Analysis and Design of Control Laws for Advanced Driver-Assistance Systems
Theory and Applications

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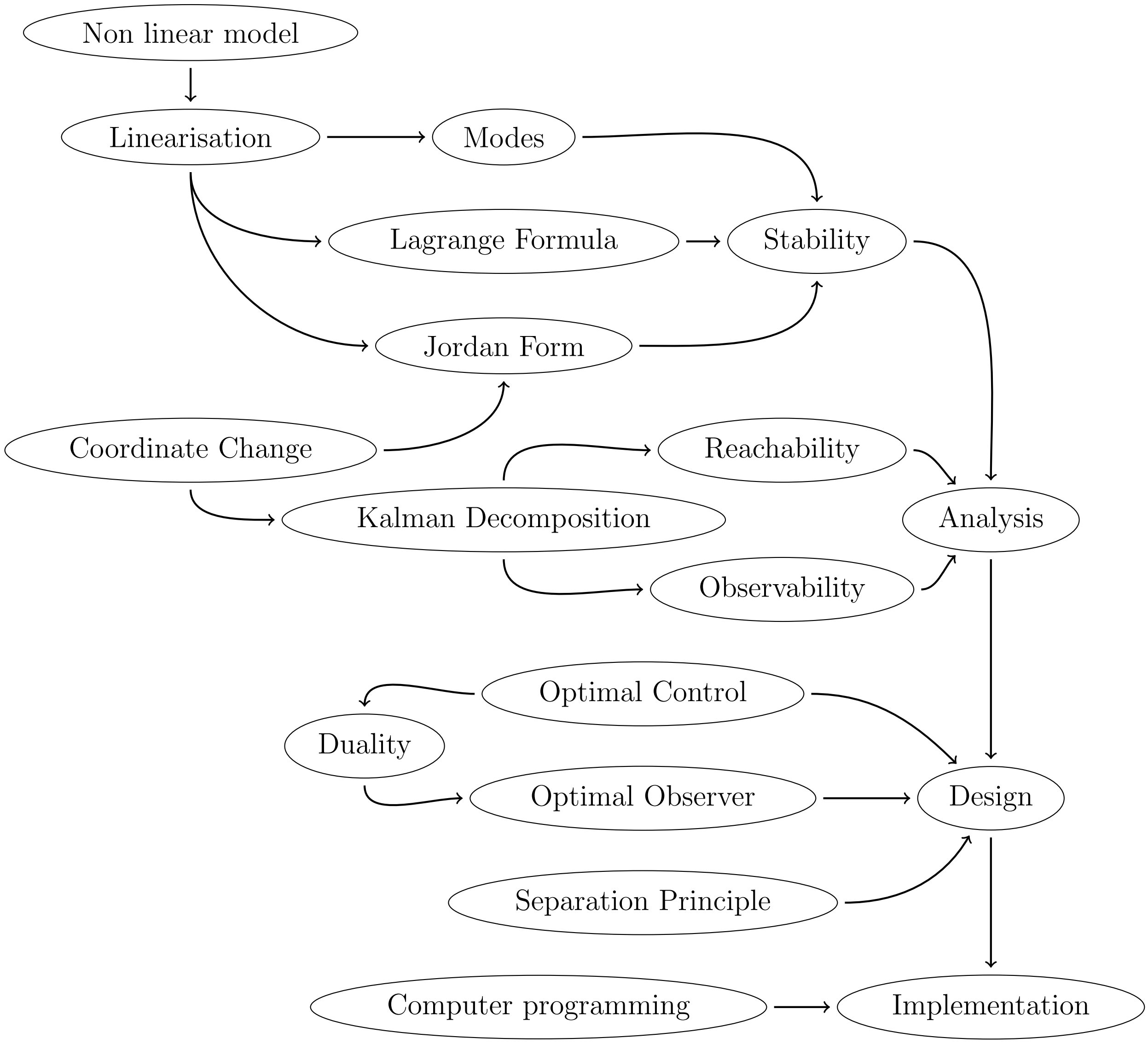


Figure 1: Control system design procedure

Contents

Notation

This book denotes sets of natural, real, and complex numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$. The elements of $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ are represented with lowercase letters, *e.g.*, $x \in \mathbb{R}$. Sets are denoted by calligraphic letters, *e.g.*, $\mathcal{X} \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$. Capital letters in outline, like $\mathbb{V}(\mathbb{C})$, are adopted to denote vector spaces. Vectors and matrices are denoted by bold lowercase and uppercase letters, *e.g.*, let n, $m \in \mathbb{N}$, then $\mathbf{x} \in \mathbb{R}^n$ denotes a vector whereas $\mathbf{X} \in \mathbb{R}^{n \times m}$ represents a matrix.

Let n matrices $\mathbf{X}_i \in \mathbb{R}^{n_i \times m}$ with n, n_i, $m \in \mathbb{N}$. Then, the column operator $\mathrm{col}(\cdot) : \mathbb{R}^{n_1 \times m} \times \cdots \times \mathbb{R}^{n_n \times m} \to \mathbb{R}^{\left(\sum_{i=1}^{n} n_i\right) \times m}$ is defined as

$$\mathrm{col}(\mathbf{X}_1, \ldots, \mathbf{X}_n) := \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{bmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} = \mathrm{col}(x_1, \ldots, x_n)$ and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$. Then, the Jacobian of $\mathbf{f}$ is denoted as $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} \in \mathbb{R}^{m \times n}$ with $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} := \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \ldots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix}$.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{A}$, and $\mathbf{B}$ be two vectors and two matrices of proper dimensions. Then, to reduce the notation complexity, this book denotes the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ with $\mathbf{x}^\top \mathbf{y}$ and the dot products $\mathbf{A} \cdot \mathbf{x}$ and $\mathbf{A} \cdot \mathbf{B}$ with $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{B}$.

Let $\mathbf{x} \in \mathbb{R}^n$, then the Euclidean norm of $\mathbf{x}$ is denoted by $\|\mathbf{x}\| := \sqrt{\mathbf{x}^\top \mathbf{x}}$.

The lower and upper bounds of matrices are defined through $\underline{\sigma}$, $\overline{\sigma} : \mathbb{R}^{n \times m} \to \mathbb{R}^+$ such that, for any $\mathbf{X} \in \mathbb{R}^{n \times m}$, we have

$$\underline{\sigma}(\mathbf{X}) = \inf_{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| = 1} \frac{\|\mathbf{X}\,\mathbf{x}\|}{\|\mathbf{x}\|}, \qquad \overline{\sigma}(\mathbf{X}) = \sup_{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| = 1} \frac{\|\mathbf{X}\,\mathbf{x}\|}{\|\mathbf{x}\|}.$$

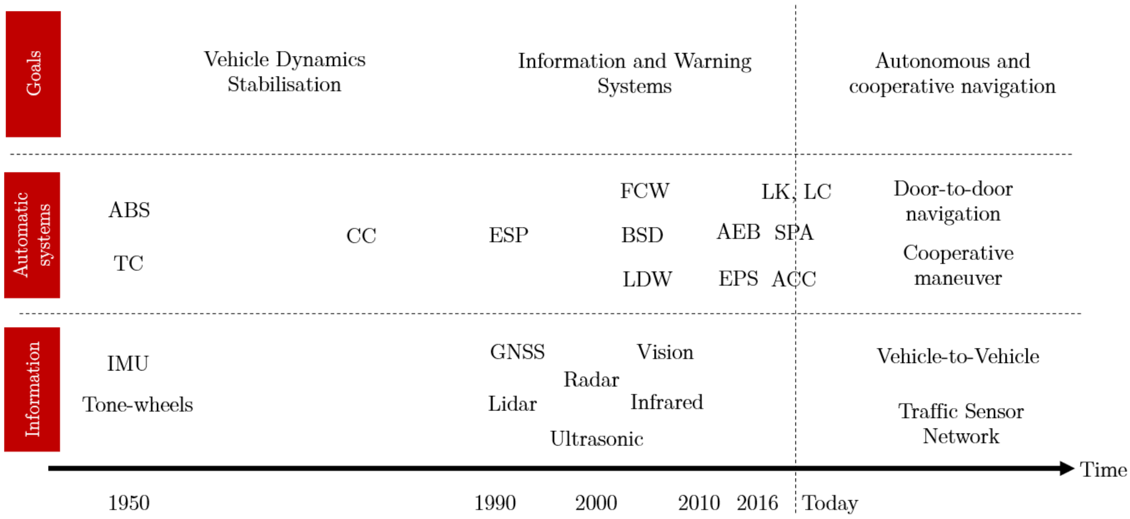


Figure 1.2: Historical evolution of automatic control system in automotive.

1.2 Plant Models

In the context of this textbook, an automotive system is conceived as a plant composed of a vehicle, a driver, actuators, and sensors, see Fig. 1.3. The suite of sensors collects information from both the plant (IMU, GNSS, etc.) and the driver (cameras, steering column torque meters, etc.). Information and warning systems elaborate these data to create higher-level information displayed to the driver (information messages, lamps, steering wheel vibrations, warning sounds, etc.). The driver, whose sensors are kinaesthetic perception and senses, can control the vehicle directly and through the automatic control systems. On the other hand, these latter elaborate control actions on the basis of the information provided by the sensor suite and accordingly to the driver inputs. Dedicated actuators complete the automatic control system. They affect the vehicle's dynamics by applying the control inputs. It is worth noting that for safety reasons, the control inputs of the driver and those of the automatic system actuators are parallel. This parallelism lets the driver take over control in case of malfunctions.

The goal of any control system is that of *modifying* the natural behavior of the **plant** under investigation [1, 11, 13, 20, 26, 27]. But, what does **plant** mean in the context of control systems?

We conceive the **plant** as a system, in which we have identified two groups of signals, namely **inputs** and **outputs**, as described in Fig. 1.4. Therefore, the plant represents the link between inputs (causes) and outputs (effects). Moreover, the inputs can be further classified as **controls** and **exogenous signals**. The controls are manipulable input signals (exploitable to modify the system's behavior), whereas

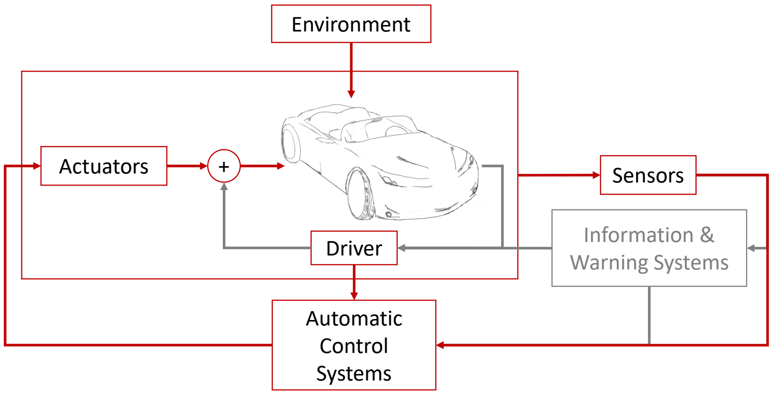


Figure 1.3: Typical control system architecture for automotive applications. The grey parts are out of the scope of this textbook.

the exogenous signals cannot be governed to change the plant's behavior. Like the inputs, the outputs can be divided into two clusters, namely **measurable outputs** (or simply outputs) and **controlled outputs**. The first group represents the set of all the data available thanks to the sensors installed in the plant. The second denotes the variables that should be modified through the control action (often, the controlled outputs are a subset of the measured output).

It is worth noting that the exogenous signals do not depend on the system's behavior. Instead, they collect disturbances, namely $\mathbf{d}$, sensor noises, denoted by $\boldsymbol{\nu}$, and the reference signal the controlled output should track, identified by the symbol $\mathbf{r}$.

Let p, r, q, m $\in \mathbb{N}$ be positive integers, then declare with $\mathbf{u}$: $\mathbb{R} \rightarrow \mathbb{R}^p$ the control vector, with $\mathbf{w}$: $\mathbb{R} \rightarrow \mathbb{R}^r$ the exogenous vector, usually defined as $\mathbf{w} = \mathrm{col}(\mathbf{d}, \boldsymbol{\nu}, \mathbf{r})$, with $\mathbf{y}$: $\mathbb{R} \rightarrow \mathbb{R}^q$ the output vector of the plant, and with $\mathbf{e}$: $\mathbb{R} \rightarrow \mathbb{R}^m$ the controlled outputs. Based on the dimensions of input and output vectors, we classify the plants as Multi-Input Multi-Output (MIMO) if p, $q > 1$ and Single-Input Single-Output (SISO) if $p = q = 1$, with MISO and SIMO representing intermediate configurations.

A mathematical model, usually given by a set of ordinary differential nonlinear equations in the so-called Input-State-Output (or State Space) representation, describes the plant's behavior. In more detail, given the inputs $\mathbf{u}$, the exogenous $\mathbf{w}$, and the outputs $\mathbf{y}$ and $\mathbf{e}$, the plant dynamics are modeled as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{e} &= \mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w}) \end{aligned} \tag{1.1}$$

where $\mathbf{x}$: $\mathbb{R} \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$, is the **state** vector and $t_0 \in \mathbb{R}$. For example, ground vehicles are systems whose inputs are the throttle, the steering angle, and the brake

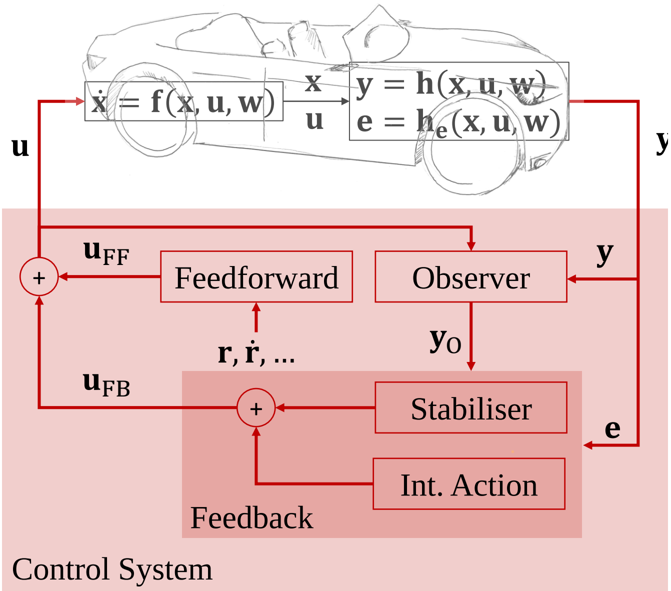
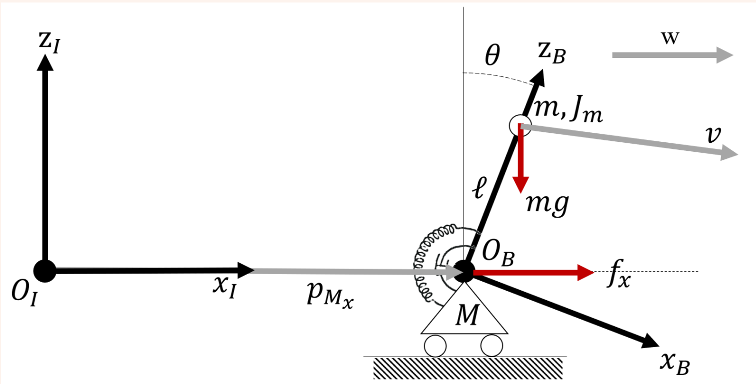


Figure 1.4: Ground vehicles as a plant with inputs, disturbances, states, and outputs. The control system architecture consists of three blocks. First, the output $\mathbf{y}$ and the input $\mathbf{u}$ are elaborated through an observer that provides $\mathbf{y}_O$ which represents a proxy of $\mathbf{x}$. This information and the couple $(\mathbf{y}, \mathbf{e})$ are passed to the feedback control which elaborates the control $\mathbf{u}_{\text{FB}}$. On the other hand, the feed-forward control takes as input the reference $\mathbf{r}$, and eventually its derivatives, and generates the signal $\mathbf{u}_{\text{FF}}$. The two control signal are summed to create $\mathbf{u}$.

pressure, and whose exogenous signals are the wind field, the road inclination, the reference speed, the noises affecting the sensors, etc. On the other hand, outputs are the measurements of tachometer, GNSS receiver, accelerometers, gyroscopes, magnetometers, potentiometers, engine tone-wheels, engine pressures, airflow sensors, etc. Moreover, the controlled outputs could be the vehicle speed, the turning rate, the wheel speed, the chassis vibrations, etc. Finally, the state could (indeed it depends on the mathematical representation adopted for the description of the dynamics) be represented by inertial positions and speeds, attitude, angular speeds of body, wheels, engine, etc.

Example 1.1 (The cart-pole model)**.** This example aims to describe the identification process, which leads to the definition of the plant model. To achieve this goal identify the state $\mathbf{x}$, the control $\mathbf{u}$, the exogenous signals $\mathbf{w}$, the output $\mathbf{y}$ as well as the functions $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$, $\mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$, and $\mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w})$. This example focuses on the cart-pole model obtained through a Lagrangian mechanics approach, see [8]. Introduce two reference frames, the first conceived as inertial and identified by the axes $x_I - O_I - z_I$, and the second attached to the base of the pole and denoted with $x_B - O_B - z_B$. The plant represents a cart of mass $M > 0$ and a pole of mass $m > 0$, and inertia $J_m > 0$. The link between these two masses is rigid, and its length is $\ell > 0$. The cart position is p_{M_x}, whereas the attitude of the pole to the vertical axis is θ. The system is subject to the gravity acceleration g and two external forces. The first represents the aerodynamic drag acting on the pole. In contrast, f_x is a horizontal force applied to the cart. The aerodynamic drag is function of the air density $\rho > 0$, the pole cross-section $S > 0$, and the drag coefficient $C_D > 0$. Finally, $\mathtt{w}$ denotes the horizontal wind speed.



The nonlinear system

$$
\begin{bmatrix} m+M & m\ell\cos\theta \\ m\ell\cos\theta & J_m + m\ell^2 \end{bmatrix} \begin{bmatrix} \ddot{p}_{M_x} \\ \ddot{\theta} \end{bmatrix} =
\begin{bmatrix} m\ell\dot{\theta}^2\sin\theta + f_x - \dfrac{1}{2}\rho S v C_D(\dot{p}_{M_x} - \mathtt{w} + \ell\cos\theta\dot{\theta}) \\ \ell g m \sin\theta - k\theta - \mu\dot{\theta} - \dfrac{1}{2}\rho S v C_D \ell(\cos\theta(\dot{p}_{M_x} - \mathtt{w}) + \ell\dot{\theta}) \end{bmatrix} \tag{1.2}
$$

represents the dynamics of the cart-pole with

$$
v = \sqrt{(\dot{p}_{M_x} + \cos\theta\ell\dot{\theta} - \mathtt{w})^2 + (\sin\theta\ell\dot{\theta})^2}.
$$

Let $\dot{p}_{M_x} = v_{M_x}$ and $\dot{\theta} = \omega$ be changes of variables, then define the state $\mathbf{x} \in \mathbb{R}^4$ as the vector $\mathbf{x} := \mathrm{col}(p_{M_x}, v_{M_x}, \theta, \omega)$, and the control input $u \in \mathbb{R}$ as $u := f_x$. The sensor suite is composed of an exteroceptive sensor providing p_{M_x}, an odometer measuring v_{M_x}, and a potentiometer providing θ. Moreover, we assume that the sensors are affected by the noises ν_p, ν_v, and $\nu_\theta \in \mathbb{R}$. Let $\boldsymbol{\nu} :=$

$\mathrm{col}(\nu_p, \nu_v, \nu_\theta)$, then the output vector $\mathbf{y} \in \mathbb{R}^3$ is defined as $\mathbf{y} := \mathrm{col}(p_{M_x}, v_{M_x}, \theta) + \boldsymbol{\nu}$. Furthermore, the regulated output is $e = (p_{M_x} + \nu_p) + \ell \sin(\theta + \nu_\theta) - p_R$, where $p_{M_x} + \ell \sin\theta$ represents the horizontal position of the pole and p_R is the reference to be tracked. The exogenous $\mathbf{w} \in \mathbb{R}^4$ is $\mathbf{w} := \mathrm{col}(\mathtt{w}, \boldsymbol{\nu}, p_R)$. Thanks to these definitions, Eq. (1.2) is rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & m\ell\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & m\ell\cos\theta & 0 & J_m + m\ell^2 \end{bmatrix} \begin{bmatrix} \dot{p}_{M_x} \\ \dot{v}_{M_x} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v_{M_x} \\ m\ell\dot{\theta}^2 \sin\theta + f_x - \frac{1}{2}\rho S v C_D(\dot{p}_{M_x} - \mathtt{w} + \ell\cos\theta\dot{\theta}) \\ \omega \\ \ell g m \sin\theta - k\theta - \mu\dot{\theta} - \frac{1}{2}\rho S v C_D \ell(\cos\theta(\dot{p}_{M_x} - \mathtt{w}) + \ell\dot{\theta}) \end{bmatrix}. \tag{1.3}$$

Define

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & m\ell\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & m\ell\cos\theta & 0 & J_m + m\ell^2 \end{bmatrix}$$

and write (1.3) in the form (1.1) with

$$\mathbf{f}(\mathbf{x}, u, \mathbf{w}) := \mathbf{F}^{-1}(\mathbf{x}) \begin{bmatrix} v_{M_x} \\ m\ell\omega^2 \sin\theta + f_x - \frac{1}{2}\rho S v C_D(v_{M_x} - \mathtt{w} + \ell\cos\theta\omega) \\ \omega \\ \ell g m \sin\theta - k\theta - \mu\omega - \frac{1}{2}\rho S v C_D \ell(\cos\theta(v_{M_x} - \mathtt{w}) + \ell\omega) \end{bmatrix}$$

$$\mathbf{h}(\mathbf{x}, u, \mathbf{w}) := \begin{bmatrix} p_{M_x} + \nu_p \\ v_{M_x} + \nu_v \\ \theta + \nu_\theta \end{bmatrix},$$

$$h_e(\mathbf{x}, u, \mathbf{w}) := (p_{M_x} + \nu_p) + \ell\sin(\theta + \nu_\theta) - p_R,$$

and

$$\mathbf{F}^{-1}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{J_m+m\ell^2}{\Delta} & 0 & -\frac{m\ell\cos\theta}{\Delta} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{m\ell\cos\theta}{\Delta} & 0 & \frac{m+M}{\Delta} \end{bmatrix}$$

with $\Delta = (J_m + m\ell^2)(m+M) - (m\ell\cos\theta)^2 > 0$.

1.2.1 Linearization

Let (1.1) be a nonlinear system. This textbook assumes that $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^n$ and $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^q$ are smooth and locally Lipschitz in $\mathbf{x}$ (the latter assumption is needed to guarantee the existence and uniqueness of solutions of (1.1)). Moreover, let $\mathbf{u}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^p$ and $\mathbf{w}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^r$ be the reference input and exogenous signals and assume that there exists a unique integral curve

$$\mathbf{x}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^n \tag{1.4}$$

such that

$$\begin{aligned} \dot{\mathbf{x}}^\star &= \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) \qquad \mathbf{x}^\star(t_0) = \mathbf{x}_0^\star \\ \mathbf{y}^\star &= \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) \\ \mathbf{0} &= \mathbf{h}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star). \end{aligned} \tag{1.5}$$

Let the functions $\mathbf{u} : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^p$, $\mathbf{w} : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^r$, the state $\mathbf{x}$, and the output $\mathbf{y}$ be defined in agreement with (1.1), and introduce the errors

$$\tilde{\mathbf{x}}_{\mathrm{NL}} := \mathbf{x} - \mathbf{x}^\star, \quad \tilde{\mathbf{u}} := \mathbf{u} - \mathbf{u}^\star \quad \tilde{\mathbf{w}} := \mathbf{w} - \mathbf{w}^\star. \tag{1.6}$$

The dynamics of $\tilde{\mathbf{x}}_{\mathrm{NL}}$ is given by

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_{\mathrm{NL}} &= \dot{\mathbf{x}}_{\mathrm{NL}} - \dot{\mathbf{x}}^\star = \\ &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) - \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) = \\ &= \mathbf{f}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) - \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star). \end{aligned} \tag{1.7}$$

Since $\mathbf{f}$ is differentiable, we can write

$$\begin{aligned} &\mathbf{f}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) = \\ &\quad = \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\tilde{\mathbf{x}}_{\mathrm{NL}} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\tilde{\mathbf{u}} + \frac{\partial \mathbf{f}}{\partial \mathbf{w}}\tilde{\mathbf{w}} + O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2). \end{aligned} \tag{1.8}$$

The substitution of (1.8) into (1.7) leads to

$$\dot{\tilde{\mathbf{x}}}_{\mathrm{NL}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\tilde{\mathbf{x}}_{\mathrm{NL}} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\tilde{\mathbf{u}} + \frac{\partial \mathbf{f}}{\partial \mathbf{w}}\tilde{\mathbf{w}} + O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2). \tag{1.9}$$

On the other hand, the output error $\tilde{\mathbf{y}}_{\mathrm{NL}} := \mathbf{y} - \mathbf{y}^\star$ is written as

$$\tilde{\mathbf{y}} = \mathbf{h}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) - \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) \tag{1.10}$$

where, as previously done for $\mathbf{f}$, $\mathbf{h}$ can also be written via a Taylor polynomial as

$$\begin{aligned} &\mathbf{h}(\mathbf{x}^\star + \tilde{\mathbf{x}}_{\mathrm{NL}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) = \\ &\quad \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\tilde{\mathbf{x}}_{\mathrm{NL}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\tilde{\mathbf{u}} + \frac{\partial \mathbf{h}}{\partial \mathbf{w}}\tilde{\mathbf{w}} + O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2). \end{aligned} \tag{1.11}$$

The output error is then rewritten by substituting (1.11) into (1.10):

$$\tilde{\mathbf{y}}_{\mathrm{NL}} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\tilde{\mathbf{x}}_{\mathrm{NL}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\tilde{\mathbf{u}} + \frac{\partial \mathbf{h}}{\partial \mathbf{w}}\tilde{\mathbf{w}} + O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2). \tag{1.12}$$

Moreover, also the regulated output error, *i.e.*, $\tilde{\mathbf{e}}_{\mathrm{NL}} := \mathbf{e} - \mathbf{0}$, can be formally expressed as

$$\tilde{\mathbf{e}}_{\mathrm{NL}} = \frac{\partial \mathbf{h}_e}{\partial \mathbf{x}}\tilde{\mathbf{x}}_{\mathrm{NL}} + \frac{\partial \mathbf{h}_e}{\partial \mathbf{u}}\tilde{\mathbf{u}} + \frac{\partial \mathbf{h}_e}{\partial \mathbf{w}}\tilde{\mathbf{w}} + O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2) \tag{1.13}$$

exploiting the same strategy used to obtain (1.12). For small values of errors (1.6), the linearization of (1.1) in the neighborhood of $\mathbf{x}^\star$ is defined as [6]

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{x}} + \mathbf{B}_1(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{u}} + \mathbf{B}_2(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{w}} \\ \tilde{\mathbf{y}} &= \mathbf{C}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{x}} + \mathbf{D}_1(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{u}} + \mathbf{D}_2(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{w}} \\ \tilde{\mathbf{e}} &= \mathbf{C}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{x}} + \mathbf{D}_{e_1}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{u}} + \mathbf{D}_{e_2}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)\tilde{\mathbf{w}} \\ \tilde{\mathbf{x}}(t_0) &= \tilde{\mathbf{x}}_0 \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star} \\ \mathbf{B}_1(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star} \\ \mathbf{B}_2(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star} \end{aligned}$$

$$\begin{aligned} \mathbf{C}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star}, & \mathbf{C}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}_e}{\partial \mathbf{x}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star} \\ \mathbf{D}_1(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star}, & \mathbf{D}_{e_1}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}_e}{\partial \mathbf{u}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star} \\ \mathbf{D}_2(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star}, & \mathbf{D}_{e_2}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) &:= \left.\frac{\partial \mathbf{h}_e}{\partial \mathbf{w}}\right|_{\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star}. \end{aligned}$$

Remark 1.1. The subscript NL appearing in Equations (1.6)-(1.13) stand for *non-linear* and has been introduced to stress the difference between the non-linear errors $\tilde{\mathbf{x}}_{\mathrm{NL}}$, $\tilde{\mathbf{y}}_{\mathrm{NL}}$, and $\tilde{\mathbf{e}}_{\mathrm{NL}}$ and the linearised errors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, and $\tilde{\mathbf{e}}$. It is worth remembering that $\tilde{\mathbf{x}} \approx \tilde{\mathbf{x}}_{\mathrm{NL}}$, $\tilde{\mathbf{y}} \approx \tilde{\mathbf{y}}_{\mathrm{NL}}$, and $\tilde{\mathbf{e}} \approx \tilde{\mathbf{e}}_{\mathrm{NL}}$ only if $\|(\tilde{\mathbf{x}}_{\mathrm{NL}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}})\|$ is small enough to neglect the terms $O(\|\tilde{\mathbf{x}}_{\mathrm{NL}}\|^2, \|\tilde{\mathbf{u}}\|^2, \|\tilde{\mathbf{w}}\|^2)$.

Remark 1.2. It is worth noting that the matrices $\mathbf{A}$, $\mathbf{B}_1$, etc. appearing in (1.14) are time-varying if at least one element of the triplet $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ is time-varying. Thus, the mentioned matrices are constant if and only if all the elements of the triplet $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ are constant. Let us assume that there exist three constant vectors $\mathbf{u}_0^\star$, $\mathbf{w}_0^\star$, and $\mathbf{x}_0^\star$ such that

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}_0^\star, \mathbf{u}_0^\star, \mathbf{w}_0^\star) \qquad \mathbf{x}^\star(t_0) = \mathbf{x}_0^\star \\ \mathbf{y}_0^\star &= \mathbf{h}(\mathbf{x}_0^\star, \mathbf{u}_0^\star, \mathbf{w}_0^\star) \\ \mathbf{0} &= \mathbf{h}(\mathbf{x}_0^\star, \mathbf{u}_0^\star, \mathbf{w}_0^\star). \end{aligned} \tag{1.15}$$

Then, the triplet $(\mathbf{x}_0^\star, \mathbf{u}_0^\star, \mathbf{w}_0^\star)$ is called ***equilibrium triplet*** and the associated system (1.14) is called **Linear Time Invariant (LTI)**.

For the remaining of this book, the dependency on the triplet $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ is omitted and the notation of (1.14) is shortened as follows:

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{\mathbf{u}} + \mathbf{B}_2\tilde{\mathbf{w}} \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \\ \tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1\tilde{\mathbf{u}} + \mathbf{D}_2\tilde{\mathbf{w}} \\ \tilde{\mathbf{e}} &= \mathbf{C}_e\tilde{\mathbf{x}} + \mathbf{D}_{e_1}\tilde{\mathbf{u}} + \mathbf{D}_{e_2}\tilde{\mathbf{w}}. \end{aligned} \tag{1.16}$$

Example 1.2 (Linearization of the cart-pole model)**.** Given the nonlinear system of Example 1.1, assume that there exist an initial condition $\mathbf{x}_0^\star = \mathrm{col}(p_0,\ v_0,\ \theta_0,\ 0)$ and a $u^\star(t) : \mathbb{R} \to \mathbb{R}$ such that, for $\mathbf{w}^\star = \mathbf{0}$, the following equality holds for all $t \geq t_0$:

$$\begin{bmatrix} v_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{f}(\mathbf{x}_0, u^\star, \mathbf{0}).$$

Then, there exists a reference trajectory $\mathbf{x}^\star : \mathbb{R} \to \mathbb{R}^4$, which is defined as a solution of

$$\begin{aligned} \dot{\mathbf{x}}^\star &= \mathbf{f}(\mathbf{x}^\star, u^\star, \mathbf{0}) \qquad \mathbf{x}^\star(t_0) = \mathbf{x}_0^\star \\ \mathbf{y}^\star &= \mathbf{h}(\mathbf{x}^\star, u^\star, \mathbf{0}) \\ \mathbf{0} &= \mathbf{h}_e(\mathbf{x}^\star, u^\star, \mathbf{0}). \end{aligned}$$

Remark. *In this simple case, we have that* $\dot{\mathbf{x}}^\star = \mathrm{col}(v_0, 0, 0, 0)$ *and* $\mathbf{x}^\star(t) =$

$\mathrm{col}(p_0 + v_0(t - t_0), v_0, \theta_0, 0)$. *Note that $\theta_0 \neq 0$ is due to the aerodynamic drag which creates a torque statically balanced by the rotational spring reaction.*

The linearized system, obtained approximating the nonlinear dynamics (1.2) in a neighborhood of the reference trajectory $\mathbf{x}^\star$ is given by (1.14) in which the following matrices appear:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix}, \ \mathbf{B}_1 = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_2 = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \rho S C_D v_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathbf{D}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{D}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_e = \begin{bmatrix} 1 & 0 & \ell\cos\theta^\star & 0 \end{bmatrix}, \ D_{e1} = 0, \ D_{e2} = \begin{bmatrix} 1 & 0 & \ell\cos\theta^\star & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} 1 \\ A_{22} \\ 0 \\ A_{42} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 1 \\ -\rho S C_D v_0 \\ 0 \\ -\rho S C_D v_0 \ell \cos\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ A_{23} \\ 0 \\ A_{43} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ell m g \cos\theta_0 - k + \frac{1}{2}\rho S C_D v_0^2 \sin\theta_0 \ell \end{bmatrix}$$
$$+ \frac{\partial \mathbf{F}^{-1}}{\partial \theta} \begin{bmatrix} v_0 \\ 0 + u^\star - \frac{1}{2}\rho S v_0^2 C_D \\ 0 \\ \ell g m \sin\theta_0 - k\theta_0 - \frac{1}{2}\rho S v_0^2 C_D \ell \cos\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ A_{24} \\ 1 \\ A_{44} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ -\rho S C_D \ell \cos\theta_0 \\ 1 \\ -\mu - \frac{1}{2}\rho S C_D \ell^2 v_0 (1+\cos^2\theta_0) \end{bmatrix}$$

$$\frac{\partial \mathbf{F}^{-1}}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m\ell\sin\theta}{\Delta_0} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{m\ell\sin\theta}{\Delta_0} & 0 & 0 \end{bmatrix} + 2m\ell\cos\theta_0\sin\theta_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{J_m+m\ell^2}{\Delta_0^2} & 0 & \frac{m\ell\cos\theta}{\Delta_0^2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m\ell\cos\theta}{\Delta_0^2} & 0 & -\frac{m+M}{\Delta_0^2} \end{bmatrix}.$$

1.3 Control via Linearization

The control system's design can be successfully achieved via linearization when vehicles work in the neighborhood of a stationary point. This section aims to formalize the control problem and present the architecture of linear control systems [10, 15], specialization of Fig. 1.4. Then, this control architecture is compared with the PID, which represents one of the most adopted industrial control systems. Finally, this section ends with some comments about the limitations of the control systems designed via linearization.

1.3.1 Control Problem Formalization

Assume the model (1.16) is known and time-invariant. Then, make the following assumptions to formalize the control problem.

Assumption 1.1 (Disturbance and reference)**.**

1. The disturbance is not *observable*, *i.e.*, it is not possible to reconstruct d. This assumption forces the creation of a control system that is *robust* to plant uncertainties;

2. The disturbance is bounded. This assumption is necessary to guarantee the existence of a bounded control action able to achieve the control goals G1 and G2 described later;

3. The reference $\mathbf{r}(t)$ and its time derivatives up to the order $r_{\max} > 0$ are known. Moreover, $d^i/dt^i\mathbf{r}(t)$ is continuous and bounded for all $i = 0, \ldots, r_{\max}$.

Assumption 1.2 (Input redundancy)**.** The number of inputs is greater than or equal to the number of regulated outputs, *i.e.*, $p \geq m$. As described in Section 4.6, this assumption is necessary (but not sufficient) to determine a control input able to steer $\mathbf{e}$ to zero in the presence of a time-varying reference $\mathbf{r}$.

Assumption 1.3 (Regulated output readability)**.** The regulated output $\mathbf{e}$ is *readable* from $\mathbf{y}$, *i.e.*, a surjective map $\mathbf{E} : \mathbb{R}^q \to \mathbb{R}^m$ exists, with $q \geq m$, such that $\mathbf{e} = \mathbf{E}(\mathbf{y})$. Moreover, the q measurements are linearly independent. This assumption is necessary to solve the control problem G2, which is described later.

Then, the control system is designed to:

G1) assure the existence of a non-empty compact set $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\bar{w} > 0$ such that, for any $\mathbf{x}(t_0) \in \mathcal{X}_0$ and $\|\mathbf{w}(t)\|_\infty < \bar{w}$, the state $\mathbf{x}(t)$, the regulated output $\mathbf{e}(t)$, and the control $\mathbf{u}(t)$ remain bounded for all $t \geq t_0$;

G2) ensures $\limsup\limits_{t\to\infty} \|\tilde{\mathbf{e}}(t)\| = 0$ in the case of constant disturbances and the absence of measurement noises.

This book adopts the hybrid **closed/open-loop** control system architecture, highlighted in the light red-filled box in Fig. 1.4, to achieve G1 and G2. It comprises a feedback line that provides the outputs $\mathbf{y}$ and $\mathbf{e}$ to the controller. The latter generates a bounded control action $\mathbf{u}$ that both limits $\mathbf{x}$ and asymptotically steers $\mathbf{e}$ close to zero.

The internal structure of the control system depicted in Fig. 1.4 is complex, but the presence of each block, *i.e.*, the observer, the feedback, and the feed-forward, can be motivated as follows.

The observer represents a tool to gain knowledge about the plant, which is more than a possible direct inversion of $\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$. Indeed, assuming that the plant model (1.1) in known, the observer exploits the signals $\mathbf{u}$ and $\mathbf{y}$ to generate $\mathbf{y}_O$, which is supplementary information related to $\mathbf{x}$.

The information extracted from the plant through the observer is fused with the current controlled output $\mathbf{e}$ via the feedback. This block aims to solve the first part of the control problem, *i.e.*, keeping the state $\mathbf{x}$ and the error $\mathbf{e}$ bounded, at any time, through the stabilizer. Moreover, the presence of an integral action provides robustness to constant external disturbances.

On the other hand, the feed-forward exploits the knowledge of the reference and its time derivative to generate control actions needed to compensate for future variations of $\mathbf{e}$. In doing so, the feed-forward does not rely on either the current values of $\mathbf{y}$ or the supplementary $\mathbf{y}_O$.

Note

Roughly, feedback and feed-forward are associated with the concepts of stability and robustness (the former) and performance (the latter). In the following, these intuitive connections are given through the description of the control actions we undertake while performing the same turn at higher and higher speeds in the presence of a lateral wind. Let us now assume the ideal trajectory is known (by experience).

Performing a turn at low speed can explain the correlation between feedback, stabilization, and robustness. In doing so, we turn the steering wheel, then we feel the acceleration, and we observe our vehicle's position and direction. The first implicit task is keeping the vehicle under control, i.e., avoiding the car from swerving off the road. While performing this task, we compensate for extra accelerations, drifts, and yaw (measured by our sensory apparatus). Second, we correct wrong vehicle positions and alignments to keep the difference between the actual and the ideal trajectory close to zero. Moreover, the wind induces a side speed which leads to a gradual drift. To compensate for this effect, we rotate the steering wheel to create a lateral force that cancels the wind effects and eliminates the trajectory tracking error accumulated (i.e., integrated) up to that time. As a result, the ideal trajectory is tracked with bounded errors whose upper values depend on our driving ability. In this experiment, we act as a feedback controller that tries to keep both the state and the tracking errors bounded.

Let us now repeat the turn at higher and higher speeds. Intuitively, we should act on the steering wheel more aggressively to keep the trajectory tracking errors confined within the same bounds. Moreover, performing this aggressive maneuver relying on a continuous feedback policy would require our senses and brain to be more responsive. Consequently, the inherent limitations of our sensory apparatus and the finite responsiveness of our brain imply that a pure feedback policy cannot guarantee good performance at high speeds.

On the other hand, let us assume we can foresee the evolution of the tracking error (through a model of the phenomenon we built by experience). Then, without waiting for the evidence of the tracking error, we anticipate our actions on the steering wheel to keep the tracking error at zero. With this control policy, we do not rely on the continuous check of the tracking error and do not require the brain to elaborate feedback based on this information. Then, in principle, this control policy, named feed-forward, may lead to a superior performance. However, the drawback of the feed-forward is that its reliability is directly associated with our ability to foresee the future, which, commonly, is questionable.

To conclude, feedback is necessary to assure stability and to face uncertainties (unknown vehicle mass and inertia, ground conditions, wind, etc.). Still, its trajectory tracking performances are constrained by the bounded ability to elaborate the output and the trajectory tracking error. On the other hand, feed-forward is necessary to guarantee high trajectory tracking performances, but a non-perfect knowledge of the process constrains its efficacy.

The design of observer, feedback, and feed-forward is the subject matter of automatic controls. Among all possible design strategies, this textbook focuses on the so-called design via linearization. The following section investigates the control architecture of Fig. 1.4 for creating linear observers and controllers.

1.3.2 Control System Architecture

In the context of linear systems, the controller is not directly designed on the model (1.1) but rather on its linear approximation (1.16). In more detail, the nonlinear control law $\mathbf{u}$ is decomposed as the sum of the reference $\mathbf{u}^\star$ plus the linearized control $\tilde{\mathbf{u}}$, with the former designed on (1.5) and the latter designed on (1.14).

Since the controller is designed on the approximation (1.14), to implement it, it is necessary to make available to the controller the same outputs $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$ of the system (1.14). For this reason, the feedback line consists in a subtraction node that, taking the values $\mathbf{y}$, $\mathbf{e}$ and $\mathbf{y}^\star$, generates the signals $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$. Then, $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$ are sent to the controller to elaborate $\tilde{\mathbf{u}}$.

