{\mathbf{y}}_{\text{NL}} := \mathbf{y} - \mathbf{y}^\star, \quad \tilde{\mathbf{e}}_{\text{NL}} := \mathbf{e} - \mathbf{e}^\star$$

Linearisation

We compute the dynamics of $\tilde{\mathbf{x}}_{\mathrm{NL}}$, exploiting $\dot{\mathbf{x}}^\star = \mathbf{0}$, $\mathbf{x} = \mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}$, $\mathbf{u} = \mathbf{u}^\star + \tilde{\mathbf{u}}$, and $\mathbf{w} = \mathbf{w}^\star + \tilde{\mathbf{w}}$, as

$$\dot{\tilde{\mathbf{x}}}_{\mathrm{NL}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}^\star = \mathbf{f}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}), \quad \tilde{\mathbf{x}}(\mathbf{t_0}) = \tilde{\mathbf{x}}_\mathbf{0} := \mathbf{x_0} - \mathbf{x}^\star \tag{2a}$$

and we rewrite the output and error variations as

$$\begin{aligned} \tilde{\mathbf{y}}_{\mathrm{NL}} &= \mathbf{h}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) \\ \tilde{\mathbf{e}}_{\mathrm{NL}} &= \mathbf{h}_e(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) \end{aligned} \tag{2b}$$

Now, define the following Jacobians

$$\mathbf{A} := \left.\frac{\partial\, \mathbf{f}(\mathbf{s}, \mathbf{u}^\star, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{x}^\star} \qquad \mathbf{B}_1 := \left.\frac{\partial\, \mathbf{f}(\mathbf{x}^\star, \mathbf{s}, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{u}^\star} \qquad \mathbf{B}_2 := \left.\frac{\partial\, \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{s})}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{w}^\star}$$

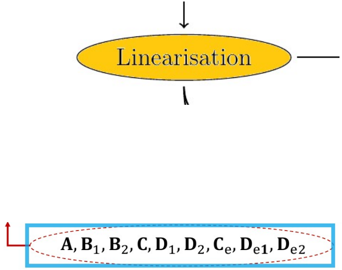
$$\mathbf{C} := \left.\frac{\partial\, \mathbf{h}(\mathbf{s}, \mathbf{u}^\star, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{x}^\star} \qquad \mathbf{D}_1 := \left.\frac{\partial\, \mathbf{h}(\mathbf{x}^\star, \mathbf{s}, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{u}^\star} \qquad \mathbf{D}_2 := \left.\frac{\partial\, \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{s})}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{w}^\star}$$

$$\mathbf{C}_e := \left.\frac{\partial\, \mathbf{h}_e(\mathbf{s}, \mathbf{u}^\star, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{x}^\star} \qquad \mathbf{D}_{e1} := \left.\frac{\partial\, \mathbf{h}_e(\mathbf{x}^\star, \mathbf{s}, \mathbf{w}^\star)}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{u}^\star} \qquad \mathbf{D}_{e2} := \left.\frac{\partial\, \mathbf{h}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{s})}{\partial\, \mathbf{s}}\right|_{\mathbf{s}=\mathbf{w}^\star}$$

Linearisation

and rewrite the variational system (2) as

$$
\begin{aligned}
\dot{\tilde{\mathbf{x}}}_{\mathrm{NL}} &= \mathbf{A}\cdot\tilde{\mathbf{x}}_{\mathrm{NL}} + \mathbf{B}_1\cdot\tilde{\mathbf{u}} + \mathbf{B}_2\cdot\tilde{\mathbf{w}} + O(\|(\tilde{\mathbf{x}}_{\mathrm{NL}},\tilde{\mathbf{u}},\tilde{\mathbf{w}})\|^2), \qquad \tilde{\mathbf{x}}_{\mathrm{NL}}(t_0) = \tilde{\mathbf{x}}_0\\
\tilde{\mathbf{y}}_{\mathrm{NL}} &= \mathbf{C}\cdot\tilde{\mathbf{x}}_{\mathrm{NL}} + \mathbf{D}_1\cdot\tilde{\mathbf{u}} + \mathbf{D}_2\cdot\tilde{\mathbf{w}} + O(\|(\tilde{\mathbf{x}}_{\mathrm{NL}},\tilde{\mathbf{u}},\tilde{\mathbf{w}})\|^2)\\
\tilde{\mathbf{e}}_{\mathrm{NL}} &= \mathbf{C}_e\cdot\tilde{\mathbf{x}}_{\mathrm{NL}} + \mathbf{D}_{e1}\cdot\tilde{\mathbf{u}} + \mathbf{D}_{e2}\cdot\tilde{\mathbf{w}} + O(\|(\tilde{\mathbf{x}}_{\mathrm{NL}},\tilde{\mathbf{u}},\tilde{\mathbf{w}})\|^2)
\end{aligned}
$$



Finally, we define the so-called **design model** as

$$
\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \mathbf{A}\cdot\tilde{\mathbf{x}} + \mathbf{B}_1\cdot\tilde{\mathbf{u}} + \mathbf{B}_2\cdot\tilde{\mathbf{w}} \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0\\
\tilde{\mathbf{y}} &= \mathbf{C}\cdot\tilde{\mathbf{x}} + \mathbf{D}_1\cdot\tilde{\mathbf{u}} + \mathbf{D}_2\cdot\tilde{\mathbf{w}}\\
\tilde{\mathbf{e}} &= \mathbf{C}_e\cdot\tilde{\mathbf{x}} + \mathbf{D}_{e1}\cdot\tilde{\mathbf{u}} + \mathbf{D}_{e2}\cdot\tilde{\mathbf{w}}
\end{aligned}
$$

Remark

The subscript *NL* stands for **non-linear**. We have that $\tilde{\mathbf{x}} \approx \tilde{\mathbf{x}}_{\mathrm{NL}}$, $\tilde{\mathbf{y}} \approx \tilde{\mathbf{y}}_{\mathrm{NL}}$, and $\tilde{\mathbf{e}} \approx \tilde{\mathbf{e}}_{\mathrm{NL}}$ only if $\|(\tilde{\mathbf{x}}_{\mathrm{NL}},\tilde{\mathbf{u}},\tilde{\mathbf{w}})\|$ is sufficiently small to neglect the terms $O(\|(\tilde{\mathbf{x}}_{\mathrm{NL}},\tilde{\mathbf{u}},\tilde{\mathbf{w}})\|^2)$.

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