ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

LTI Systems Stability Analysis

Nicola Mimmo

Department of Electrical, Electronics and Information Engineering
University of Bologna

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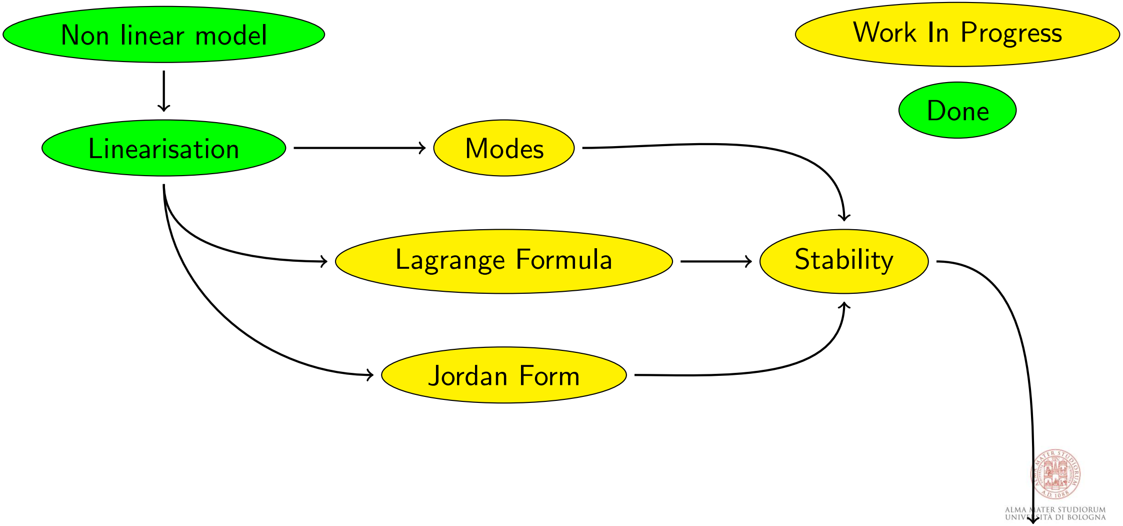
Motivations and Goals

- In this course, we (should) learn how to control LTI systems to achieve two goals:
  - G1) boundedness of signals
  - G2) tracking performance
- Regarding G1), we should learn when and why the evolutions of LTI's state and output are bounded (provided that the inputs are bounded)

Where are we?

Regarding the course contents ...



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Premise

We are too lazy for reporting "$\tilde{}$" and " $\cdot$ " to denote the variational quantities and the "row-by-column" product. Therefore, from this slide on, we write

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{w} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \\
\mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_{e1}\mathbf{u} + \mathbf{D}_{e2}\mathbf{w}
\end{aligned}$$

to denote

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \mathbf{A}\cdot\tilde{\mathbf{x}} + \mathbf{B}_1\cdot\tilde{\mathbf{u}} + \mathbf{B}_2\cdot\tilde{\mathbf{w}} \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \\
\tilde{\mathbf{y}} &= \mathbf{C}\cdot\tilde{\mathbf{x}} + \mathbf{D}_1\cdot\tilde{\mathbf{u}} + \mathbf{D}_2\cdot\tilde{\mathbf{w}} \\
\tilde{\mathbf{e}} &= \mathbf{C}_e\cdot\tilde{\mathbf{x}} + \mathbf{D}_{e1}\cdot\tilde{\mathbf{u}} + \mathbf{D}_{e2}\cdot\tilde{\mathbf{w}}
\end{aligned}$$

Basic Mathematics Notions

Integral linearity

Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function and $c \in \mathbb{R}$ be a constant. Then

$$\int c\,\mathbf{f}(\mathbf{x})d\,\mathbf{x} = c\int \mathbf{f}(\mathbf{x})d\,\mathbf{x}$$

Integral Norm Inequality

Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. Then

$$\left\|\int \mathbf{f}(\mathbf{x})d\,\mathbf{x}\right\| \leq \int \|\mathbf{f}(\mathbf{x})\|\,d\,\mathbf{x}$$

Basic Mathematics Notions

Definition - Induced Matrix Norm

Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ be a matrix. Then, we define

$$\| \mathbf{A} \| := \sup\{\| \mathbf{A}{\cdot}\mathbf{x} \|, \mathbf{x} \in \mathbb{R}^m : \| \mathbf{x} \| = 1\}$$

Matrix-Product Norm Inequality

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of functions of compatible sizes such that $\mathbf{A} \cdot \mathbf{B}$ is well-posed. Then

$$\|\mathbf{A} \cdot \mathbf{B}\| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|$$