In the context of linear control systems, it is natural to conceive observer, feedback, and feed-forward algorithms as further linear systems [14]. In particular, let the observer be defined as

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{O}} &= \mathbf{A}_{\mathrm{O}}\mathbf{x}_{\mathrm{O}} + \mathbf{B}_{\mathrm{O}}\tilde{\mathbf{u}} + \mathbf{K}_{\mathrm{O}}\tilde{\mathbf{y}} \\ \mathbf{y}_{\mathrm{O}} &= \mathbf{C}_{\mathrm{O}}\mathbf{x}_{\mathrm{O}} + \mathbf{D}_{\mathrm{O}}\tilde{\mathbf{y}}, \end{aligned} \tag{1.17a}$$

and let the feedback control law be

$$\mathbf{u}_{\mathrm{FB}} := \mathbf{u}_{\mathrm{S}} + \mathbf{u}_{\mathrm{IA}} \tag{1.17b}$$

where $\mathbf{u}_{\mathrm{S}}$ denotes the stabilizer law

$$\mathbf{u}_{\mathrm{S}} := \mathbf{K}_{\mathrm{S}}\mathbf{y}_{\mathrm{O}}. \tag{1.17c}$$

On the other hand, $\mathbf{u}_{\mathrm{IA}}$ defines the integral action, which represents the output of the following dynamic system:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{e} \\ \mathbf{u}_{\mathrm{IA}} &= \mathbf{K}_{\mathrm{I}}\boldsymbol{\eta}. \end{aligned} \tag{1.17d}$$

Let the feed-forward control law be defined as

$$\begin{aligned}\dot{\mathbf{x}}_{\mathrm{FF}} &= \mathbf{A}_{\mathrm{FF}}\mathbf{x}_{\mathrm{FF}} + \sum_{i}^{r_{\max}} \mathbf{B}_{\mathrm{FF}_i}\frac{d^i}{dt^i}\mathbf{r} \\ \mathbf{u}_{\mathrm{FF}} &= \mathbf{C}_{\mathrm{FF}}\mathbf{x}_{\mathrm{FF}} + \sum_{i}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i}\frac{d^i}{dt^i}\mathbf{r}\end{aligned} \tag{1.17e}$$

for some finite $r_{\max} \in \mathbb{N}$.

The architecture of a control system designed via linearization is depicted in Fig. 1.5 where the matrices, constituting the observer, the feedback, and the feed-forward, are designed exploiting $\mathbf{A}, \mathbf{B}_1, \ldots$ of (1.14). The rationale behind (1.17) will be evident in the next section where a comparison with classic PIDs is provided.

Important

A straight implementation of the control law (1.17c) would lead to an algebraic loop. Indeed, $\mathbf{u}_{\mathrm{S}}$ depends on $\tilde{\mathbf{y}}$, which, in turn, depends on $\mathbf{u}_{\mathrm{S}}$. Either $\mathbf{D}_1 = \mathbf{0}$ or $\mathbf{D}_{\mathrm{O}} = \mathbf{0}$ break this algebraic loop. This section introduces $\mathbf{D}_{\mathrm{O}}$ to compare (1.17) with the classic PIDs (see the following section), although it is not necessary to achieve the control goals G1 and G2. As detailed in Section 4.4, the matrix $\mathbf{D}_{\mathrm{O}} \neq \mathbf{0}$ is introduced when $\mathbf{D}_1 = \mathbf{0}$ to reduce the size of $\mathbf{x}_{\mathrm{O}}$.

1.3.3 Comparison with Classic PIDs

The PID controller is widespread in industrial applications [7, 28]. It is composed of a parallel of three control actions, the proportional, the integral, and the derivative. They all elaborate the regulated output, arranged as in Fig. 1.6.

The classic time-domain formulation of a PID controller is given by

$$\dot{\mathbf{w}} = \mathbf{A}_{\mathrm{O}}\mathbf{w} + \mathbf{K}_{\mathrm{O}}\dot{\mathbf{e}} \tag{1.18a}$$

$$\mathbf{u}_{\mathrm{PID}} = \mathbf{K}_{\mathrm{P}}\mathbf{e} + \mathbf{K}_{\mathrm{D}}\mathbf{w} + \mathbf{K}_{\mathrm{I}}\int_0^t \mathbf{e}(\tau)d\tau + \mathbf{u}^{\star} \tag{1.18b}$$

where it is worth noting that $\mathbf{w}$ represents the output of the dynamic system necessary to let the control system be causal. Moreover, $\mathbf{u}^{\star}$ denotes the initial condition of the integral action. To compare the PID control architecture with that proposed in Fig. 1.5, note that $\mathbf{e} := \mathbf{r} - \mathbf{y}$, assume $\mathbf{r}^{\star} = \mathbf{y}^{\star}$, then subtract $\mathbf{r}^{\star} - \mathbf{y}^{\star}$ from $\mathbf{e}$ to obtain

$$\mathbf{e} = \mathbf{r} - \mathbf{y} = \tilde{\mathbf{r}} - \tilde{\mathbf{y}}$$

Chapter 3

LTI System Analysis

Contents

This chapter introduces the study of the trajectories of linear systems. In particular, the trajectory boundedness is linked to the eigenvalues of matrix **A**. In more detail, Section 3.1 introduces an instrumental mathematical tool called *Jordan canonical form.* This canonical form is then exploited in Section 3.2 to compute the solution to a set of linear ordinary differential equations. The section investigates these solutions and links them with the eigenvalues of **A**. Concerning the control goal G1, this link is exploited in Section 3.3 to state a stability criterion, which assures the boundedness of trajectories. Finally, these theoretical tools are exploited to investigate the linearized plants introduced in Chapter 2.

3.1 Jordan Canonical Form

This section presents the design of a change of coordinate suitable to study the behavior of the state of LTI systems [2]. This change of coordinates transforms the original linear plant into the so-called *Jordan canonical form*. Two ingredients are needed to achieve this result, *i.e.*, a change of coordinates and a design criterion. As detailed in Section 3.1.2, the eigenvectors steer the design criterion. Section 3.1.1 defines the transformation of coordinates, while Section 3.1.3 specializes it to study the dynamics of LTI systems.

3.1.1 Change of Coordinates

Let $\mathbb{V}(\mathbb{C})$ be an *n-dimensional* vector space with $n \in \mathbb{N}$; see Section A.6. Then, a basis of $\mathbb{V}(\mathbb{C})$ is a finite set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, $\mathbf{b}_i \in \mathbb{V}(\mathbb{C})$, with $i = 1, \dots, n$, such that any vector $\mathbf{v} \in \mathbb{V}(\mathbb{C})$ can be represented as a linear combination of $\mathbf{b}_1, \dots, \mathbf{b}_n$. In other words, n constants $\beta_i \in \mathbb{C}$ exist such that

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

The term β_i means the i-th component of $\mathbf{v}$ on $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Define $\mathbf{u} = \mathrm{col}(\beta_1, \dots, \beta_n)$ and let $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a second basis, such that the vector $\mathbf{v}$ has components $\mathbf{w} := \mathrm{col}(\gamma_1, \dots, \gamma_n)$ on $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$, *i.e.*, such that $\mathbf{v} = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \mathbf{w}$. Vectors $\mathbf{u}$ and $\mathbf{w}$ are related through a linear function $\mathbf{T} : \mathbb{V}(\mathbb{C}) \to \mathbb{V}(\mathbb{C})$, called *change of coordinates*, defined as $\mathbf{T} := \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$ and such that

$$\mathbf{u} = \mathbf{T}\,\mathbf{w}.$$

The columns of $\mathbf{T}$ represent the components of vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$ on $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Since the representation of a vector on a given basis is unique, coordinate changes are bijections, and so they are invertible such that

$$\mathbf{w} = \mathbf{T}^{-1}\mathbf{u}.$$

Let $\mathbf{A} : \mathbb{V}(\mathbb{C}) \to \mathbb{V}(\mathbb{C})$ be a linear function with $\mathbf{A} \in \mathbb{C}^{n \times n}$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{V}(\mathbb{C})$ be two vectors, both defined on the basis $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ and such that

$$\mathbf{y} = \mathbf{A}\mathbf{x}. \tag{3.1}$$

Then, let

$$\boldsymbol{\chi} := \mathbf{T}\mathbf{x} \qquad \boldsymbol{\mu} := \mathbf{T}\mathbf{y},$$

pre-multiply both sides of (3.1) by $\mathbf{T}$, and exploit $\mathbf{x} = \mathbf{T}^{-1}\boldsymbol{\chi}$ to obtain

$$\boldsymbol{\mu} = \bar{\mathbf{A}}\boldsymbol{\chi}$$

where $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ represents $\mathbf{A}$ described on $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. This result is directly exploited in Section 3.1.3 to build the Jordan canonical form.

This section ends with some details, which are instrumental for the results proposed in Chapter 4.

Let $\mathbb{S}(\mathbb{C}) \subseteq \mathbb{V}(\mathbb{C})$ be a p-dimensional subspace of $\mathbb{V}(\mathbb{C})$ and let $\{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ be a basis for $\mathbb{S}(\mathbb{C})$. Then, the orthogonal complement of $\mathbb{S}(\mathbb{C})$, denoted with $\mathbb{S}^{\perp}(\mathbb{C})$, is defined as a subspace of $\mathbb{V}(\mathbb{C})$ such that

$$\mathbb{V}(\mathbb{C}) = \mathbb{S}(\mathbb{C}) \oplus \mathbb{S}^{\perp}(\mathbb{C}).$$

Remark 3.1. The direct sum $\oplus$ means that $\mathbb{S}(\mathbb{C}) \cup \mathbb{S}^{\perp}(\mathbb{C}) = \mathbb{V}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C}) \cap \mathbb{S}^{\perp}(\mathbb{C}) = \{0\}$, or equivalently that $\mathbb{S}(\mathbb{C})$ and $\mathbb{S}^{\perp}(\mathbb{C})$ share only the origin but their union corresponds to $\mathbb{V}(\mathbb{C})$.

Moreover, let $\{\mathbf{s}_1^{\star}, \ldots, \mathbf{s}_q^{\star}\}$, with $q = n - p$, be a basis for $\mathbb{S}^{\perp}(\mathbb{C})$. Then, the set $\{\mathbf{s}_1, \ldots, \mathbf{s}_p, \mathbf{s}_1^{\star}, \ldots, \mathbf{s}_q^{\star}\}$ represents a basis for $\mathbb{V}(\mathbb{C})$.

Let $\mathbb{X}(\mathbb{C})$ and $\mathbb{Y}(\mathbb{C})$ be two linear vector spaces and let $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ be a linear function. Then, the *image* of $\mathbf{A}$ is the subspace of $\mathbb{Y}$ defined as

$$\mathrm{im}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Y} \,:\, \mathbf{y} = \mathbf{A}\mathbf{x},\, \mathbf{x} \in \mathbb{X}\}$$

with $\mathrm{im}(\mathbf{A}) \subseteq \mathbb{Y}$. The *kernel* of $\mathbf{A}$ is the subspace of $\mathbb{X}$ defined as

$$\ker(\mathbf{A}) = \{\mathbf{x} \in \mathbb{X} \,:\, \mathbf{0} = \mathbf{A}\mathbf{x}\}$$

with $\ker(\mathbf{A}) \subseteq \mathbb{X}$. The kernel and the image are such that $\ker(\mathbf{A}) = \big(\mathrm{im}(\mathbf{A}^{\top})\big)^{\perp}$. Finally, the following equalities hold true:

$$\begin{aligned}\mathbb{X} &= \ker(\mathbf{A}) \oplus (\ker(\mathbf{A}))^{\perp} = \ker(\mathbf{A}) \oplus \mathrm{im}(\mathbf{A}^{\top})\\ \mathbb{Y} &= \mathrm{im}(\mathbf{A}) \oplus (\mathrm{im}(\mathbf{A}))^{\perp} = \mathrm{im}(\mathbf{A}) \oplus \ker(\mathbf{A}^{\top}).\end{aligned}$$

3.1.2 Eigenvalues and Eigenvectors

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix and let

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{3.2}$$

be a LTI system. It is interesting finding all the non-trivial solutions to (3.2) for which there exists $\lambda \in \mathbb{C}$ such that [[7],§3]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{3.3}$$

i.e., those vectors whose direction does not change over time. Then, to solve this problem, rearrange $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{3.4}$$

where the non-trivial solutions are those belonging to $\ker(\mathbf{A} - \lambda\mathbf{I})$. Consequently, the values of λ that make $\mathbf{A} - \lambda\mathbf{I}$ singular (*i.e.*, those making $\ker(\mathbf{A} - \lambda\mathbf{I})$ non-trivial) are needed. These particular λ, called **eigenvalues of A** [[1],§2], are found as solutions to

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

where $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial in λ, called **characteristic polynomial**. Let λ_i be the i-th eigenvalue, then the **algebraic multiplicity**, denoted with a_i, is defined as the order of the root λ_i. Thanks to this definition, and considering p distinct eigenvalues, it follows that

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{a_i}$$

with $\sum_{i=1}^{p} a_i = n$.

Remark 3.2. The eigenvalues of block triangular matrices correspond to those of the matrices on the main diagonal. Chapter 4 uses this result.

The vectors $\mathbf{v}$ satisfying

$$(\mathbf{A} - \lambda_i\mathbf{I})^{a_i}\mathbf{v} = \mathbf{0} \tag{3.5}$$

are the **eigenvectors** associated with λ_i [[1],§2]. The **geometric multiplicity** related to λ_i, denoted with g_i, corresponds to the dimension of $\ker(\mathbf{A} - \lambda_i\mathbf{I})$. To find the eigenvectors associated with $\mathbf{A}$, rewrite (3.5) through the recursive definition detailed hereafter. Let $\mathbf{v}_{i,j,0} := \mathbf{0}$ and $\mathbf{v}_{i,j,k-1} := (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_{i,j,k}$ for $i = 1, \ldots, p$, $j = 1, \ldots, g_i$, and $k = 1, \ldots, q_{i,j}$ with $1 \le q_{i,j} \le a_i$ such that $\sum_{j=1}^{g_i} q_{i,j} = a_i$. Then, (3.5) is equivalent to

$$\begin{aligned}
(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_{i,j,1} &= \mathbf{0} \\
(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_{i,j,2} &= \mathbf{v}_{i,j,1} \\
&\vdots \\
(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_{i,j,q_{i,j}} &= \mathbf{v}_{i,j,q_{i,j}-1}.
\end{aligned} \tag{3.6}$$

The term $q_{i,j}$ denotes the **length of the chain of the eigenvectors** related to $\mathbf{v}_{i,j,1}$ and represents the largest integer such that the vectors $\mathbf{v}_{i,j,q_{i,j}}$ and $\mathbf{v}_{i,j,q_{i,j}-1}$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \mathbf{0} = \begin{cases} \lambda_1 \\ \vdots \\ \lambda_i & \begin{cases} \mathbf{v}_{i,1,1} & \cdots & \mathbf{v}_{i,1,q_{i,1}} \\ \vdots & & \\ \mathbf{v}_{i,g_i,1} & \cdots & \mathbf{v}_{i,g_i,q_{i,g_i}} \end{cases} \\ \vdots \\ \lambda_p \end{cases}$$

Figure 3.1: Organisation of eigenvectors of matrix $\mathbf{A}$. First, the p distinct eigenvalues are defined. Second, the g_i eigenvectors are associated with each eigenvalue. Third, a chain of eigenvectors of length $q_{i,j}$ is associated with each eigenvector $\mathbf{v}_{i,j,1}$.

are linearly independent. For instance, Fig. 3.1 depicts the organization of the eigenvectors of $\mathbf{A}$.

Remark 3.3. The length of chains of eigenvectors is equal to 1 when the geometric multiplicity is equal to the algebraic multiplicity. Indeed, assuming $g_i = a_i$ and $q_{i,j} \geq 1$ we have $\sum_{j=1}^{a_i} q_{i,j} = a_i$ if and only if $q_{i,j} = 1$ for each $j = 1, \ldots, g_i$.

Example 3.1 (Eigenvalues and Eigenvectors)**.** Let s_1, s_2, s_3 be real numbers such that $s_1 \neq s_2 \neq s_3 \neq 0$, and define

$$\mathbf{A} = \begin{bmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 1 \\ 0 & 0 & 0 & 0 & s_3 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are the roots of $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$, where

$$\begin{aligned} \det(\mathbf{A}-\lambda\mathbf{I}) &= \det\left(\begin{bmatrix} s_1-\lambda & 0 & 0 & 0 & 0 \\ 0 & s_2-\lambda & 0 & 0 & 0 \\ 0 & 0 & s_2-\lambda & 0 & 0 \\ 0 & 0 & 0 & s_3-\lambda & 1 \\ 0 & 0 & 0 & 0 & s_3-\lambda \end{bmatrix}\right) \\ &= (s_1-\lambda)(s_2-\lambda)^2(s_3-\lambda)^2. \end{aligned}$$

There are three independent eigenvalues (thus $p = 3$) which are $\lambda_1 = s_1$, $\lambda_2 = s_2$, and $\lambda_3 = s_3$. Their algebraic multiplicities are $a_1 = 1$, $a_2 = 2$, and $a_3 = 2$. Note that $\sum_{i=1}^{3} a_i = 5$. The eigenvector associated with λ_1 is found by solving

$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v}=\mathbf{0}$. In particular, let $\mathbf{v} := \mathrm{col}(x_1,\cdots,x_5)$ and compute

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_2-s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2-s_1 & 0 & 0 \\ 0 & 0 & 0 & s_3-s_1 & 1 \\ 0 & 0 & 0 & 0 & s_3-s_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ (s_2-s_1)x_2 \\ (s_2-s_1)x_3 \\ (s_3-s_1)x_4+x_5 \\ (s_3-s_1)x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is worth noting that the assumption $s_1 \neq s_2 \neq s_3 \neq 0$ implies $x_2 = x_3 = x_4 = x_5 = 0$. Then, the eigenvector associated with λ_1 is found as

$$\mathbf{v}_{1,1,1} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with $x_1 \neq 0$ free (one possible choice is $x_1 = 1$). The eigenvalues related to λ_2 are found by solving

$$(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{v} = \begin{bmatrix} s_1-s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3-s_2 & 1 \\ 0 & 0 & 0 & 0 & s_3-s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (s_1-s_2)x_1 \\ 0 \\ 0 \\ (s_3-s_2)x_4+x_5 \\ (s_3-s_2)x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which leads to $x_1 = x_4 = x_5 = 0$ and $(x_2, x_3) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, it is possible to find two eigenvectors (geometric multiplicity $g_2 = 2$), namely $\mathbf{v}_{2,1,1}$ and $\mathbf{v}_{2,2,1}$, as

$$\mathbf{v}_{2,1,1} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_{2,2,1} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvectors associated with λ_3 are the solution to

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{v} =$$

$$\begin{bmatrix} s_1 - s_3 & 0 & 0 & 0 & 0 \\ 0 & s_2 - s_3 & 0 & 0 & 0 \\ 0 & 0 & s_2 - s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} =$$

$$\begin{bmatrix} (s_1 - s_3)x_1 \\ (s_2 - s_3)x_2 \\ (s_2 - s_3)x_3 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which leads to $x_1 = x_2 = x_3 = x_5 = 0$. These equalities imply the existence of only one eigenvector, namely $\mathbf{v}_{3,1,1}$, with geometric multiplicity $g_3 = 1$, given by

$$\mathbf{v}_{3,1,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix}.$$

Since $a_3 - g_3 > 0$, there exists a chain of eigenvectors with length $q_{3,1} = 2$. These eigenvectors are found as solution to $(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{v}_{3,1,2} = \mathbf{v}_{3,1,1}$. In detail, let $\mathbf{v}_{3,1,2} := \mathrm{col}(y_1, \cdots, y_5)$ and compute

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{v}_{3,1,2} =$$

$$\begin{bmatrix} s_1 - s_3 & 0 & 0 & 0 & 0 \\ 0 & s_2 - s_3 & 0 & 0 & 0 \\ 0 & 0 & s_2 - s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} =$$

$$\begin{bmatrix} (s_1 - s_3)y_1 \\ (s_2 - s_3)y_2 \\ (s_2 - s_3)y_3 \\ y_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix}$$

which implies $y_1 = y_2 = y_3 = 0$ and $y_5 = x_4$ leading to

$$\mathbf{v}_{3,1,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_4 \\ x_4 \end{bmatrix}.$$

3.1.3 Jordan Transformation

This section presents a change of coordinates, based on eigenvectors, to investigate the dynamics of LTI systems in form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} \in \mathbb{R}^{n \times n}$.

Define the set of distinct eigenvalues of $\mathbf{A}$ with

$$\{\lambda_1, \ldots, \lambda_p\} \qquad p \leq n.$$

Let a_i and g_i be the algebraic and the geometric multiplicity associated with λ_i. On the one hand, let $q_{i,j}$ be the length of the chain of eigenvectors related to the eigenvector $\mathbf{v}_{i,j,1}$. Let

$$\mathbf{V}_{i,j} := \begin{bmatrix} \mathbf{v}_{i,j,1} & \mathbf{v}_{i,j,2} & \cdots & \mathbf{v}_{i,j,q_{i,j}} \end{bmatrix} \tag{3.7}$$

be the matrix obtained as composition of the $q_{i,j}$ eigenvectors associated with $\mathbf{v}_{i,j,1}$. On the other hand, for each $i = 1, \ldots, p$, the following relations are true (see Eq. (3.6))

$$\mathbf{A}\mathbf{v}_{i,j,k} = \lambda_i \mathbf{v}_{i,j,k} + \mathbf{v}_{i,j,k-1} \qquad j = 1, \ldots, g_i, \quad k = 1, \ldots, q_{i,j},$$

which can be written in compact form as

$$\mathbf{A}\mathbf{V}_{i,j} = \mathbf{V}_{i,j}\mathbf{J}_{i,j}$$

where

$$\mathbf{J}_{i,j} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{q_{i,j} \times q_{i,j}}.$$

Moreover, let

$$\mathbf{V}_i := \begin{bmatrix} \mathbf{V}_{i,1} & \mathbf{V}_{i,2} & \cdots & \mathbf{V}_{i,g_i} \end{bmatrix}$$

be the compositions of all the chains of eigenvectors related to λ_i. Let

$$\mathbf{V} := \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_p \end{bmatrix}$$

be the composition of all the eigenvectors of $\mathbf{A}$. It is worth noting that $\mathbf{V}$ represents a basis for $\mathbb{C}^n$ thanks to the linear independence of $\mathbf{V}_i$ and $\mathbf{V}_j$, for each $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Define

$$\mathbf{J}_i := \text{blckdiag}\left(\mathbf{J}_{i,1}, \ldots, \mathbf{J}_{i,g_i}\right) \qquad \mathbf{J} := \text{blckdiag}\left(\mathbf{J}_1, \ldots, \mathbf{J}_p\right),$$

and use $\mathbf{J}$ to write

$$\mathbf{AV} = \mathbf{VJ}.$$

Since $\mathbf{V}$ is composed of linearly independent vectors, its inverse is well-posed. Consequently, it can be exploited to define the following transformation

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{J}, \tag{3.8}$$

called *Jordan canonical form* [[1],§2] and representing one of the primary tools for investigating LTI system dynamics.

The remaining of this section presents a procedure to reduce $\mathbf{V}$ to a real-valued matrix.

Let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ be an eigenvector associated with λ. Assume $\lambda \in \mathbb{C}$ and $\mathbf{v} \in \mathbb{C}^n$. Then, there exists $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ such that

$$\begin{aligned} \lambda &= \alpha + \mathrm{i}\beta \quad \lambda^\star = \alpha - \mathrm{i}\beta \\ \mathbf{v} &= \mathbf{a} + \mathrm{i}\mathbf{b} \quad \mathbf{v}^\star = \mathbf{a} - \mathrm{i}\mathbf{b} \end{aligned}$$

where the starred quantities denote complex conjugates. Then, through the same arguments used in Section 3.1.2, we have

$$\mathbf{A}\begin{bmatrix} \mathbf{v} & \mathbf{v}^\star \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{v}^\star \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^\star \end{bmatrix}, \tag{3.9}$$

which involves complex conjugate quantities. We note that

$$\begin{aligned}[\mathbf{A}-\lambda\mathbf{I}]\,\mathbf{v} &= [\mathbf{A}-(\alpha+\mathrm{i}\beta)\,\mathbf{I}]\,[\mathbf{a}+\mathrm{i}\mathbf{b}] \\ &= [\mathbf{A}-\alpha\mathbf{I}-\mathrm{i}\beta\mathbf{I}]\,\mathbf{a}+[\mathbf{A}-\alpha\mathbf{I}-\mathrm{i}\beta\mathbf{I}]\,\mathrm{i}\mathbf{b} \\ &= [\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{a}-\mathrm{i}\beta_i\mathbf{I}\mathbf{a}+[\mathbf{A}-\alpha\mathbf{I}]\,\mathrm{i}\mathbf{b}-\mathrm{i}^2\beta_i\mathbf{I}\mathbf{b} \\ &= [\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{a}+\beta_i\mathbf{I}\mathbf{b}+\mathrm{i}\,\{[\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{b}-\beta\mathbf{I}\mathbf{a}\}=0\end{aligned} \tag{3.10}$$

and, similarly,

$$[\mathbf{A}-\lambda^\star\mathbf{I}]\,\mathbf{v}^\star = [\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{a}+\beta_i\mathbf{I}\mathbf{b}-\mathrm{i}\,\{[\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{b}-\beta\mathbf{I}\mathbf{a}\}=0. \tag{3.11}$$

Then, we impose the most right equalities of (3.10) and (3.11) being zero by forcing the real and the imaginary parts to be zero individually

$$\begin{cases}[\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{a}+\beta\mathbf{I}\mathbf{b}=0\\ [\mathbf{A}-\alpha\mathbf{I}]\,\mathbf{b}-\beta\mathbf{I}\mathbf{a}=0\end{cases} \iff \begin{cases}\mathbf{A}\mathbf{a}=\alpha\mathbf{a}-\beta\mathbf{b}\\ \mathbf{A}\mathbf{b}=\alpha\mathbf{b}+\beta\mathbf{a},\end{cases}$$

which is compacted in the following matrix form

$$\mathbf{A}\begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}=\begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix},$$

which is alternative to (3.9). Let $\lambda_i \in \mathbb{C}$ be an eigenvalue of $\mathbf{A}$ and let $\alpha_i, \beta_i \in \mathbb{R}$ such that $\lambda_i = \alpha_i + \mathrm{i}\beta_i$. Let $\mathbf{V}_{i,j}$ be the chain of eigenvectors defined in (3.7) and let $\mathbf{a}_{i,j,k}$, $\mathbf{b}_{i,j,k} \in \mathbb{R}^n$ such that for $j = 1, \ldots, g_i$ and $k = 1, \ldots, q_{i,j}$ we have $\mathbf{v}_{i,j,k} = \mathbf{a}_{i,j,k} + \mathrm{i}\mathbf{b}_{i,j,k}$. Let

$$\bar{\mathbf{V}}_{i,j} := \begin{bmatrix}\mathbf{a}_{i,j,1} & \mathbf{b}_{i,j,1} & \mathbf{a}_{i,j,2} & \mathbf{b}_{i,j,2} & \ldots & \mathbf{a}_{i,j,q_{i,j}} & \mathbf{b}_{i,j,q_{i,j}}\end{bmatrix}$$

$$\bar{\mathbf{J}}_{i,j} := \begin{bmatrix}\begin{matrix}\alpha_i & \beta_i\\ -\beta_i & \alpha_i\end{matrix} & \mathbf{I} & \ldots & \mathbf{0}\\ \mathbf{0} & \begin{matrix}\alpha_i & \beta_i\\ -\beta_i & \alpha_i\end{matrix} & \ddots & \mathbf{0}\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ldots & \begin{matrix}\alpha_i & \beta_i\\ -\beta_i & \alpha_i\end{matrix}\end{bmatrix},$$

then, $\mathbf{A}\bar{\mathbf{V}}_{i,j} = \bar{\mathbf{V}}_{i,j}\bar{\mathbf{J}}_{i,j}$ holds true.

Example 3.2 (Study of the cart-pole)**.** Investigate the matrix $\mathbf{A}$, defined in Example 1.2, by assuming

$$(m,\, M,\, \ell,\, g,\, J_m,\, k,\, \mu) = (1000,\, 100,\, 1,\, 9.81,\, 100,\, 16000,\, 1000)$$

(these numerical values model the longitudinal dynamics of a car). Moreover, assume as a linearization triplet $(\mathbf{x}^\star, u^\star, d^\star) = (\mathbf{0}, 0, 0)$. On the one hand, as a consequence of the latter assumption, the matrix becomes

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}$$

with

$$\begin{aligned} A_{23} &= \frac{\ell m(k - \ell m g)}{Mm\ell^2 + J_m(M+m)} & A_{24} &= \frac{\ell k m}{Mm\ell^2 + J_m(M+m)} \\ A_{43} &= -\frac{(M+m)(k - \ell m g)}{Mm\ell^2 + J_m(M+m)} & A_{44} &= -\frac{\mu(M+m)}{Mm\ell^2 + J_m(M+m)}. \end{aligned}$$

The eigenvalues of this matrix and their algebraic multiplicity, obtained by solving the problem $\det(\mathbf{A} - \lambda \mathbf{I}) = \mathbf{0}$, are

$$\begin{aligned} \lambda_1 &= 0 & a_1 &= 2 \\ \lambda_{2,3} &= \frac{A_{44} \pm \sqrt{A_{44}^2 + 4A_{43}}}{2} & a_{2,3} &= 1. \end{aligned}$$

On the other hand, a numerical computation of the eigenvalues leads to $\lambda_{2,3}$ complex conjugate, *i.e.*, $\lambda_2 = \lambda_3^\star$. Moreover, it is easy to check that the geometric multiplicity of λ_1 is $g_1 = 1$, so there exist a chain a generalized eigenvectors (related to λ_1) of length $a_1 - g_1 = 1$. Let $\alpha_2, \beta_2 \in \mathbb{R}$ be such that $\lambda_2 = \alpha_2 + \mathrm{i}\beta_2$. Then, eigenvalues and eigenvectors uniquely define the Jordan matrix as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{J}}_2 \end{bmatrix}, \ \mathbf{J}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \bar{\mathbf{J}}_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix}.$$

3.2 Dynamics of LTI systems

This section exploits the Jordan transformation to study the dynamics of a LTI systems. The goal is to transform the system from the original coordinates to some particular coordinates in which the system dynamics correspond to a composition of independent sub-dynamics, for which an explicit description is available.

Let

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.12}$$

be an LTI system, let $\mathbf{V}$ be the set of the eigenvectors of $\mathbf{A}$, and assume $\mathbf{V}$ is in the real form. Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a change of coordinates such that $\mathbf{T} = \mathbf{V}^{-1}$. Define

$\mathbf{z} = \mathbf{T}\mathbf{x}$. Then, the dynamics of $\mathbf{z}$ are obtained by pre-multiplying both members of (3.12) by $\mathbf{T}$ and by exploiting $\mathbf{x} = \mathbf{T}^{-1}\mathbf{z}$

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}\mathbf{u} \quad \mathbf{z}(t_0) = \mathbf{T}\mathbf{x}_0, \tag{3.13}$$

where $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ and $\bar{\mathbf{B}} = \mathbf{T}\mathbf{B}$.

Since the new coordinates are chosen according to the Jordan canonical form, matrix $\bar{\mathbf{A}}$ assumes the following block diagonal form:

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{J}_1 & 0 & \dots & 0 \\ 0 & \mathbf{J}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{J}_p \end{bmatrix}, \mathbf{J}_i = \begin{bmatrix} \mathbf{J}_{i,1} & 0 & \dots & 0 \\ 0 & \mathbf{J}_{i,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{J}_{i,g_i} \end{bmatrix}. \tag{3.14}$$

Redefine

$$\mathbf{z} := \mathrm{col}(\mathbf{z}_{1,1}, \dots, \mathbf{z}_{1,g_1}, \mathbf{z}_{2,1}, \dots, \mathbf{z}_{2,g_2}, \dots, \mathbf{z}_{p,1}, \dots, \mathbf{z}_{p,g_p})$$

and exploit (3.14) to rewrite (3.13) as

$$\begin{aligned} \dot{\mathbf{z}}_{i,j} &= \mathbf{J}_{i,j}\mathbf{z}_{i,j} + \bar{\mathbf{B}}_{i,j}\mathbf{u} \quad \mathbf{z}_{i,j}(t_0) = \mathbf{V}_{i,j}\mathbf{x}_0 & i = 1, \dots, p \\ \mathbf{y} &= \bar{\mathbf{C}}_{i,j}\mathbf{z}_{i,j} + \mathbf{D}\mathbf{u} & j = 1, \dots, g_i, \end{aligned} \tag{3.15}$$

where $\bar{\mathbf{B}}_{i,j}$ and $\bar{\mathbf{C}}_{i,j}$ are proper sub-parts of $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$.

The solution to (3.15) is [1, 7]

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}(t - t_0))\mathbf{z}_{i,j}(t_0) + \int_{t_0}^{t} \exp(\mathbf{J}_{i,j}(t - \tau))\bar{\mathbf{B}}_{i,j}\mathbf{u}(\tau)d\tau, \tag{3.16}$$

in which $\exp(\mathbf{J}_{i,j}t) := \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{J}_{i,j}^k t^k$ is called *transition matrix* [[1],§2].

Remark 3.4. With $\mathbf{z}(t)$ at hand, the solution to (3.12) is obtained as $\mathbf{x}(t) = \mathbf{V}\,\mathbf{z}(t)$, for all $t \geq t_0$. Let $\mathbf{z} := \mathrm{col}(z_1, \dots, z_n)$ and assume $\mathbf{V}$ is in the real form. Denote with $v_{i,j}$ the (i, j)-th entry of $\mathbf{V}$. Then, the time behavior of the i-th state component is obtained as a linear combination of independent dynamics

$$x_i(t) = \sum_{j=1}^{n} v_{i,j} z_j(t), \tag{3.17}$$

where $x_i(t)$ is thought as a weighted mean of $\mathbf{z}(t)$ with weights $v_{i,j}$. Fig. 3.2 provides a graphical representation of (3.17). With abuse of terminology, we define the elements of $\mathbf{z}(t)$ as *modes*.

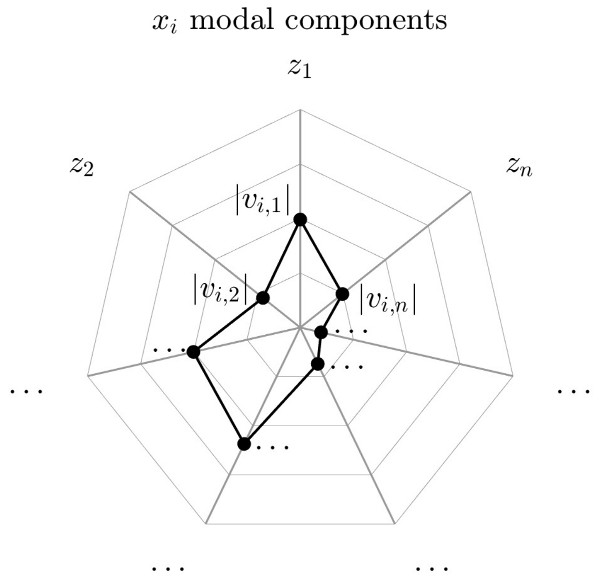


Figure 3.2: The i-th state dynamics are a linear combination of $\mathbf{z}$. This diagram provides graphical support to understand which are if any, the components of $\mathbf{z}$ that principally describe x_i.

It is interesting to note that the evolution of $\mathbf{z}_{i,j}(t)$ can be seen as a superposition of two parts [5]

- *free evolution*, only dependent on the initial conditions,

$$\mathbf{z}_{i,j}^{\text{free}}(t) = \exp(\mathbf{J}_{i,j}(t - t_0))\mathbf{z}_{i,j}(t_0);$$

- *forced evolution*, only dependent on the input history,

$$\mathbf{z}_{i,j}^{\text{forced}}(t) = \int_{t_0}^{t} \exp(\mathbf{J}_{i,j}(t - \tau))\bar{\mathbf{B}}_{i,j}\mathbf{u}(\tau)d\tau.$$

The term $\mathbf{z}_{i,j}^{\text{free}}(t)$ is null if $\mathbf{z}_{i,j}(t_0) = \mathbf{0}$, whereas the term $\mathbf{z}_{i,j}^{\text{forced}}(t) = \mathbf{0}$ if $\mathbf{u}(\tau) = \mathbf{0}$ $\forall \tau \in [t_0, t]$. To compute $\exp(\mathbf{J}_{i,j}t)$, note that

$$\mathbf{J}_{i,j} = \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}$$

can be written as $\mathbf{J}_{i,j} = \lambda_i \mathbf{I} + \mathbf{N}_{i,j}$ with

$$\mathbf{N}_{i,j} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then, the transition matrix is

$$\exp(\mathbf{J}_{i,j}t) = \exp((\lambda_i \mathbf{I} + \mathbf{N}_{i,j})t) = \exp(\lambda_i t)\exp(\mathbf{N}_{i,j}t)$$

and

$$\begin{aligned}\exp(\mathbf{N}_{i,j}t) &= \sum_{k=0}^{\infty} \frac{1}{k!}\bar{\mathbf{N}}_{i,j}^k t^k = \mathbf{I} + \mathbf{N}_{i,j}t + \frac{1}{2}\mathbf{N}_{i,j}^2 t^2 + \dots \\ &= \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!} \\ 0 & 1 & t & \ddots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!} \\ 0 & 0 & 1 & \ddots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.\end{aligned}$$

Infobox 3.1 (Powers of $\mathbf{N}_{i,j}$)**.** The k-th power of $\mathbf{N}_{i,j}$ is a null matrix with the k-th upper diagonal composed of "1" elements. For example,

$$\mathbf{N}_{i,j}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \mathbf{N}_{i,j}^{q_{i,j}-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let $q_{i,j}$ be the dimension of $\mathbf{N}_{i,j}$, then $\mathbf{N}_{i,j}^k = \mathbf{0}$ for all $k \geq q_{i,j}$. As a consequence, $\mathbf{N}_{i,j}$ is said to be **nilpotent** of order $q_{i,j}$.

In the case of complex conjugate eigenvalues, the Jordan block $\mathbf{J}_{i,j}$ is reformulated as

$$\mathbf{J}_{i,j} = \begin{bmatrix} \mathbf{M} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{M} \end{bmatrix},$$

where $\mathbf{M} = \alpha_i \mathbf{I} + \beta_i \mathbf{S}$ with $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore, $\mathbf{J}_{i,j} = \mathbf{D}_{i,j} + \mathbf{N}_{i,j}$ where

$$\mathbf{D}_{i,j} = \begin{bmatrix} \alpha_i \mathbf{I} + \beta_i \mathbf{S} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \alpha_i \mathbf{I} + \beta_i \mathbf{S} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \alpha_i \mathbf{I} + \beta_i \mathbf{S} \end{bmatrix}$$

and

$$\mathbf{N}_{i,j} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The exponential of $\mathbf{D}_{i,j}$ is composed of the exponential of $\alpha_i \mathbf{I} + \beta_i \mathbf{S}$, where

$$\begin{aligned} \exp\left(\left(\alpha_i \mathbf{I} + \beta_i \mathbf{S}\right) t\right) &= \exp\left(\alpha_i \mathbf{I} t\right) \exp\left(\beta_i \mathbf{S} t\right) \\ &= \exp\left(\alpha_i t\right) \exp\left(\beta_i \mathbf{S} t\right). \end{aligned}$$

It can be demonstrated that

$$\exp\left(\beta_i \mathbf{S} t\right) = \begin{bmatrix} \cos(\beta_i t) & \sin(\beta_i t) \\ -\sin(\beta_i t) & \cos(\beta_i t) \end{bmatrix}$$

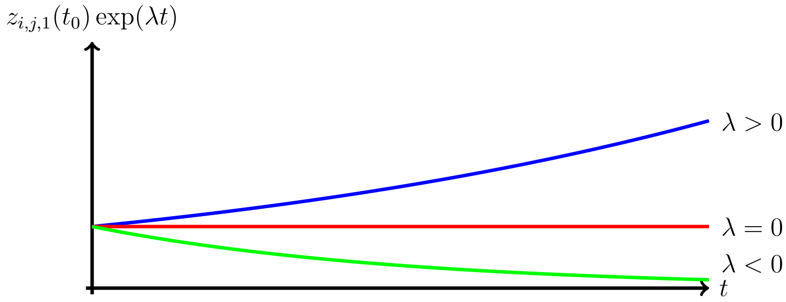


Figure 3.3: Time behavior of $z_{i,j,1}(t_0)\exp(\lambda t)$

and

$$\exp(\mathbf{N}_{i,j}t) = \begin{bmatrix} \mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\mathbf{I} \\ \mathbf{0} & \mathbf{I} & t\mathbf{I} & \ddots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \ddots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!}\mathbf{I} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{I} & t\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

In conclusion, the exponential of $\mathbf{J}_{i,j}$ in case of complex conjugate eigenvalues is

$$\begin{aligned} \exp(\mathbf{J}_{i,j}t) &= \exp(\mathbf{D}_{i,j}t + \mathbf{N}_{i,j}t) = \exp(\mathbf{D}_{i,j}t)\exp(\mathbf{N}_{i,j}t) \\ &= [\exp(\alpha_i t)\,\mathbf{I} \otimes \exp(\beta_i \mathbf{S}t)]\ \exp(\mathbf{N}_{i,j}t) \end{aligned}$$

where $\otimes$ represents the Kroneker product, see Appendix A.2.

The following paragraphs investigate the time behavior associated with the transition matrix of a single block $\mathbf{J}_{i,j}$ for different eigenvalues.

Real eigenvalues

In the case of real λ_i, the transition matrix related to $\mathbf{J}_{i,j}$ is given by

$$\begin{aligned} \exp(\mathbf{J}_{i,j}t) &= \exp(\lambda_i t)\exp(\mathbf{N}_{i,j}t) \\ &= \begin{bmatrix} \exp(\lambda_i t) & t\exp(\lambda_i t) & \dots & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\exp(\lambda_i t) \\ 0 & \exp(\lambda_i t) & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\exp(\lambda_i t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(\lambda_i t) \end{bmatrix}. \end{aligned}$$

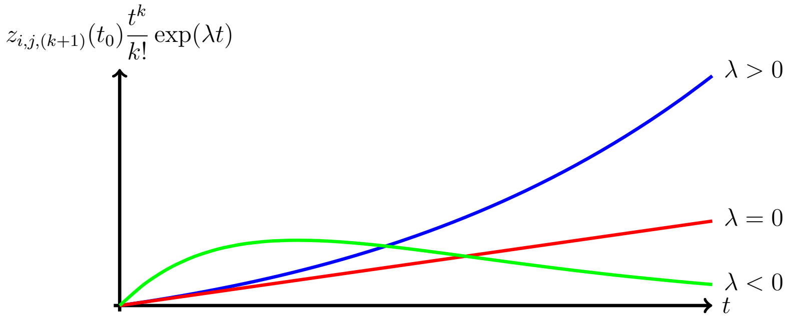


Figure 3.4: Time behavior of $z_{i,j,(k+1)}(t_0)\dfrac{t^k}{k!}\exp(\lambda t)$ for $k > 0$.

If $\mathbf{u} = \mathbf{0}$, the dynamics of $\mathbf{z}_{i,j}$ are given by

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}t)\mathbf{z}_{i,j}(t_0)$$

where, in particular, the dynamics of the first element of $\mathbf{z}_{i,j}$, namely $z_{i,j,1}$, are

$$z_{i,j,1}(t) = \exp(\lambda_i t)\sum_{k=0}^{q_{i,j}-1}\frac{t^k}{k!}z_{i,j,(k+1)}(t_0).$$

This dynamics are bounded if, for each k, there exists a finite real M_k such that

$$\lim_{t\to\infty}\exp(\lambda_i t)t^k < M_k \in \mathbb{R},\ \forall k = 0,\dots,q_{i,j}-1.$$

The latter inequality leads to the following conditions:

$$\mathbf{z}_{i,j}(t)\begin{cases} \text{exponentially convergent to zero} & \lambda_i < 0,\ q_{i,j} \geq 1 \\ \text{constant} & \lambda_i = 0,\ q_{i,j} = 1 \\ \text{polinomially divergent from } \mathbf{z}_{i,j}(t_0) & \lambda_i = 0,\ q_{i,j} > 1 \\ \text{exponentially divergent from } \mathbf{z}_{i,j}(t_0) & \lambda_i > 0,\ q_{i,j} \geq 1 \end{cases}.$$

Figures 3.3-3.4 depict the time behavior of the term $\exp(\lambda_i t)\dfrac{t^k}{k!}z_{i,j,(k+1)}(t_0)$ in case of real λ_i for $k = 0$ and $k > 0$ respectively.

Complex conjugate eigenvalues

In the case of complex conjugate eigenvalues, the exponential of $\mathbf{J}_{i,j}t$ becomes

$$\begin{aligned}\exp(\mathbf{J}_{i,j}t) &= [\exp(\alpha_i t)\,\mathbf{I} \otimes \exp(\beta_i \mathbf{S} t)]\ \exp(\mathbf{N}_{i,j}t) = \\ &= \exp(\alpha_i t)\,\mathbf{I} \otimes \begin{bmatrix} \cos(\beta_i t) & \sin(\beta_i t) \\ -\sin(\beta_i t) & \cos(\beta_i t) \end{bmatrix}. \\ &\begin{bmatrix} \mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \frac{t^3}{3!}\mathbf{I} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\mathbf{I} \\ \mathbf{0} & \mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \dots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & t\mathbf{I} & \dots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!}\mathbf{I} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & t\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \end{bmatrix}.\end{aligned}$$

Assuming $\mathbf{u} = \mathbf{0}$, the dynamics of $\mathbf{z}_{i,j}$ is given by

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}t)\mathbf{z}_{i,j}(t_0)$$

where, in particular, the dynamics of the first two elements of $\mathbf{z}_{i,j}$, *i.e.*, $\mathrm{col}\,(z_{i,j,1}(t), z_{i,j,2}(t))$, are

$$\begin{bmatrix} z_{i,j,1}(t) \\ z_{i,j,2}(t) \end{bmatrix} = \exp(\alpha_i t) \begin{bmatrix} \cos(\beta_i t) & \sin(\beta_i t) \\ -\sin(\beta_i t) & \cos(\beta_i t) \end{bmatrix} \sum_{k=0}^{q_{i,j}-1} \frac{t^k}{k!} \begin{bmatrix} z_{i,j,2k+1}(t_0) \\ z_{i,j,2k+2}(t_0) \end{bmatrix}. \tag{3.18}$$

The elements of $\mathrm{col}\,(z_{i,j,1}(t), z_{i,j,2}(t))$ are bounded if there exists finite $M > 0$ such that

$$\lim_{t\to\infty} \exp(\alpha_i t)\, t^k < M_k \in \mathbb{R}, \quad \forall k = 0, \dots, q_{i,j} - 1.$$

This inequality leads to the following conditions:

$$\mathbf{z}_{i,j}(t) \begin{cases} \text{exponentially convergent to zero} & \alpha_i < 0,\ q_{i,j} \geq 1 \\ \text{constant} & \alpha_i = 0,\ q_{i,j} = 1 \\ \text{polynomially divergent from } \mathbf{z}_{i,j}(t_0) & \alpha_i = 0,\ q_{i,j} > 1 \\ \text{exponentially divergent from } \mathbf{z}_{i,j}(t_0) & \alpha_i > 0,\ q_{i,j} \geq 1. \end{cases}$$

Figures 3.6-3.8 depict the time behavior of the terms appearing in (3.18) for both $k = 0$ and $k > 0$.

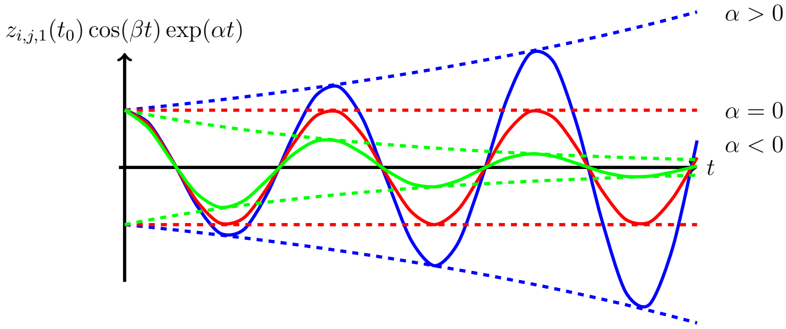


Figure 3.5: Time behavior of $z_{i,j,1}(t_0)\cos(\beta t)\exp(\alpha t)$

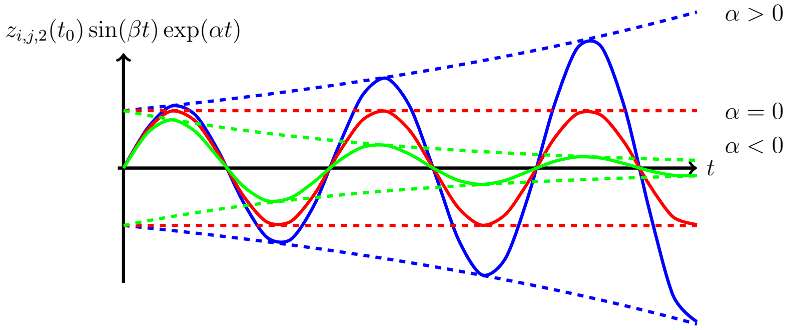


Figure 3.6: Time behavior of $z_{i,j,2}(t_0)\sin(\beta t)\exp(\alpha t)$

Infobox 3.2. Regarding the frequency analysis, the values α_i and β_i, representing the real and the imaginary part of a complex eigenvalue, are usually exploited to define the pulsation ω_i and the damping ratio δ_i as

$$\omega_i = \sqrt{\alpha_i^2 + \beta_i^2}, \quad \delta_i = \tan^{-1}(\alpha_i/\beta_i).$$

Example 3.3 (Cart-pole modes)**.** Let $\mathbf{J}$ be the Jordan matrix associated with the matrix $\mathbf{A}$ of Exercise 3.2. Then, the plant trajectories are described by

$$\exp(\mathbf{J}t) = \begin{bmatrix} \exp(\mathbf{J}_1 t) & \mathbf{0} \\ \mathbf{0} & \exp(\mathbf{J}_2 t) \end{bmatrix}, \exp(\mathbf{J}_1 t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
$$\exp(\mathbf{J}_2 t) = \exp(\alpha_2 t) \begin{bmatrix} \cos(\beta_2 t) & \sin(\beta_2 t) \\ -\sin(\beta_2 t) & \cos(\beta_2 t) \end{bmatrix}.$$

Since the numerical values specified in Exercise 3.2 lead to $\alpha_2 < 0$, the system

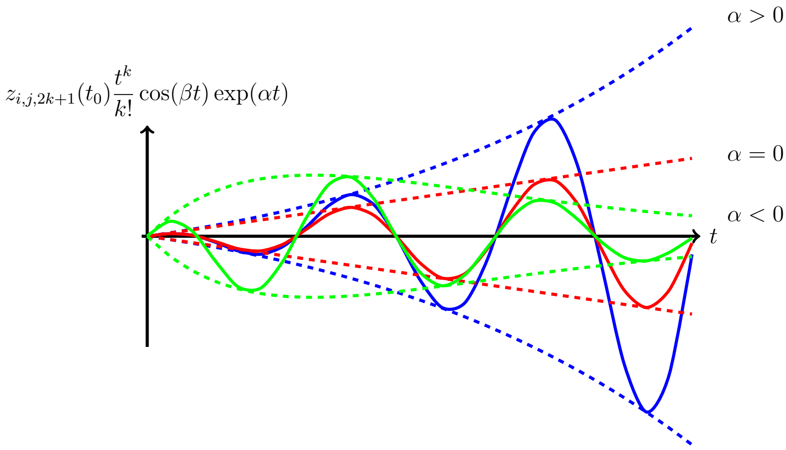


Figure 3.7: Time behavior of $z_{i,j,2k+1}(t_0)\dfrac{t^k}{k!}\cos(\beta t)\exp(\alpha t)$ for $k > 0$

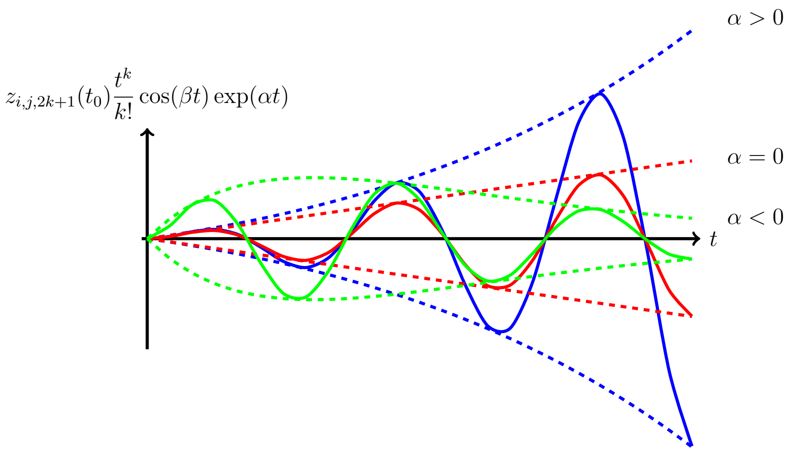


Figure 3.8: Time behavior of $z_{i,j,2k+1}(t_0)\dfrac{t^k}{k!}\sin(\beta t)\exp(\alpha t)$ for $k > 0$

(*i.e.*, the linearization of the cart-pole around the static equilibrium) has one divergent mode (associated with the first line of $\mathbf{J}_1$), one constant mode (associated with the second line of $\mathbf{J}_1$), and two dumped oscillating modes (associated with $\mathbf{J}_2$).

3.3 BIBS Stability

As previously described, the eigenvalues and the lengths of the chains of eigenvectors associated with matrix $\mathbf{A}$ entirely describe the dynamics of LTI systems [1, 2]. In particular, the trajectories of $\mathbf{x}$ are convergent to the origin if and only if the real parts of the eigenvalues are all strictly negative. In addition, eigenvalues with null or positive real parts may lead to constant or divergent behaviors, depending on the length of the eigenvector chains. This section exploits these results to state a stability criterion that ensures trajectory boundedness [3, 7].

Defintion 3.1. Bounded-Input-Bounded-State (BIBS) Stability. We define

$$\chi_{\mathbf{u}}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}\mathbf{u}(\tau) d\tau$$

as the solution to LTI systems in form (3.12). Then, the system is BIBS-stable if $\forall \varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\forall \, \delta\mathbf{u}(\cdot) \, : \, \|\delta\mathbf{u}(t)\| \leq \delta_\varepsilon \implies \|\chi_{\mathbf{u}+\delta\mathbf{u}}(t, \mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t, \mathbf{x}(t_0))\| \leq \varepsilon \quad \forall \, t \geq t_0.$$

The definition of BIBS stability links the perturbation of the input $\mathbf{u}$, *i.e.*, $\delta\mathbf{u}$, with the evolution of the state $\mathbf{x}$, see Fig. 3.9.

Note

Roughly, the BIBS stability requires that for all possible choices of $\varepsilon > 0$ (*i.e.*, from the smallest to the largest greater than zero), there exists a bounded acceptable input variation δ_ε (which is positive, finite, and possibly related to ε) such that the perturbed and unperturbed trajectories remain close forever.

The remaining of this section demonstrates that having negative the real part of all eigenvalues implies BIBS stability. Without loss of generality, assume $\mathbf{x}(t_0) = \mathbf{0}$ and calculate the perturbed and unperturbed trajectories

$$\begin{aligned} \chi_{\mathbf{u}+\delta\mathbf{u}}(t, \mathbf{x}(t_0)) &= \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau)) \, \mathbf{B} \, (\mathbf{u}(\tau) + \delta\mathbf{u}(\tau)) \, d\tau \\ \chi_{\mathbf{u}}(t, \mathbf{x}(t_0)) &= \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau)) \, \mathbf{B} \, \mathbf{u}(\tau) d\tau \end{aligned}$$

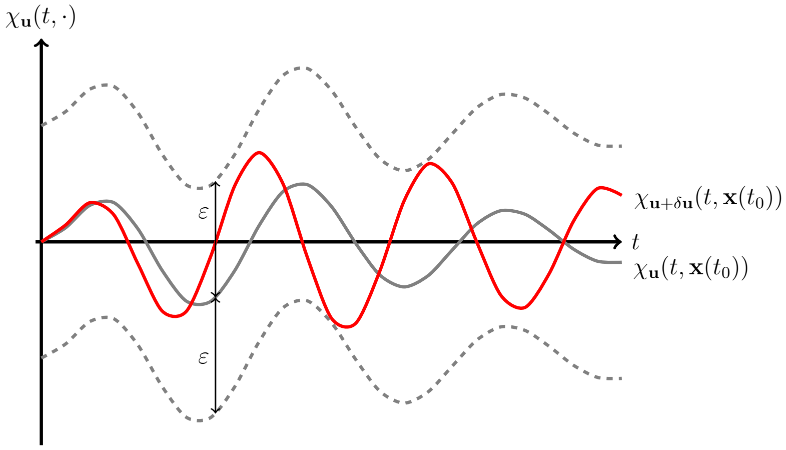


Figure 3.9: Case of BIBS-stable system

whose difference is

$$\chi_{\mathbf{u}+\delta\mathbf{u}}(t,\mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t,\mathbf{x}(t_0)) = \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\,\mathbf{B}\,\delta\mathbf{u}(\tau)d\tau.$$

Let $\mathbf{V}$ and α_i be the matrix of eigenvectors and the real part of the ith eigenvalue of $\mathbf{A}$. Then, using $\bar{\mathbf{A}} := \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ and the results of Section 3.2, we can write

$$\begin{gathered}\|\chi_{\mathbf{u}+\delta\mathbf{u}}(t,\mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t,\mathbf{x}(t_0))\| = \left\| \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\delta\mathbf{u}(\tau)d\tau \right\| \leq \\ \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\mathbf{B}\delta\mathbf{u}(\tau)\|\, d\tau \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| d\tau\, \|\mathbf{B}\|\delta_\varepsilon \leq \\ \|\mathbf{V}\| \int_{t_0}^{t} \max_{\substack{i=1,\dots,p \\ j=1,\dots,g_i}} \sum_{k=0}^{q_{i,j}-1} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right| d\tau \|\mathbf{V}^{-1}\| \|\mathbf{B}\| \delta_\varepsilon\end{gathered}$$

Therefore, LTI systems (3.12) are BIBS-stable if for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\delta_\varepsilon \leq \frac{\varepsilon}{\|\mathbf{V}\| \int_{t_0}^{t} \max_{\substack{i=1,\dots,p \\ j=1,\dots,g_i}} \sum_{k=0}^{q_{i,j}} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right| d\tau \|\mathbf{V}^{-1}\| \|\mathbf{B}\|}.$$

The integral appearing at the denominator of the right-hand side is finite if

$$\max_{\substack{i=1,\dots,p \\ j=1,\dots,a_i}} \sum_{k=0}^{q_{i,j}-1} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right|$$

is absolutely integrable which, in turn, is true if the real part of all the eigenvalues of $\mathbf{A}$ is strictly negative. Such a matrix is called Hurwitz.

Example 3.4 (Cart-pole Stability)**.** The cart-pole is not BIBS-stable in agreement with the results of Example 3.3.

Note

The BIBS stability implies (but is not implied by) the boundedness of trajectories with perturbed initial conditions. In detail, set $\mathbf{u} = \mathbf{0}$ without loss of generality, define $\delta\mathbf{x}_0$ as the initial condition perturbation, and let

$$\begin{aligned}\chi_{\mathbf{0}}(t, \mathbf{x}(t_0) + \delta\mathbf{x}_0) &= \exp(\mathbf{A}(t - t_0))(\mathbf{x}(t_0) + \delta\mathbf{x}_0) \\ \chi_{\mathbf{0}}(t, \mathbf{x}(t_0)) &= \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0)\end{aligned}$$

be the system trajectories with and without perturbations of the initial conditions. Then, we use $\bar{\mathbf{A}} := \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ and the results of Section 3.2 to bound the norm of the trajectory difference as

$$\begin{aligned}\|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0) + \delta\mathbf{x}_0) - \chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| \leq& \|\exp(\mathbf{A}(t - t_0))\| \, \|\delta\mathbf{x}_0\| \\ \leq& \|\mathbf{V}\| \max_{\substack{i = 1, \ldots, p \\ j = 1, \ldots, a_i}} \sum_{k=0}^{q_{i,j}-1} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right| \|\mathbf{V}^{-1}\| \|\delta\mathbf{x}_0\|\end{aligned}$$

Then, we note that the boundedness of the latter is implied by having the real part of the eigenvalues of $\mathbf{A}$ negative, which represents a sufficient but not necessary condition. Indeed, if $q_{i,j} = 1$ the boundedness of

$$\|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0) + \delta\mathbf{x}_0) - \chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\|$$

holds also for $\alpha_i = 0$.

3.4 ADAS Analysis

This section investigates the stability properties of the linear plants defined in Chapter 2. The computation of eigenvectors and eigenvalues is instrumental to achieve this goal.

3.4.1 Active Suspensions

Equations (2.7) and (2.16) model the dynamics of the suspension system with active components for the single corner and the half car, respectively.

Chapter 4

Control System Architecture

Contents

Goal G1 requires the controller to keep states $\mathbf{x}$, inputs $\mathbf{u}$, and regulated outputs $\mathbf{e}$ bounded for all times under the assumption of bounded exogenous signals (*i.e.*, bounded disturbances, noises, and references). Chapter 3 has linked the boundedness of these signals to BIBS stability through the eigenvalues of the linearized system. This chapter incrementally obtains the controller architecture starting from the simplest control system, *i.e.*, a *stabilizer* based on a *state feedback*, see Section 4.2. Section 4.3 enhances the basic controller by presenting the robust output regulation of set points in case of unknown but constant disturbances. Unfortunately, the whole state, often not available at measurement, is necessary to implement the state-feedback stabilizer. Therefore, Section 4.5 proposes stabilizing linear plants by employing feasible *output feedback* to overcome this limitation. To achieve this result, the concept of *state observer* described in Section 4.4 plays a fundamental role. Furthermore, Section 4.6 relies on the state-feedback stabilizer to solve the tracking of time-varying references by utilizing the feed-forward. Finally, this chapter describes the control system architecture for each case study illustrated in Chapter 2.

4.1 Closed-Loop System

This section exploits Eq.s (1.16) and (1.17) to identify an overall linear system whose properties are directly associated with achieving control goals G1 and G2.

We impose $\mathbf{D}_\mathrm{O} = \mathbf{0}$ and $\mathbf{C}_\mathrm{O} = \mathbf{I}$ to simplify the presentation of topics. Infobox 4.6 shows how to remove this assumption.

Let $\mathbf{e}_x := \mathbf{x}_\mathrm{O} - \tilde{\mathbf{x}}$ and $\boldsymbol{\chi} := \mathrm{col}(\tilde{\mathbf{x}}, \boldsymbol{\eta}, \mathbf{e}_x, \mathbf{x}_\mathrm{FF})$. Then, exploit $\mathbf{x}_\mathrm{O} = \mathbf{e}_x + \tilde{\mathbf{x}}$, Eq. (1.16) and Eq. (1.17) to compute the closed-loop dynamics

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}_\chi \boldsymbol{\chi} + \mathbf{B}_{\chi w} \tilde{\mathbf{w}} + \sum_{i=0}^{r_{\max}} \mathbf{B}_{\chi i} \frac{d^i}{dt^i} \mathbf{r} \\ \tilde{\mathbf{u}} &= \mathbf{C}_\chi \boldsymbol{\chi} + \sum_{i=0}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i} \frac{d^i}{dt^i} \mathbf{r}, \end{aligned} \tag{4.1a}$$

where

$$\mathbf{A}_\chi = \begin{bmatrix} \mathbf{A} + \mathbf{B}_1 \mathbf{K}_\mathrm{S} & \mathbf{B}_1 \mathbf{K}_\mathrm{I} & \mathbf{B}_1 \mathbf{K}_\mathrm{S} & \mathbf{B}_1 \mathbf{C}_\mathrm{FF} \\ \mathbf{C}_e + \mathbf{D}_{e_1} \mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1} \mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1} \mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1} \mathbf{C}_\mathrm{FF} \\ \mathbf{A}_\mathrm{O} + \mathbf{K}_\mathrm{O} \mathbf{C} - \mathbf{A} + \mathbf{M} \mathbf{K}_\mathrm{S} & \mathbf{M} \mathbf{K}_\mathrm{I} & \mathbf{A}_\mathrm{O} + \mathbf{M} \mathbf{K}_\mathrm{S} & \mathbf{M} \mathbf{C}_\mathrm{FF} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\mathrm{FF} \end{bmatrix}, \tag{4.1b}$$

$$\mathbf{B}_{\chi w} = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \\ \mathbf{K}_\mathrm{O} \mathbf{D}_2 - \mathbf{B}_2 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{B}_{\chi i} = \begin{bmatrix} \mathbf{B}_1 \mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{D}_{e_1} \mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{M} \mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{B}_{\mathrm{FF}_i} \end{bmatrix}, \tag{4.1c}$$

and

$$\mathbf{C}_\chi = \begin{bmatrix} \mathbf{K}_\mathrm{S} & \mathbf{K}_\mathrm{I} & \mathbf{K}_\mathrm{S} & \mathbf{C}_\mathrm{FF} \end{bmatrix} \tag{4.1d}$$

with $\mathbf{M} = \mathbf{B}_\mathrm{O} + \mathbf{K}_\mathrm{O}\mathbf{D}_1 - \mathbf{B}_1$.

First, observe that the boundedness of $\boldsymbol{\chi}$ and $\mathbf{r}$ (and all its time derivatives up to order r_max) implies the boundedness of $\tilde{\mathbf{u}}$. On the one hand, $\mathbf{r}$ is bounded by assumption, but, on the other hand, the BIBS stability of system (4.1a) implies the boundedness of $\boldsymbol{\chi}$ as described in Section 3.3. In turn, the BIBS stability of system (4.1a) is related to the eigenvalues of $\mathbf{A}_\chi$ (which must be Hurwitz). Moreover, the BIBS stability of system (4.1a) is twofold because, $\mathbf{e}$ being part of $\boldsymbol{\chi}$, the boundedness of the latter implies the boundedness of the former. Consequently, to achieve G1 and G2, the control system is designed to make $\mathbf{A}_\chi$ Hurwitz.

The remaining sections describe the design of the controller matrices with an incremental approach. First, in more detail, Section 4.2 states the conditions the plant must satisfy to guarantee the existence of $\mathbf{K}_\mathrm{S}$ such that $\mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S}$ is Hurwitz. Then, Section 4.3 introduces the integral action and designs $\mathbf{K}_\mathrm{I}$ that makes Hurwitz the sub-matrix

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix}.$$

Section 4.4 describes the conditions the plant must satisfy to allow $\mathbf{K}_\mathrm{O}$ to exist such that $\mathbf{A}-\mathbf{K}_\mathrm{O}\mathbf{C}$ is Hurwitz. Moreover, Section 4.5 designs $\mathbf{A}_\mathrm{O}$ and $\mathbf{B}_\mathrm{O}$, which make the following sub-matrix Hurwitz:

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} \\ \mathbf{A}_\mathrm{O}+\mathbf{K}_\mathrm{O}\mathbf{C}-\mathbf{A}+\mathbf{M}\mathbf{K}_\mathrm{S} & \mathbf{M}\mathbf{K}_\mathrm{I} & \mathbf{A}_\mathrm{O}+\mathbf{M}\mathbf{K}_\mathrm{S} \end{bmatrix}.$$

Finally, Section 4.6 designs the feed-forward control (with $\mathbf{A}_\mathrm{FF}$ Hurwitz) and allows the tracking of time-varying references.

4.2 State-Feedback Stabiliser

As introduced in Section 4.1, the achievement of control goals G1 and G2 is obtained through a design strategy whose first step consists of making Hurwitz the matrix $\mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S}$. But, first, it is essential to understand which assumptions the plant must verify to let such $\mathbf{K}_\mathrm{S}$ exist. The so-called *reachability decomposition* represents the tool to perform these analyses. To keep the notation lighter, refer to the linearized vectors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{u}}$, and $\tilde{\mathbf{w}}$ by hiding the accent $\tilde{}$.

Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}_1\mathbf{u}+\mathbf{B}_2\mathbf{w} \quad \mathbf{x}(t_0)=\mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x}+\mathbf{D}_1\mathbf{u}+\mathbf{D}_2\mathbf{w} \end{aligned} \tag{4.2}$$

be the LTI system obtained by picking the first two equations of system (1.16). Then,

$$\chi_u(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau. \tag{4.3}$$

represents the integral curve of (4.2) with $\mathbf{w}=\mathbf{0}$. Assume that the initial condition is identically null, *i.e.*, $\mathbf{x}(t_0) = \mathbf{0}$, and denote with $\mathcal{R}(t)$ the set of *reachable states from the origin* thanks to any control $\mathbf{u} : [t_0,\, t] \to \mathbb{R}^p$ [[1],§3]

$$\mathcal{R}(t) := \{\mathbf{x} \in \mathbb{R}^n \;:\; \mathbf{x} = \chi_u(t, \mathbf{0})\}\,.$$

Note

In practice, the set $\mathcal{R}(t)$ represents the set of states reached from the origin thanks to the whole set of time-varying control signals $\tau \mapsto \mathbf{u}(\tau)$ with $\tau \in [t_0,\, t]$ (not just one particular control law!).

Remark 4.1. The reachability set $\mathcal{R}$ is a subspace of $\mathbb{R}^n$, *i.e.*, is a set closed to multiplication by scalar and addition. Therefore, a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_{\ell_\mathrm{R}}\}$ exists, which spans $\mathcal{R}$, where ℓ_R represents the dimension of $\mathcal{R}$. Moreover, it is impossible to reach the states which do not belong to $\mathcal{R}$ starting from the origin. Equivalently, the control $\mathbf{u}(\cdot)$ can not influence the dynamics of states out of $\mathcal{R}$.

Theorem 4.1 (Reachability [6])**.** *Let* (4.2)*, then the matrix*

$$\mathbf{R} := \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}\mathbf{B}_1 & \mathbf{A}^2\mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 \end{bmatrix} \in \mathbb{R}^{n \times np}$$

is such that im$(\mathbf{R}) = \mathcal{R}$*. The matrix* $\mathbf{R}$ *is called* ***reachability matrix****.*

Infobox 4.1 (Proof of Theorem 4.1)**.** The proof of Theorem 4.1 is founded on the *Cayley-Hamilton* theorem, which allows writing

$$\exp(\mathbf{A}t) = \sum_{i=0}^{n-1} \Phi_i(t)\mathbf{A}^i$$

where $\Phi_i : \mathbb{R} \to \mathbb{R}$ for any $i = 1, \dots, n-1$. Then, write

$$\begin{aligned}
\mathbf{x}(t) &= \int_0^t \exp\left(\mathbf{A}(t-\tau)\right) \mathbf{B}_1 \mathbf{u}(\tau)\, d\tau \\
&= \int_0^t \sum_{i=0}^{n-1} \Phi_i(t-\tau) \mathbf{A}^i \mathbf{B}_1 \mathbf{u}(\tau)\, d\tau \\
&= \sum_{i=0}^{n-1} \mathbf{A}^i \mathbf{B}_1 \int_0^t \Phi_i(t-\tau) \mathbf{u}(\tau)\, d\tau \\
&= \mathbf{R} \begin{bmatrix} \int_0^t \Phi_0(t-\tau)\mathbf{u}(\tau)\, d\tau \\ \vdots \\ \int_0^t \Phi_{n-1}(t-\tau)\mathbf{u}(\tau)\, d\tau \end{bmatrix}.
\end{aligned}$$

Conceive the vector of the convolution integrals as a degree of freedom and obtain $\mathbf{x}(t)$ as a linear combination of the elements of $\mathbf{R}$. So, $\mathbf{x}(t)$ must belong to the image of $\mathbf{R}$ to let the equality be verified, *i.e.*, to allow for the existence of a family of control laws $\mathbf{u}(t)$ that verify the equality.

Infobox 4.2 (Reachability and Controllability)**.** Control system theory developed the concept of **controllability set**. In particular, consider the LTI system (4.2), then $\bar{\mathbf{x}} \in \mathbb{R}^n$ belongs to the controllability set, namely $\mathcal{C}(t)$, if there exists a control law $\tau \in [t_0,\, t] \mapsto \mathbf{u}(\tau)$ such that

$$\mathbf{0} = \exp(\mathbf{A}(t-t_0))\bar{\mathbf{x}} + \int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau.$$

Since $\exp(\mathbf{A}(t-t_0))$ is invertible, the latter equality can be re-arranged as

$$\bar{\mathbf{x}} = -\exp(-\mathbf{A}(t-t_0)) \int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau.$$

Now, note that $\exp(-\mathbf{A}(t-t_0))$ is full-rank and

$$\int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau \in \mathcal{R}.$$

Therefore, since $\bar{\mathbf{x}} \in \mathcal{C}$ implies $\bar{\mathbf{x}} \in \mathcal{R}$ and vice-versa, the reachability and controllability subspaces are equivalent (only for continuous-time LTI systems).

For this reason, the terms **reachability** and **controllability** are interchangeable (for instance this book adopts the former while MATLAB® uses the latter, *e.g.*, in the command `ctrb`).

Important

An LTI system, or its couple $(\mathbf{A}, \mathbf{B})$ equivalently, is said to be **completely reachable** if $\text{rank}(\mathbf{R}) = n$, *i.e.*, if $\mathcal{R} = \mathbb{R}^n$.

Note

Intuitively, the reachability matrix form is derived as follows. Assume that $\mathbf{x}(t_0) = \mathbf{0}$ and take a constant control action, $\mathbf{u}$. Select p particular control inputs as

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \mathbf{u}_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Evaluate the first-order time derivatives $\dot{\mathbf{x}}_i(t_0) = \mathbf{B}_1\mathbf{u}_i$ for each control input, collecting them in the matrix

$$\mathbf{X}^{(1)} := \begin{bmatrix} \frac{d\mathbf{x}_1}{dt} & \frac{d\mathbf{x}_2}{dt} & \cdots & \frac{d\mathbf{x}_p}{dt} \end{bmatrix} = \mathbf{B}_1 \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix},$$

where it is essential to remark that $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix} = \mathbf{I}$ (hidden hereafter). On the other hand, $\frac{d^k\mathbf{x}_i}{dt^k}(t_0) = \mathbf{A}^{k-1}\mathbf{B}_1\mathbf{u}_i$ for $k = 2, \ldots, n$ and $i = 1, \ldots, p$ represent the higher–order time derivatives of $\mathbf{x}$ at time t_0. For each $k = 2, \ldots, n$, build the matrices $\mathbf{X}^{(k)}$ and organize them in a matrix as

$$\begin{bmatrix} \mathbf{X}^{(1)} & \mathbf{X}^{(2)} & \cdots & \mathbf{X}^{(n)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}\mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 \end{bmatrix}.$$

In conclusion, the reachability matrix represents the basis for describing state time derivatives up to order n.

Example 4.1 (Reachability study)**.** Let the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, where $\mathbf{x} \in \mathbb{R}^2$,

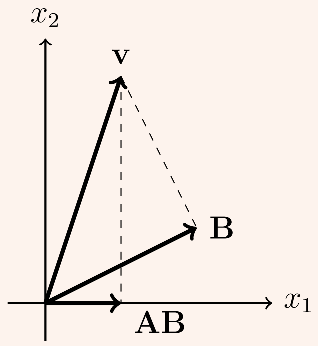
$u \in \mathbb{R}$, and

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

with a, b_1, $b_2 > 0$. The reachability matrix is then given by

$$\mathbf{R} = \left[\ \mathbf{B} \quad \mathbf{AB}\ \right] = \left[\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \begin{pmatrix} ab_2 \\ 0 \end{pmatrix} \right].$$

Draw the vectors $\mathbf{B}$ and $\mathbf{AB}$ on the plane x_1-x_2.



The vectors $\mathbf{B}$ and $\mathbf{AB}$ are linearly independent because they have different directions and thus they represent a valid base for the representation of any vector $\mathbf{v}$ in the space $x_1 - x_2$. More formally, the vectors $\mathbf{B}$ and $\mathbf{AB}$ span the whole state space $\mathbb{R}^2$ implying a fully reachable system.

Example 4.2 (Reachability study)**.** Consider the system defined in Example 4.1 and assume $b_2 = 0$. The reachability matrix becomes

$$\mathbf{R} = \left[\ \mathbf{B} \quad \mathbf{AB}\ \right] = \left[\begin{pmatrix} b_1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right],$$

which implies that the vectors $\mathbf{B}$ and $\mathbf{AB}$ do not span the whole state space; equivalently, they cannot represent any vector $\mathbf{v} \in \mathbb{R}^2$ but only those in the x_1 direction. Consequently, the system is not fully reachable and the reachable subspace is spanned by $\mathbf{B}$ (which represents the x_1 direction). The non-reachable subspace is identified by the direction orthogonal to $\mathbf{B}$, *i.e.*, by $\mathbf{w} = \mathrm{col}(0, 1)$.

Example 4.3 (Cart-pole reachability study)**.** Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & A_{23} & A_{42} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 \\ B_{12} \\ 0 \\ B_{14} \end{bmatrix}$$

be defined in Example 1.2, with the assumptions made in Example 3.2. The reachability matrix is obtained as

$$\mathbf{R} = \left[\begin{pmatrix} 0 \\ B_{12} \\ 0 \\ B_{14} \end{pmatrix} \begin{pmatrix} B_{12} \\ A_{24}B_{14} \\ B_{14} \\ A_{44}B_{14} \end{pmatrix} \begin{pmatrix} B_{12} \\ A_{23}B_{14} + A_{24}A_{44}B_{14} \\ A_{44}B_{14} \\ A_{43}B_{14} + A_{44}^2 B_{14} \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} B_{12} \\ A_{23}A_{44}B_{14} + A_{24}(A_{43}B_{14} + A_{44}^2 B_{14}) \\ A_{43}B_{14} + A_{44}^2 B_{14} \\ A_{43}A_{44}B_{14} + A_{44}(A_{43}B_{14} + A_{44}^2 B_{14}) \end{pmatrix} \right].$$

The substitution of the numerical values listed in Example 3.2 leads to a full-rank matrix $\mathbf{R}$ that implies a fully reachable system.

Since $\mathcal{R}$ is a subspace of $\mathbb{R}^n$, its orthogonal complement exists, namely $\mathcal{R}^\perp$, such that $\mathcal{R} \oplus \mathcal{R}^\perp = \mathbb{R}^n$. Since the image of $\mathbf{R}$ corresponds to $\mathcal{R}$, then $\ker\left(\mathbf{R}^\top\right) = \mathcal{R}^\perp$. Therefore, the matrices $\mathrm{im}(\mathbf{R})$ and $\ker\left(\mathbf{R}^\top\right)$ can be exploited to identify a change of coordinate, namely $\mathbf{z} = \mathbf{T}_\mathrm{R}\mathbf{x}$, which highlights the reachable part of system (4.2). In particular, the transformation matrix $\mathbf{T}_\mathrm{R}$ is called *reachability transformation* and is defined as [[2],§6]

$$\mathbf{T}_\mathrm{R}^{-1} = \left[\ \mathrm{im}\left(\mathbf{R}\right) \quad \ker\left(\mathbf{R}^\top\right) \ \right],$$

which, applied to system (4.2), leads to

$$\begin{aligned} \dot{\mathbf{z}} =& \ \mathbf{T}_\mathrm{R}\mathbf{A}\mathbf{T}_\mathrm{R}^{-1}\mathbf{z} + \mathbf{T}_\mathrm{R}\mathbf{B}_1\mathbf{u} + \mathbf{T}_\mathrm{R}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_\mathrm{R}\mathbf{x}_0 \\ \mathbf{y} =& \ \mathbf{C}\mathbf{T}_\mathrm{R}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}. \end{aligned} \tag{4.4}$$

The study of $\bar{\mathbf{A}} := \mathbf{T}_\mathrm{R}\mathbf{A}\mathbf{T}_\mathrm{R}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_\mathrm{R}\mathbf{B}_1$, $\bar{\mathbf{B}}_2 := \mathbf{T}_\mathrm{R}\mathbf{B}_2$, and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_\mathrm{R}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \quad \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11} \\ \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \end{bmatrix} \quad \bar{\mathbf{C}} = \left[\ \bar{\mathbf{C}}_1 \quad \bar{\mathbf{C}}_2 \ \right]$$

so that, if the state is defined as $\mathbf{z} = \mathrm{col}(\mathbf{z}_\mathrm{R}, \mathbf{z}_\mathrm{NR})$, the dynamics of (4.4) become

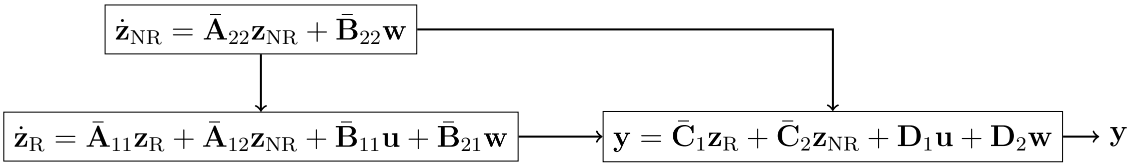


Figure 4.1: Reachability decomposition

(see Fig. 4.1)

$$
\begin{array}{ccccccccc}
\dot{\mathbf{z}}_{\mathrm{R}} & = & \bar{\mathbf{A}}_{11}\,\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{A}}_{12}\,\mathbf{z}_{\mathrm{NR}} & + & \bar{\mathbf{B}}_{11}\mathbf{u} & + & \bar{\mathbf{B}}_{21}\mathbf{w} \\
\dot{\mathbf{z}}_{\mathrm{NR}} & = & & & \bar{\mathbf{A}}_{22}\,\mathbf{z}_{\mathrm{NR}} & & & + & \bar{\mathbf{B}}_{22}\mathbf{w} \\
\mathbf{y} & = & \bar{\mathbf{C}}_{1}\,\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{C}}_{2}\,\mathbf{z}_{\mathrm{NR}} & + & \mathbf{D}_1\mathbf{u} & + & \mathbf{D}_2\mathbf{w} \\
\left[\begin{array}{c} \mathbf{z}_{\mathrm{R}}(t_0) \\ \mathbf{z}_{\mathrm{NR}}(t_0) \end{array}\right] & = & \mathbf{T}_{\mathrm{R}}\,\mathbf{x}_0. & & & & & &
\end{array}
\tag{4.5}
$$

Equation (4.5) clearly shows that the dynamics of $\mathbf{z}_{\mathrm{NR}}$ are not (and cannot be) influenced by the control $\mathbf{u}(\cdot)$. It is important to note that the properties of $\bar{\mathbf{A}}_{22}$ determine the dynamics of

$$\dot{\mathbf{z}}_{\mathrm{NR}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{NR}} + \bar{\mathbf{B}}_{22}\mathbf{w} \quad \mathbf{z}_{\mathrm{NR}}(t_0) = \mathbf{z}_{\mathrm{NR}_0}.$$

Important

Accordingly to the nature of the eigenvalues and eigenvectors of $\bar{\mathbf{A}}_{22}$, the dynamics of the non-reachable subpart of the state, $\mathbf{z}_{\mathrm{NR}}$, can be either BIBS-stable or not and we can do nothing to modify it.