Boundedness of Jacobians

Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a function and $\mathbf{f} \in \mathcal{C}^2$. Then, there exists $\overline{\sigma}_{\mathbf{f}} > 0$ such that

$$\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right\| \leq \overline{\sigma}_{\mathbf{f}}$$

Basic Mathematics Notions

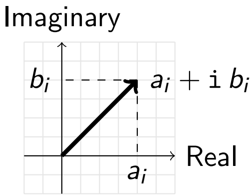
Definition - Imaginary Unit

We define the imaginary unit as the scalar $\mathtt{i} := \sqrt{-1}$

Definition - Imaginary Vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be two vectors, with $n \in \mathbb{N}$. Then, we define a complex vector as $\mathbf{x} := \mathbf{a} + \mathtt{i}\,\mathbf{b}$

Roughly, each element of $\mathbf{x}$ can be conceived as a 2D vector in the Real/Imaginary axes (complex plane). As example, let a_i and b_i be the ith element of $\mathbf{a}$ and $\mathbf{b}$. Then, we can draw the ith element of $\mathbf{x}$ as follows



Basic Mathematics Notions

Definition - Nilpotent Matrix

Let $\mathbf{N} \in \mathbb{R}^{n \times n}$, with $n \in \mathbb{N}$, be a matrix. Then, we say $\mathbf{N}$ is **nilpotent** of order $q \in \mathbb{N}$ if $\mathbf{N}^k = \mathbf{0}$ for all $k \geq q$.

Example

Let $q \in \mathbb{N}$. Then, the matrices

$$\mathbf{N} := \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ & \cdots & 0 & 1 \\ & & \cdots & 0 \end{bmatrix}}_{\mathbf{N} \in \mathbb{R}^{q \times q}}, \quad \mathbf{N} := \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ & \cdots & \mathbf{0} & \mathbf{I} \\ & & \cdots & \mathbf{0} \end{bmatrix}}_{\mathbf{I},\mathbf{0} \in \mathbb{R}^{m \times m},\, \mathbf{N} \in \mathbb{R}^{mq \times mq}}$$

are nilpotent od order q.

Definition - Rotation Matrices

We say a matrix $\mathbf{R}$ is a rotation matrix if $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$.

Example

The matrix

$$\mathbf{R} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a rotation matrix.

Basic Mathematics Notions

Definition - Matrix Exponential

Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix. Then $\exp(\mathbf{A}) := \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$ in which $\mathbf{A}^k := \underbrace{\mathbf{A} \cdot \ldots \cdot \mathbf{A}}_{k \text{ times}}$.

Basic Mathematics Notions

Definition - Matrix Exponential

Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix. Then $\exp(\mathbf{A}) := \sum_{k=0}^{\infty} \dfrac{\mathbf{A}^k}{k!}$ in which $\mathbf{A}^k := \underbrace{\mathbf{A} \cdot \ldots \cdot \mathbf{A}}_{k \text{ times}}$.

Definition - LTI systems' trajectories

Consider the following LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

Then, we denote the solutions to (1), starting from $\mathbf{x}_0$ at time t_0 and evaluated at time t with

$$\chi_{\mathbf{w}}(t, \mathbf{x}(t_0)) := \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}\mathbf{w}(\tau)d\tau \tag{2}$$

Example (1)

The solutions to the scalar LTI system, $\dot{x} = ax + bw$, with $x(0) = x_0$ and $w = \bar{w}$, with $a, b, \bar{w} \in \mathbb{R}$, are

$$\chi_{\bar{w}}(t, x(0)) = e^{at}x(0) + \bar{w}\frac{b}{a}\left(e^{at} - 1\right)$$

Example (2)

Consider the system $\dot{\mathbf{x}} = \mathbf{N}\,\mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^2$ and the nilpotent matrix

$$\mathbf{N} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$\exp(\mathbf{N}t) = \sum_{k=0}^{\infty} \frac{(\mathbf{N}t)^k}{k!} = \mathbf{I} + \mathbf{N}t$$

and

$$\chi_{\mathbf{0}}(t, \mathbf{x}(0)) = \exp(\mathbf{N}t)\,\mathbf{x}(0) = (\mathbf{I} + \mathbf{N}t)\,\mathbf{x}(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}(0)$$

Basic Mathematics Notions

Example (3)

Consider the system $\dot{\mathbf{x}} = \beta \mathbf{R}\, \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^2$, $\beta \in \mathbb{R}$, and the rotation matrix

$$\mathbf{R} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then,

$$\exp(\mathbf{R}t) = \sum_{k=0}^{\infty} \frac{(\mathbf{R}t)^k}{k!} = \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}$$

and

$$\chi_{\mathbf{0}}(t, \mathbf{x}(0)) = \exp(\mathbf{R}t)\, \mathbf{x}(0) = \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} \mathbf{x}(0)$$