Example 4.4 (Reachability decomposition)**.** The reachability matrix associated with the couple

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

is

$$\mathbf{R} = \left[\begin{array}{ccc} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{array}\right] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $\mathrm{rank}(\mathbf{R}) = 2$, the reachable subspace has dimensions two, whereas the non-reachable one is one-dimensional. Thus, one possible basis for the representation of the reachable subspace is

$$\{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In addition, a basis for the representation of the non-reachable subspace is

$$\mathbf{b}_3 := \ker\left(\begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \end{bmatrix} \right) = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

So, the reachability decomposition relies on the transformation matrix

$$\mathbf{T}_{\mathrm{R}}^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{bmatrix}.$$

Finally, matrices $\bar{\mathbf{A}} = \mathbf{T}_{\mathrm{R}}\mathbf{A}\mathbf{T}_{\mathrm{R}}^{-1}$ and $\bar{\mathbf{B}} = \mathbf{T}_{\mathrm{R}}\mathbf{B}$ are

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & | & -2 \\ 1 & 1 & | & 0 \\ -- & -- & -- & -- \\ 0 & 0 & | & -1 \end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix} 1 \\ 0 \\ -- \\ 0 \end{bmatrix}.$$

Example 4.5 (Cart-pole reachability decomposition)**.** Example 4.3 shows that the cart-pole plant of Example 1.2 is fully reachable. Consequently, the origin trivially represents the non-reachable subspace, which is zero-dimensional. Thus, the transformation $\mathbf{T}_{\mathrm{R}}^{-1} = \mathbf{I}$ means that the linear system of Example 4.3 is already in the reachability form.

As for the reachable subsystem, assume $\mathbf{z}_{\mathrm{NR}} = \mathbf{0}$, write

$$\dot{\mathbf{z}}_{\mathrm{R}} = \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{R}} + \bar{\mathbf{B}}_1\mathbf{u} \quad \mathbf{z}_{\mathrm{R}}(t_0) = \mathbf{z}_{\mathrm{R}_0}, \tag{4.6}$$

and note that system (4.6) is completely reachable by construction.

Theorem 4.2 (Existence of a stabilizing state feedback ([1],§4))**.** *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *be a fully reachable LTI system. Then, there exists a matrix* $\mathbf{K}_R$ *such that* $\mathbf{A} + \mathbf{B}\mathbf{K}_R$ *is Hurwitz.*

Infobox 4.3 (Proof sketch of Theorem 4.2)**.** To reduce the complexity of this proof, assume that the input is scalar, *i.e.*, $u \in \mathbb{R}$. Introduce the following system [6]

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}u$$

with

$$\bar{\mathbf{A}} := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}, \qquad \bar{\mathbf{B}} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and where $\alpha_0, \dots, \alpha_{n-1}$ are the coefficients of the polynomial

$$\det\left(\bar{\mathbf{A}} - \lambda\mathbf{I}\right) := \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1} + \lambda^n.$$

On the one hand, it is worth noting that these coefficients are directly linked to the eigenvalues of $\bar{\mathbf{A}}$ which are the roots of $\det\left(\bar{\mathbf{A}} - \lambda\mathbf{I}\right) = 0$. So, any change in $\alpha_0, \dots, \alpha_{n-1}$ results in a change of the eigenvalues of the plant. On the other hand, the couple $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ is fully reachable and therefore the matrix

$$\mathbf{R}_z := \left[\begin{array}{ccc} \bar{\mathbf{B}} & \cdots & \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}} \end{array}\right]$$

is full-rank. Now, any feedback $u := \mathbf{K}_{\mathrm{R}}\mathbf{z}$ with the matrix

$$\mathbf{K}_{\mathrm{R}} := \left[\begin{array}{ccc} k_0 & \cdots & k_{n-1} \end{array}\right]$$

leads to a closed-loop system $\dot{\mathbf{z}} = \left(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_{\mathrm{R}}\right)\mathbf{z}$ with

$$\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_{\mathrm{R}} := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ k_0 - \alpha_0 & k_1 - \alpha_1 & \dots & k_{n-2} - \alpha_{n-2} & k_{n-1} - \alpha_{n-1} \end{bmatrix}.$$

As a consequence, the roots of $\det\left(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_{\mathrm{R}} - \lambda\mathbf{I}\right) = 0$ can be modified, and their real parts can be made negative. The last step of this proof demonstrates that, for any scalar-input system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, fully reachable, a change of coordinates $\mathbf{T}$ exists such that $\mathbf{z} = \mathbf{T}\mathbf{x}$, $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, and $\bar{\mathbf{B}} := \mathbf{T}\mathbf{B}$. To show the existence of $\mathbf{T}$, note that if $(\mathbf{A}, \mathbf{B})$ is fully reachable, the matrix

$$\mathbf{R}_x := \left[\begin{array}{ccc} \mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{array}\right]$$

is full-rank. On the other hand, $\mathbf{R}_z$ is rewritten as

$$\mathbf{R}_z := \mathbf{T}\left[\begin{array}{ccc} \mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{array}\right],$$

which, compared with $\mathbf{R}_x$, leads to $\mathbf{R}_z = \mathbf{T}\mathbf{R}_x$. Then, define $\mathbf{T} = \mathbf{R}_z\mathbf{R}_x^{-1}$. A suitable choice of the reachability subspaces, associated with each control input, extends this proof strategy to plants with multidimensional inputs [7]. In detail, in the $\mathbf{z}$ coordinates, the $\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_\mathrm{R}$ matrix is a block diagonal matrix whose blocks have the structure of the plant exploited in this proof.

Important

The direct consequence of Theorem 4.2 is that there exists a **state-feedback control law** $\mathbf{u} = \mathbf{K}_\mathrm{R}\mathbf{z}_\mathrm{R}$ which makes the reachable system (4.6) BIBS-stable in the closed loop.

The substitution of $\mathbf{u} = \mathbf{K}_\mathrm{R}\mathbf{z}_\mathrm{R}$ into Eq. (4.6) leads to

$$\dot{\mathbf{z}}_\mathrm{R} = \left(\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11}\mathbf{K}_\mathrm{R}\right)\mathbf{z}_\mathrm{R} + \bar{\mathbf{B}}_{21}\mathbf{w} \quad \mathbf{z}_\mathrm{R}(t_0) = \mathbf{z}_{\mathrm{R}_0}, \tag{4.7}$$

whose block scheme is depicted in Fig. 4.2. Moreover, the dynamics of (4.5) subject

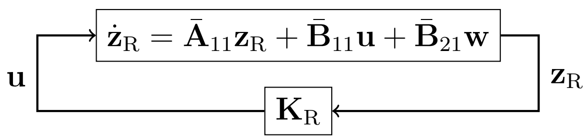


Figure 4.2: The stabilizer is based on a state-feedback architecture.

to $\mathbf{u} = \mathbf{K}_\mathrm{R}\mathbf{z}_\mathrm{R}$ are given by

$$\begin{array}{ccccccc}
\dot{\mathbf{z}}_\mathrm{R} & = & \left(\bar{\mathbf{A}}_{11} + \mathbf{B}_1\mathbf{K}_\mathrm{R}\right)\mathbf{z}_\mathrm{R} & + & \bar{\mathbf{A}}_{12}\,\mathbf{z}_\mathrm{NR} & + & \bar{\mathbf{B}}_{21}\,\mathbf{w} \\
\dot{\mathbf{z}}_\mathrm{NR} & = & & & \bar{\mathbf{A}}_{22}\,\mathbf{z}_\mathrm{NR} & + & \bar{\mathbf{B}}_{22}\,\mathbf{w} \\
\mathbf{y} & = & \left(\bar{\mathbf{C}}_1 + \mathbf{D}_1\mathbf{K}_\mathrm{R}\right)\mathbf{z}_\mathrm{R} & + & \bar{\mathbf{C}}_2\,\mathbf{z}_\mathrm{NR} & + & \bar{\mathbf{D}}_2\,\mathbf{w} \\
\left[\begin{array}{c} \mathbf{z}_\mathrm{R}(t_0) \\ \mathbf{z}_\mathrm{NR}(t_0) \end{array}\right] & = & \mathbf{T}\,\mathbf{x}_0 & & & &
\end{array} \tag{4.8}$$

in which the exogenous $\mathbf{z}_\mathrm{NR}$ influences the dynamics of the reachable part. Since the reachable subsystem is BIBS-stable, thanks to the stability properties of $\bar{\mathbf{A}}_{11}+\mathbf{B}_1\mathbf{K}_\mathrm{R}$, the state $\mathbf{z}_\mathrm{R}$ is bounded if also $\mathbf{z}_\mathrm{NR}$ is so too. On the other hand, the state $\mathbf{z}_\mathrm{NR}$ is bounded if $\bar{\mathbf{A}}_{22}$ is Hurwitz.

Important

An LTI system is said to be stabilizable if the non-reachable state is BIBS-stable, *i.e.*, if $\bar{\mathbf{A}}_{22}$ is Hurwitz.

From now on, this book adopts the following assumption.

Assumption 4.1. The plant (1.16) is stabilizable.

To conclude, with reference to (4.1a), use Assumptions 4.1, Theorem 4.2, and

$$\mathbf{u} = \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\mathrm{R}} \\ \mathbf{z}_{\mathrm{NR}} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \mathbf{T}_{\mathrm{R}}\mathbf{x}$$

to guarantee that $\mathbf{K}_{\mathrm{S}} := \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \mathbf{T}_{\mathrm{R}}$ makes $\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}}$ Hurwitz.

Before moving to the next session, it is worth noting that the control law $\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x}$ applied to system (4.2) makes BIBS-stable the closed loop

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{x} + \mathbf{B}_2\mathbf{w}$$

which, under Assumption 1.1, achieves goal G1. Ideally, this law solves all those problems in which the main control goal is stabilizing the linearization point (*e.g.*, vibration suppression via active suspensions, yaw damper, and lane keeping). But:

- Its performance could be improved to achieve asymptotic tracking, *i.e.*, to have $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$, at least for constant $\mathbf{w}$. Indeed, this scenario (called *set-point regulation*) represents one of the most common operations in automotive applications (*e.g.*, speed control, vehicle height regulation, roll control, and lane changing). Section 4.3 extends the control architecture to deal with this requirement;

- It is not implementable because it requires a perfect knowledge of $\mathbf{x}$. Section 4.5 overcomes this problem through the adoption of the observer described in Section 4.4;

- The control law can be extended to feedback also the non-reachable part, namely as $\mathbf{u} = [\mathbf{K}_{\mathrm{R}}\ \mathbf{K}_{\mathrm{NR}}]\,\mathbf{z}$. Indeed, even if it does not contribute to stability, $\mathbf{z}_{\mathrm{NR}}$ can improve the performance of the closed-loop plant. Section 5.1 provides an optimal design criterion for $\mathbf{K}_{\mathrm{NR}}$.

4.3 Integral Action

As anticipated in Section 4.1 and reiterated at the end of Section 4.2, this section increments the controller design complexity by introducing the integral control action

[8]. First, this section aims to design $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ exploiting the results of Section 4.2. Second, the benefits associated with the integral control action are highlighted. Indeed, concerning plant (4.1a) with assumption $\mathbf{e}_x=\mathbf{0}$, the integral action asymptotically steers to zero the regulated output for any constant reference $\mathbf{r}$ and despite the presence of unknown but constant disturbances $\mathbf{d}$.

Let

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{e_1} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \end{bmatrix} \mathbf{w} \tag{4.9}$$

be the LTI system obtained by picking the first and the last equations of system (1.16) with $\dot{\boldsymbol{\eta}} := \mathbf{e}$. Then, as done in Section 4.2, first study the stabilisability of system (4.9) through the reachability matrix

$$\mathbf{R}_e := \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}\mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 & \mathbf{A}^{n}\mathbf{B}_1 & \cdots & \mathbf{A}^{n+m-1}\mathbf{B}_1 \\ \mathbf{D}_{e_2} & \mathbf{C}_e\mathbf{B}_1 & \cdots & \mathbf{C}_e\mathbf{A}^{n-2}\mathbf{B}_1 & \mathbf{C}_e\mathbf{A}^{n-1}\mathbf{B}_1 & \cdots & \mathbf{C}_e\mathbf{A}^{n+m-2}\mathbf{B}_1 \end{bmatrix}.$$

Then, if system (4.9) is stabilizable there exists a couple $(\mathbf{K}_\mathrm{S}, \mathbf{K}_\mathrm{I})$ that makes Hurwitz

$$\begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix}. \tag{4.10}$$

Therefore, define

$$\mathbf{u} = \mathbf{K}_\mathrm{S}\mathbf{x} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta} \tag{4.11}$$

and demonstrate that it asymptotically steers $\mathbf{e}$ to zero if $\mathbf{w}$ is constant, as follows. Substitute the control law (4.11) into system (4.9)

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \end{bmatrix} \mathbf{w} \tag{4.12}$$

and assume this system is BIBS-stable. Then, a bounded $t \mapsto \mathbf{w}(t)$ leads to limited trajectories $t \mapsto (\mathbf{x}(t), \boldsymbol{\eta}(t))$. In particular, the integral curve of system (4.12) evaluated at $\mathbf{w}$ constant is such that

$$\lim_{t\to\infty} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} = - \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \end{bmatrix} \mathbf{w} \tag{4.13}$$

which is well defined and bounded because matrix (4.10) is Hurwitz (and thus invertible) by assumption. As a consequence $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}(t) = \lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$.

Example 4.6 (PI control)**.** Let

$$\begin{aligned} \dot{x} &= ax + b_1 u + b_2\, d \\ y &= cx \end{aligned}$$

be an LTI system with x, y, u, d, a, b_1, b_2, $c \in \mathbb{R}$, b_1, b_2, $c \neq 0$, and $\dot{d} = 0$. The goal is to asymptotically steer $x \to 0$ despite the presence of d. Start solving this problem by noting that the couple (a, b_1) is fully reachable. So, design $u = k_{\mathrm{S}}\, x + k_{\mathrm{I}}\, \eta$ with $\dot{\eta} = y$. Then, the controller takes the form

$$\begin{aligned}\dot{\eta} &= y\\ u &= k_{\mathrm{S}}\, x + k_{\mathrm{I}}\, \eta,\end{aligned}$$

representing a standard Proportional–Integral (PI) controller.

Example 4.7 (Cart-pole set-point regulation)**.** The cart-pole model defined in Example 1.2 is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1 u + \mathbf{B}_2 d,$$

in which $\mathbf{x} := \mathrm{col}(p - p_0 - v_0(t - t_0), v - v_0, \theta - \theta_0, 0)$, $u = f_x - f_{x0}$, $d = \mathtt{w}$, and

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & A_{22} & A_{23} & A_{24}\\ 0 & 0 & 0 & 1\\ 0 & A_{42} & A_{43} & A_{44}\end{bmatrix}, \mathbf{B}_1 := \begin{bmatrix} 0\\ B_{12}\\ 0\\ B_{14}\end{bmatrix}, \mathbf{B}_2 := \begin{bmatrix} 0\\ B_{22}\\ 0\\ B_{24}\end{bmatrix}.$$

Let $p_{\mathrm{R}}(t)$ be the reference position and $e := p - p_{\mathrm{R}}(t)$ the tracking error, define $\dot{\eta} = e$, $\mathbf{C}_e = \begin{bmatrix} 1 & 0 & 0 & 0\end{bmatrix}$, and $\mathbf{w} := \mathrm{col}(\mathtt{w}, p_{\mathrm{R}}(t))$. Extend the plant as

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\eta}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C}_e & 0\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{B}_1\\ 0\end{bmatrix} u + \begin{bmatrix}\mathbf{B}_2 & \mathbf{0}\\ 0 & -1\end{bmatrix}\mathbf{w}.$$

Then, since this system is fully reachable, there exists a control law $u = [\mathbf{K}_{\mathrm{S}}\ k_{\mathrm{I}}]\,\mathrm{col}(\mathbf{x}, \eta)$ that makes Hurwitz

$$\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C}_e & 0\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1\\ 0\end{bmatrix}\begin{bmatrix}\mathbf{K}_{\mathrm{S}} & k_{\mathrm{I}}\end{bmatrix}.$$

4.4 State Observer

Section 4.2 showed that the control law $\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x}$ guarantees the stability of the closed-loop plant. However, the same section pointed out that this control law could not be implemented because $\mathbf{x}$ may be unknown. So then, is it possible to "estimate" $\mathbf{x}$ starting from the available information $\mathbf{y}$ and $\mathbf{u}$? This section introduces a further dynamic system, called *observer*, which answers this question [6]. The design of this system is achieved through a change of coordinates (dual to the reachability one) which reveals which parts of $\mathbf{x}$ can be "estimated" and which cannot.

Let (4.2) be the plant and assume $\mathbf{u}(\tau), \mathbf{w}(\tau)=\mathbf{0}$ for all $\tau \in [t_0, t]$. As a consequence, the integral curve is only given by the free evolution, *i.e.*,

$$\mathbf{x}(t) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0), \tag{4.14}$$

which leads to

$$\mathbf{y}(t) = \mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0). \tag{4.15}$$

It is interesting to understand if (and eventually, which part of) $\mathbf{x}(t_0)$ can be reconstructed from the output $\mathbf{y}(\tau)$ with $\tau \in [t_0, t]$. With this aim, define the *unobservability subspace at time* $t > t_0$, namely $\mathcal{E}(t)$, as [[1],§3]

$$\mathcal{E}(t) = \{\mathbf{x} \in \mathbb{R}^n \, : \, \mathbf{C}\exp(\mathbf{A}(\tau - t_0))\mathbf{x}=\mathbf{0}, \, \forall \tau \in [t_0, t]\} \, .$$

Note

In practice, $\mathcal{E}(t)$ represents the set of initial conditions that lead to a null output for any time belonging to the interval $[t_0, t]$. If $\mathbf{x} \in \mathcal{E}(t)$, then $\mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{x} = \mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{0} = \mathbf{0}$. Hence, $\mathbf{x}$ is not "distinguishable" from the origin because they produce the same (null) output.

Theorem 4.3 (Unobservability [6])**.** *Let* (4.2) *be the plant, then its unobservability subspace corresponds to* $\ker(\mathbf{O})$*, where*

$$\mathbf{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is called ***observability matrix****.*

Infobox 4.4 (Proof sketch of Theorem 4.3)**.** Theorem 4.3 is proved with the joint use of the arguments of the proof of Theorem 4.1, detailed in the Infobox 4.1, and of the concept of duality, presented in Section 5.2.

Important

An LTI system, or its couple $(\mathbf{A}, \mathbf{C})$ equivalently, is said to be **completely observable** if $\ker(\mathbf{O}) = \{\mathbf{0}\}$

Note

Intuitively, the observability matrix is derived through the following procedure. First, let the output be $\mathbf{y} = \mathbf{C}\mathbf{x}$ and assume $\mathbf{u}=\mathbf{0}$. Then consider n particular vectors, namely $\mathbf{x}_i$ with $i = 1, \dots, n$, as

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Next, collect the outputs $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$ in

$$\mathbf{Y}^{(0)} := \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}.$$

and note that $\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} = \mathbf{I}$. Compute $\dfrac{d^k \mathbf{y}_i}{dt^k} = \mathbf{C}\mathbf{A}^k \mathbf{x}_i$, for $i = 1, \dots, n$ and $k = 0, \dots, n-1$. Now, collect terms $\dfrac{d^k \mathbf{y}_i}{dt^k}$ in matrices $\mathbf{Y}^{(k)}$ and sequentially collect these matrices in

$$\begin{bmatrix} \mathbf{Y}^{(0)} \\ \mathbf{Y}^{(1)} \\ \vdots \\ \mathbf{Y}^{(n-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}.$$

Thus, intuitively speaking, the kernel of the observability matrix represents the basis for the description of the states that make null the first $n-1$ time derivatives of $\mathbf{y}$.

By looking at this rough explanation and assuming $(\mathbf{A}, \mathbf{C})$ fully observable, one can be tempted to use $\mathrm{col}(\mathbf{y},\ d\mathbf{y}/dt,\ \dots,\ d^{n-1}\mathbf{y}/\,dt^{n-1})$ to estimate $\mathbf{x}$ employing the pseudo-inverse of $\mathbf{O}$ as

$$\hat{\mathbf{x}} = (\mathbf{O}^\top \mathbf{O})^{-1}\mathbf{O}^\top \begin{bmatrix} \mathbf{y} \\ d\mathbf{y}/dt \\ \vdots \\ d^{n-1}\mathbf{y}/dt^{n-1} \end{bmatrix}.$$

This procedure is feasible only if $\mathbf{y}$ is noise-free. Indeed, assume a scalar output $y = \mathbf{C}\mathbf{x} + \nu(t)$ affected by the noise $\nu(t) := \sum_{i=1}^{\infty} \bar{\nu}_i \sin(i\,\omega t)$ with $\omega > 0$ and $\bar{\nu}_i \geq 0$ such that $\sum_i^{\infty} \bar{\nu}_i^2$ is finite. So, for any $k \geq 1$, the kth-order time derivative of y is

$$\frac{d^k y}{dt^k} = \mathbf{C}\mathbf{A}^k\mathbf{x} + \frac{d^k \nu}{dt^k}.$$

The application of the pseudo-inverse of $\mathbf{O}$ to $\mathrm{col}(y,\ dy/dt,\ \ldots,\ d^{n-1}y/\,dt^{n-1})$ leads to

$$\begin{aligned} \hat{\mathbf{x}} &= (\mathbf{O}^\top \mathbf{O})^{-1}\mathbf{O}^\top \left(\mathbf{O}\mathbf{x} + \begin{bmatrix} \nu \\ d\nu/dt \\ \vdots \\ d^{n-1}\nu/dt^{n-1} \end{bmatrix} \right) \\ &= \mathbf{x} + (\mathbf{O}^\top \mathbf{O})^{-1}\mathbf{O}^\top \begin{bmatrix} \nu \\ d\nu/dt \\ \vdots \\ d^{n-1}\nu/dt^{n-1} \end{bmatrix}. \end{aligned}$$

This expression demonstrates that $\hat{\mathbf{x}}$ is proportionally affected by the magnitude of the time derivatives of ν. For any $k \geq 1$ these magnitudes are upper bounded by

$$\left\| \frac{d^k \nu}{dt^k} \right\|_\infty = \sum_{i=1}^{\infty} \bar{\nu}_i (i\,\omega)^k,$$

which is an increasing function of i and k (for all $i \in \mathbb{N} : i\omega > 1$).

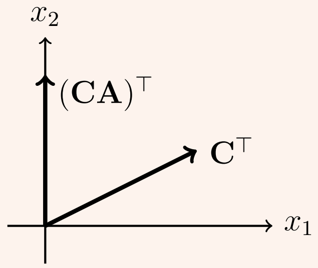
Example 4.8 (Observability study)**.** Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $y = \mathbf{C}\mathbf{x}$, with $\mathbf{x} \in \mathbb{R}^2$, and $y \in \mathbb{R}$ be the plant model. Define $\mathbf{A}$ and $\mathbf{C}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$

with a, c_1, $c_2 > 0$. Therefore, the observability matrix is

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 a \end{bmatrix}.$$

Now, draw the rows of $\mathbf{O}$ in the plane $x_1 - x_2$ as

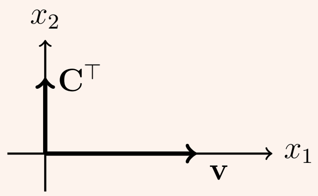


Since $\mathbf{C}^\top$ and $(\mathbf{CA})^\top$ span the whole state space $\mathbb{R}^2$, the unique vector with null projections on $\mathbf{C}^\top$ and $(\mathbf{CA})^\top$ is the trivial $\mathbf{v} = \mathbf{0}$. So then, since the set of states which create a null output is only constituted by $\mathbf{0}$, the system is fully observable.

Example 4.9 (Observability Study)**.** Let the plant be defined in Example 4.8 and consider $c_1 = 0$. The observability matrix becomes

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix},$$

whose row vectors are drawn as



Then, since $\mathbf{C}^\top$ does not span the whole state space $\mathbb{R}^2$, the system is not fully observable. Indeed, the unobservable set is composed of vectors, like $\mathbf{v} = \mathrm{col}(s, 0)$ with $s \in \mathbb{R}$, which have a null projection in direction $\mathbf{C}^\top$.

Example 4.10 (Cart-pole observability)**.** The couple $(\mathbf{A}, \mathbf{C})$ of the linearized cart-pole model of Example 1.2 are reported as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The observability matrix associated with this couple is then obtained as

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & o_{52} & o_{53} & o_{54} \\ 0 & A_{42} & A_{43} & A_{44} \\ 0 & o_{72} & o_{73} & o_{74} \\ 0 & o_{82} & o_{83} & o_{84} \end{bmatrix},$$

where

$$\begin{aligned} o_{52} &= A_{22}^2 + A_{24}A_{42} \\ o_{53} &= A_{22}A_{23} + A_{24}A_{43} \\ o_{54} &= A_{22}A_{24} + A_{23} + A_{24}A_{44} \\ o_{72} &= o_{52}A_{22} + o_{54}A_{42} \\ o_{73} &= o_{52}A_{23} + o_{54}A_{43} \\ o_{74} &= o_{52}A_{24} + o_{53} + o_{54}A_{44} \\ o_{82} &= A_{42}A_{22} + A_{44}A_{42} \\ o_{83} &= A_{42}A_{23} + A_{44}A_{43} \\ o_{84} &= A_{42}A_{24} + A_{43} + A_{44}^2. \end{aligned}$$

Since the first column of $\mathbf{O}$ is null and the transpose of the first, second, and fourth rows span $\mathbb{R}^3$, the kernel of $\mathbf{O}$ is one-dimensional and given by

$$\ker(\mathbf{O}) = \mathrm{col}(1, 0, 0, 0).$$

Consequently, the system of Example 1.2 is not fully observable.

Let $\mathcal{E}$ be an unobservability subspace, then its orthogonal complement $\mathcal{E}^\perp$ is such that $\mathcal{E} \oplus \mathcal{E}^\perp = \mathbb{R}^n$. In terms of basis, if $\ker(\mathbf{O})$ represents a basis for $\mathcal{E}$, then $(\ker(\mathbf{O}))^\perp = \mathrm{im}\left(\mathbf{O}^\top\right)$ defines a basis for $\mathcal{E}^\perp$. The remaining of this section relies on these bases to determine a change of coordinates that decomposes the state into unobservable and observable subparts. The so-called "decomposition of

$$
\begin{tikzcd}
\boxed{\dot{\mathbf{z}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{NO}} + \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w}} & & \\
\boxed{\dot{\mathbf{z}}_{\mathrm{O}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{21}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w}} \arrow[u] \arrow[r] & \boxed{\mathbf{y} = \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{O}} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{u}} \arrow[r] & \mathbf{y}
\end{tikzcd}
$$

Figure 4.3: Observability decomposition

observability” [[2],§6] is defined by introducing the transformation $\mathbf{z} = \mathbf{T}_{\mathrm{O}}\mathbf{x}$ where

$$\mathbf{T}_{\mathrm{O}}^{-1} = \begin{bmatrix} \ker(\mathbf{O}) & \operatorname{im}\left(\mathbf{O}^{\top}\right) \end{bmatrix} \tag{4.16}$$

which applied to system (4.2) leads to

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{T}_{\mathrm{O}}\mathbf{A}\mathbf{T}_{\mathrm{O}}^{-1}\mathbf{z} + \mathbf{T}_{\mathrm{O}}\mathbf{B}_1\mathbf{u} + \mathbf{T}_{\mathrm{O}}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_{\mathrm{O}}\mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{T}_{\mathrm{O}}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}. \end{aligned} \tag{4.17}$$

A study of $\bar{\mathbf{A}} := \mathbf{T}_{\mathrm{O}}\mathbf{A}\mathbf{T}_{\mathrm{O}}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_{\mathrm{O}}\mathbf{B}_1$, $\bar{\mathbf{B}}_2 := \mathbf{T}_{\mathrm{O}}\mathbf{B}_2$, and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_{\mathrm{O}}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \quad \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11} \\ \bar{\mathbf{B}}_{12} \end{bmatrix} \quad \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \end{bmatrix} \quad \bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{0}} & \bar{\mathbf{C}}_2 \end{bmatrix}.$$

So, define $\mathbf{z} = \operatorname{col}(\mathbf{z}_{\mathrm{NO}}, \mathbf{z}_{\mathrm{O}})$ to make the dynamics of system (4.17) become

$$\begin{array}{ccccccccc} \dot{\mathbf{z}}_{\mathrm{NO}} & = & \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{NO}} & + & \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{O}} & + & \bar{\mathbf{B}}_{11}\mathbf{u} & + & \bar{\mathbf{B}}_{21}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{O}} & = & & & \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{O}} & + & \bar{\mathbf{B}}_{21}\mathbf{u} & + & \bar{\mathbf{B}}_{22}\mathbf{w} \\ \mathbf{y} & = & & & \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{O}} & + & \mathbf{D}_1\mathbf{u} & + & \mathbf{D}_2\mathbf{w} \\ \begin{bmatrix} \mathbf{z}_{\mathrm{NO}}(t_0) \\ \mathbf{z}_{\mathrm{O}}(t_0) \end{bmatrix} & = & \mathbf{T}_{\mathrm{O}}\mathbf{x}_0. & & & & & & \end{array} \tag{4.18}$$

Equation (4.18) shows that $\mathbf{z}_{\mathrm{O}}$ represents the subpart of the system available at the output. It is worth noting that $\mathbf{z}_{\mathrm{NO}}$ does not influence the dynamics of $\mathbf{z}_{\mathrm{O}}$. Therefore, assuming $\mathbf{u}, \mathbf{w}=0$, the output $\mathbf{y}$ is null for any time if and only if $\mathbf{z}_{\mathrm{O}}=\mathbf{0}$. Fig. 4.3 depicts system (4.18).

Example 4.11 (Observability decomposition)**.** The observability matrix associated with the couple

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

is

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The kernel of $\mathbf{O}$ provides a basis for the unobservable subspace

$$\mathbf{b}_1 := \ker(\mathbf{O}) = \ker\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

In addition, the image of $\mathbf{O}$ leads to a basis of the observable subspace

$$\{\mathbf{b}_2,\, \mathbf{b}_3\} := \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Finally, the observability decomposition is obtained through the transformation matrix $\mathbf{T}_{\mathrm{O}}^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, that leads to

$$\bar{\mathbf{A}} = \mathbf{T}_{\mathrm{O}}\mathbf{A}\mathbf{T}_{\mathrm{O}}^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \mathbf{C}\mathbf{T}_{\mathrm{O}}^{-1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Example 4.12 (Cart-pole observability decomposition)**.** As described in Example 4.10, the system of Example 1.2 is not fully observable. Then, the kernel of $\mathbf{O}$ provides the basis of the unobservable subspace $\mathbf{b}_1 := \ker(\mathbf{O}) = \mathrm{col}(1,\, 0,\, 0,\, 0)$, whereas the basis of the observable subspace is

$$\{\mathbf{b}_2,\, \mathbf{b}_3,\, \mathbf{b}_4\} := \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Finally, the observability transformation is $\mathbf{T}_{\mathrm{O}}^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix} = \mathbf{I}$ meaning that the system of Example 1.2 is already in form of observability.

Now, focus on the observable subsystem of (4.18)

$$\begin{aligned} \dot{\mathbf{z}}_{\mathrm{O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \quad \mathbf{z}_{\mathrm{O}}(t_0) = \mathbf{z}_{O_0} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{O}} + \mathbf{D}_2\mathbf{w}, \end{aligned} \tag{4.19}$$

which is completely observable, by definition, and for which the following result can be stated.

Theorem 4.4 (Existence of a stabilizing output feedback ([1],§4))**.** *Let*

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}\\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

be a completely observable LTI system. Then, there exists a matrix $\mathbf{K}_O$ *such that* $\mathbf{A}-\mathbf{K}_O\mathbf{C}$ *is Hurwitz.*

Infobox 4.5 (Proof sketch of Theorem 4.4)**.** Theorem 4.4 is proved through the same strategy adopted to prove Theorem 4.2 (see Infobox 4.3) with the support of the concept of *duality* presented in Section 5.2.

Important

The direct consequence of Theorem 4.4 is that it is possible to design a BIBS-stable dynamic system that provides an estimation of $\mathbf{z}_\mathrm{O}$ in the following form [4]:

$$\begin{aligned}\dot{\hat{\mathbf{z}}}_\mathrm{O} &= \bar{\mathbf{A}}_{22}\hat{\mathbf{z}}_\mathrm{O} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{K}}_\mathrm{O}\left(\mathbf{y}-\hat{\mathbf{y}}\right) \quad \hat{\mathbf{z}}_\mathrm{O}(t_0) = \mathbf{0}\\ \hat{\mathbf{y}} &= \bar{\mathbf{C}}_2\hat{\mathbf{z}}_\mathrm{O} + \mathbf{D}_1\mathbf{u}.\end{aligned} \tag{4.20}$$

Let $\mathbf{e}_\mathrm{O} := \mathbf{z}_\mathrm{O} - \hat{\mathbf{z}}_\mathrm{O}$ be the estimation error; then, the estimator (4.20), whose scheme is illustrated in Fig. 4.4, makes $\mathbf{e}_\mathrm{O}$ bounded. Indeed

$$\dot{\mathbf{e}}_\mathrm{O} = \left(\bar{\mathbf{A}}_{22} - \bar{\mathbf{K}}_\mathrm{O}\bar{\mathbf{C}}_2\right)\mathbf{e}_\mathrm{O} + (\bar{\mathbf{B}}_{22} + \bar{\mathbf{K}}_\mathrm{O}\mathbf{D}_2)\mathbf{w} \quad \mathbf{e}_\mathrm{O}(t_0) = \mathbf{z}_\mathrm{O}(t_0) - \hat{\mathbf{z}}_\mathrm{O}(t_0), \tag{4.21}$$

whose state asymptotically converges to a neighborhood of the origin, provided that $\bar{\mathbf{A}}_{22} - \bar{\mathbf{K}}_\mathrm{O}\bar{\mathbf{C}}_2$ is Hurwitz.

Infobox 4.6 (Reduced-Order Observer)**.** The implementation of observer (4.20) requires the computation of a dynamic system of the same order as the plant. This section proposes a strategy to reduce the observer dimension [2, 6]. Let

$$\begin{aligned}\dot{\mathbf{z}}_\mathrm{O} &= \bar{\mathbf{A}}_{22}\mathbf{z}_\mathrm{O} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w}\\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_\mathrm{O} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}\end{aligned}$$

be the observable part of system (4.18). Define $\mathbf{N}$ such that

$$\mathbf{T} := \begin{bmatrix}\bar{\mathbf{C}}_2\\ \mathbf{N}\end{bmatrix}$$

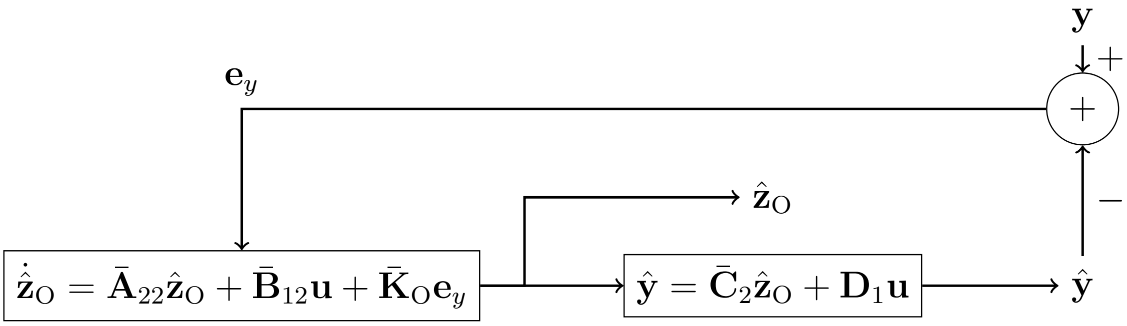


Figure 4.4: State Observer

constitutes a change of coordinates. Introduce $\boldsymbol{\zeta} = \mathbf{T}\mathbf{z}_{\mathrm{O}}$, with $\boldsymbol{\zeta} := \mathrm{col}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)$ and $\boldsymbol{\zeta}_1 \in \mathbb{R}^q$. Then, the dynamics of $\boldsymbol{\zeta}$ are

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_1 &= \underline{\mathbf{A}}_{11}\boldsymbol{\zeta}_1 + \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + \underline{\mathbf{B}}_{11}\mathbf{u} + \underline{\mathbf{B}}_{21}\mathbf{w} \\
\dot{\boldsymbol{\zeta}}_2 &= \underline{\mathbf{A}}_{21}\boldsymbol{\zeta}_1 + \underline{\mathbf{A}}_{22}\boldsymbol{\zeta}_2 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{B}}_{22}\mathbf{w} \\
\mathbf{y} &= \boldsymbol{\zeta}_1 + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w},
\end{aligned}$$

where $\underline{\mathbf{A}}_{ij}$ and $\underline{\mathbf{B}}_{ij}$, with $i, j \in \{1, 2\}$, denote subparts of $\underline{\mathbf{A}} := \mathbf{T}\bar{\mathbf{A}}_{22}\mathbf{T}^{-1}$, $\underline{\mathbf{B}}_1 := \mathbf{T}\bar{\mathbf{B}}_1$, and $\underline{\mathbf{B}}_2 := \mathbf{T}\bar{\mathbf{B}}_2$. As a consequence, since $\hat{\boldsymbol{\zeta}}_1 := \mathbf{y} - \mathbf{D}_1\mathbf{u} = \boldsymbol{\zeta}_1 + \mathbf{D}_2\mathbf{w}$ provides a direct (noisy) estimation of $\boldsymbol{\zeta}_1$, only $\boldsymbol{\zeta}_2$ needs to be estimated. To this end, conceive

$$\boldsymbol{\mu} := \dot{\hat{\boldsymbol{\zeta}}}_1 - \underline{\mathbf{A}}_{11}\hat{\boldsymbol{\zeta}}_1 - \underline{\mathbf{B}}_{11}\mathbf{u} = \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + (\underline{\mathbf{B}}_{21} - \underline{\mathbf{A}}_{11}\mathbf{D}_2)\mathbf{w} + \mathbf{D}_2\dot{\mathbf{w}}$$

as a potential output and rewrite the plant as

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_2 &= \underline{\mathbf{A}}_{22}\boldsymbol{\zeta}_2 + \underline{\mathbf{A}}_{21}\boldsymbol{\zeta}_1 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{B}}_{22}\mathbf{w} \\
\boldsymbol{\mu} &= \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + (\underline{\mathbf{B}}_{21} - \underline{\mathbf{A}}_{11}\mathbf{D}_2)\mathbf{w} + \mathbf{D}_2\dot{\mathbf{w}}.
\end{aligned}$$

Since the couple $(\underline{\mathbf{A}}_{22}, \underline{\mathbf{A}}_{12})$ is fully observable, $\underline{\mathbf{K}}_{\mathrm{O}}$ exists such that $\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{A}}_{12}$ is Hurwitz and

$$\dot{\hat{\boldsymbol{\zeta}}}_2 = (\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{A}}_{12})\hat{\boldsymbol{\zeta}}_2 + \underline{\mathbf{A}}_{21}\hat{\boldsymbol{\zeta}}_1 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{K}}_{\mathrm{O}}\boldsymbol{\mu}$$

converges to a neighborhood of $\boldsymbol{\zeta}_2$. To make this observer implementable, introduce a change of variables that removes the derivative of $\hat{\boldsymbol{\zeta}}_1$ (appearing in $\boldsymbol{\mu}$).

Let $\hat{\mathbf{v}} := \hat{\boldsymbol{\zeta}}_2 - \underline{\mathbf{K}}_{\mathrm{O}}\hat{\boldsymbol{\zeta}}_1$, exploit $\hat{\boldsymbol{\zeta}}_2 = \hat{\mathbf{v}} + \underline{\mathbf{K}}_{\mathrm{O}}\hat{\boldsymbol{\zeta}}_1$, and rewrite the observer as

$$\begin{aligned}\dot{\hat{\mathbf{v}}} = &(\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{A}}_{12})\hat{\mathbf{v}} + ((\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{A}}_{12})\underline{\mathbf{K}}_{\mathrm{O}} + \underline{\mathbf{A}}_{21} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{A}}_{11})\hat{\boldsymbol{\zeta}}_1 \\ &+ (\underline{\mathbf{B}}_{12} - \underline{\mathbf{K}}_{\mathrm{O}}\underline{\mathbf{B}}_{11})\mathbf{u}.\end{aligned}$$

Then, the estimation of $\mathbf{z}_{\mathrm{O}}$ is obtained as

$$\hat{\mathbf{z}}_{\mathrm{O}} = \mathbf{T}^{-1}\begin{bmatrix}\hat{\boldsymbol{\zeta}}_1 \\ \hat{\boldsymbol{\zeta}}_2\end{bmatrix} = \mathbf{T}^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \underline{\mathbf{K}}_{\mathrm{O}} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\zeta}}_1 \\ \hat{\mathbf{v}}\end{bmatrix}.$$

To conclude this section, focus on the estimation of the non-observable states. First, introduce the subsequent definition.

Important

The LTI system of equation (4.18) is said to be **detectable** if the $\bar{\mathbf{A}}_{11}$ is Hurwitz.

Then, adopt the next assumption.

Assumption 4.2. The system (4.2) is detectable.

Therefore, the non-observable states $\mathbf{z}_{\mathrm{NO}}$ are estimated by the so-called *identity observer* simply consisting of a copy of the plant

$$\dot{\hat{\mathbf{z}}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\hat{\mathbf{z}}_{\mathrm{NO}} + \bar{\mathbf{A}}_{12}\hat{\mathbf{z}}_{\mathrm{O}} + \bar{\mathbf{B}}_{12}\mathbf{u} \quad \hat{\mathbf{z}}_{\mathrm{NO}}(t_0) = \hat{\mathbf{z}}_{\mathrm{NO}_0}. \tag{4.22}$$

To analyze this algorithm, let $\mathbf{e}_{\mathrm{NO}} := \hat{\mathbf{z}}_{\mathrm{NO}} - \mathbf{z}_{\mathrm{NO}}$ be the estimation error. Then, compute its dynamics exploiting Eq.s (4.18) and (4.22) as

$$\dot{\mathbf{e}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\mathbf{e}_{\mathrm{NO}} - \bar{\mathbf{A}}_{12}\mathbf{e}_{\mathrm{O}} - \bar{\mathbf{B}}_{22}\mathbf{w} \qquad \mathbf{e}_{\mathrm{NO}}(t_0) = \hat{\mathbf{z}}_{\mathrm{NO}}(t_0) - \mathbf{z}_{\mathrm{NO}}(t_0). \tag{4.23}$$

It is worth noting that the dynamics of $\mathbf{e}_{\mathrm{NO}}$ are bounded because $\bar{\mathbf{A}}_{11}$ is Hurwitz, but the identity observer can not tune them. Adopt the following strategy to estimate the state:

$$\hat{\mathbf{x}} = \mathbf{T}_{\mathrm{O}}^{-1}\begin{bmatrix}\hat{\mathbf{z}}_{\mathrm{NO}} \\ \hat{\mathbf{z}}_{\mathrm{O}}\end{bmatrix}. \tag{4.24}$$

4.5 Output-Feedback Stabilizer

As described at the end of Section 4.2, the state feedback $\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x}$ could not be implementable because $\mathbf{x}$ may be unavailable. On the other hand, Section 4.4 provides a strategy to estimate $\mathbf{x}$. Consequently, an implementable control law is [[2],§9]

$$\mathbf{u}_{\mathrm{S}} = \mathbf{K}_{\mathrm{S}}\hat{\mathbf{x}}, \tag{4.25}$$

which matches the architecture of Eq. (1.17). Concerning Eq. (4.1a), this section demonstrates that using control law (4.25), as well as the integral action defined in Section 4.3, makes Hurwitz the matrix

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} \\ \mathbf{A}_\mathrm{O}+\mathbf{K}_\mathrm{O}\mathbf{C}-\mathbf{A}+\mathbf{M}\mathbf{K}_\mathrm{S} & \mathbf{M}\mathbf{K}_\mathrm{I} & \mathbf{A}_\mathrm{O}+\mathbf{M}\mathbf{K}_\mathrm{S} \end{bmatrix}, \tag{4.26}$$

where $\mathbf{M}=\mathbf{B}_\mathrm{O}+\mathbf{K}_\mathrm{O}\mathbf{D}_1-\mathbf{B}_1$.

First, introduce a change of coordinates to exploit the observability decomposition. Let $\mathbf{T}_\mathrm{O}$ be as in Eq. (4.16) and define $\mathbf{T}=\texttt{blckdiag}(\mathbf{I},\mathbf{I},\mathbf{T}_\mathrm{O})$. Premultiply and post-multiply Eq. (4.26) by $\mathbf{T}$ and $\mathbf{T}^{-1}$

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S}\mathbf{T}_\mathrm{O}^{-1} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S}\mathbf{T}_\mathrm{O}^{-1} \\ \mathbf{T}_\mathrm{O}(\mathbf{A}_\mathrm{O}+\mathbf{K}_\mathrm{O}\mathbf{C}-\mathbf{A}+\mathbf{M}\mathbf{K}_\mathrm{S}) & \mathbf{T}_\mathrm{O}\mathbf{M}\mathbf{K}_\mathrm{I} & \mathbf{T}_\mathrm{O}(\mathbf{A}_\mathrm{O}+\mathbf{M}\mathbf{K}_\mathrm{S})\mathbf{T}_\mathrm{O}^{-1} \end{bmatrix}. \tag{4.27}$$

Second, use observers (4.20) and (4.22) to define

$$\dot{\hat{\mathbf{z}}}=\mathbf{A}_\mathbf{z}\hat{\mathbf{z}}+\mathbf{B}_\mathbf{z}\mathbf{u}+\mathbf{K}_\mathbf{z}\mathbf{y}, \tag{4.28}$$

where $\hat{\mathbf{z}}=\mathrm{col}(\hat{\mathbf{z}}_\mathrm{NO},\hat{\mathbf{z}}_\mathrm{O})$ and

$$\mathbf{A}_z=\begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22}-\bar{\mathbf{K}}_\mathrm{O}\bar{\mathbf{C}}_2 \end{bmatrix}, \quad \mathbf{B}_z=\mathbf{T}_\mathrm{O}\mathbf{B}_1-\bar{\mathbf{K}}_\mathrm{O}\mathbf{D}_1, \quad \mathbf{K}_z=\begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{K}}_\mathrm{O} \end{bmatrix}.$$

Then, define $\mathbf{x}_\mathrm{O}:=\hat{\mathbf{x}}$ and compare Eq.s (1.17a) and (4.28) to identify $\mathbf{A}_\mathrm{O}=\mathbf{T}_\mathrm{O}^{-1}\mathbf{A}_z\mathbf{T}_\mathrm{O}$, $\mathbf{B}_\mathrm{O}=\mathbf{T}_\mathrm{O}^{-1}\mathbf{B}_z$, and $\mathbf{K}_\mathrm{O}=\mathbf{T}_\mathrm{O}^{-1}\mathbf{K}_z$. As a consequence, $\mathbf{M}=\mathbf{0}$ and $\mathbf{A}_\mathrm{O}=\mathbf{A}-\mathbf{K}_\mathrm{O}\mathbf{C}$, which reduce matrix (4.27) to

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S}\mathbf{T}_\mathrm{O}^{-1} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S}\mathbf{T}_\mathrm{O}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_z \end{bmatrix}. \tag{4.29}$$

This matrix is Hurwitz because its eigenvalues correspond to the union of eigenvalues of the main block diagonals. Indeed, $\mathbf{A}_z$ is Hurwitz thanks to Assumption 4.2 and the design of observer (4.20). Moreover,

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix}$$

is Hurwitz thanks to Assumption 4.1 and the design of stabilizer (4.11).

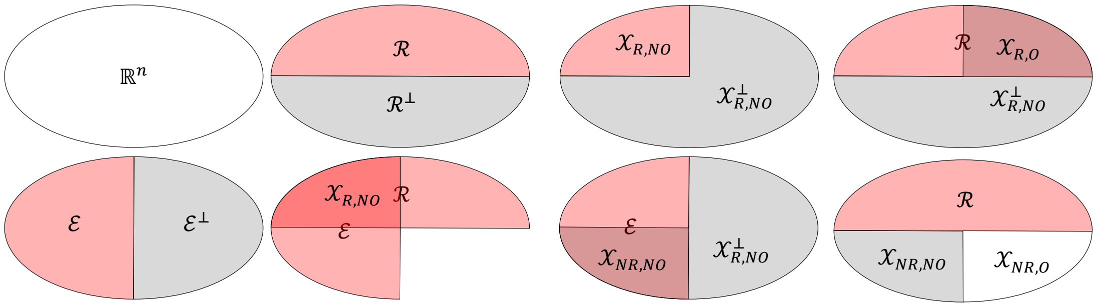


Figure 4.5: Graphical representation of the sequence of intersections exploited to find the ultimate Kalman decomposition of LTI systems.

Important

The triangular structure of (4.29) lets the design of matrices $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ be independent of $\bar{\mathbf{K}}_\mathrm{O}$. This design feature is called *Separation Principle*.

4.5.1 Minimal Stabilizer

The output-feedback stabilizer (4.24), (4.25), and (4.28) requires implementing a dynamic system of the same order as the plant. Moreover, Assumptions 4.1 and 4.2 allow the state feedback to be simpler to assure the stability of the reachable and observable subpart of $\mathbf{x}$ only. In particular, thanks to the introduction of a change of coordinates called *ultimate Kalman decomposition* [[2],§6], this section highlights that goal G1 is achieved with observer (4.20) (eventually reduced as in the Infobox 4.6) and by a suitable subpart of $\mathbf{K}_\mathrm{R}$.

The concepts of reachability and unobservability and their relative decompositions can be jointly exploited to find a transformation, namely $\mathbf{z} = \mathbf{T}_\mathrm{K}\mathbf{x}$, which highlights the reachable, unreachable, unobservable, and observable subsystems of (4.2). Let us identify the following subspaces:

- reachable and unobservable $\mathcal{X}_{\mathrm{R,NO}} := \mathcal{R} \cap \mathcal{E}$ with basis $X_{\mathrm{R,NO}}$;
- reachable and observable $\mathcal{X}_{\mathrm{R,O}} := \mathcal{X}_{\mathrm{R,NO}}^\perp \cap \mathcal{R}$ with basis $X_{\mathrm{R,O}}$;
- unreachable and unobservable $\mathcal{X}_{\mathrm{NR,NO}} := \mathcal{X}_{\mathrm{R,NO}}^\perp \cap \mathcal{E}$ with basis $X_{\mathrm{NR,NO}}$;
- unreachable and observable $\mathcal{X}_{\mathrm{NR,O}} := (\mathcal{R} \cup \mathcal{X}_{\mathrm{NR,NO}})^\perp$ with basis $X_{\mathrm{NR,O}}$,

whose derivation is supported by the Venn diagram of Fig. 4.5.

The transformation $\mathbf{T}_\mathrm{K}$ is then defined as

$$\mathbf{T}_\mathrm{K}^{-1} = \begin{bmatrix} X_{\mathrm{R,NO}} & X_{\mathrm{R,O}} & X_{\mathrm{NR,NO}} & X_{\mathrm{NR,O}} \end{bmatrix},$$

which, applied to system (4.2), leads to

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{T}_{\mathrm{K}}\mathbf{A}\mathbf{T}_{\mathrm{K}}^{-1}\mathbf{z} + \mathbf{T}_{\mathrm{K}}\mathbf{B}_1\mathbf{u} + \mathbf{T}_{\mathrm{K}}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_{\mathrm{K}}\mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{T}_{\mathrm{K}}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}.\end{aligned} \tag{4.30}$$

A study of $\bar{\mathbf{A}} := \mathbf{T}_{\mathrm{K}}\mathbf{A}\mathbf{T}_{\mathrm{K}}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_{\mathrm{K}}\mathbf{B}$, and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_{\mathrm{K}}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} & \mathbf{0} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{33} & \bar{\mathbf{A}}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{44} \end{bmatrix}, \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11} \\ \bar{\mathbf{B}}_{12} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \\ \bar{\mathbf{B}}_{23} \\ \bar{\mathbf{B}}_{24} \end{bmatrix}$$
$$\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{C}}_2 & \mathbf{0} & \bar{\mathbf{C}}_4 \end{bmatrix}.$$

Let the state $\mathbf{z}$ be divided in four parts as

$$\mathbf{z} = \mathrm{col}\left(\mathbf{z}_{\mathrm{R,NO}}, \mathbf{z}_{\mathrm{R,O}}, \mathbf{z}_{\mathrm{NR,NO}}, \mathbf{z}_{\mathrm{NR,O}}\right),$$

then the dynamics of $\mathbf{z}$ is given by (Fig. 4.6)

$$\begin{aligned}\dot{\mathbf{z}}_{\mathrm{R,NO}} &= \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{R,NO}} + \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{13}\mathbf{z}_{\mathrm{NR,NO}} + \bar{\mathbf{A}}_{14}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{24}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,NO}} &= \bar{\mathbf{A}}_{33}\mathbf{z}_{\mathrm{NR,NO}} + \bar{\mathbf{A}}_{34}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{23}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} &= \bar{\mathbf{A}}_{44}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{24}\mathbf{w} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{C}}_4\mathbf{z}_{\mathrm{NR,O}} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}.\end{aligned}$$

Example 4.13 (Ultimate Kalman decomposition)**.** Let matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be those of Examples 4.4 and 4.11. Then, the basis of reachable and unobservable subspaces are

$$\{\mathbf{b}_{R_1}, \mathbf{b}_{R_2}\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathbf{b}_{\mathrm{NO}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

As illustrated in the following picture, the intersection of the reachable and non-observable subspaces is trivial and corresponds to the origin, *i.e.*, $\mathcal{X}_{\mathrm{R,NO}} = \mathcal{R} \cap \mathcal{E} = \{0\}$. Consequently, the ultimate Kalman decomposition does not highlight any reachable but non-observable subsystem. Moreover, since $\mathcal{X}_{\mathrm{R,NO}}^{\perp} = \mathbb{R}^3$, it is $\mathcal{X}_{\mathrm{R,O}} = \mathcal{X}_{\mathrm{R,NO}}^{\perp} \cap \mathcal{R} = \mathcal{R}$ and $\mathcal{X}_{\mathrm{NR,NO}} = \mathcal{X}_{\mathrm{R,NO}}^{\perp} \cap \mathcal{E} = \mathcal{E}$. In conclusion, since $\mathcal{R} \cup \mathcal{X}_{\mathrm{NR,NO}} = \mathbb{R}^3$, the subspace $\mathcal{X}_{\mathrm{NR,O}} = \{0\}$ and then

$$\mathbf{T}_{\mathrm{K}}^{-1} = \begin{bmatrix} \mathbf{b}_{R_1} & \mathbf{b}_{R_2} & \mathbf{b}_{\mathrm{NO}} \end{bmatrix}.$$

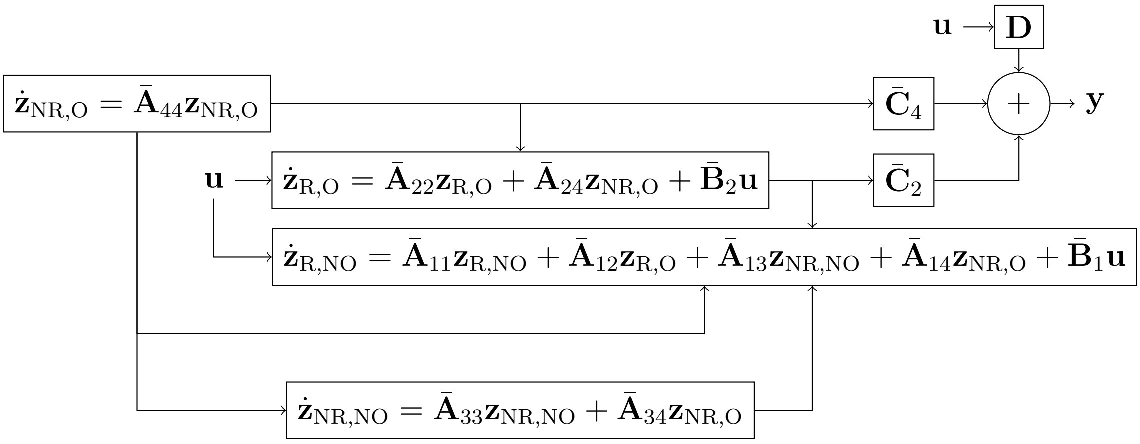
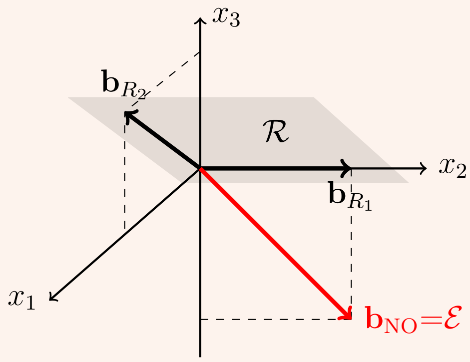


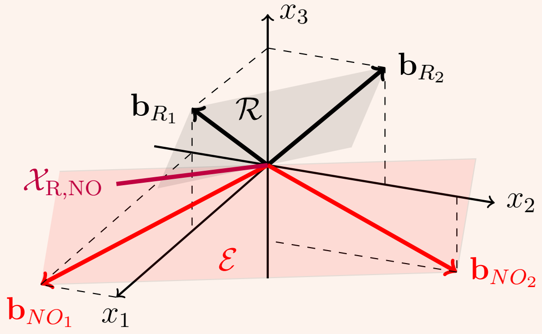
Figure 4.6: The ultimate Kalman decomposition highlights the reachable and observable parts of systems. The input affects only the reachable parts, whereas the observable ones contribute only to the output.



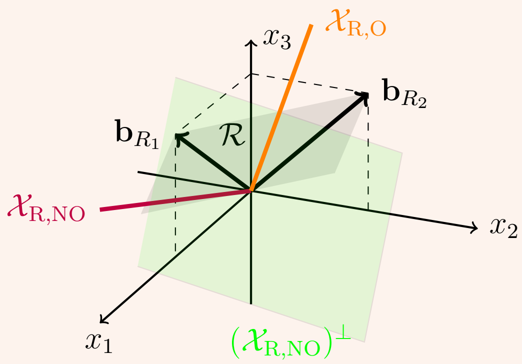
The application of $\mathbf{T}_{\mathrm{K}}$ to matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ defined in Examples 4.4 and 4.11, leads to

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Example 4.14 (Ultimate Kalman decomposition)**.** Assume that the reachability and unobservability subspaces of an LTI system have basis $\mathrm{im}(\mathbf{R}) = [\mathbf{b}_{R_1}\ \mathbf{b}_{R_1}]$ and $\ker(\mathbf{O}) = [\mathbf{b}_{NO_1}\ \mathbf{b}_{NO_1}]$, respectively. The following figure depicts the subspaces geometry.

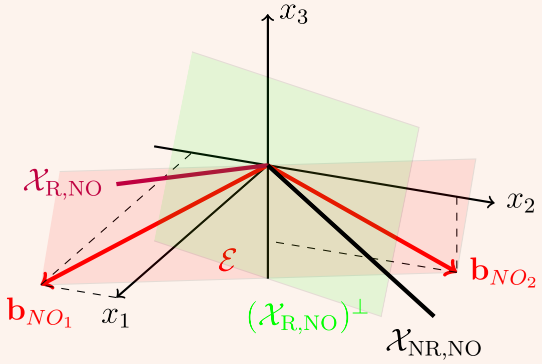


The intersection $\mathcal{X}_{\mathrm{R,NO}} = \mathcal{R} \cap \mathcal{E}$ represents the unique line that belongs to both $\mathcal{R}$ and $\mathcal{E}$. Moreover, the orthogonal to $\mathcal{X}_{\mathrm{R,NO}}$, drawn as a green plane in the following figure, intersects the reachable set and provides $\mathcal{X}_{\mathrm{R,O}} = (X_{\mathrm{R,NO}})^{\perp} \cap \mathcal{R}^{+}$, whose basis is $X_{\mathrm{R,O}}$.



Finally, the intersection between $(\mathcal{X}_{\mathrm{R,NO}})^{\perp}$ and $\mathcal{E}$ generates $\mathcal{X}_{\mathrm{NR,NO}}$, whose basis is $X_{\mathrm{NR,NO}}$. Then, the transformation matrix is

$$\mathbf{T}_{\mathrm{K}}^{-1} = \left[\begin{array}{ccc} X_{\mathrm{R,NO}} & X_{\mathrm{R,O}} & X_{\mathrm{NR,NO}} \end{array} \right].$$



Example 4.15 (Ultimate Kalman decomposition)**.** The following matrices identify an LTI system:

$$\mathbf{A} = \begin{bmatrix} 7 & 5 & -1 \\ -3 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

This system is neither fully reachable nor fully observable because the reachability and observability matrices,

$$\mathbf{R} = \begin{bmatrix} 2 & 8 & 24 \\ -1 & -6 & -20 \\ 1 & 2 & 4 \end{bmatrix}, \qquad \mathbf{O} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 4 & 4 & -4 \end{bmatrix},$$

have rank $= 2$ and rank $= 1$, respectively. Define $\text{im}(\mathbf{R}) = [\mathbf{b}_{R_1}\, \mathbf{b}_{R_2}]$ and $\ker(\mathbf{O}) = [\mathbf{b}_{NO_1}\, \mathbf{b}_{NO_2}]$ with

$$[\mathbf{b}_{R_1}\, \mathbf{b}_{R_2}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad [\mathbf{b}_{NO_1}\, \mathbf{b}_{NO_2}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since these two bases are equal, the reachable and unobservable subspace coincide, *i.e.*, $\mathcal{R}=\mathcal{E}$ and thus $\mathcal{X}_{\text{R,NO}} = \mathcal{R} \cap \mathcal{E} = \mathcal{R}$. As a consequence $\mathcal{X}_{\text{R,O}} = (\mathcal{X}_{\text{R,NO}})^{\perp} \cap \mathcal{R} = \{0\}$ as well as $\mathcal{X}_{\text{NR,NO}} = (\mathcal{X}_{\text{R,NO}})^{\perp} \cap \mathcal{E} = \{0\}$. Therefore, the only non-trivial subspace further than $\mathcal{X}_{\text{R,NO}}$ is $\mathcal{X}_{\text{NR,O}} = (\mathcal{X}_{\text{R,NO}})^{\perp}$. In this case, $X_{\text{NR,O}} = \text{col}(1,\, 1,\, -1)$ represents a basis for $\mathcal{X}_{\text{NR,O}}$. Finally, the transformation matrix is

$$\mathbf{T}_{\text{K}}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

that applied to matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ leads to

$$\bar{\mathbf{A}} = \begin{bmatrix} 6 & 4 & 11 \\ -4 & -2 & -5 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}.$$

Example 4.16 (Cart-pole ultimate Kalman decomposition)**.** The bases of reachability and unobservability subspaces of the cart-pole, modeled in Example

1.2, have been studied in Examples 4.3 and 4.12. In more detail, they are

$$\mathrm{im}(\mathbf{R}) = \mathbf{I} \in \mathbb{R}^{4\times 4}, \quad \ker(\mathbf{O}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the $X_{\mathrm{R,NO}}$ represents the intersection between line $\ker(\mathbf{O})$ and the whole 4D space. This intersection leads to

$$X_{\mathrm{R,NO}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X^{\perp}_{\mathrm{R,NO}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also, the intersections of $\mathcal{X}^{\perp}_{\mathrm{R,NO}}$ with $\mathcal{R}$ and $\mathcal{E}$ lead to

$$X_{\mathrm{R,O}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_{\mathrm{NR,NO}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, $\mathcal{X}_{\mathrm{NR,O}}$ is orthogonal to the union of $\mathcal{R}$ with $\mathcal{X}_{\mathrm{NR,NO}}$, whose basis is $\mathrm{im}(\mathcal{R} \cup \mathcal{X}_{\mathrm{NR,NO}}) = \mathbf{I} \in \mathbb{R}^{4\times 4}$. Then, the basis of its orthogonal is represented by the origin, *i.e.*, $X_{\mathrm{NR,O}} = \mathbf{0}$. The ultimate Kalman decomposition is finally obtained as

$$\mathbf{T} = \begin{bmatrix} X_{\mathrm{R,NO}} & X_{\mathrm{R,O}} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix}.$$

The subsystem

$$\begin{aligned} \dot{\mathbf{z}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{24}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} &= \bar{\mathbf{A}}_{44}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{24}\mathbf{w} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{C}}_4\mathbf{z}_{\mathrm{NR,O}} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{4.31}$$

is of interest because it is the only part that influences the output and, thanks to Assumption 1.3, the regulated output.

Infobox 4.7 (Transfer Matrix)**.** Transfer functions only consider the reachable and observable part of LTI systems [1, 6]. Indeed, the term $\mathbf{z}_{\mathrm{NR,O}}$ is regarded as an external input for system (4.31), which is not involved in the input-output relation between $\mathbf{u}$ and $\mathbf{y}$. So, the transfer matrix is

$$\mathbf{G}(s) := \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \bar{\mathbf{C}}_2 \left(s\mathbf{I} - \bar{\mathbf{A}}_{22}\right)^{-1} \bar{\mathbf{B}}_{12} + \mathbf{D}_1$$

It is worth noting that $\mathbf{z}_{\mathrm{R,O}}$ and $\mathbf{z}_{\mathrm{NR,O}}$ influence the output. $\mathbf{z}_{\mathrm{NR,O}}$ represents a known signal (thanks to the existence of the observer) whose dynamics cannot be modified by the control action. Since the couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{B}}_{12})$ is fully reachable by definition, a matrix $\mathbf{K}_{\mathrm{R,O}}$ exists such that $\bar{\mathbf{A}}_{22} + \bar{\mathbf{B}}_{12}\mathbf{K}_{\mathrm{R,O}}$ is Hurwitz, thus implying that the state $\mathbf{z}_{\mathrm{R,O}}$ is bounded if also $\mathbf{z}_{\mathrm{NR,O}}$ is so too. In turn, this is true if Assumption 4.1 is verified. Then, the minimal stabilizer becomes

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{z}}_{\mathrm{R,O}} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} \end{bmatrix} &= \left(\begin{bmatrix} \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \bar{\mathbf{A}}_{44} \end{bmatrix} - \bar{\mathbf{K}}_{\mathrm{O}} \begin{bmatrix} \bar{\mathbf{C}}_2 & \bar{\mathbf{C}}_4 \end{bmatrix} \right) \begin{bmatrix} \mathbf{z}_{\mathrm{R,O}} \\ \mathbf{z}_{\mathrm{NR,O}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{12} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \bar{\mathbf{K}}_{\mathrm{O}}\mathbf{y} \\ \mathbf{u}_{\mathrm{S}} &= \mathbf{K}_{\mathrm{R,O}}\mathbf{z}_{\mathrm{R,O}}. \end{aligned} \tag{4.32}$$

Important

The overall closed-loop system can be made BIBS-stable if matrices $\bar{\mathbf{A}}_{11}$, $\bar{\mathbf{A}}_{33}$, and $\bar{\mathbf{A}}_{44}$ are Hurwitz, *i.e.*, if the plant is detectable and stabilizable.

Example 4.17 (Cart-pole stabilizer)**.** The ultimate Kalman decomposition of Example 4.16 applied to the plant of Example 1.2 shows that the first state is reachable but not observable. In contrast, the last three states are reachable and observable. Formally, the system, in the ultimate Kalman decomposition, is

$$\begin{aligned} \begin{bmatrix} \dot{z}_{\mathrm{R,NO}} \\ \dot{\mathbf{z}}_{\mathrm{R,O}} \end{bmatrix} &= \begin{bmatrix} \bar{a}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{R,NO}} \\ \mathbf{z}_{\mathrm{R,O}} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\mathbf{B}}_2 \end{bmatrix} u \\ \mathbf{y} &= \begin{bmatrix} 0 & \bar{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} z_{\mathrm{R,NO}} \\ \mathbf{z}_{\mathrm{R,O}} \end{bmatrix} \end{aligned}$$

in which the contribution of the disturbances has been neglected. The output-feedback stabilizer is given by

$$\begin{aligned}\dot{\hat{\mathbf{z}}}_{\mathrm{R,O}} &= (\bar{\mathbf{A}}_{22} - \mathbf{K}_{\mathrm{O}}\bar{\mathbf{C}}_2 + \bar{\mathbf{B}}_2\mathbf{K}_{\mathrm{S}})\hat{\mathbf{z}}_{\mathrm{R,O}} + \mathbf{K}_{\mathrm{O}}\mathbf{y} \\ u &= \mathbf{K}_{\mathrm{R,O}}\hat{\mathbf{z}}_{\mathrm{R,O}}.\end{aligned}$$

4.5.2 Robustness to Disturbance and Noise

Previous sections demonstrate that controller (4.24), (4.25), and (4.28), extended with the integral action (4.11), makes the closed-loop plant BIBS-stable. Moreover, assuming a fully reachable and observable plant, the eigenvalues of system (4.2) can be assigned through the design of the matrices $\mathbf{K}_{\mathrm{S}}$, $\mathbf{K}_{\mathrm{I}}$, and $\mathbf{K}_{\mathrm{O}}$. Thus, one could be tempted to push on the feedback gains to make the control system as reactive as desired and to reduce the asymptotic bound of the regulated output (see control goal G2). But unfortunately, the presence of measurement noises deeply impacts the design of $\mathbf{K}_{\mathrm{S}}$, $\mathbf{K}_{\mathrm{I}}$, and $\mathbf{K}_{\mathrm{O}}$, as well as the asymptotic values of state and regulated output. In particular, let

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{d} \\ \dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{K}_{\mathrm{O}}\mathbf{C})\hat{\mathbf{x}} + (\mathbf{B}_1 - \mathbf{K}_{\mathrm{O}}\mathbf{D}_1)\mathbf{u} + \mathbf{K}_{\mathrm{O}}\mathbf{y} \\ \dot{\boldsymbol{\eta}} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_{e1}\mathbf{u} + \mathbf{E}\boldsymbol{\nu} \\ \mathbf{u} &= \mathbf{K}_{\mathrm{S}}\hat{\mathbf{x}} + \mathbf{K}_{\mathrm{I}}\boldsymbol{\eta} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \boldsymbol{\nu}\end{aligned} \tag{4.33}$$

be a fully observable and reachable LTI plant, subject to disturbances and measurement noises, controlled by an output-feedback stabilizer plus an integral action where $\mathbf{d}$ denotes the exogenous disturbance and $\boldsymbol{\nu}$ is the measurement noise. Define the estimation error as $\mathbf{e}_x = \hat{\mathbf{x}} - \mathbf{x}$, let

$$\mathbf{A}_{\mathrm{cl}} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} \\ \mathbf{C}_e + \mathbf{D}_{e1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e1}\mathbf{K}_{\mathrm{I}} \end{bmatrix},$$

and use the expressions of $\mathbf{y}$ and $\mathbf{u}$ to rewrite system (4.33) as

$$\begin{aligned}\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} &= \mathbf{A}_{\mathrm{cl}}\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1\mathbf{K}_{\mathrm{S}} \\ \mathbf{D}_{e1}\mathbf{K}_{\mathrm{S}} \end{bmatrix}\mathbf{e}_x + \begin{bmatrix} \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}\begin{bmatrix} \mathbf{d} \\ \boldsymbol{\nu} \end{bmatrix} \\ \dot{\mathbf{e}}_x &= (\mathbf{A} - \mathbf{K}_{\mathrm{O}}\mathbf{C})\mathbf{e}_x - \mathbf{B}_2\mathbf{d} + \mathbf{K}_{\mathrm{O}}\boldsymbol{\nu},\end{aligned}$$

whose solution is

$$\begin{aligned}\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} &= \int_0^t e^{\mathbf{A}_{\mathrm{cl}}(t-\tau)}\left(\begin{bmatrix} \mathbf{B}_1\mathbf{K}_{\mathrm{S}} \\ \mathbf{D}_{e1}\mathbf{K}_{\mathrm{S}} \end{bmatrix}\mathbf{e}_x(\tau) + \begin{bmatrix} \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}\begin{bmatrix} \mathbf{d}(\tau) \\ \boldsymbol{\nu}(\tau) \end{bmatrix}\right) d\tau \\ \mathbf{e}_x(t) &= \int_0^t e^{(\mathbf{A}-\mathbf{K}_{\mathrm{O}}\mathbf{C})(t-\tau)}\left(-\mathbf{B}_2\mathbf{d}(\tau) + \mathbf{K}_{\mathrm{O}}\boldsymbol{\nu}(\tau)\right) d\tau,\end{aligned}$$

where, without loss of generality, $(\mathbf{x}(0), \boldsymbol{\eta}(0)) := (\mathbf{0}, \mathbf{0})$ and $\mathbf{e}_x(0) := \mathbf{0}$. Then, assume that the norms of disturbance and noise are uniformly bounded by $\bar{d}$ and $\bar{\nu}$ respectively, *i.e.*, $\|\mathbf{d}(t)\| \leq \bar{d}$ and $\|\boldsymbol{\nu}(t)\| \leq \bar{\nu}$ for all $t \geq 0$. Then, the norm of $\mathbf{x}(t)$, $\boldsymbol{\eta}(t)$, and $\mathbf{e}_x(t)$ are bounded as

$$\begin{aligned} \left\| \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} \right\| &\leq \frac{1}{\underline{\sigma}\left(\mathbf{A}_{\text{cl}}\right)} \left(\overline{\sigma} \left(\begin{bmatrix} \mathbf{B}_1 \mathbf{K}_{\text{S}} \\ \mathbf{D}_{e1} \mathbf{K}_{\text{S}} \end{bmatrix} \right) \bar{e} + \overline{\sigma}\left(\mathbf{B}_2\right) \bar{d} + \overline{\sigma}\left(\mathbf{E}\right) \bar{\nu} \right) \\ \|\mathbf{e}_x(t)\| &\leq \bar{e}, \end{aligned} \tag{4.34}$$

with $\bar{e} = [\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\text{O}}\mathbf{C})]^{-1} \left(\overline{\sigma}(\mathbf{B}_2)\bar{d} + \overline{\sigma}(\mathbf{K}_{\text{O}})\bar{\nu}\right)$. A more aggressive observer feedback $\mathbf{K}_{\text{O}}$ cannot attenuate the effects of $\boldsymbol{\nu}$ on $\mathbf{e}_x$. Indeed, a larger $\mathbf{K}_{\text{O}}$ (possibly) leads to larger $\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\text{O}}\mathbf{C})$ and $\overline{\sigma}(\mathbf{K}_{\text{O}})$. Then, increasing $\mathbf{K}_{\text{O}}$ such that $\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\text{O}}\mathbf{C}) \approx \underline{\sigma}(\mathbf{K}_{\text{O}}\mathbf{C})$ leads to $\bar{e} \approx [\underline{\sigma}(\mathbf{K}_{\text{O}}\mathbf{C})]^{-1} \left(\overline{\sigma}(\mathbf{B}_2)\bar{d} + \overline{\sigma}(\mathbf{K}_{\text{O}})\bar{\nu}\right)$, which cannot be arbitrarily reduced with a suitable design of $\mathbf{K}_{\text{O}}$ because of $\underline{\sigma}(\mathbf{K}_{\text{O}}) \leq \overline{\sigma}(\mathbf{K}_{\text{O}})$. The same argument demonstrates that increasing $\mathbf{K}_{\text{S}}$ does not attenuate the effects of $\boldsymbol{\nu}$ on $\|(\mathbf{x}(t), \boldsymbol{\eta}(t))\|$. Conversely, more aggressive feedback $\mathbf{K}_{\text{O}}$, $\mathbf{K}_{\text{S}}$, and $\mathbf{K}_{\text{I}}$ could attenuate the effects of $\mathbf{d}$ on the norm of $\mathbf{e}_x$, $\mathbf{x}$, and $\boldsymbol{\eta}$ because, accordingly to the first of Eq.s (4.34), the disturbance amplification is inversely proportional to $\overline{\sigma}(\mathbf{B}_2)/\underline{\sigma}\left(\mathbf{A}_{\text{cl}}\right)$.

In conclusion, the BIBS stability properties guaranteed by the output feedback (4.11), (4.24), (4.25), and (4.28) makes the state and the estimation error bounded even in the presence of disturbances and noises. Unfortunately, $\|(\mathbf{x}(t), \boldsymbol{\eta}(t))\|$ and $\|\mathbf{e}_x(t)\|$ cannot be arbitrarily reduced through an high-gain policy. In detail, more aggressive $\mathbf{K}_{\text{O}}$, $\mathbf{K}_{\text{S}}$, and $\mathbf{K}_{\text{I}}$ could attenuate the effects of disturbances. Conversely, increasing $\mathbf{K}_{\text{S}}$, $\mathbf{K}_{\text{I}}$, and $\mathbf{K}_{\text{O}}$ may not lead to any noise attenuation. Chapter 5 presents an optimal criterion for designing $\mathbf{K}_{\text{S}}$, $\mathbf{K}_{\text{I}}$, and $\mathbf{K}_{\text{O}}$.

4.5.3 Limitations on the Stabilization of Nonlinear Systems

The output-feedback controller (4.11), (4.24), (4.25), and (4.28) guarantees $\tilde{\mathbf{x}} = \mathbf{0}$ to be a globally (*i.e.*, for any initial condition) exponentially stable equilibrium point for the linearized plant (1.14). On the other hand, it is worth remembering that system (1.14) represents a linearization of system (1.1) in the neighborhood of $\mathbf{x}_0$. Then, it seems natural to try to transfer the stability properties of the origin of the linearized system to the equilibrium of the nonlinear one. Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{e} &= \mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w}) \end{aligned} \tag{4.35}$$

be a nonlinear plant which is assumed completely observable and reachable, as for the definitions of earlier in this section. Then, for systems such as (4.35), which are locally completely reachable and observable, it is possible to transfer the stability properties of the origin of the linearized system to the equilibrium point of the nonlinear system. This result is constrained by the domains of initial conditions and exogenous signals whose size depends on the controller parameters [[5], §4.3].