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Boundedness

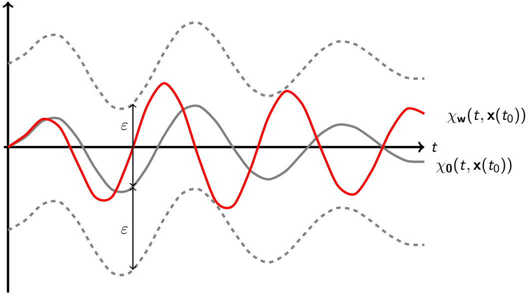
Definition - Bounded-Input-Bounded-State (BIBS) Stability

Consider the following LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w} \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

whose solutions are $\chi_{\mathbf{w}}(t, \mathbf{x}(t_0))$. Then, the system is BIBS-stable if $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\ \mathbf{w}\ :\ \|\mathbf{w}(t)\| \leq \delta,\ \forall t \geq t_0 \implies \|\chi_{\mathbf{w}}(t, \mathbf{x}(t_0)) - \chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| \leq \varepsilon,\ \forall t \geq t_0$$



Boundedness

We are interested in understanding which are the conditions the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w}$ must posses for being BIBS-stable. Therefore, we explicitly use the solutions $\chi_{\mathbf{w}}(t, \mathbf{x}(t_0))$ and $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))$, see eq. (2), and the BIBS-stability criterion

$$\chi_{\mathbf{w}}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}\mathbf{w}(\tau)d\tau$$

$$\chi_{\mathbf{0}}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0)$$

from which

$$\|\chi_{\mathbf{w}}(t, \mathbf{x}(t_0)) - \chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| = \left\| \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}\mathbf{w}(\tau)d\tau \right\|$$

The idea is to upper bound the latter for conservativeness, and later force this upper bound to match the BIBS-stability criterion (see next slide)

Boundedness

$$\|\chi_{\mathbf{w}}(t,\mathbf{x}(t_0)) - \chi_{\mathbf{0}}(t,\mathbf{x}(t_0))\| = \left\| \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{w}(\tau)d\tau \right\|$$

$$\text{1) integral norm inequality} \implies \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{w}(\tau)\|\, d\tau$$

$$\text{2) matrix product norm inequality} \implies \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\|\, \|\mathbf{B}\|\, \|\mathbf{w}(\tau)\|\, d\tau$$

$$\text{3) } \|\,\mathbf{w}(t)\| \leq \delta \;\forall\, t \geq t_0 \implies \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\|\, \|\mathbf{B}\|\, \delta d\tau$$

$$\text{4) Jacob. bound. } \exists\, \overline{\sigma}_{\mathbf{B}} > 0 : \|\,\mathbf{B}\,\| \leq \overline{\sigma}_{\mathbf{B}} \implies \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\|\, \overline{\sigma}_{\mathbf{B}} \delta d\tau$$

$$\text{5) integral linearity} \implies \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\|\, d\tau \overline{\sigma}_{\mathbf{B}} \delta \leq \varepsilon$$

Boundedness

From the latter inequality

$$\delta \leq \frac{\varepsilon}{\displaystyle\int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| \, d\tau \overline{\sigma}_{\mathbf{B}}}$$

According with the BIBS-Stability " ... *system is BIBS-stable if* $\forall\, \varepsilon > 0$ *there exists* $\delta > 0$ *such that ...*" Then, we enforce $\delta > 0$

$$0 < \delta \leq \frac{\varepsilon}{\displaystyle\int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| \, d\tau \overline{\sigma}_{\mathbf{B}}}$$

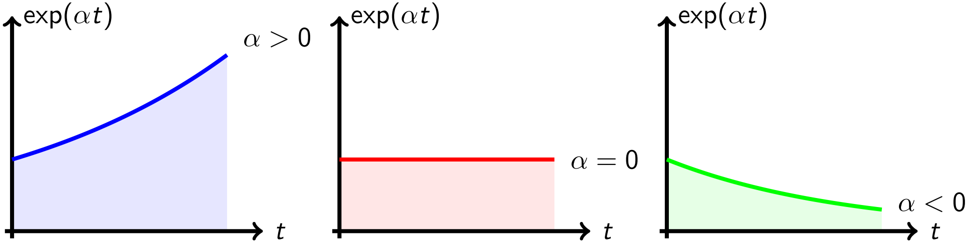
from which

$$\int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| \, d\tau$$

must be bounded for the system being BIBS-Stable.