In detail, assume to control system (4.35) with (4.11), (4.24), (4.25), and (4.28) designed on the equilibrium tuple $(\mathbf{x}_0, \mathbf{u}_0, \mathbf{w}_0, \mathbf{y}_0)$. First, change the coordinates as

$$\tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}_0,\ \tilde{\mathbf{u}} := \mathbf{u} - \mathbf{u}_0,\ \tilde{\mathbf{w}} := \mathbf{w} - \mathbf{w}_0,\ \tilde{\mathbf{y}} := \mathbf{y} - \mathbf{y}_0$$

and rewrite system (4.35) as

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \\ \tilde{\mathbf{y}} &= \tilde{\mathbf{h}}(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) \\ \mathbf{e} &= \tilde{\mathbf{h}}_e(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}), \end{aligned} \tag{4.36}$$

in which

$$\begin{aligned} \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) &:= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}_0,\ \tilde{\mathbf{u}} + \mathbf{u}_0,\ \tilde{\mathbf{w}} + \mathbf{w}_0) \\ \tilde{\mathbf{h}}(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) &:= \mathbf{h}(\tilde{\mathbf{x}} + \mathbf{x}_0,\ \tilde{\mathbf{u}} + \mathbf{u}_0,\ \tilde{\mathbf{w}} + \mathbf{w}_0) \\ \tilde{\mathbf{h}}_e(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) &:= \mathbf{h}_e(\tilde{\mathbf{x}} + \mathbf{x}_0,\ \tilde{\mathbf{u}} + \mathbf{u}_0,\ \tilde{\mathbf{w}} + \mathbf{w}_0). \end{aligned}$$

Second, write the closed loop as

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) \\ \dot{\boldsymbol{\eta}} &= \tilde{\mathbf{h}}_e(\tilde{\mathbf{x}},\ \tilde{\mathbf{u}},\ \tilde{\mathbf{w}}) \\ \dot{\hat{\tilde{\mathbf{x}}}} &= \mathbf{A}\hat{\tilde{\mathbf{x}}} + \mathbf{B}_1\tilde{\mathbf{u}} + \mathbf{K}_{\mathrm{O}}(\tilde{\mathbf{y}} - \hat{\tilde{\mathbf{y}}}) \\ \tilde{\mathbf{u}} &= \mathbf{K}_{\mathrm{S}}\hat{\tilde{\mathbf{x}}} + \mathbf{K}_{\mathrm{I}}\boldsymbol{\eta} \\ \hat{\tilde{\mathbf{y}}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1\tilde{\mathbf{u}}. \end{aligned} \tag{4.37}$$

Define $\mathbf{e}_x = \hat{\tilde{\mathbf{x}}} - \tilde{\mathbf{x}}$, exploit the formulation of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$, and add and subtract $\mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{\mathbf{u}}$ and $\mathbf{C}_e\tilde{\mathbf{x}} + \mathbf{D}_{e1}\tilde{\mathbf{u}}$ to the first and second line of Eq. (4.37), respectively. Let $\bar{\mathbf{x}} := \mathrm{col}(\tilde{\mathbf{x}},\ \boldsymbol{\eta},\ \mathbf{e}_x)$ and compute its dynamics as

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}), \tag{4.38}$$

where

$$\bar{\mathbf{A}} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} & \mathbf{B}_1\mathbf{K}_{\mathrm{S}} \\ \mathbf{C}_{e1} + \mathbf{D}_{e1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e1}\mathbf{K}_{\mathrm{I}} & \mathbf{D}_{e1}\mathbf{K}_{\mathrm{S}} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} - \mathbf{K}_{\mathrm{O}}\mathbf{C} \end{bmatrix}$$

and

$$\mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}) := \begin{bmatrix} \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{B}_1\tilde{\mathbf{u}} \\ \tilde{\mathbf{h}}_e(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) - \mathbf{C}_e\tilde{\mathbf{x}} - \mathbf{D}_{e1}\tilde{\mathbf{u}} \\ \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{\mathbf{u}} - \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{K}_{\mathrm{O}} \end{bmatrix} \left(\tilde{\mathbf{h}}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}) - \mathbf{C}\tilde{\mathbf{x}} - \mathbf{D}_1\tilde{\mathbf{u}}\right).$$

Define with $n_{\bar{x}}$ and n_w the dimensions of $\bar{\mathbf{x}}$ and $\tilde{\mathbf{w}}$. Then, since $\mathbf{f}$, $\mathbf{h}$, and $\mathbf{h}_e$ are assumed locally Lipschitz, for any $\epsilon > 0$, $\mathbf{K}_{\mathrm{S}} \in \mathbb{R}^{p\times n}$, and $\mathbf{K}_{\mathrm{I}} \in \mathbb{R}^{p\times m}$, there exist $L_1, L_2 > 0$ such that, for any $(\bar{\mathbf{x}}, \tilde{\mathbf{w}}) \in \mathbb{R}^{n_{\bar{x}}} \times \mathbb{R}^{n_w}$ such that $\|(\bar{\mathbf{x}}, \tilde{\mathbf{w}})\| \leq \epsilon$, $\mathbf{G}$ is bounded as

$$\|\mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}})\| \leq (L_1 + L_2\overline{\sigma}(\mathbf{K}_{\mathrm{O}}))\left(\|\bar{\mathbf{x}}\|^2 + \|\tilde{\mathbf{w}}\|\right).$$

To compute a bound for the trajectories of system (4.38), rely on the support function $V(\bar{\mathbf{x}}) := \bar{\mathbf{x}}^\top \mathbf{P}\bar{\mathbf{x}}$ in which $\mathbf{P} = \mathbf{P}^\top \succ 0$ represents the solution to the so-called Lyapunov equation

$$\mathbf{P}\bar{\mathbf{A}} + \bar{\mathbf{A}}^\top \mathbf{P} = -2\lambda I$$

for some $\lambda > 0$. Then, exploit Eq. (4.38) to calculate the time derivative of V as

$$\dot{V} = \frac{\partial V}{\partial \bar{\mathbf{x}}}\dot{\bar{\mathbf{x}}} = \dot{\bar{\mathbf{x}}}^\top \mathbf{P}\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top \mathbf{P}\dot{\bar{\mathbf{x}}} = -\lambda\|\bar{\mathbf{x}}\|^2 + 2\bar{\mathbf{x}}^\top \mathbf{P}\mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}) \tag{4.39}$$

As a consequence, exploiting $V \leq \|\bar{\mathbf{x}}\|^2\overline{\sigma}(\mathbf{P})$ and assuming $\bar{w} > 0$ such that $\|\tilde{\mathbf{w}}(t)\| \leq \bar{w}$ for all $t \geq 0$, $\dot{V}$ can be bounded from above as

$$\dot{V} \leq 2\alpha V + \|\bar{\mathbf{x}}\|(L_1 + L_2\overline{\sigma}(\mathbf{K}_{\mathrm{O}}))\bar{w}, \tag{4.40}$$

where $\alpha := (L_1 + L_2\overline{\sigma}(\mathbf{K}_{\mathrm{O}}))\epsilon^2 - \lambda/\overline{\sigma}(\mathbf{P})$. Let $q \in (0, \lambda/\overline{\sigma}(\mathbf{P}))$, then ϵ is chosen to have $\alpha \in (-\lambda/\overline{\sigma}(\mathbf{P}), -q]$, *i.e.*, $\epsilon^2 \leq (\lambda/\overline{\sigma}(\mathbf{P}) - q)/(L_1 + L_2\overline{\sigma}(\mathbf{K}_{\mathrm{O}}))$. Then, equation (4.40) represents a linear system whose solution is

$$V(t) \leq V(\bar{\mathbf{x}}(0))e^{2\alpha t} + \epsilon\frac{L_1 + L_2\overline{\sigma}(\bar{\mathbf{K}}_{\mathrm{O}})}{2\alpha}\bar{w}. \tag{4.41}$$

Use $\|\bar{\mathbf{x}}\| \leq \sqrt{V/\underline{\sigma}(\mathbf{P})}$ and $V(\bar{\mathbf{x}}(0)) \leq \|\bar{\mathbf{x}}_0\|^2\overline{\sigma}(\mathbf{P})$ to bound the trajectories of $\bar{\mathbf{x}}$ as

$$\begin{aligned} \|\bar{\mathbf{x}}(t)\| &\leq \sqrt{\overline{\sigma}(\mathbf{P})V(\bar{\mathbf{x}}(0))e^{2\alpha t} + \epsilon\frac{L_1 + L_2\overline{\sigma}(\bar{\mathbf{K}}_{\mathrm{O}})}{\alpha}\bar{w}} \\ &\leq \sqrt{\frac{\overline{\sigma}(\mathbf{P})}{\underline{\sigma}(\mathbf{P})}}\|\bar{\mathbf{x}}_0\|e^{\alpha t} + \sqrt{\epsilon\frac{L_1 + L_2\overline{\sigma}(\bar{\mathbf{K}}_{\mathrm{O}})}{\alpha}\bar{w}}. \end{aligned} \tag{4.42}$$

To assure $\|\bar{\mathbf{x}}(t)\| \leq \epsilon$ for all $t \geq 0$ (and thus to make valid this investigation), define $\rho \in (0, \epsilon/2)$ and impose the following bounds on $\bar{\mathbf{x}}_0$ and $\tilde{\mathbf{w}}$:

$$\|\bar{\mathbf{x}}_0\| \leq \rho\sqrt{\frac{\underline{\sigma}(\mathbf{P})}{\overline{\sigma}(\mathbf{P})}}, \quad \bar{w} \leq \min\left\{\frac{\rho^2}{\epsilon}\frac{\alpha}{L_1 + L_2\overline{\sigma}(\mathbf{K}_{\mathrm{O}})}, \epsilon\right\}.$$

To conclude, this section demonstrates that if the nonlinear plant is locally completely observable and reachable and in form (4.35), then the control via linearization makes the equilibrium $\mathbf{x}_0$ locally exponentially stable with restrictions on initial conditions and exogenous signals. Therefore, the trajectories of system (4.38) are guaranteed to live into a bounded domain containing $\mathbf{x}_0$.

4.6 Feed-Forward Control

As described in Section 1.2, a feed-forward action completes the control system. The feed-forward forces the plant to *anticipate* the reference variation independently of the current state [3]. This section presents a design strategy applicable to BIBS-stable plants or plants made BIBS-stable by a stabilizer. Let the stabilizer be pure state feedback to reduce the topic presentation complexity without losing generality. Thus, let

$$\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x} + \mathbf{u}_{\mathrm{FF}} \tag{4.43}$$

be the complete control law whose $\mathbf{K}_{\mathrm{S}}$ is designed accordingly to Section 4.2. Assuming $\mathbf{d} = \mathbf{0}$, the goal is to create $\mathbf{u}_{\mathrm{FF}}$ to assure $\lim_{t\to\infty}\mathbf{e}(t) = \mathbf{0}$. To achieve this result, introduce a further hypothesis, Assumption 4.3. If this latter is not verified, the set of trackable references is reduced, as detailed in Infobox 4.9.

Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{r} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{r} \end{aligned} \tag{4.44}$$

be a stabilizable LTI system. Let e_i, with $i = 1, \ldots, m$, be the i-th entry of $\mathbf{e}$. Then, for each $i = 1, \ldots, m$, determine $r_{\max_i}$ as the smallest integer such that

$$\frac{d^{r_{\max_i}}}{dt^{r_{\max_i}}}e_i \propto \mathbf{m}_i\mathbf{u}$$

for some $\mathbf{m}_i \neq \mathbf{0}$.

We develop the remaining of this section under the assumptions $\mathbf{D}_1 = \mathbf{0}$ and $\sum_{i=1}^{m} r_{\max_i} = n$, formalised in Assumption 4.3, to reduce the arguments' complexity presentation. Nevertheless, Infoboxes 4.9 and 4.10 extend the results to the cases not covered in Assumption 4.3.

Assumption 4.3 (Vector Relative Degree)**.** Let (4.44) be an LTI plant with $\mathbf{D}_1 = \mathbf{0}$ and $\sum_{i=1}^{m} r_{\max_i} = n$.

Then, substitute (4.43) into (4.44)

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{x} + \mathbf{B}_1\mathbf{u}_\mathrm{FF} \ + \ \mathbf{B}_2\mathbf{r} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_2\mathbf{r}.\end{aligned} \tag{4.45}$$

Let $\mathbf{c}_{e,i}$ be the i-th row of $\mathbf{C}_e$ and define $\boldsymbol{\zeta} = \mathbf{T}_\zeta\mathbf{x}$ with

$$\mathbf{T}_\zeta := \mathrm{col}(\mathbf{c}_{e,1}, \cdots, \mathbf{c}_{e,1}\mathbf{A}^{r_{\max_1}-1}, \ldots, \mathbf{c}_{e,m}, \cdots, \mathbf{c}_{e,m}\mathbf{A}^{r_{\max_m}-1}). \tag{4.46}$$

Apply the change of coordinates (4.46) to system (4.45) to obtain

$$\begin{aligned}\dot{\boldsymbol{\zeta}} =& \ \mathbf{T}_\zeta(\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{T}_\zeta\mathbf{B}_1\mathbf{u}_\mathrm{FF} \ + \ \mathbf{T}_\zeta\mathbf{B}_2\mathbf{r} \quad \boldsymbol{\zeta}(t_0) = \mathbf{T}_\zeta\mathbf{x}_0 \\ \mathbf{e} =& \ \mathbf{C}_e\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{D}_2\mathbf{r}.\end{aligned} \tag{4.47}$$

Let

$$\boldsymbol{\zeta} := \mathrm{col}(\boldsymbol{\zeta}_1, \cdots, \boldsymbol{\zeta}_m) \tag{4.48}$$

and, for each $i = 1, \ldots, m$, define

$$\boldsymbol{\zeta}_i = \mathrm{col}(\zeta_{i,1}, \cdots, \zeta_{i,r_{\max_i}}). \tag{4.49}$$

Then, let $\mathbf{d}_{2,i}$ be the i-th row of $\mathbf{D}_2$ and compute the dynamics of $\boldsymbol{\zeta}_i$ as

$$\begin{aligned}\dot{\zeta}_{i,1} &= \zeta_{i,2} + \mathbf{c}_{e,i}\mathbf{B}_2\mathbf{r} \\ \dot{\zeta}_{i,2} &= \zeta_{i,3} + \mathbf{c}_{e,i}\mathbf{A}\mathbf{B}_2\mathbf{r} \\ &\vdots \\ \dot{\zeta}_{i,r_{\max_i}-1} &= \zeta_{i,r_{\max_i}} + \mathbf{c}_{e,i}\mathbf{A}^{r_{\max_i}-1}\mathbf{B}_2\mathbf{r} \\ \dot{\zeta}_{i,r_{\max_i}} &= \mathbf{c}_{e,i}\mathbf{A}^{r_{\max_i}-1}((\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{B}_1\mathbf{u}_\mathrm{FF} + \mathbf{B}_2\mathbf{r}) \\ e_i &= \zeta_{i,1} + \mathbf{d}_{2,i}\mathbf{r}.\end{aligned} \tag{4.50}$$

Exploit Eq. (4.50) to define the reference state $\boldsymbol{\zeta}_i^\star$ such that $\boldsymbol{\zeta}_i = \boldsymbol{\zeta}_i^\star \implies e_i = 0$, *i.e.*,

$$\begin{aligned}\zeta_{i,1}^\star &:= -\mathbf{d}_{2,i}\mathbf{r} \\ \zeta_{i,2}^\star &:= -\mathbf{d}_{2,i}\frac{d}{dt}\mathbf{r} - \mathbf{c}_{e,i}\mathbf{B}_2\mathbf{r} \\ &\vdots \\ \zeta_{i,r_{\max_i}}^\star &:= -\mathbf{d}_{2,i}\frac{d^{r_{\max_i}-1}}{dt^{r_{\max_i}-1}}\mathbf{r} - \sum_{k=0}^{r_{\max_i}-2} \mathbf{c}_{e,i}\mathbf{A}^k\mathbf{B}_2\frac{d^{r_{\max_i}-2-k}}{dt^{r_{\max_i}-2-k}}\mathbf{r}.\end{aligned} \tag{4.51}$$

To conclude, let $\boldsymbol{\zeta}^\star := \mathrm{col}(\boldsymbol{\zeta}_1^\star, \cdots, \boldsymbol{\zeta}_m^\star)$, define $\mathbf{H}$ such that

$$\mathbf{H}\dot{\boldsymbol{\zeta}}^\star = \mathrm{col}(\dot{\zeta}^\star_{1,r_{\max_1}}, \ldots, \dot{\zeta}^\star_{m,r_{\max_m}}),$$

take

$$\overline{\mathbf{M}} := \begin{bmatrix} \mathbf{c}_{e,1}\mathbf{A}^{r_{\max_1}-1} \\ \vdots \\ \mathbf{c}_{e,m}\mathbf{A}^{r_{\max_m}-1} \end{bmatrix},$$

and exploit Eq. (4.50), for $i = 1, \ldots, m$, to write

$$\mathbf{H}\dot{\boldsymbol{\zeta}}^\star = \overline{\mathbf{M}}((\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta}^\star + \mathbf{B}_1\mathbf{u}_\mathrm{FF} + \mathbf{B}_2\mathbf{r}). \tag{4.52}$$

Assumption 4.4. The matrix $\mathbf{M} := \overline{\mathbf{M}}\mathbf{B}_1$ has m rows linearly independent. As a consequence, the Moore-Penrose right pseudo-inverse $\mathbf{M}^+ := \mathbf{M}^\top(\mathbf{M}\mathbf{M}^\top)^{-1}$ is well defined.

If Assumption 4.4 is verified, define $r_{\max} = \max\{r_{\max_1}, \ldots, r_{\max_m}\}$ and compute the feed-forward law as

$$\begin{aligned} \mathbf{u}_\mathrm{FF} &= \sum_{i=1}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i}\frac{d^i}{dt^i}\mathbf{r} \\ &:= \mathbf{M}^+\left(\mathbf{H}\dot{\boldsymbol{\zeta}}^\star - \overline{\mathbf{M}}\left((\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta}^\star + \mathbf{B}_2\mathbf{r}\right)\right). \end{aligned} \tag{4.53}$$

Remark 4.2. The feed-forward control (4.53) represents an algebraic sum of the reference and its time derivatives. It also constitutes a subpart of system (1.17e), whose remaining elements are detailed in Infobox 4.9.

Infobox 4.8 (Implementation in the original coordinates)**.** Define $\mathbf{x}^\star = \mathbf{T}_\zeta{}^{-1}\boldsymbol{\zeta}^\star$, exploit $\mathbf{M}^+\overline{\mathbf{M}}\mathbf{B}_1 = (\overline{\mathbf{M}}\mathbf{B}_1)^+\overline{\mathbf{M}}\mathbf{B}_1 = \mathbf{I}$, and rewrite Eq. (4.53) as

$$\mathbf{u}_\mathrm{FF} = \mathbf{M}^+\left(\mathbf{H}\mathbf{T}_\zeta\dot{\mathbf{x}}^\star - \overline{\mathbf{M}}\left(\mathbf{A}\mathbf{x}^\star + \mathbf{B}_2\mathbf{r}\right)\right) - \mathbf{K}_\mathrm{S}\mathbf{x}^\star.$$

Then, the overall control law, which is the sum of state-feedback stabilizer and feed-forward, becomes

$$\mathbf{u} = \mathbf{K}_\mathrm{S}(\mathbf{x} - \mathbf{x}^\star) + \mathbf{M}^+\left(\mathbf{H}\mathbf{T}_\zeta\dot{\mathbf{x}}^\star - \overline{\mathbf{M}}\left(\mathbf{A}\mathbf{x}^\star + \mathbf{B}_2\mathbf{r}\right)\right).$$

Example 4.18. Let

$$\begin{aligned}\dot{x} &= ax + bu\\ y &= x\\ e &= cx - dr(t)\end{aligned}$$

be an LTI system where a, b, c, $d > 0$, x is the state, y the output, e the controlled output, and $r(t)$ represents the known reference value. Since the goal is steering $e \to 0$, it must be $x^\star = (d/c)\, r(t)$. Note that $r_{\max} = 1$ and Assumption 4.3 is verified. Therefore, define $u = k_{\mathrm{R}} x + v$, with $k_{\mathrm{R}} \,:\, a + b_1 k_{\mathrm{R}} < 0$, and v as

$$v = -[c(a + bk_{\mathrm{R}})^{-1} b]^{-1} dr(t) + b^{-1}\dot{x}^\star.$$

Infobox 4.9 (Zero dynamics and reference generator)**.** This note shows how to solve the asymptotic tracking problem when Assumption 4.3 is not verified but $\mathbf{D}_1 = \mathbf{0}$. As detailed hereafter, a modified change of coordinates represents the right tool for the problem solution. Consequently, the feed-forward describes the output of a dynamic system called zero dynamics. Moreover, we design the reference to make the zero dynamics BIBS-stable under some stabilization conditions. Let $\mathbf{T}_\zeta$ be as in Eq. (4.46) and introduce $\mathbf{T}_{\zeta_\perp}$ such that $\mathbf{T} := \mathrm{col}(\mathbf{T}_\zeta,\, \mathbf{T}_{\zeta_\perp})$ is invertible and $\mathbf{T}_{\zeta_\perp}\mathbf{B}_1 = \mathbf{0}$. Then, let $\mathrm{col}(\boldsymbol{\zeta},\, \boldsymbol{\zeta}_\perp) = \mathbf{T}\mathbf{x}$, define

$$\begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix} = \mathbf{T}(\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{T}^{-1},$$

and compute the dynamics of $\boldsymbol{\zeta}$, $\boldsymbol{\zeta}_\perp$ as

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= \bar{\mathbf{A}}_{11}\boldsymbol{\zeta} + \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}_\perp + \mathbf{T}_\zeta\mathbf{B}_1\mathbf{u}_{\mathrm{FF}}\\ \dot{\boldsymbol{\zeta}}_\perp &= \bar{\mathbf{A}}_{21}\boldsymbol{\zeta} + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}_\perp\\ \mathbf{e} &= \mathbf{C}_e\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{D}_2\mathbf{r}.\end{aligned}$$

Exploit this system to generate the references $\boldsymbol{\zeta}^\star$ and $\boldsymbol{\zeta}^\star_\perp$. Let $\boldsymbol{\zeta}^\star := \mathrm{col}(\boldsymbol{\zeta}^\star_1, \cdots, \boldsymbol{\zeta}^\star_m)$ with $\boldsymbol{\zeta}^\star_i$ be defined in Eq. (4.51), for $i = 1, \ldots, m$. Then, $\boldsymbol{\zeta} = \boldsymbol{\zeta}^\star$ and $\boldsymbol{\zeta}_\perp = \boldsymbol{\zeta}^\star_\perp$ imply

$$\dot{\boldsymbol{\zeta}}^\star_\perp = \bar{\mathbf{A}}_{21}\boldsymbol{\zeta}^\star + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}^\star_\perp,$$

which represents the zero dynamics. If the couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{A}}_{21})$ is stabilizable, there exists $\mathbf{K}$ such that $\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$ is Hurwitz. Therefore, a reference $\mathbf{r}$ such that $\boldsymbol{\zeta}^\star = \mathbf{K}\boldsymbol{\zeta}^\star_\perp + \mathbf{v}$, where $\mathbf{v}$ represents the actual reference to be tracked, makes BIBS-stable the dynamics $\boldsymbol{\zeta}^\star_\perp$. To conclude, use the same steps exploited in Eq.s

(4.52)-(4.53) to obtain system (1.17e) with $\mathbf{x}_{\mathrm{FF}} := \boldsymbol{\zeta}^{\star}_{\perp}$, $\mathbf{A}_{\mathrm{FF}} := \bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$, $\sum_{i=1}^{r_{\max}} \mathbf{B}_{\mathrm{FF}_i} d^i\mathbf{r}/dt^i := \bar{\mathbf{A}}_{21}\mathbf{v}$,

$$\mathbf{C}_{\mathrm{FF}} := \mathbf{M}^{+}\mathbf{H}\left[\mathbf{K}\left(\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}\right) - \bar{\mathbf{A}}_{12} - \bar{\mathbf{A}}_{11}\mathbf{K}\right],$$

and

$$\sum_{i=1}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i} \frac{d^i}{dt^i}\mathbf{r} := \mathbf{M}^{+}\mathbf{H}\left[\dot{\mathbf{v}} + \left(\mathbf{K}\bar{\mathbf{A}}_{21} - \bar{\mathbf{A}}_{11}\right)\mathbf{v}\right].$$

Example 4.19 (Cart-pole: speed tracking)**.** As shown in Example 4.17, the reachable and observable subsystem of (1.2) is

$$\dot{\mathbf{z}}_{\mathrm{R,O}} = \mathbf{A}\mathbf{z}_{\mathrm{R,O}} + \mathbf{B}u,$$

in which $\mathbf{d} = \mathbf{0}$ and

$$\mathbf{A} = \begin{bmatrix} A_{22} & A_{23} & A_{24} \\ 0 & 0 & 1 \\ A_{42} & A_{43} & A_{44} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix}.$$

The goal is to design $u = \mathbf{K}_{\mathrm{S}}\mathbf{z}_{\mathrm{R,O}} + u_{\mathrm{FF}}$ that makes the cart follow a known speed profile, namely $\dot{p}^{\star}(t)$. As first step, define the stabilizing feedback $\mathbf{K}_{\mathrm{s}} = \begin{bmatrix} k_{\mathrm{S}_1} & k_{\mathrm{S}_2} & k_{\mathrm{S}_3} \end{bmatrix}$. Then, let the speed error be $e := \mathbf{C}_e\mathbf{z}_{\mathrm{R,O}} - \dot{p}^{\star}(t)$, where $\mathbf{C}_e := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Compute

$$\dot{e} = \mathbf{C}_e\dot{\mathbf{z}}_{\mathrm{R,O}} - \ddot{p}^{\star}(t) = \mathbf{C}_e\mathbf{A}\mathbf{z}_{\mathrm{R,O}} + \mathbf{C}_e\mathbf{B}u - \dot{p}^{\star}(t)$$

and observe that $\bar{m} := \mathbf{C}_e\mathbf{B} = b_1 \neq 0$. Define with $n_{\mathrm{R,O}}$ the dimension of $\mathbf{z}_{\mathrm{R,O}}$, then $r_{\max} = 1 < n_{\mathrm{R,O}}$ implies that Assumption 4.3 is not verified. Thus, proceed accordingly to Infobox 4.9 and identify

$$\mathbf{T}_{\zeta} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{T}_{\zeta_{\perp}} = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & b_1 \end{bmatrix}$$

such that $\mathbf{T} := \mathrm{col}(\mathbf{T}_\zeta, \mathbf{T}_{\zeta_\perp})$ is full-rank and $\mathbf{T}_{\zeta_\perp}\mathbf{B} = \mathbf{0}$. Compute

$$\begin{aligned}
\begin{bmatrix} \bar{A}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix} &:= \mathbf{T}(\mathbf{A} + \mathbf{B}\mathbf{K}_{\mathrm{S}})\mathbf{T}^{-1} \\
&= \mathbf{T} \begin{bmatrix} A_{22} + b_1 k_{\mathrm{S}_1} & A_{23} + b_1 k_{\mathrm{S}_2} & A_{24} + b_1 k_{\mathrm{S}_3} \\ 0 & 0 & 1 \\ A_{42} + b_2 k_{\mathrm{S}_1} & A_{43} + b_2 k_{\mathrm{S}_2} & A_{44} + b_2 k_{\mathrm{S}_3} \end{bmatrix} \mathbf{T}^{-1}
\end{aligned}$$

with

$$\begin{aligned}
\bar{A}_{11} &= A_{22} + b_1 k_{\mathrm{S}_1} + \frac{(A_{24} + b_1 k_{\mathrm{S}_3})\, b_3}{b_1} \\
\bar{\mathbf{A}}_{12} &= \begin{bmatrix} A_{23} + b_1 k_{\mathrm{S}_2} & \frac{A_{24} + b_1 k_{\mathrm{S}_3}}{b_1} \\ 0 & \frac{1}{b_1} \end{bmatrix} \\
\bar{\mathbf{A}}_{21} &= \begin{bmatrix} \frac{b_3}{b_1} \\ (A_{42} + b_2 k_{\mathrm{S}_1})\, b_1 - (A_{22} + b_1 k_{\mathrm{S}_1})\, b_3 - \frac{b_3 \left((A_{24} + b_1 k_{\mathrm{S}_3})\, b_3 - (A_{44} + b_2 k_{\mathrm{S}_3})\, b_1 \right)}{b_1} \end{bmatrix} \\
\bar{\mathbf{A}}_{22} &= \begin{bmatrix} 0 & \frac{1}{b_1} \\ (A_{43} + b_2 k_{\mathrm{S}_2})\, b_1 - (A_{23} + b_1 k_{\mathrm{S}_2})\, b_3 & -\frac{(A_{24} + b_1 k_{\mathrm{S}_3})\, b_3 - (A_{44} + b_2 k_{\mathrm{S}_3})\, b_1}{b_1} \end{bmatrix}.
\end{aligned}$$

Note that couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{A}}_{21})$ is fully reachable, and thus, $\mathbf{K}$ exists such that $\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$ is Hurwitz. Define $\mathrm{col}(\zeta, \boldsymbol{\zeta}_\perp) = \mathbf{T}\mathbf{z}_{\mathrm{R,O}}$ whose dynamics are

$$\begin{aligned}
\dot{\zeta} &= \bar{A}_{11}\zeta + \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}_\perp + b_{21} u_{\mathrm{FF}} \\
\dot{\boldsymbol{\zeta}}_\perp &= \bar{\mathbf{A}}_{21}\zeta + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}_\perp \\
\mathbf{e} &= \zeta - \dot{p}^\star(t).
\end{aligned}$$

Let $v : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function and define the reference generator as

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_\perp^\star &= (\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K})\boldsymbol{\zeta}_\perp^\star + \bar{\mathbf{A}}_{21} v(t) \\
\zeta^\star &= \mathbf{K}\boldsymbol{\zeta}_\perp^\star + v(t).
\end{aligned}$$

To conclude, the feed-forward control law is

$$\begin{aligned}
u_{\mathrm{FF}} &= b_1^{-1} \left(\dot{\zeta}^\star - \bar{A}_{11}\zeta^\star - \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}_\perp^\star \right) \\
&= b_1^{-1} \left(\mathbf{K} \left(\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K} \right) - \bar{A}_{11}\mathbf{K} - \bar{\mathbf{A}}_{12} \right) \boldsymbol{\zeta}_\perp^\star \\
&\quad + b_1^{-1} \left(\dot{v}(t) + \left(\mathbf{K}\bar{\mathbf{A}}_{21} - \bar{A}_{11} \right) v(t) \right).
\end{aligned}$$

Infobox 4.10 (The case of zero vector relative degree)**.** In the case Assumption 4.3 does not apply because $\mathbf{D}_1 \neq \mathbf{0}$, we have that the computations (4.45)-(4.53) can not be used. Now, we show that there exist two changes of coordinates, namely $\boldsymbol{\varepsilon} := \mathbf{T}_e\mathbf{e}$ and $\boldsymbol{\mu} := \mathbf{T}_u\mathbf{u}$, with $\mathbf{T}_e \in \mathbb{R}^{m\times m}$ and $\mathbf{T}_u \in \mathbb{R}^{p\times p}$, such that Assumption 4.3 is recovered for a subpart of the original system (4.44). In particular, we define $\mathbf{T}_e^{-1} := \left[\begin{array}{cc}\ker(\mathbf{D}_1^\top) & \mathrm{im}(\mathbf{D}_1)\end{array}\right]$ and $\mathbf{T}_u^{-1} := \left[\begin{array}{cc}\ker(\mathbf{D}_1) & \mathrm{im}(\mathbf{D}_1^\top)\end{array}\right]$. Let us define with $n_1 \in \mathbb{N}$ the dimension of $\mathrm{im}(\mathbf{D}_1)$. Then, we define $\boldsymbol{\varepsilon}_1 \in \mathbb{R}^{m-n_1}$, $\boldsymbol{\varepsilon}_2 \in \mathbb{R}^{n_1}$, $\boldsymbol{\mu}_1 \in \mathbb{R}^{p-n_1}$, and $\boldsymbol{\mu}_2 \in \mathbb{R}^{n_1}$ such that $\boldsymbol{\varepsilon} := \mathrm{col}(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$, $\boldsymbol{\mu} := \mathrm{col}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$. With all these definitions at hand, we obtain

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \bar{\mathbf{B}}_{11}\boldsymbol{\mu}_1 + \bar{\mathbf{B}}_{12}\boldsymbol{\mu}_2 + \mathbf{B}_2\mathbf{r}\\
\boldsymbol{\varepsilon}_1 &= \bar{\mathbf{C}}_{e1}\mathbf{x} + \bar{\mathbf{D}}_{21}\mathbf{r}\\
\boldsymbol{\varepsilon}_2 &= \bar{\mathbf{C}}_{e2}\mathbf{x} + \bar{\mathbf{D}}\boldsymbol{\mu}_2 + \bar{\mathbf{D}}_{22}\mathbf{r}
\end{aligned}
$$

where

$$
\begin{bmatrix} \bar{\mathbf{C}}_{e1} \\ \bar{\mathbf{C}}_{e2} \end{bmatrix} = \mathbf{T}_e\mathbf{C}_e, \quad \begin{bmatrix} \bar{\mathbf{D}}_{21} \\ \bar{\mathbf{D}}_{22} \end{bmatrix} = \mathbf{T}_e\mathbf{D}_2, \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{D}} \end{bmatrix} = \mathbf{T}_e\mathbf{D}_1\mathbf{T}_u^{-1}
$$

and $\left[\begin{array}{cc}\bar{\mathbf{B}}_{11} & \bar{\mathbf{B}}_{12}\end{array}\right] = \mathbf{B}_1\mathbf{T}_u^{-1}$. Since $\bar{\mathbf{D}}$ is invertible by definition, we define

$$
\boldsymbol{\mu}_{2_{\mathrm{FF}}} := -\bar{\mathbf{D}}^{-1}\left(\bar{\mathbf{C}}_{e2}\mathbf{x} + \bar{\mathbf{D}}_{22}\mathbf{r}\right)
$$

as a solution to $\boldsymbol{\varepsilon}_2 = \mathbf{0}$. Then, we enforce $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_{2_{\mathrm{FF}}}$ into the dynamics of $\mathbf{x}$ to obtain

$$
\begin{aligned}
\dot{\mathbf{x}} &= (\mathbf{A} - \bar{\mathbf{B}}_{12}\bar{\mathbf{D}}^{-1}\bar{\mathbf{C}}_{e2})\mathbf{x} + \bar{\mathbf{B}}_{11}\boldsymbol{\mu}_1 + (\mathbf{B}_2 - \bar{\mathbf{B}}_{12}\bar{\mathbf{D}}^{-1}\bar{\mathbf{D}}_{22})\mathbf{r}\\
\boldsymbol{\varepsilon}_1 &= \bar{\mathbf{C}}_{e1}\mathbf{x} + \bar{\mathbf{D}}_{21}\mathbf{r}.
\end{aligned}
$$

Steps (4.45)-(4.53) apply to this subsystem because it matches Assumption 4.3.

4.7 ADAS Architecture

This section translates into practice the theoretical achievements presented in Sections 4.2-4.6. In particular, this section aims to define the system architecture to achieve the control goals described in Chapter 2 through reachability and observability analyses.

Chapter 5

Optimal Control and Kalman Filter

Contents

Chapter 4 proved the existence of the matrices required to implement a linearization-based control system. Accordingly, this chapter presents a technique for designing these matrices.

In more detail, Section 5.1 designs the state feedback matrices through the so-called *optimal control technique* [2, 12, 13]. Then, Section 5.2 presents the concept of duality, through which Section 5.3 reuses all the results of Section 5.1 to design the observer matrix. The optimal control design technique is then adopted to solve the problems developed in Chapters 2-4. Finally, the performance of the controllers is evaluated in simulation.

5.1 Robust Stationary Optimal Control

Sections 4.2 and 4.3 detailed the feedback control structure as the composition of an ideal state feedback plus an integral action. In more detail, let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_{e1}\mathbf{u} \end{aligned} \tag{5.1}$$

be a stabilizable LTI system, then there exist two matrices $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ such that

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{e} \\ \mathbf{u} &= \mathbf{K}_\mathrm{S}\mathbf{x} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta} \end{aligned} \tag{5.2}$$

makes plant (5.1) BIBS-stable. This section provides an optimal criterion for the design of $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$. Let $\mathbf{x}_e := \mathrm{col}(\mathbf{x}, \boldsymbol{\eta})$,

$$\mathbf{A}_e := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_e := \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{e1} \end{bmatrix},$$

and substitute Eq. (5.2) into Eq. (5.1) to obtain

$$\dot{\mathbf{x}}_e = \mathbf{A}_e\mathbf{x}_e + \mathbf{B}_e\mathbf{u} \qquad \mathbf{x}_e(t_0) = \mathbf{x}_{e0}. \tag{5.3}$$

To keep the notation clean, this chapter drops the subscripts from $\mathbf{x}_e$, $\mathbf{A}_e$, and $\mathbf{B}_e$.

Then, to introduce a robust optimality criterion [3, 9], alter system (5.3) and define a cost function $J > 0$ as follows:

$$\dot{\mathbf{x}} = (\mathbf{A} + \alpha\mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{5.4a}$$

$$\boldsymbol{\epsilon} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5.4b}$$

$$J = \int_{t_0}^{\infty} \boldsymbol{\epsilon}^\top\mathbf{Q}\boldsymbol{\epsilon} + \mathbf{u}^\top\mathbf{R}\mathbf{u}\, dt \tag{5.4c}$$

in which $\alpha \geq 0$, $\boldsymbol{\epsilon}$ denotes any linear combination of states and control to be penalized, $\mathbf{Q} = \mathbf{Q}^\top \succeq \mathbf{0}$ and $\mathbf{R} = \mathbf{R}^\top \succeq \mathbf{0}$ constitute the cost to pay if $\boldsymbol{\epsilon}(t) \neq \mathbf{0}$ and $\mathbf{u}(t) \neq \mathbf{0}$.

Important

The error $\boldsymbol{\epsilon}$ is not necessarily a measurable output and therefore does not need to be either a part of $\mathbf{y}$ or obtainable from $\mathbf{y}$. The matrices $\mathbf{C}$ and $\mathbf{D}$ are not necessarily those defined in model (1.16).

Problem 5.1. Let (5.4) be an optimization problem, then design a control law $\mathbf{u}(t)$ that minimizes the cost (5.4c) under the constraint (5.4a).

Note

Roughly, the tunable parameters appearing in problem (5.4) can be interpreted as follows.

To make the error $\boldsymbol{\epsilon}(t)$ stay close to the origin, increase $\mathbf{Q}$ to associate the condition $\boldsymbol{\epsilon}(t) \neq \mathbf{0}$ with a higher cost. Conversely, to let $\boldsymbol{\epsilon}(t)$ possibly go far from the origin, take $\mathbf{Q} = \mathbf{0}$. Moreover, the cost $\mathbf{Q}$ must be $\succeq \mathbf{0}$ to be well-posed. Indeed, a negative cost would mean a "gain" which could encourage $\boldsymbol{\epsilon}(t)$ to be infinite.

As for $\mathbf{R}$, the price to pay for using $\mathbf{u}$ is directly proportional to $\bar{\mathbf{R}} := \mathbf{D}^\top \mathbf{Q} \mathbf{D} + \mathbf{R}$. Hence, increase the magnitude of $\mathbf{R}$ to limit the control action. On the other hand, if the control law is for free (modeled with $\bar{\mathbf{R}} \succeq \mathbf{0}$), an infinite-magnitude control law can be designed. Since this is unfeasible due to the finite power of actuators, the cost $\mathbf{R}$ must be such that $\bar{\mathbf{R}} \succ \mathbf{0}$.

To conclude, α is introduced to design the control system on an apparently less stable plant. This trick makes the control robust to model uncertainties and extreme selection of gains $\mathbf{Q}$ and $\mathbf{R}$.

The computation of $\mathbf{u}^\star$ that minimizes cost (5.4c) under constraint (5.4a) represents, in general, a challenging task. To face this complex assignment, a good practice consists of rewriting this constrained minimization problem as an unconstrained one using Lagrange multipliers. Introduce the trivial term $\boldsymbol{\lambda}^\top ((\mathbf{A} + \alpha\mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\mathbf{x}})$ in Eq. (5.4c) to obtain

$$J = \int_{t_0}^{\infty} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) - \boldsymbol{\lambda}^\top \dot{\mathbf{x}}\, dt, \tag{5.5}$$

where

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) := \boldsymbol{\epsilon}^\top \mathbf{Q}\, \boldsymbol{\epsilon} + \mathbf{u}^\top \mathbf{R}\, \mathbf{u} + \boldsymbol{\lambda}^\top ((\mathbf{A} + \alpha\mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{5.6}$$

is called Hamiltonian function and $\boldsymbol{\lambda} : \mathbb{R} \to \mathbb{R}^n$ represents the Lagrange multiplier, also called *co-state*. Integrate Eq. (5.5) by parts as

$$J = -\, \boldsymbol{\lambda}^\top \mathbf{x}\Big|_{t_0}^{\infty} + \int_{t_0}^{\infty} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}}^\top \mathbf{x}\, dt \tag{5.7}$$

and introduce the variation of J with respect to $\mathbf{x}$ and $\mathbf{u}$ as follows:

$$\begin{aligned} \delta J = &-[\boldsymbol{\lambda}(\infty)]^\top \delta\mathbf{x}(\infty) + [\boldsymbol{\lambda}(t_0)]^\top \delta\mathbf{x}(t_0) + \\ &\int_{t_0}^{\infty} \frac{\partial H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})}{\partial \mathbf{u}} \delta\mathbf{u} + \left(\frac{\partial H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^\top \right) \delta\mathbf{x}\, dt. \end{aligned} \tag{5.8}$$

The J variation can be made independent of $\delta\mathbf{x}$ during the transient from t_0 to ∞ if

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{x}}\right)^\top, \qquad \boldsymbol{\lambda}(\infty) = \mathbf{0} \tag{5.9}$$

is imposed. Condition (5.9) simplifies Eq. (5.8) as follows:

$$\delta J = [\boldsymbol{\lambda}(t_0)]^\top \delta\mathbf{x}(t_0) + \int_{t_0}^{\infty} \frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}} \delta\mathbf{u}\, dt. \tag{5.10}$$

Then, if $\mathbf{u}$ evaluated at $\mathbf{u}^\star$ is a minimizer for J, then it must be such that

$$\int_{t_0}^{\infty} \left.\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^\star} \delta\mathbf{u}^\star\, d\tau = \mathbf{0}, \tag{5.11}$$

which leads to

$$\left.\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^\star} = 2\mathbf{x}^\top\mathbf{C}^\top\mathbf{Q}\mathbf{D} + \boldsymbol{\lambda}^\top\mathbf{B} + 2[\mathbf{u}^\star]^\top\bar{\mathbf{R}} = \mathbf{0} \tag{5.12}$$

where $\bar{\mathbf{R}} := \mathbf{D}^\top\mathbf{Q}\mathbf{D} + \mathbf{R}$. From Eq. (5.12), we find

$$\mathbf{u}^\star = -\frac{1}{2}\bar{\mathbf{R}}^{-1}\left(2\mathbf{D}^\top\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{B}^\top\boldsymbol{\lambda}\right). \tag{5.13}$$

Moreover, to make $\mathbf{u}^\star$ linearly dependent on $\boldsymbol{\lambda}$, design the Lagrange multiplier as a linear function of $\mathbf{x}$. Let $\boldsymbol{\lambda} := 2\mathbf{S}\mathbf{x}$ with $\mathbf{S} = \mathbf{S}^\top \succ \mathbf{0}$ and write Eq. (5.13) as

$$\begin{aligned} \mathbf{u}^\star &= \mathbf{K}\mathbf{x} \\ \mathbf{K} &= -\bar{\mathbf{R}}^{-1}\left(\mathbf{D}^\top\mathbf{Q}\mathbf{C} + \mathbf{B}^\top\mathbf{S}\right). \end{aligned} \tag{5.14}$$

An expression for $\mathbf{S}$ is found to conclude the optimal control design. Indeed, the solution of the stationary optimal control problem is found by noting that $\dot{\boldsymbol{\lambda}} = 2\mathbf{S}\dot{\mathbf{x}}$ must agree with Eq. (5.9) evaluated at $\mathbf{u}^\star$, *i.e.*, $\mathbf{S}$ must verify [[1],§9]

$$\begin{aligned} &\mathbf{S}\mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{B}^\top\mathbf{S} - \mathbf{S}(\mathbf{A} + \alpha\mathbf{I} - \mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}\mathbf{C}) \\ &- (\mathbf{A} + \alpha\mathbf{I} - \mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}\mathbf{C})^\top\mathbf{S} - \mathbf{C}^\top\mathbf{Q}[\mathbf{I} - \mathbf{D}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}]\mathbf{C} = \mathbf{0}. \end{aligned} \tag{5.15}$$

Important

It is worth noting that the solution to Problem 5.1 must also stabilize the plant.

Expression (5.15), called Algebraic Riccati Equation (ARE), has one stabilizing positive definite solution, namely $\mathbf{S}_\infty$, if $\mathbf{Q}$, $\mathbf{R} \succ \mathbf{0}$, the couple $(\mathbf{A}+\alpha\mathbf{I}, \mathbf{B})$ is stabilizable and the couple $(\mathbf{A} + \alpha\mathbf{I}, \mathbf{C})$ is detectable (these three conditions must be jointly verified!).

Note

Let

$$\begin{aligned}\dot{x} &= (a+\alpha)x + bu \\ \varepsilon &= cx \\ 0 &= s^2\frac{b^2}{r} - 2(a+\alpha)s - c^2q\end{aligned}$$

be a scalar plant with the associated ARE [10]. The solutions to the ARE are

$$s_{1,2} = \frac{(a+\alpha)r}{b^2} \pm \sqrt{\frac{(a+\alpha)^2r^2}{b^4} + \frac{c^2qr}{b^2}},$$

with $s_1 \geq 0$ and $s_2 \leq 0$. The optimal control law is $u^\star = ks_1$ with $k := -r^{-1}bs_1$. Then, the solution s_1 approaches zero for $r \to 0$ while

$$\lim_{r\to 0} k = -\lim_{r\to 0}\frac{bs_1}{r} = -\lim_{r\to 0}\left(\frac{a+\alpha}{b} + b\sqrt{\frac{(a+\alpha)^2}{b^4} + \frac{c^2q}{rb^2}}\right) = -\infty,$$

which confirms the intuition that if r is trivial, the control can be infinite. On the other hand, for $r \to \infty$ the solution s_1 goes to infinite but k remains finite

$$\begin{aligned}\lim_{r\to\infty} k &= -\lim_{r\to\infty}\frac{bs_1}{r} = -\lim_{r\to\infty}\left(\frac{a+\alpha}{b} + b\sqrt{\frac{(a+\alpha)^2}{b^4} + \frac{c^2q}{rb^2}}\right) \\ &= -\frac{a+\alpha+\sqrt{(a+\alpha)^2}}{b}.\end{aligned}$$

In practice, when $r \to \infty$, the cost associated with the control is so much higher than q (the cost associated with ϵ) that the latter can be neglected. In this configuration, the main objectives are reducing the control action and the stability of the closed-loop system. As a consequence, $k = 0$ for any $a \leq -\alpha$ because the system is already BIBS-stable with eigenvalue less or equal to $-\alpha$. On the other hand, for any $a > -\alpha$, the feedback gain is $k = -2(a+\alpha)/b$, which assures a closed-loop eigenvalue less or equal to $-(a + 2\alpha)$.

Infobox 5.1. The stationary optimal control based on the quadratic cost function represents one of the most attractive control solutions of linear systems. Indeed, as described in Appendix C, this control strategy possesses surprising robustness to model uncertainties that, in the frequency domain, is translated into a phase margin of at least 60^o and a gain margin between 1/2 and ∞.

This section ends by investigating the solution to Problem 5.1 concerning the detectability of the couple $(\mathbf{A}+\alpha\mathbf{I},\mathbf{C})$ and the stabilisability of the couple $(\mathbf{A}+\alpha\mathbf{I},\mathbf{B})$. Let $\mathbf{T}_\mathrm{K}$ be the transformation that changes Eq.s (5.4a)-(5.4b) into the ultimate Kalman decomposition form. Then, define $\mathbf{z}=\mathbf{T}_\mathrm{K}\mathbf{x}$ and determine the dynamics of $\mathbf{z}$ as follows:

$$
\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{A}^\star\mathbf{z}+\mathbf{B}^\star\mathbf{u} \quad \mathbf{x}(t_0)=\mathbf{x}_0\\
\boldsymbol{\epsilon} &= \mathbf{C}^\star\mathbf{z}+\mathbf{D}\mathbf{u},
\end{aligned}
$$

in which $\mathbf{z}:=\mathrm{col}(\mathbf{z}_{\mathrm{R,NO}},\mathbf{z}_{\mathrm{R,O}},\mathbf{z}_{\mathrm{NR,NO}},\mathbf{z}_{\mathrm{NR,O}})$, $\mathbf{A}^\star:=\mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, $\mathbf{B}^\star:=\mathbf{T}\mathbf{B}$ and $\mathbf{C}^\star:=\mathbf{C}\mathbf{T}^{-1}$. Let $\mathbf{S}^\star$ be partitioned in 16 parts as

$$
\mathbf{S}^\star=\begin{bmatrix}\mathbf{S}^\star_{11} & \mathbf{S}^\star_{12} & \mathbf{S}^\star_{13} & \mathbf{S}^\star_{14}\\ (\mathbf{S}^\star_{12})^\top & \mathbf{S}^\star_{22} & \mathbf{S}^\star_{23} & \mathbf{S}^\star_{24}\\ (\mathbf{S}^\star_{13})^\top & (\mathbf{S}^\star_{23})^\top & \mathbf{S}^\star_{33} & \mathbf{S}^\star_{34}\\ (\mathbf{S}^\star_{14})^\top & (\mathbf{S}^\star_{24})^\top & (\mathbf{S}^\star_{34})^\top & \mathbf{S}^\star_{44}\end{bmatrix}.
$$

Then, we demonstrate that $\mathbf{S}^\star$ reduces to

$$
\mathbf{S}^\star=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^\star_{22} & \mathbf{0} & \mathbf{S}^\star_{24}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & (\mathbf{S}^\star_{24})^\top & \mathbf{0} & \mathbf{S}^\star_{44}\end{bmatrix}. \tag{5.16}
$$

To prove Eq. (5.16), calculate the closed-loop $\mathbf{A}^\star-\mathbf{B}^\star\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star)^\top\mathbf{S}^\star$ and note that only $\mathbf{S}^\star_{11}$, $\mathbf{S}^\star_{12}$, $\mathbf{S}^\star_{13}$, $\mathbf{S}^\star_{23}=0$ keep the ultimate Kalman decomposition. As consequence, the solution of the ARE imposes $\mathbf{S}^\star_{14}$, $\mathbf{S}^\star_{33}$, $\mathbf{S}^\star_{34}=0$.

The substitution of Eq. (5.16) into Eq. (5.14) leads to

$$
\mathbf{u}^\star=-\bar{\mathbf{R}}^{-1}\begin{bmatrix}\mathbf{0} & (\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{22} & \mathbf{0} & (\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{24}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{\mathrm{R,NO}}\\ \mathbf{z}_{\mathrm{R,O}}\\ \mathbf{z}_{\mathrm{NR,NO}}\\ \mathbf{z}_{\mathrm{NR,O}}\end{bmatrix}. \tag{5.17}
$$

Finally, the closed-loop dynamics are given by

$$
\begin{bmatrix}\dot{\mathbf{z}}_{\mathrm{R,NO}}\\ \dot{\mathbf{z}}_{\mathrm{R,O}}\\ \dot{\mathbf{z}}_{\mathrm{NR,NO}}\\ \dot{\mathbf{z}}_{\mathrm{NR,O}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}^\star_{11}+\alpha\mathbf{I} & \mathbf{A}^\star_{12}-\mathbf{B}^\star_1\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{22} & \mathbf{A}^\star_{13} & \mathbf{A}^\star_{14}-\mathbf{B}^\star_1\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{24}\\ \mathbf{0} & \mathbf{A}^\star_{22}+\alpha\mathbf{I}-\mathbf{B}^\star_2\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{22} & \mathbf{0} & \mathbf{A}^\star_{24}-\mathbf{B}^\star_2\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{24}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}^\star_{33}+\alpha\mathbf{I} & \mathbf{A}^\star_{34}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}^\star_{44}+\alpha\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{\mathrm{R,NO}}\\ \mathbf{z}_{\mathrm{R,O}}\\ \mathbf{z}_{\mathrm{NR,NO}}\\ \mathbf{z}_{\mathrm{NR,O}}\end{bmatrix},
$$

which must be stabilizable and detectable to let the ARE having a stabilizing solution. Since $\mathbf{A}^\star_{22}+\alpha\mathbf{I}-\mathbf{B}^\star_2\bar{\mathbf{R}}^{-1}(\mathbf{B}^\star_2)^\top\mathbf{S}^\star_{22}$ can be made Hurwitz because $(\mathbf{A}^\star_{22}+\alpha\mathbf{I},\mathbf{B}^\star_2)$ is fully reachable and observable, $\alpha>0$ is upper bounded by the smallest absolute value of the real part of the eigenvalues of $\mathbf{A}^\star_{11}$, $\mathbf{A}^\star_{33}$, and $\mathbf{A}^\star_{44}$.

Example 5.1 (Non detectable model)**.** Consider a point with mass $m > 0$ that moves on a straight path. Define $\mathbf{x} = \mathrm{col}(p, v)$ as the state in which $p, v \in \mathbb{R}$ are positions and speed, respectively. Let $u \in \mathbb{R}$ be the force acting on the mass and $y \in \mathbb{R}$ be the speed measurement. Then, the following linear system models the plant dynamics:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \mathbf{x}(0) = \mathbf{x}_0 \\ e &= \mathbf{C}\mathbf{x}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It is worth noting that the couple $(\mathbf{A}, \mathbf{B})$ is fully reachable, whereas the couple $(\mathbf{A}, \mathbf{C})$ is not fully observable. Moreover, the unobservable part of the system is not BIBS-stable thus leading to a non-detectable system. Let $\alpha = 0$, $r > 0$, $q \geq 0$, and

$$\mathbf{S} := \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix},$$

then the ARE associated with this problem becomes

$$\frac{1}{r\, m^2} \begin{bmatrix} s_{12}^2 & s_{12}s_{22} \\ s_{12}s_{22} & s_{22}^2 \end{bmatrix} - \begin{bmatrix} 0 & s_{11} \\ s_{11} & 2s_{12} + q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which leads to a positive semi-definite $\mathbf{S}$

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & m\sqrt{q\, r} \end{bmatrix}.$$

Also, the optimal control law is $u = \mathbf{K}\mathbf{x}$ with

$$\mathbf{K} = -\frac{1}{r}[0 \ m^{-1}] \begin{bmatrix} 0 & 0 \\ 0 & m\sqrt{q\, r} \end{bmatrix} = - \begin{bmatrix} 0 & \sqrt{q/r} \end{bmatrix}.$$

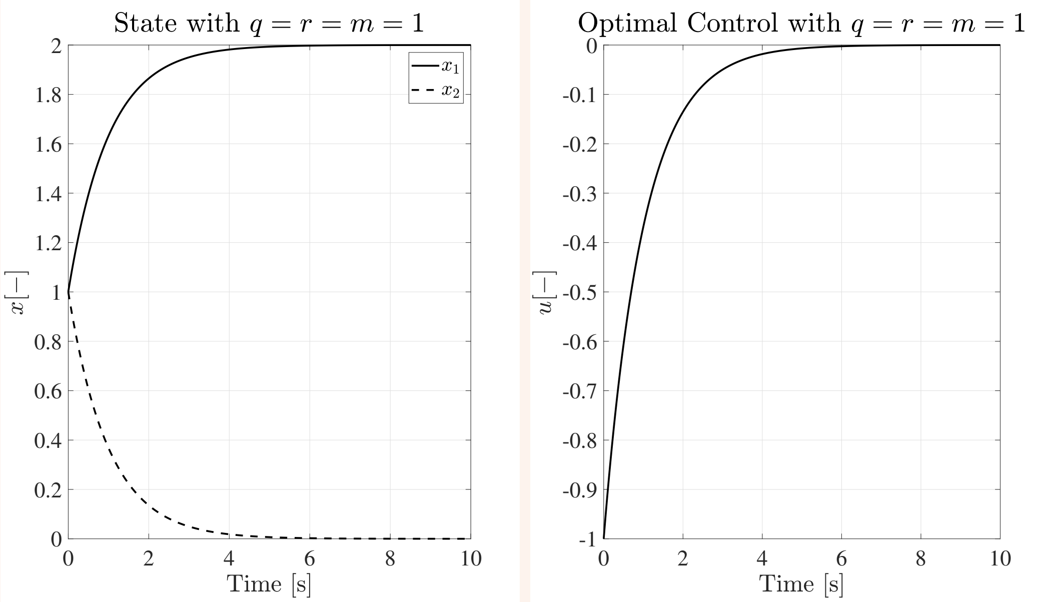
Finally, the closed-loop system has dynamics described by

$$\mathbf{A} + \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ 0 & -\sqrt{q/r}/m \end{bmatrix}$$

which is not BIBS-stable. The time evolution of the state is obtained utilizing the Lagrange formula (3.16) that, applied to $\mathbf{A} + \mathbf{BK}$, leads to

$$x_1(t) = \frac{m}{\sqrt{q/r}} \left(1 - e^{-t\sqrt{q/r}/m}\right) x_2(0) + x_1(0)$$
$$x_2(t) = e^{-t\sqrt{q/r}/m} x_2(0)$$

The following figures depict the time evolution of states and control.



Example 5.2 (Non stabilizable model)**.** With reference to the plant of Example 5.1, assume that $\mathbf{C} = [1\ 0]$ and $\mathbf{B} = \text{col}(1,\, 0)$. Due to these variations, the plant becomes fully observable but not fully reachable. Moreover, the non-reachable subsystem is characterized by a null eigenvalue, so the system is not stabilizable. Assuming $\alpha = 0$, the ARE associated with the optimal control problem becomes

$$\begin{bmatrix} s_{11}^2/r - q & s_{11}(s_{12}/r - 1) \\ s_{11}(s_{12}/r - 1) & s_{12}(s_{12}/r - 2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which has no solutions for $q > 0$. Therefore, assume $q = 0$ to obtain $s_{11} = 0$,

$s_{22} \in \mathbb{R}$, and $s_{12} \in \{0, 2r\}$. Observe that

$$\det(\mathbf{S} - \lambda I) = \det\left(\begin{bmatrix} -\lambda & s_{12} \\ s_{12} & s_{22} - \lambda \end{bmatrix}\right) = \lambda^2 - \lambda s_{22} - s_{12}^2 = 0 \implies$$

$$\lambda_{1,2} = \frac{s_{22} \pm \sqrt{s_{22}^2 + 4s_{12}^2}}{2}.$$

The roots $\lambda_{1,2}$ are opposite in sign and different from zero for any $s_{22} \in \mathbb{R}$. Hence, the ARE is solved by an $\mathbf{S}$ which is semi-positive definite. The optimal control is obtained as $u = \mathbf{Kx}$ with

$$\mathbf{K} = -\frac{1}{r}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = -\frac{1}{r}\begin{bmatrix} 0 & s_{12} \end{bmatrix}.$$

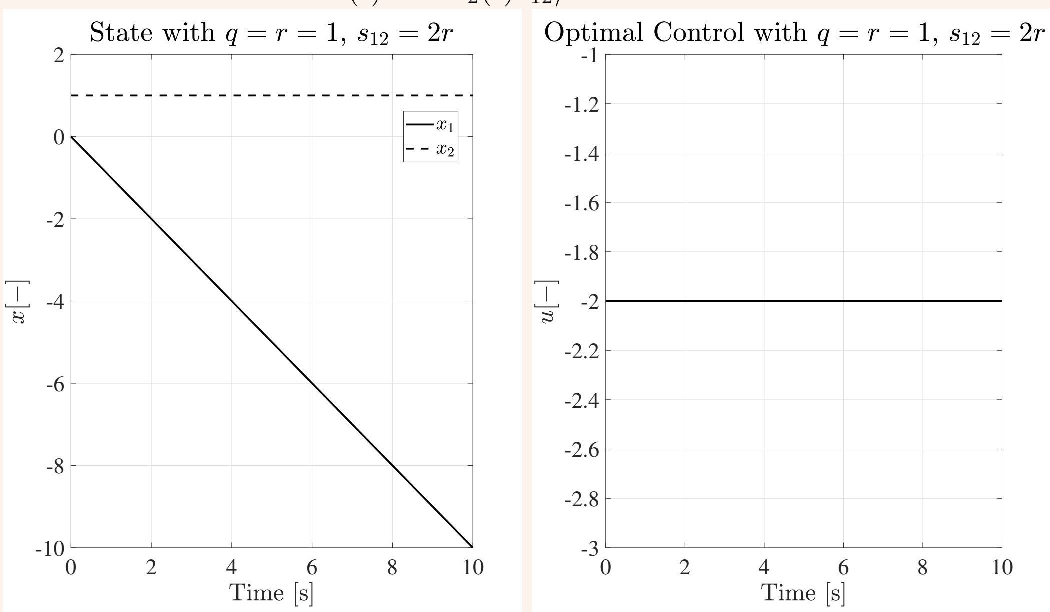
Finally, the closed-loop transition matrix becomes

$$\mathbf{A} + \mathbf{BK} = \begin{bmatrix} 0 & 1 - s_{12}/r \\ 0 & 0 \end{bmatrix}$$

which indeed is not BIBS-stable. In conclusion, the time evolution of the state is described as

$$x_1(t) = (1 - s_{12}/r)x_2(0)\,t + x_1(0)$$
$$x_2(t) = x_2(0)$$

that leads to the control $u(t) = -x_2(t)s_{12}/r$.



Example 5.3 (Observable and reachable model)**.** With reference to the plant of Example 5.1, assume that $\mathbf{C} = [1\ 0]$ which makes $(\mathbf{A}, \mathbf{C})$ fully observable. The reachability of $(\mathbf{A}, \mathbf{B})$ assures that there exists a stabilizing optimal control $u = -r^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x}$, with $\mathbf{S}$ solving the following ARE:

$$\begin{bmatrix} s_{12}^2/(m^2 r) - q & s_{12}s_{22}/(m^2 r) - s_{11} \\ s_{12}s_{22}/(m^2 r) - s_{11} & s_{22}^2/(m^2 r) - 2s_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let

$$s_{11} = \sqrt{2m}\, q^{3/4}\, r^{1/4}, \quad s_{12} = m\sqrt{rq}, \quad s_{22} = \sqrt{2}\, m^{3/2}\, r^{3/4}\, q^{1/4}.$$

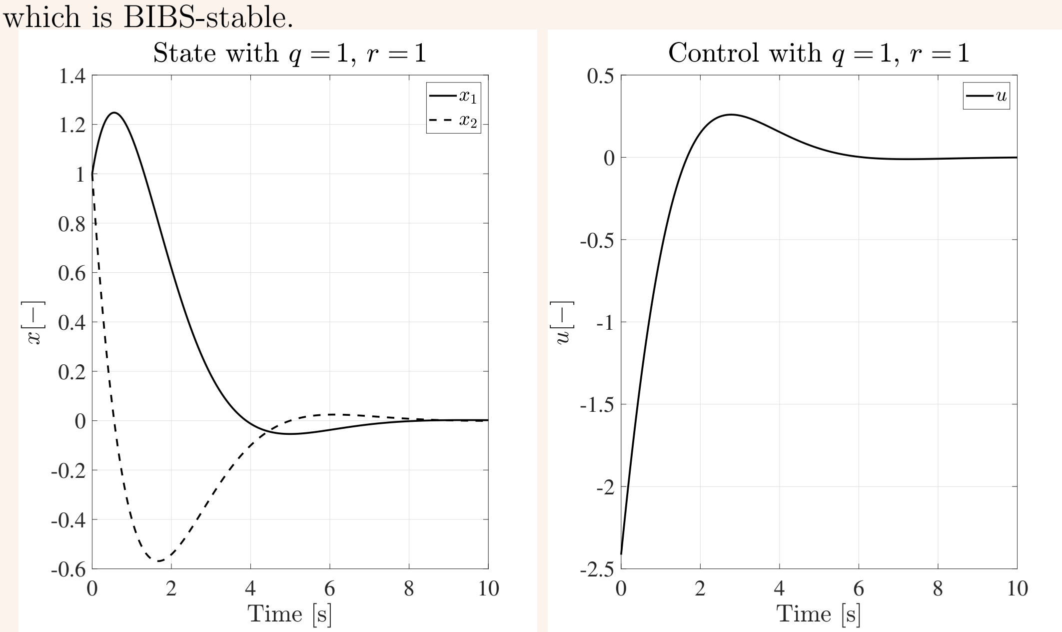
be the solution of the ARE, then the state-feedback control is $u = \mathbf{K}\mathbf{x}$ with

$$\mathbf{K} = -\frac{1}{r}[0\ m^{-1}] \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = -\frac{1}{rm} \begin{bmatrix} s_{12} & s_{22} \end{bmatrix}.$$

Finally, the closed-loop system has dynamics described by the matrix

$$\mathbf{A} + \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -\dfrac{s_{12}}{rm} & -\dfrac{s_{22}}{rm} \end{bmatrix}$$

which is BIBS-stable.



Example 5.4 (Cart-pole set point regulator design)**.** Constants $\mathbf{K}_{\text{R,O}}$ and k_{I}, defined in Sections 4.5.1 and 4.3, are designed through the stationary optimal control technique. Let the system be (see Example 4.17)

$$\begin{bmatrix} \dot{\mathbf{z}}_{\text{R,O}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{22} & \mathbf{0} \\ \bar{\mathbf{C}}_e & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\text{R,O}} \\ \eta \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_2 \\ 0 \end{bmatrix} u$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\text{R,O}} \\ \eta \end{bmatrix}$$

and define the cost function

$$J = \int_0^\infty \boldsymbol{\epsilon}^\top \mathbf{Q} \boldsymbol{\epsilon} + u^2 R \, dt,$$

in which $\mathbf{Q} = \mathbf{Q}^\top \succ 0$ and $R > 0$. Assume

$$\mathbf{S} := \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^\top & s_{22} \end{bmatrix},$$

in which $\mathbf{S}_{12} \in \mathbb{R}^{1\times 3}$ and $\mathbf{S}_{11} \in \mathbb{R}^{3\times 3}$. Then, the stationary optimal control law is

$$u^\star = -R^{-1} \begin{bmatrix} \bar{\mathbf{B}}_2^\top & 0 \end{bmatrix} \mathbf{S} \begin{bmatrix} \mathbf{z}_{\text{R,O}} \\ \eta \end{bmatrix} = -R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{11} \mathbf{z}_{\text{R,O}} - R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{12}^\top \eta$$

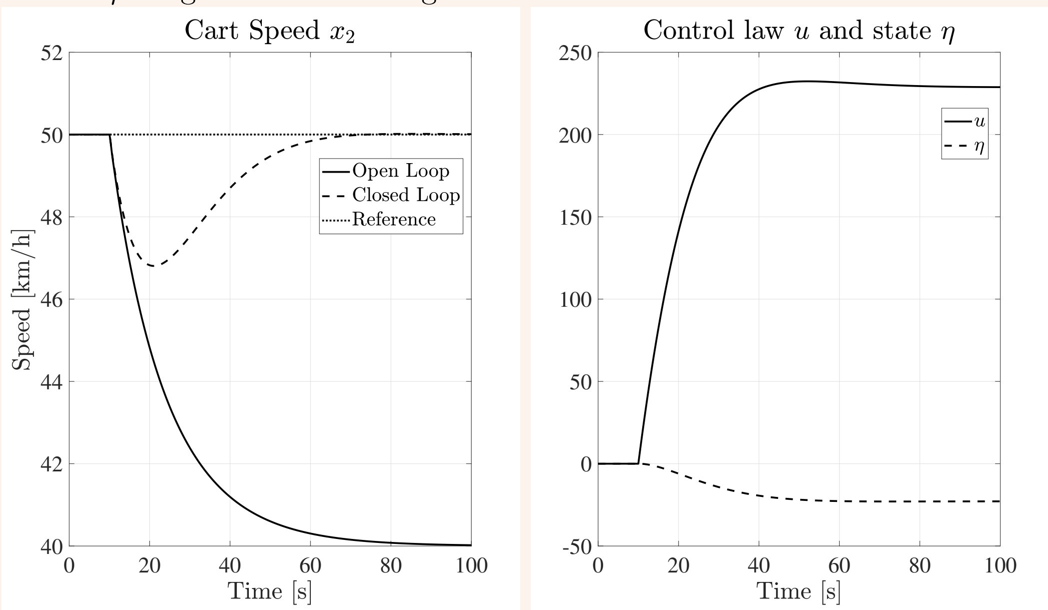
where $\mathbf{K}_{\text{R,O}} = -R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{11}$ and $k_{\text{I}} = -R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{12}^\top$. The matrices $\mathbf{S}_{12}$ and $\mathbf{S}_{11}$ can be found via numerical solvers. As an example, exploit the following MATLAB® listing:

```
% Plant
Ac = [barA22 zeros(3,1); barCe 0];
Bc = [barB2; 0];
Cc = eye(4);

% Costs
Q = % any 4x4 matrix positive definite
R = % any positive scalar

% Solve the ARE
[K,S,e] = lqr(Ac,Bc,Q,R,zeros(4,1)); % WARNING! The control law is u = -K*x. S is the positive definite solution of the ARE.
```

The following results have been obtained by imposing $\mathbf{Q} = 100\,\mathbf{I}$ and $R = 1$. The cart starts the simulation at the equilibrium speed of 50 km/h. At $t = 10s$, a constant wind of -10 km/h appears and brakes the system. Thanks to the control action, the target speed is recovered after a transient. In this period, the variable η integrates the tracking error.



5.1.1 Gain Selection

Matrices $\mathbf{Q}$ and $\mathbf{R}$ represent tunable gains at the disposal of the designer. The desire to limit the error $\boldsymbol{\epsilon}(t)$ and the control $\mathbf{u}(t)$ drives a rule of thumb for designing these matrices (other approaches are found in [4, 5, 11, 14]). Let

$$\boldsymbol{\epsilon} := \mathrm{col}(\epsilon_1, \dots, \epsilon_m), \quad \mathbf{u} := \mathrm{col}(u_1, \dots, u_p)$$

and define $\epsilon_{i_{\max}}, u_{j_{\max}} > 0$, for $i = 1, \dots, m$ and $j = 1, \dots, p$. Conceive

- $|\epsilon_i(t)| \leq \epsilon_{i_{\max}} \in \mathbb{R}_{>0}$, for $i = 1, \dots, m$ and
- $|u_i(t)| \leq u_{i_{\max}} \in \mathbb{R}_{>0}$, for $i = 1, \dots, p$

as soft constraints the controlled plant should verify. Then, the matrices $\mathbf{Q}$ and $\mathbf{R}$ are defined as

$$\mathbf{Q}^{-1} = m \begin{bmatrix} \epsilon^2_{1_{\max}} & 0 & \dots & 0 \\ 0 & \epsilon^2_{2_{\max}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon^2_{q_{\max}} \end{bmatrix}$$

$$\mathbf{R}^{-1} = p \begin{bmatrix} u^2_{1_{\max}} & 0 & \dots & 0 \\ 0 & u^2_{2_{\max}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u^2_{p_{\max}} \end{bmatrix}.$$

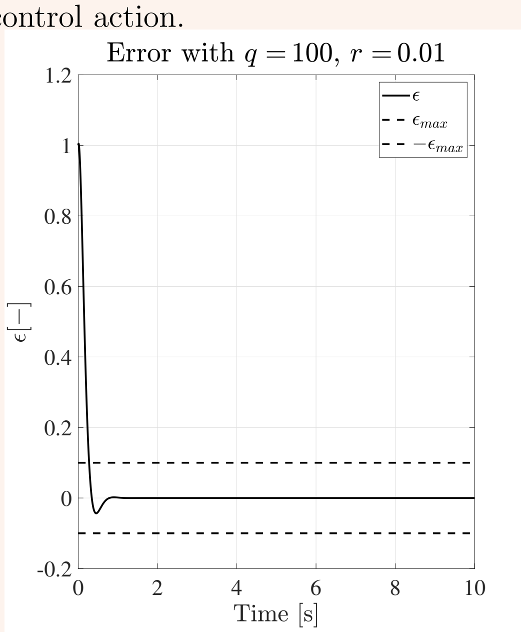
It is worth noting that the above-defined $\mathbf{Q}$ and $\mathbf{R}$ do not guarantee that $\boldsymbol{\epsilon}$ and $\mathbf{u}$ remain bounded within the prescribed constraints. Parameters m and p are normalization weights that make matrices $\mathbf{Q}$ and $\mathbf{R}$ independent of the dimensions of vectors $\boldsymbol{\epsilon}$ and $\mathbf{u}$.

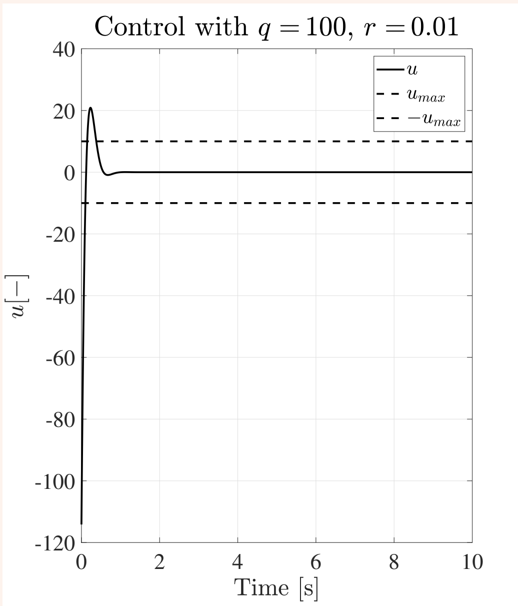
Example 5.5 (Tuning of $\mathbf{Q}$ and $\mathbf{R}$)**.** With reference to the model of Example 5.3, impose $m = 1$ and try different settings, one for each combination of $\epsilon_{\max}$, $u_{\max} \in \{0.1, 10\}$. First of all, note that

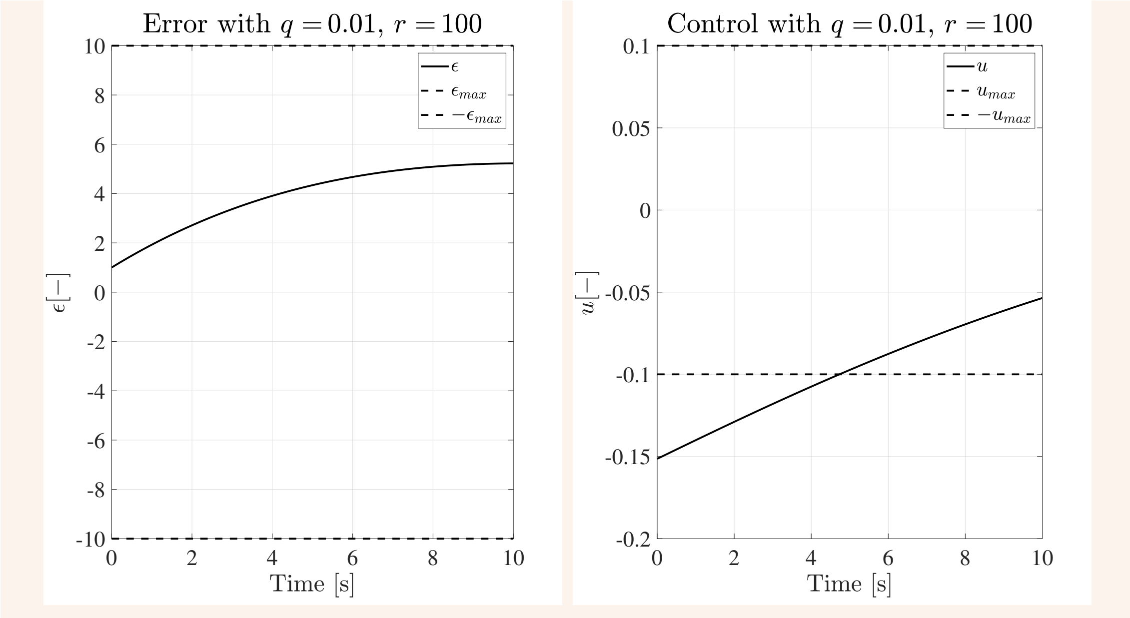
$$\frac{s_{12}}{mr} = \sqrt{q/r}, \qquad \frac{s_{22}}{mr} = \sqrt{2m}\,(q/r)^{1/4}.$$

Then, the closed-loop system behaves the same for any q/r constant. For this reason, the plots for $\epsilon_{\max} = u_{\max}$ are not shown.

Moreover, concerning the time evolution of ϵ and u of Example 5.3, observe that increasing q and decreasing r leads to a faster response that relies on larger control values, which can rapidly steer the error to zero. Conversely, reducing q and increasing r induces the controller to tolerate larger errors with a "lazier" control action.







5.2 Duality

Section 5.1 presented a constructive procedure for designing the state-feedback control gain $\mathbf{K}_S$. It is possible to apply the same approach for the design of the observer matrix $\mathbf{K}_O$ through the duality properties of linear systems.

Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{5.18}$$

be an LTI system with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^p$ and $\mathbf{y} \in \mathbb{R}^q$. Then, the *dual system* associated with the *primary system* (5.18) is defined as

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}^\top \boldsymbol{\chi} + \mathbf{C}^\top \boldsymbol{\upsilon} \\ \boldsymbol{\mu} &= \mathbf{B}^\top \boldsymbol{\chi} + \mathbf{D}^\top \boldsymbol{\upsilon} \end{aligned} \tag{5.19}$$

with $\boldsymbol{\chi} \in \mathbb{R}^n$, $\boldsymbol{\upsilon} \in \mathbb{R}^q$ and $\boldsymbol{\mu} \in \mathbb{R}^p$.

The primary and the dual systems are two equivalent representations of the same mathematical model and, thus, any property possessed by the primary system can be translated into an equivalent property of its dual.

Lemma 5.1. *Let the primary system* (5.18) *be fully reachable. Then, the dual system is fully observable and vice versa.*

Proof. The reachability subset of the primary system (5.18) has as one of its bases the image of the reachability matrix

$$\mathbf{R}_p := \left[\begin{array}{ccccc} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{array} \right],$$

whereas the unobservability subspace of the dual system $\mathcal{E}_d$ has as one of its bases the kernel of the observability matrix

$$\mathbf{O}_d := \begin{bmatrix} \mathbf{B}^\top \\ \mathbf{B}^\top\mathbf{A}^\top \\ \mathbf{B}^\top\left(\mathbf{A}^\top\right)^2 \\ \cdots \\ \mathbf{B}^\top\left(\mathbf{A}^\top\right)^{n-1} \end{bmatrix}.$$

If the primary system is fully reachable, then $\mathrm{im}(\mathbf{R}_p) = \mathbb{R}^n$, whereas if the dual system is fully observable, then $\ker(\mathbf{O}_d) = \{\mathbf{0}\}$ or equivalently $(\ker(\mathbf{O}_d))^\perp = \mathbb{R}^n$. In particular,

$$(\ker(\mathbf{O}_d))^\perp = \mathrm{im}\left(\mathbf{O}_d^\top\right) = \mathrm{im}\left(\mathbf{R}_p\right),$$

which demonstrates that if the primary system is fully reachable, the dual system is fully observable and vice versa. □

Lemma 5.2. *Let the primary system* (5.18) *be fully observable. Then, the dual system is fully reachable and vice versa.*

Proof. The reachability subset of the dual system (5.18) has as one of its bases the image of the reachability matrix

$$\mathbf{R}_d := \begin{bmatrix} \mathbf{C}^\top & \mathbf{A}^\top\mathbf{C}^\top & (\mathbf{A}^\top)^2\mathbf{C}^\top & \dots & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix},$$

whereas the unobservability subspace of the primary system $\mathcal{E}_p$ has as one of its bases the kernel of the observability matrix

$$\mathbf{O}_p := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \cdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

If the dual system is fully reachable, then $\mathrm{im}(\mathbf{R}_d) = \mathbb{R}^n$, whereas if the primary system is fully observable, then $\ker(\mathbf{O}_p) = \{\mathbf{0}\}$ or equivalently $(\ker(\mathbf{O}_p))^\perp = \mathbb{R}^n$. In particular,

$$(\ker(\mathbf{O}_p))^\perp = \mathrm{im}\left(\mathbf{O}_p^\top\right) = \mathrm{im}\left(\mathbf{R}_d\right),$$

which demonstrates that if the dual system is fully reachable, the primary system is fully observable and vice versa. □

Lemma 5.3. *Let the primary model* (5.18) *be BIBS-stable. Then also the dual model* (5.19) *is BIBS-stable and vice versa.*

Proof. If the primary model is BIBS-stable, the real part of the eigenvalues of the matrix $\mathbf{A}$ is strictly negative. Let $\sigma_{\mathbf{A}}$ be the set of the eigenvalues of $\mathbf{A}$ and let $\text{Real}(\sigma_{\mathbf{A}})$ denote the real part of each eigenvalue of $\mathbf{A}$. Since $\mathbf{A}$ and its transpose have the same eigenvalues, it is

$$\text{Real}(\sigma_{\mathbf{A}}) < 0 \iff \text{Real}(\sigma_{\mathbf{A}^\top}) < 0.$$

□

This section extends the duality properties also to control and observation problems so that the existence of a solution to one problem implies the existence of a solution to its dual.

Let the primary system (5.18) be fully observable. Then in agreement with the results of Section 4.4, $\mathbf{K}_{\mathrm{O}_p}$ exists such that the state estimation error $\mathbf{e}_{\mathbf{x}} := \hat{\mathbf{x}} - \mathbf{x}$ is governed by the dynamics

$$\dot{\mathbf{e}}_{\mathbf{x}} = (\mathbf{A} - \mathbf{K}_{\mathrm{O}_p}\mathbf{C})\mathbf{e}_{\mathbf{x}} \tag{5.20}$$

whose dual is

$$\dot{\boldsymbol{\varepsilon}} = (\mathbf{A}^\top - \mathbf{C}^\top\mathbf{K}_{\mathrm{O}_p}^\top)\boldsymbol{\varepsilon}. \tag{5.21}$$

Now, since system (5.19) is fully reachable, there exists $\mathbf{K}_{\mathrm{S}_d}$ such that $\boldsymbol{\upsilon} := \mathbf{K}_{\mathrm{S}_d}\boldsymbol{\chi}$ leads to

$$\dot{\boldsymbol{\chi}} = \left(\mathbf{A}^\top + \mathbf{C}^\top\mathbf{K}_{\mathrm{S}_d}\right)\boldsymbol{\chi}. \tag{5.22}$$

Compare Eq.s (5.21) and (5.22) to note that matrices $\mathbf{A}^\top - \mathbf{C}^\top\mathbf{K}_{\mathrm{O}_p}^\top$ and $\mathbf{A}^\top + \mathbf{C}^\top\mathbf{K}_{\mathrm{S}_d}$ coincide if

$$\mathbf{K}_{\mathrm{O}_p} = -\mathbf{K}_{\mathrm{S}_d}^\top.$$

Then, the solution to the observation problem associated with the primary system is implied by the solution to the control problem associated with the dual system.

5.3 Kalman Filter

Section 5.2 shows that the control and observation properties (and problems) associated with linear systems are dual. Therefore, the solution of the observation problem of the primary system is implied by the solution of the control problem of the dual system. Also, Section 5.1 shows how to solve an optimal control problem by designing a state-feedback gain. This section formulates and solves the optimal observation problem for the primary system through the statement and the solution of the optimal control problem for the dual system. Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{w} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{5.23}$$

be a detectable LTI system for which the observer has to be designed. In agreement with Section 4.4, define an identity observer as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}_1\mathbf{u} \\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}} + \mathbf{D}_1\mathbf{u}.\end{aligned} \tag{5.24}$$

Let $\mathbf{e}_x := \hat{\mathbf{x}} - \mathbf{x}$ and $\tilde{\mathbf{y}} := \hat{\mathbf{y}} - \mathbf{y}$, and compute the dynamics of the estimation error as

$$\begin{aligned}\dot{\mathbf{e}}_x &= \mathbf{A}\mathbf{e}_x - \mathbf{B}_2\mathbf{w} \\ \tilde{\mathbf{y}} &= \mathbf{C}\mathbf{e}_x - \mathbf{D}_2\mathbf{w}.\end{aligned} \tag{5.25}$$

Let

$$\begin{aligned}\dot{\boldsymbol{\chi}} &= \mathbf{A}^\top\boldsymbol{\chi} + \mathbf{C}^\top\boldsymbol{\upsilon} \\ \boldsymbol{\mu} &= \mathbf{B}_2^\top\boldsymbol{\chi} + \mathbf{D}_2^\top\boldsymbol{\upsilon}\end{aligned} \tag{5.26}$$

be the dual model associated with system (5.25). Define and solve the robust optimal control problem for plant (5.26) through the steps depicted in Section 5.1. In more detail, let $\alpha > 0$, alter model (5.26) as

$$\begin{aligned}\dot{\boldsymbol{\chi}} &= (\mathbf{A}^\top + \alpha\mathbf{I})\boldsymbol{\chi} + \mathbf{C}^\top\boldsymbol{\upsilon} \quad \boldsymbol{\chi}(t_f) = \boldsymbol{\chi}_f \\ \boldsymbol{\mu} &= \mathbf{B}_2^\top\boldsymbol{\chi} + \mathbf{D}_2^\top\boldsymbol{\upsilon},\end{aligned} \tag{5.27a}$$

and define the following cost function:

$$J_d = \int_{t_0}^{\infty} \boldsymbol{\mu}^\top\mathbf{Q}_d\boldsymbol{\mu} + \boldsymbol{\upsilon}^\top\mathbf{R}_d\boldsymbol{\upsilon}\, dt, \tag{5.27b}$$

where $\mathbf{Q}_d \succeq 0$ and $\mathbf{R}_d \succeq 0$. Apply steps (5.5)-(5.15) to the constrained optimization problem (5.27) to obtain $\boldsymbol{\upsilon}^\star = \mathbf{K}_{\mathrm{S}_d}\boldsymbol{\chi}$, where

$$\begin{aligned}\mathbf{K}_{\mathrm{S}_d} =&\ -\bar{\mathbf{R}}_d^{-1}\left(\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top + \mathbf{C}\mathbf{S}\right) \\ \mathbf{0} =&\ \mathbf{S}\mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{C}\mathbf{S} - \mathbf{S}(\mathbf{A}^\top + \alpha\mathbf{I} - \mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top) \\ &- (\mathbf{A}^\top + \alpha\mathbf{I} - \mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top)^\top\mathbf{S} - \mathbf{B}_2\mathbf{Q}_d[\mathbf{I} - \mathbf{D}_2^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d]\mathbf{B}_2^\top\end{aligned} \tag{5.28}$$

and $\bar{\mathbf{R}}_d = \mathbf{D}_2\mathbf{Q}_d\mathbf{D}_2^\top + \mathbf{R}_d$. Then, as described in Section 5.2, the observer for model (5.23) is given by [9]

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}_1\mathbf{u} - \mathbf{K}_{\mathrm{S}_d}^\top(\mathbf{y} - \hat{\mathbf{y}}) \quad \hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}} + \mathbf{D}_1\mathbf{u},\end{aligned} \tag{5.29}$$

where $\mathbf{K}_{\mathrm{S}_d}$ is defined in Eq. (5.28). Solution (5.28) can be specialized by noting that, in most practical cases, $\mathbf{B}_2 := \begin{bmatrix} \mathbf{B}_{21} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathbf{D}_2 := \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$ (remember from Section 1.2 that $\mathbf{w} := \mathrm{col}(\mathbf{d}, \boldsymbol{\nu}, \mathbf{r})$). Let $\mathbf{Q}_d$ be divided into nine parts as follows:

$$\mathbf{Q}_d = \begin{bmatrix} \mathbf{Q}_{d_{11}} & \mathbf{Q}_{d_{12}} & \mathbf{Q}_{d_{13}} \\ \mathbf{Q}_{d_{12}}^\top & \mathbf{Q}_{d_{22}} & \mathbf{Q}_{d_{23}} \\ \mathbf{Q}_{d_{13}}^\top & \mathbf{Q}_{d_{23}}^\top & \mathbf{Q}_{d_{33}} \end{bmatrix}.$$

Then,

$$\begin{aligned}\bar{\mathbf{R}}_d &= \mathbf{Q}_{d_{22}} + \mathbf{R}_d \\ \mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top &= \mathbf{Q}_{d_{12}}^\top \mathbf{B}_{21}^\top \\ \mathbf{B}_2\mathbf{Q}_d\mathbf{B}_2^\top &= \mathbf{B}_{21}\mathbf{Q}_{d_{11}}\mathbf{B}_{21}^\top .\end{aligned}$$

Infobox 5.2 (Kalman Filter)**.** It has been demonstrated [6, 7] that the dynamic observer constituted by the composition of Eq.s (5.28)–(5.29) represents an *optimal observer* in stochastic terms. In detail, this observer minimizes the expectation of the squared estimation error

$$\mathrm{E}[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^\top \mathbf{M}(\mathbf{x}(t) - \hat{\mathbf{x}}(t))],$$

for some $\mathbf{M} = \mathbf{M}^\top \succ 0$, if the following conditions are verified. Let $\delta(\cdot) : \mathbb{R} \to \mathbb{R}$ be the unit impulse, then:

- $\mathbf{d}(t)$ and $\boldsymbol{\nu}(t)$ are white stochastic processes with covariance kernel

$$\mathrm{E}[\mathbf{d}(t)\mathbf{d}^\top(\tau)] = \mathbf{Q}_{d_{11}}\delta(t-\tau) \quad \forall\, t, \tau \geq t_0$$

$$\mathrm{E}[\boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(\tau)] = \mathbf{Q}_{d_{22}}\delta(t-\tau) \quad \forall\, t, \tau \geq t_0;$$

- the processes $\mathbf{d}(t)$ and $\boldsymbol{\nu}(t)$ are correlated by

$$\mathrm{E}[\mathbf{d}(t)\boldsymbol{\nu}^\top(\tau)] = \mathbf{Q}_{d_{12}}\delta(t-\tau) \quad \forall\, t, \tau \geq t_0;$$

- the state vector $\mathbf{x}_0$ is a random variable with mean and covariance given by

$$\mathrm{E}[\mathbf{x}_0] = \bar{\mathbf{x}}_0, \qquad \mathrm{E}[(\mathbf{x}_0 - \bar{\mathbf{x}}_0)(\mathbf{x}_0 - \bar{\mathbf{x}}_0)^\top] = \mathbf{P}_0;$$

- the stochastic processes $\mathbf{d}(t)$ and $\boldsymbol{\nu}(t)$ are not correlated with respect to the random variable $\mathbf{x}_0$, *i.e.*,

$$\mathrm{E}[\mathbf{x}_0\boldsymbol{\nu}^\top(t)] = \mathbf{0}, \qquad \mathrm{E}[\mathbf{x}_0\mathbf{d}^\top(t)] = \mathbf{0} \qquad \forall\, t \geq t_0.$$

Note

Roughly, the gain matrix $\mathbf{K}_{\mathrm{S}_d}$ represents a compromise between two opposite approaches. On the one hand, the observer exploits model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to make a prediction, but on the other hand, the observer corrects the prediction made with $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ through an output-feedback loop. These two pieces of information, *i.e.*, model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and measurement $\mathbf{y}$, are corrupted by the unknowns $\mathbf{d}$ and $\boldsymbol{\nu}$, respectively. So, the gain $\mathbf{K}_{\mathrm{S}_d}$ represents a compromise between the model's reliability and the measurement's trustworthiness. Suppose the model is "perfect" or the measurement is totally unreliable. In this case, the gain $\mathbf{K}_{\mathrm{S}_d}$ can be "any" in the family of matrices stabilizing the couple $(\mathbf{A}^\top + \alpha\mathbf{I}, \mathbf{C}^\top)$. Conversely, if the model is inconsistent or the measurement is perfect, $\mathbf{K}_{\mathrm{S}_d}$ should become infinite. To confirm this intuition, it is worth noting that the matrix $\mathbf{K}_{\mathrm{S}_d}$ is directly proportional to $\mathbf{Q}_{d_{11}}$ (through $\mathbf{S}$) and $(\mathbf{Q}_{d_{22}} + \mathbf{R}_d)^{-1}\mathbf{C}$. So, high (low) magnitudes of $(\mathbf{Q}_{d_{22}} + \mathbf{R}_d)^{-1}\mathbf{C}\mathbf{B}_{21}\mathbf{Q}_{d_{11}}\mathbf{B}_{21}^\top$ mean that the model is less (more) reliable than the output. Also, the term $\mathbf{Q}_{d_{22}}^{-1}\mathbf{C}$ can be interpreted as a signal-to-noise ratio between the output $\mathbf{y}$ and the noise $\boldsymbol{\nu}$. To conclude, the designer can tune $\mathbf{Q}_{d_{11}}$ and $\mathbf{R}_d$ to balance the exploitation of the model and the measurement and to make $\bar{\mathbf{R}}_d$ invertible.

Example 5.6 (Optimal observer)**.** Design an optimal observer for the following LTI system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_2\mathbf{w} \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

First, write the dual

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}^\top\boldsymbol{\chi} + \mathbf{C}^\top\upsilon \\ \boldsymbol{\mu} &= \mathbf{B}^\top\boldsymbol{\chi} + \mathbf{D}^\top\upsilon \end{aligned}$$

and introduce the cost

$$J = \int_0^\infty \boldsymbol{\mu}^\top \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\mu} + r\upsilon^2\, dt.$$

Second, let $\mathbf{S} := [s_{ij}]$, with $s_{ij} \in \mathbb{R}$ for $i,\, j\, = 1, 2$. Impose $s_{21} = s_{12}$ and find s_{11}, s_{12}, and s_{22} by solving the following ARE:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_{11}^2/r + q + 2s_{12} \\ s_{22} - s_{11} - s_{11}s_{12}/r \\ -s_{12}^2/r - 2s_{12} \end{bmatrix}.$$

In particular, the solutions are $s_{12} = 0$, $s_{11} = \sqrt{(q + 2s_{12})r}$, and $s_{22} = s_{11} + s_{11}s_{12}/r$, Also, the observer matrix given by the first of Eq. (5.28) is

$$\mathbf{K}_O = -\mathbf{K}_{\mathrm{S}_d}^{\top} = \frac{1}{r}\begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}.$$

Then, write the observer as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -s_{11}/r & 1 \\ -1 - s_{12}/r & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} + \frac{1}{r}\begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} y.$$

It is worth noting that the eigenvalues of

$$\begin{bmatrix} -s_{11}/r & 1 \\ -1 - s_{12}/r & 0 \end{bmatrix},$$

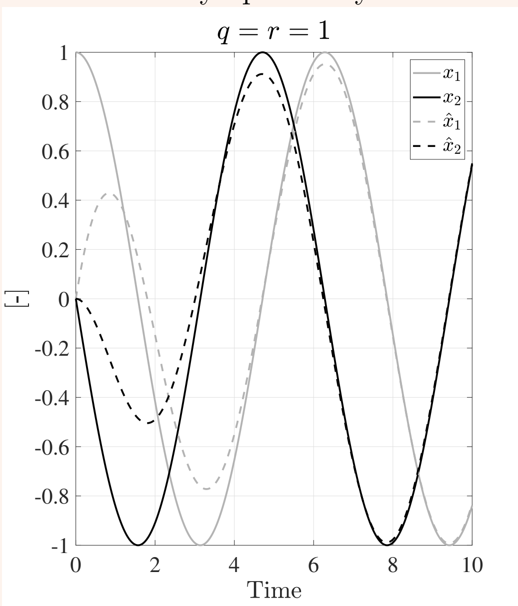
given as the solution to

$$\lambda^2 + \lambda s_{11}/r + 1 + s_{12}/r = 0,$$

are

$$\lambda_{1,2} = \frac{-s_{11} \pm \sqrt{s_{11}^2 - 4(r + s_{12})}}{2r}$$

whose real part is negative for $s_{11} > 0$ and $s_{12} > -r$. Finally, the following figure shows that the estimation asymptotically tracks the state.



Example 5.7 (Tuning of $\mathbf{Q}_{d_{11}}$)**.** This example investigates the tuning of $\mathbf{Q}_{d_{11}}$ interpreted in the sense of Kalman as a covariance matrix. To make clear how $\mathbf{Q}_{d_{11}}$ affects the estimation performance, deal with the scalar system

$$\begin{aligned} \dot{x} &= ax + bu + w \\ y &= cx + \nu \end{aligned}$$

with covariances

$$\begin{aligned} \mathrm{E}[w(t)w(\tau)] &= q_{d_{11}}\delta(t-\tau) \\ \mathrm{E}[\nu(t)\nu(\tau)] &= q_{d_{22}}\delta(t-\tau) \\ \mathrm{E}[w(t)\nu(\tau)] &= 0. \end{aligned}$$

The observer is given as

$$\begin{aligned} \dot{\hat{x}} &= a\hat{x} + bu + k_O(y - \hat{y}) \\ \hat{y} &= c\hat{x} \end{aligned}$$

with the gain

$$\begin{aligned} k_O &= sc/q_{d_{22}} \\ s &= \frac{aq_{d_{22}}}{c^2} + \sqrt{\frac{a^2 q_{d_{22}}^2}{c^4} + \frac{q_{d_{22}} q_{d_{11}}}{c^2}}. \end{aligned}$$

obtained thanks to the solution to an optimal observation problem. The feedback gain is directly proportional to $q_{d_{11}}$ and this is interpreted (in the sense of Kalman) as: "model $\dot{x} = ax + bu$ becomes increasingly uncertain as $q_{d_{11}}$ increases; thus, it is better to rely on the output y by increasing the feedback gain k_O."

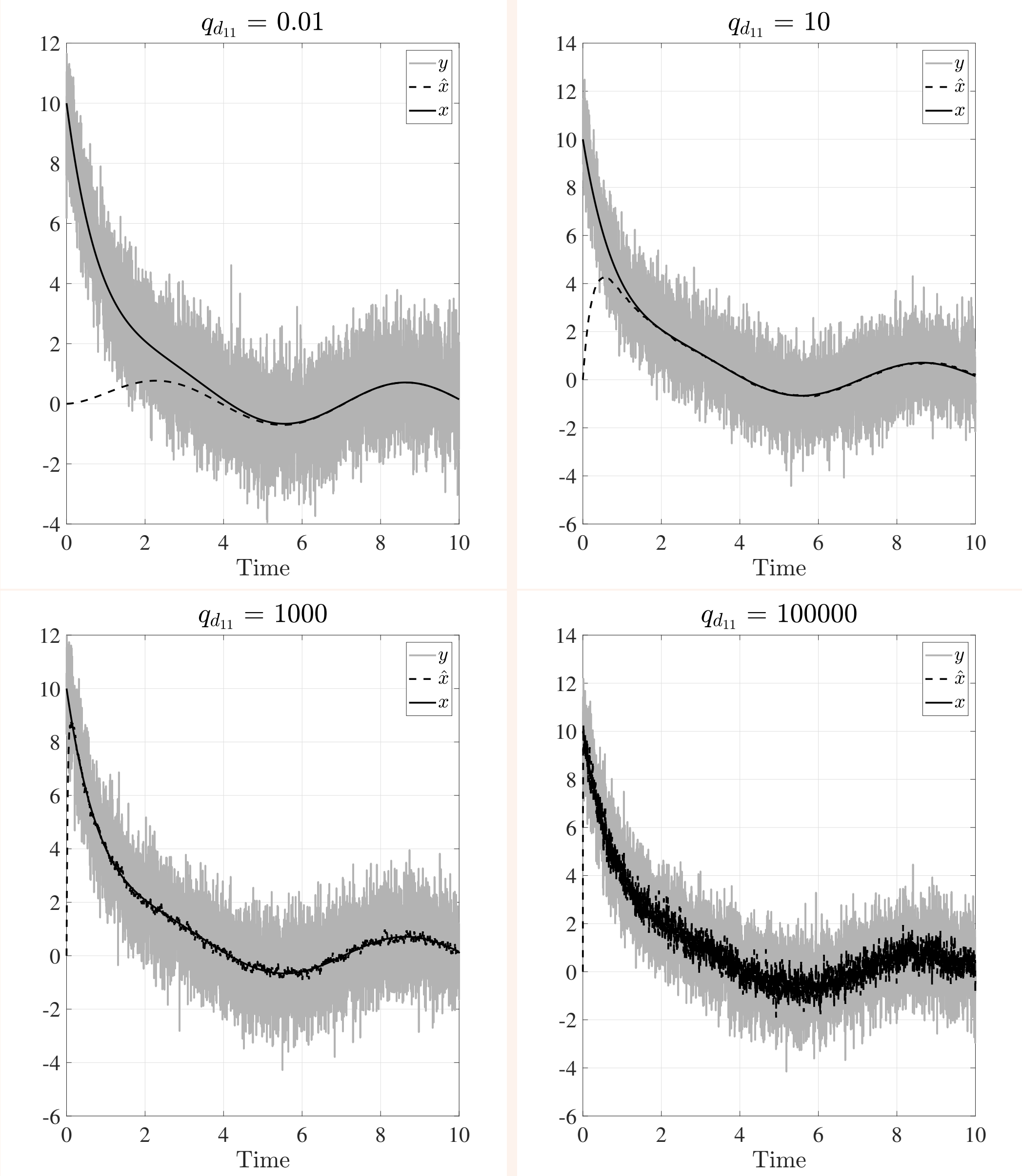
On the other hand, the dynamics of the observation error $e := x - \hat{x}$ are

$$\dot{e} = (a - k_O c)e + k_O \nu,$$

whose eigenvalue is

$$\lambda = a - k_O c = -\sqrt{a^2 + \frac{q_{d_{11}}}{q_{d_{22}}} c^2},$$

which is strictly negative and whose norm is directly proportional to $q_{d_{11}}$. So, a larger $q_{d_{11}}$ leads to a larger k_O which, on the one hand, makes the observer more *reactive* (thanks to a more negative λ). But, conversely, a larger $q_{d_{11}}$ magnifies the effects of the measurement noise ν. For this reason, the compromise between the observer's reactiveness and the noise sensitivity could guide an intuitive tuning for $q_{d_{11}}$. The following figures report observation performance for different settings, obtained by imposing $a = -1$, $b = c = q_{d_{22}} = 1$, and $u = \sin t$.



Example 5.8 (Cart-pole observer design)**.** Design an observer for

$$
\begin{aligned}
\dot{\mathbf{z}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{B}}_{12}u + \bar{\mathbf{B}}_{22}d \\
\mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{R,O}} + \boldsymbol{\nu},
\end{aligned}
$$

which is the dynamics of the reachable and observable part of the model presented in Example 1.2. First, define the dual system

$$\begin{aligned}\dot{\boldsymbol{\chi}} &= \bar{\mathbf{A}}_{22}^\top \boldsymbol{\chi} + \bar{\mathbf{C}}_2^\top \boldsymbol{\upsilon} \\ \boldsymbol{\mu} &= \begin{bmatrix} \bar{\mathbf{B}}_{22} & \mathbf{0} & \mathbf{0} \end{bmatrix}^\top \boldsymbol{\chi} + \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}^\top \boldsymbol{\upsilon}\end{aligned}$$

and introduce the cost function

$$J_d = \int_0^\infty \boldsymbol{\mu}^\top \mathbf{Q}_d \boldsymbol{\mu} + \boldsymbol{\upsilon}^\top \mathbf{R}_d \boldsymbol{\upsilon}\, dt,$$

in which $\mathbf{Q}_d = \mathbf{Q}_d^\top \succeq 0$ and $\mathbf{R}_d = \mathbf{R}_d^\top \succeq 0$. The optimal control law for the dual is obtained as

$$\boldsymbol{\upsilon}^\star = -\bar{\mathbf{R}}_d^{-1} \bar{\mathbf{C}}_2 \mathbf{S} \boldsymbol{\chi}$$

while the observer is

$$\begin{aligned}\dot{\hat{\mathbf{z}}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22} \hat{\mathbf{z}}_{\mathrm{R,O}} + \bar{\mathbf{B}}_{12} u + \mathbf{S} \bar{\mathbf{C}}_2^\top \bar{\mathbf{R}}_d^{-1} (\mathbf{y} - \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} &= \bar{\mathbf{C}}_2 \hat{\mathbf{z}}_{\mathrm{R,O}}.\end{aligned}$$

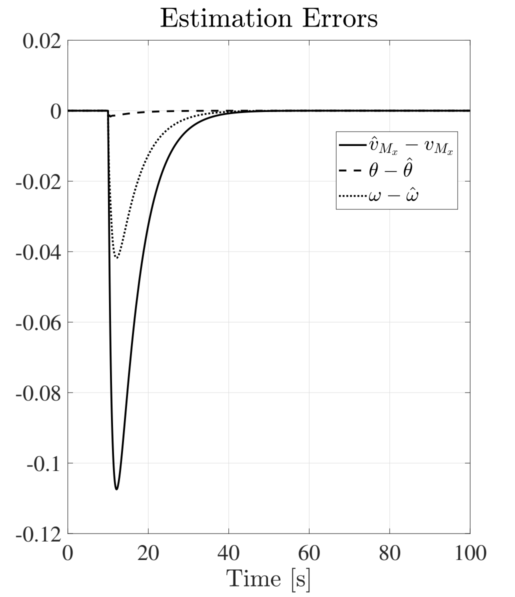
The matrix $\mathbf{S}$ can be found through numerical solvers. For example, exploit the following MATLAB® listing:

```
% Plant

% Costs
Qm = barB22*Qd11*barB22.'% any 3x3 semi-positive definite matrix
Rm = Qd22+Rd% any 2x2 positive definite matrix

% Solve the ARE
[K,S,e] = lqr(barA22.',barC2.',Qm,Rm,zeros(3,2)); % WARNING! The
    control law is u = -Kc*x. S is the positive definite solution of
    the ARE.
% Use duality results
Ko = K.';
```

With $\mathbf{Q}_{d_{11}} = 1$, $\mathbf{Q}_{d_{22}} = I$ and $\mathbf{R}_d = \mathbf{0}$ the observer demonstrates the performance depicted in the following figure, under the same simulation conditions of Example 5.4. In more detail, in agreement with the results of Examples 4.10 and 4.12 and thanks to the definition of the plant in Example 1.1, $\mathbf{z}_{\mathrm{R,O}} := \mathrm{col}(v_{M_x}, \theta, \omega)$.



It is worth noting that the estimation errors are not vanishing and this is due to the model mismatch. Indeed, since the model is highly nonlinear, the linearisation matrices $\bar{\mathbf{A}}_{22}$ and $\bar{\mathbf{B}}_{22}$ do not accurately approximate the plant.

5.4 ADAS Design

This section aims to set up optimal control and observation problems. Then, concerning the control architectures identified in Section 4.7, matrices $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{I}$, and $\mathbf{K}_\mathrm{O}$ are designed numerically. Lastly, the performance of the control systems is investigated in simulations.

5.4.1 Active Suspensions

As described in Section 2, the AS can be applied to control the cabin's vertical movement (single corner) and rotation (half car). In the following, these two case studies are investigated.

Single-Corner Model

Section 4.7.1 identifies the control architecture (4.54) in which the matrices $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{O}$, and the scalar k_I have to be designed.

Start with the design of $\mathbf{K}_\mathrm{S}$ and k_I and let $\mathbf{A}$, $\mathbf{B}_1$, and $\mathbf{C}_e$ be given by Eq. (2.11). Then, define $\mathbf{x}_e = \mathrm{col}(\mathbf{x}, \eta)$,

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B}_e = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

Appendix A

Basic Mathematics

Contents

This appendix recalls the base notions of matrices, vectors, linear spaces, and the operations between matrices and vectors exploited in this book. These notions are provided beside MATLAB® listings.

Before moving to those definitions, it is worth recalling that a **set** is a collection of objects, which are called *elements* of the set. In this book, sets are usually denoted with calligraphic letters, *e.g.*, $\mathcal{X}$. However, we refer to the sets of real, natural, and complex numbers with $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$.

We use the following mathematical symbols throughout this chapter:

- The colon symbol $:$ and the arrow $\rightarrow$ are read as *such that* and *goes to*;
- The symbol $:=$ reads as *defined as*;
- We use $\exists$ for *exists*;
- We read the symbols $>$, $\geq$, $<$, and $\leq$ as *greater than*, *greater than or equal to*, *less than*, and *less than or equal to*, respectively;

- We read $\implies$, $\not\Longrightarrow$, and $\iff$ as *implies*, *does not imply*, and *is equivalent to*;
- The symbol $\in$ reads as *belongs to*;
- We use $\succ$ and $\succeq$ to say *strictly successive to* and *non-strictly successive to.*

A.1 Matrices and Vectors

Let $a_{ij} \in \mathbb{C}$, with $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $n, m \in \mathbb{N}$, then

$$\mathbf{A} := \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix} \tag{A.1}$$

is called *matrix*. A matrix of m rows and n columns, whose entries belong to $\mathbb{C}$, represents an element of the space $\mathbb{C}^{n\times m}$, *i.e.*, $\mathbf{A} \in \mathbb{C}^{n\times m}$. Matrices are called *square* if $n = m$.

```
% Declaration and assignment of a m-by-n matrix, A, of
    random real numbers
m = randi(10); % number of rows as a random integert between
     1 and 10
n = randi(10); % number of columns as a random integert
    between 1 and 10
A = rand(m,n); % Matrix declaration and assignment
```

When $m = 1$, a matrix is said to be a *vector* (or column vector), and it is denoted with $\mathbf{x} \in \mathbb{C}^n$.

Commonly, we graphically represent vectors as arrows in the $\mathbb{R}^n$ space. For example, let $\mathbf{e}_1 := \mathrm{col}(1, 0, 0)$, $\mathbf{e}_2 := \mathrm{col}(0, 1, 0)$, and $\mathbf{e}_3 := \mathrm{col}(0, 0, 1)$ be three orthonormal vectors belonging to $\mathbb{R}^3$. Moreover, let $x, y, z \in \mathbb{R}$ be three real numbers collected to form the vector $\mathbf{v} := \mathrm{col}(x, y, z)$. Then, we depict $\mathbf{v}$ in Figure A.1 by highlighting that x, y, and z represent the projections of $\mathbf{v}$ on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively.

Let $\mathbf{A}$ be a matrix with elements a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $n, m \in \mathbb{N}$. Then, its *transpose*, namely $\mathbf{A}^\top$, is defined as

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \ldots & a_{nm} \end{bmatrix} \tag{A.2}$$

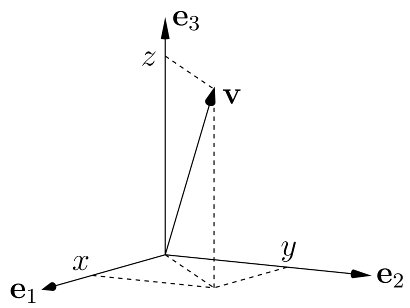


Figure A.1: Graphical representation of a vector

where $\left(\mathbf{A}^\top\right)^\top = \mathbf{A}$.

```
% Transpose of a matrix
B = A.'; % returns the nonconjugate transpose of A, that is,
    interchanges the row and column index for each element.
    If A contains complex elements, then A.' does not affect
    the sign of the imaginary parts.
```

A matrix $\mathbf{A}$ is said to be *skew symmetric* if $\mathbf{A}^\top = -\mathbf{A}$.

Furthermore, a square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *diagonal* if $a_{ij} = 0$ for any $i, j = 1, \dots, n$ such that $i \neq j$, *i.e.*, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}. \tag{A.3}$$

```
% Diagonal matrix
v = rand(1,m); % creates a vector of m-columns of real
    random numbers
D = diag(v);  % returns a square diagonal matrix with the
    elements of vector v on the main diagonal
```

A diagonal matrix with $a_{ii} = 1$ for all $i = 1, \dots, n$ is called *identity* and is denoted with $\mathbf{I}$. *Null matrices* and *null vectors*, both denoted wih $\mathbf{0}$, are defined as matrices and vectors whose elements are all null.

```
% Null matrix
X = zeros(m,n); % returns an n-by-m matrix of zeros
```

A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *upper (lower) triangular* if the elements below (above) the principal diagonal are all null, *i.e.*,

$$\mathbf{A}_{\text{up}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mn} \end{bmatrix} \qquad \mathbf{A}_{\text{low}} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

A.2 Matrix Sum and Product

Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$, with $m, n \in \mathbb{N}$, be matrices whose elements are a_{ij} and b_{ij} respectively. Then, the sum $\mathbf{A}+\mathbf{B}$ is defined as

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}.$$

```
% Sum of matrices
C = A+B; % is the matrix sum of A and B. If A and B are an n
    -by-m matrices, then C is an n-by-m matrix
```

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, with $m, n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$, then the product $\alpha\mathbf{A}$ is defined as

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}.$$

```
% Multiplication by a scalar
alpha = rand(1); % generates a random real number
B = alpha*A; % is the matrix product of alpha and A.
```

The operator "$\cdot$" defines the dot matrix product, representative of the well-known rule *row-by-column*. In details, let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times p}$, and $\mathbf{C} \in \mathbb{C}^{m\times p}$, with $m, n, p \in \mathbb{N}$. Let a_{ik}, b_{kj}, and c_{ij} be the elements of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, with $i = 1, \ldots, m$, $k = 1, \ldots, n$, and $j = 1, \ldots, p$. Then, $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ if

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{A.4}$$

Remark A.1. It is worth noting that the matrix product is well-posed if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

```
% Matrices multiplications
C = A*B; % is the matrix product of A and B. If A is an m-by
    -n and B is a n-by-p matrix, then C is an m-by-p matrix
```

Let $\alpha \in \mathbb{C}$ and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{m \times n}$, then the following equalities are true:

$$\begin{gathered}
\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} \\
(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\
\alpha\,(\mathbf{A} \cdot \mathbf{B}) = (\alpha \mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\alpha \mathbf{B}) \\
\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \\
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\
(\mathbf{B} + \mathbf{C}) \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{A} \\
(\mathbf{A} \cdot \mathbf{B})^{\top} = \mathbf{B}^{\top} \cdot \mathbf{A}^{\top} \\
(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top}.
\end{gathered}$$

Anyway, it is worth noting that in general

- $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$;
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} \;\not\Longrightarrow\; \mathbf{B} = \mathbf{C}$;
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{0} \;\not\Longrightarrow\; \mathbf{B} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$.

The identity matrix represents the neutral element for the dot product, *i.e.*, it is such that

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{A} \qquad \mathbf{I} \cdot \mathbf{A} = \mathbf{A}.$$

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, then its k-th power is defined as

$$\mathbf{A}^k = \prod_{i=1}^{k} \mathbf{A} = \mathbf{A} \cdot \mathbf{A} \cdot \ldots \cdot \mathbf{A} \qquad k \text{ times}.$$

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$. Let a_{ij} be the entries of $\mathbf{A}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, then the *Kroneker product* is denoted by the operator $\otimes$ and is such that

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}.$$

```
% Kroneker product
K = kron(A,B) % returns the Kroneker tensor product of matrices A and B. If A is an m-by-n matrix and B is a p-by-q matrix, then kron(A,B) is an m*p-by-n*q matrix formed by taking all possible products between the elements of A and the matrix B.
```

Let $\mathbf{A}_i, \mathbf{A} \in \mathbb{C}^{m \times n}$, with $i = 1, \dots, p$. Then $\mathbf{A}$ is said to be a *linear combination* of $\{\mathbf{A}_1, \dots, \mathbf{A}_p\}$ if it can be obtained through

$$\mathbf{A} = \alpha_1 \mathbf{A}_1 + \cdots + \alpha_p \mathbf{A}_p$$

for some $\alpha_i \in \mathbb{C}$. The matrices in the set $\{\mathbf{A}_1, \dots, \mathbf{A}_p\}$ are said to be *linearly independent* if neither of them can be written as linear combination of the remainder. Equivalently, $\mathbf{A}_1, \dots, \mathbf{A}_p$ are linearly independent if and only if

$$\alpha_1 \mathbf{A}_1 + \cdots + \alpha_p \mathbf{A}_p = \mathbf{0}$$

implies $\alpha_i = 0$ for any $i = 1, \dots, p$.

A.3 Vector Products

Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^n$ be two vectors, then we define the *inner product* $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \in \mathbb{C}$ with

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle := \mathbf{u}_1^\top \cdot \mathbf{u}_2 = \sum_k u_{1k} u_{2k}.$$

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, then the inner product between vectors satisfies the following properties:

- distribution with respect to the sum: $\langle \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 + \mathbf{u}_2 \rangle$;
- distribution with respect to the scalar product: $\alpha \langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \alpha \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \alpha \mathbf{u}_2 \rangle$;
- commutation: $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_2, \mathbf{u}_1 \rangle$;
- positive definition: $\langle \mathbf{u}_1, \mathbf{u}_1 \rangle > 0$, $\forall\, \mathbf{u}_1 \neq \mathbf{0}$.

```
% Inner product
u3 = dot(u1,u2); %returns the scalar dot product of the
    vectors u1 and u2 which must have the same length
```

A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said *(semi-)positive definite*, and is denoted as $A(\succeq) \succ 0$, if for any vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, the scalar $\langle \mathbf{x}, \mathbf{A}\cdot\mathbf{x}\rangle\,(\geq) > 0$.

Let $\mathbf{u} := \mathrm{col}(u_1, u_2, u_3) \in \mathbb{R}^3$ and define the function $S : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ with

$$\mathbf{S}(\mathbf{u}) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

Let $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{R}^3$, then the *outer product* is defined as

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{S}(\mathbf{u}_1)\cdot\mathbf{u}_2$$

The cross product is

- non–commutative: $\mathbf{u}_1 \times \mathbf{u}_2 = -\left(\mathbf{u}_2 \times \mathbf{u}_1\right)$;
- distributive with respect to the sum: $\mathbf{u}_1 \times (\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{u}_1 \times \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_3$;
- distributive with respect to the product by a scalar: $\alpha\,(\mathbf{u}_1 \times \mathbf{u}_2) = (\alpha\mathbf{u}_1) \times \mathbf{u}_2 = \mathbf{u}_1 \times (\alpha\mathbf{u}_2)$;
- non–associative: $\mathbf{u}_1 \times (\mathbf{u}_2 \times \mathbf{u}_3) \neq (\mathbf{u}_1 \times \mathbf{u}_2) \times \mathbf{u}_3$.

```
% cross product of vectors
u3 = cross(u1,u2); % returns the cross product of the
    vectors u1 and u2 which must have a length of 3.
```

A.4 Matrix Inverse

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, then the square matrix obtained from $\mathbf{A}$ by the cancellation of the i-th row and the j-th column is denoted with $\mathbf{A}^{(ij)} \in \mathbb{R}^{n-1\times n-1}$. It is worth observing that the i-th row and the j-th column share one common element, which is a_{ij}. The determinant $\det(\mathbf{A}) : \mathbb{R}^{n\times n} \mapsto \mathbb{R}$ is defined as

$$\det(\mathbf{A}) := \sum_j a_{ij}\, C_{ij}\, (-1)^{i+j}$$

where C_{ij} is the *cofactor* associated to a_{ij}. Iteratively, the cofactors C_{ij} are defined as

$$C_{ij} = \det\left(\mathbf{A}^{(ij)}\right).$$

The following properties hold true:

- $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \det(\mathbf{B})$;
- $\det(\mathbf{A}) = 0$ if the matrix $\mathbf{A}$ has a row (column) which is a linear combination of the remaining rows (columns);
- $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$ if the matrix is either triangular or diagonal.

If $\det(\mathbf{A}) = 0$ the matrix $\mathbf{A}$ is said to be *singular*.

```
% Determinant of a matrix
d = det(A); % returns the determinant of square matrix A.
```

The **rank** of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is defined as a function $\rho : \mathbb{R}^{n \times n} \mapsto \mathbb{N}$ and corresponds to the maximum integer, namely p, for which there exists at least one sub–matrix of dimensions p (built selecting p rows and p columns of $\mathbf{A}$) whose determinant is not null. The following properties are valid:

- $\rho(\mathbf{A} \cdot \mathbf{B}) \leq \rho(\mathbf{A}) \cdot \rho(\mathbf{B})$;
- $\rho\left(\mathbf{A}^\top\right) = \rho(\mathbf{A})$.

Furthermore, if $\rho(\mathbf{A}) = n$ the matrix $\mathbf{A}$ is said to be *full rank* or, equivalently, *invertible*.

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, the *adjoint matrix* is defined as

$$\mathrm{Adj}(\mathbf{A}) = \begin{bmatrix} C_{11}(-1)^{1+1} & C_{12}(-1)^{1+2} & \dots & C_{1n}(-1)^{1+n} \\ C_{21}(-1)^{2+1} & C_{22}(-1)^{2+2} & \dots & C_{2n}(-1)^{1+n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(-1)^{n+1} & C_{n2}(-1)^{n+2} & \dots & C_{nn}(-1)^{n+n} \end{bmatrix}^\top .$$

The *matrix inversion* is an operator which applies to square matrices. In particular, let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be invertible. Then, its inverse is defined as

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \mathrm{Adj}(\mathbf{A})$$

with

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}.$$

Finally, let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, then $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$.

```
        % Inverse of a matrix
        Y = inv(X); % computes the inverse of square matrix
          X.
```

A.5 Matrix Pseudo-Inverses (Moore-Penrose)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $\rho(\mathbf{A}) = \min\{m, n\}$, then the *left* and *right pseudo-inverse of* $\mathbf{A}$ are defined as follows:

$$\mathbf{A}^+ = \begin{cases} \left(\mathbf{A}^\top \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^\top & \rho(\mathbf{A}) = n \quad \text{left pseudo-inverse} \\ \mathbf{A}^\top \cdot \left(\mathbf{A} \cdot \mathbf{A}^\top\right)^{-1} & \rho(\mathbf{A}) = m \quad \text{right pseudo-inverse} \end{cases}$$

```
% Pseudo-Inverse of a matrix
B = pinv(A); % returns the Moore-Penrose Pseudoinverse of
    matrix A.
```

A.6 Vector Spaces, Norm, and Normed Vector Spaces

Let $\mathbb{V}(\mathbb{C}) := \{\mathbf{x} \in \mathbb{C}^n\}$ with $n \in \mathbb{N}$. Then $\mathbb{V}$ is a *vector space* if the following conditions hold true:

1. for any $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{V}$ then
$$(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3);$$
2. for any $\mathbf{u} \in \mathbb{V}$ the null vector $\mathbf{0}$ is such that
$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u};$$
3. for any $\mathbf{u}_1 \in \mathbb{V}$ there exists $\mathbf{u_2} \in \mathbb{V}$ such that
$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{0} \implies \mathbf{u}_2 = -\mathbf{u}_1;$$
4. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$
$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_2 + \mathbf{u}_1;$$
5. for any $\mathbf{u} \in \mathbb{V}$ and for any $\alpha, \beta \in \mathbb{C}$
$$\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u};$$
6. there exists a neutral element, namely 1, such that
$$1\mathbf{u} = \mathbf{u};$$

7. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$ and for any $\alpha \in \mathbb{C}$

$$\alpha(\mathbf{u}_1 + \mathbf{u}_2) = \alpha\mathbf{u}_1 + \alpha\mathbf{u}_2;$$

8. for any $\mathbf{u} \in \mathbb{V}$ and any $\alpha, \beta \in \mathbb{C}$

$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets. Then, a *function* $\mathbf{f}$ from $\mathcal{X}$ to $\mathcal{Y}$ is a relation that assigns to each element of $\mathcal{X}$ exactly one element of $\mathcal{Y}$. The set $\mathcal{X}$ is called the *domain* of the function and the set $\mathcal{Y}$ is called the *codomain* of the function. We encapsulate these concepts in the symbol $\mathbf{f} : \mathcal{X} \to \mathcal{Y}$. Moreover, $\mathbf{f}(\mathbf{x}) \in \mathcal{Y}$ for any $\mathbf{x} \in \mathcal{X}$.

Let $\mathcal{V}$ be a set. Then, the *norm* is a real-valued function denoted with $\|\cdot\|$ and such that $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathcal{V}$ and $\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$. Let $\mathbf{v} \in \mathbb{R}^n$ be a vector. Then, we define $\|\mathbf{v}\| := \sqrt{\mathbf{v}^\top \cdot \mathbf{v}}$ as the Euclidean norm.

```
% Norm of vectors
n = norm(u); % returns the Euclidean norm of vector u. This
    norm is also called the 2-norm, vector magnitude, or
    Euclidean length.
```

Finally, a normed vector space is a vector space equipped with a norm.

A.7 Linear Functions and Matrices

Let $\mathbb{U}(\mathbb{C})$ and $\mathbb{V}(\mathbb{C})$ be to vector spaces, which are sets by definition, then a function $\mathbf{f} : \mathbb{U} \to \mathbb{V}$ is said to be *linear* if for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{U}$ and for any $\alpha \in \mathbb{C}$ the following relations hold:

$$\mathbf{f}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{f}(\mathbf{u}_1) + \mathbf{f}(\mathbf{u}_2)$$
$$\mathbf{f}(\alpha\mathbf{u}_1) = \alpha\mathbf{f}(\mathbf{u}_1).$$

Let $\mathbf{x} \in \mathbb{U}$ and $\mathbf{y} \in \mathbb{V}$, then a linear transformation between the two vector spaces can be represented through a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, as follows

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}.$$

A.8 Continuity and differentiability

We start with the definition of subset. A set $\mathcal{X}$ is a *subset* of a set $\mathcal{Y}$ if all elements of $\mathcal{X}$ are also elements of $\mathcal{Y}$. We denote this relationship with the symbol $\mathcal{X} \subseteq \mathcal{Y}$

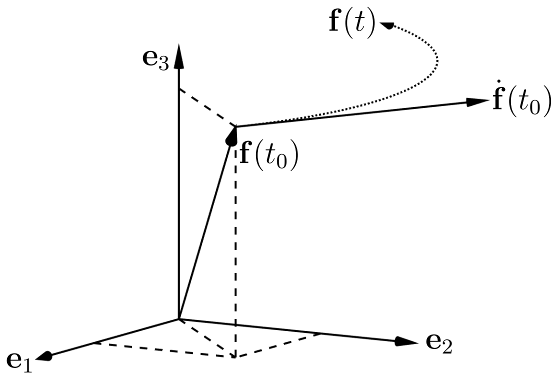


Figure A.2: The dotted line graphically represents the function $\mathbf{f}$. The arrow at the end of the dotted line denotes the time evolution direction. The evaluations $\mathbf{f}(t)$ and $\dot{\mathbf{f}}(t)$, at time $t = t_0$, are represented as vectors. Note that $\dot{\mathbf{f}}(t)$ is tangent to $\mathbf{f}(t)$ for all t.

and we stress that it is possible for $\mathcal{X}$ and $\mathcal{Y}$ to be equal. If they are unequal, then $\mathcal{X}$ is called a *proper subset* of $\mathcal{Y}$, and we write $\mathcal{X} \subset \mathcal{Y}$.

With the definition of subset at hand, we let $\mathbf{f} : \mathcal{X} \to \mathbb{R}^n$ be a function defined on $\mathcal{X} \subseteq \mathbb{R}^m$. Then, we say that $\mathbf{f}$ is *continuous* on $\mathcal{X}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that all $\mathbf{s} \in \mathcal{X}$ satisfying $\|\mathbf{s} - \mathbf{x}\| < \delta$ will also satisfy $\|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{x})\| < \varepsilon$ for all $\mathbf{x} \in \mathcal{X}$.

Moreover, let $\mathbf{f} : \mathcal{X} \to \mathbb{R}^n$ be a function. It is said to be *differentiable* at point $\mathbf{x}_0 \in \mathcal{X}$ if there exists a linear function $\mathbf{J} : \mathcal{X} \to \mathbb{R}^n$ such that

$$\lim_{\mathbf{h} \to 0} \frac{\|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{J}(\mathbf{h})\|}{\|\mathbf{h}\|} = \mathbf{0}.$$

It is worth noting that differentiability implies continuity.

We can now introduce the class $\mathcal{C}^1$, which consists of all differentiable functions whose derivatives are continuous; such functions are called *continuously differentiable.*

Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{f} \in \mathcal{C}^1$ and let $t \in \mathbb{R}$ be the time. Then, we define the *time derivative* of $\mathbf{f}$ as

$$\dot{\mathbf{f}} := \frac{d}{dt}\mathbf{f}(t).$$

Figure A.2 reports the graphical representation of the derivative $\dot{\mathbf{f}}$.

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a function, with $n, m \in \mathbb{N}$. Then, we say that $\mathbf{f}$ is *locally Lipschitz-continuous* if for any finite $\mathcal{X} \subset \mathbb{R}^n$ there exists $L > 0$, which depends on $\mathcal{X}$, such that $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| \leq L\|\mathbf{y} - \mathbf{x}\|$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

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