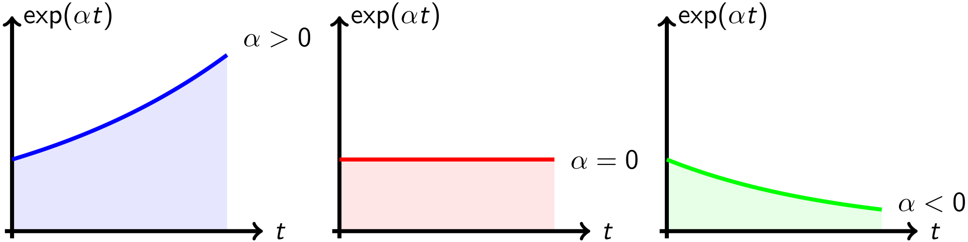
Boundedness

- Roughly, $\int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\|\, d\tau$ bounded means that the area below $\|\exp(\mathbf{A}(t-t_0))\|$ is bounded. Moreover, this area is bounded for all $t \geq t_0$ if and only if $\lim_{t\to\infty} \|\exp(\mathbf{A}(t-t_0))\| \to 0$



Boundedness

- Roughly, $\int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| \, d\tau$ bounded means that the area below $\|\exp(\mathbf{A}(t-t_0))\|$ is bounded. Moreover, this area is bounded for all $t \geq t_0$ if and only if $\lim_{t\to\infty} \|\exp(\mathbf{A}(t-t_0))\| \to 0$



- But, we saw $\boldsymbol{\chi_0}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t-t_0))\,\mathbf{x}(t_0)$
- Therefore, $\lim_{t\to\infty} \|\exp(\mathbf{A}(t-t_0))\| \to 0$ implies $\lim_{t\to\infty} \|\boldsymbol{\chi_0}(t, \mathbf{x}(t_0))\| \to 0$

Boundedness

Take-home Message

For LTI systems $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{w}$, with bounded matrices $\mathbf{B}$, the BIBS-stability properties are embedded into $\mathbf{A}$!

Therefore, $\mathbf{A}$ deserves to be studied in detail. In the following, we introduce

1. the concept of eigenvalues of (square) matrices
2. a result linking an eigenvalues' property to $\lim_{t\to\infty} \|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| \to 0$

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Definition - (i, j)-Minor

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix. Then, we define $\mathbf{A}^{(i,j)}$ as the sub-matrix of $\mathbf{A}$ obtained by cancelling the ith row and the jth column. Consequently $\mathbf{A}^{(i,j)} \in \mathbb{R}^{(n-1)\times(n-1)}$.

Definition - Determinant of a square Matrix

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix, whose (i, j)th element is $a_{i,j}$. Then, the determinant $\det(\mathbf{A}) : \mathbb{R}^{n \times n} \to \mathbb{R}$ is defined as

$$\det(\mathbf{A}) := \sum_j a_{i,j}\, C_{i,j}\, (-1)^{i+j}$$

where $C_{i,j}$ is the **cofactor** associated to $a_{i,j}$. Iteratively, the cofactors $C_{i,j}$ are defined as

$$C_{i,j} := \det\left(\mathbf{A}^{(i,j)}\right) \text{ with } \det(a_{i,j}) = a_{i,j}$$

Eigenvalues

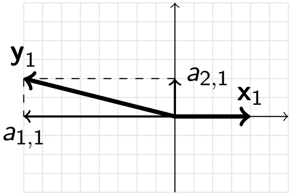
Linear Transformations - Intuition

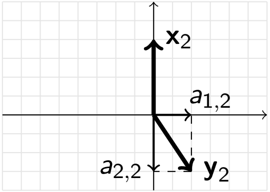
Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a matrix and $\mathbf{x} \in \mathbb{R}^n$ be a vector. Intuitively, $\mathbf{A}$ transforms $\mathbf{A} \cdot \mathbf{x}$ into another vector. Then, $\mathbf{y} := \mathbf{A} \cdot \mathbf{x}$ represents a vector in $\mathbb{R}^n$ obtained by rotating and scaling $\mathbf{x}$ via $\mathbf{A}$.

Example

Let $\mathbf{x}_1 := \mathrm{col}(1, 0)$ and $\mathbf{x}_2 := \mathrm{col}(0, 1)$, and take $a_{i,j} \in \mathbb{R}$ being the (i, j)th element of $\mathbf{A} \in \mathbb{R}^{2\times 2}$. Then, define $\mathbf{y}_i := \mathbf{A} \cdot \mathbf{x}_i$, with $i = 1, 2$. In detail

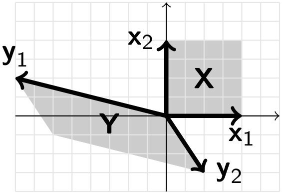
$$\mathbf{y}_1 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}, \qquad \mathbf{y}_2 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$$





Determinant - Intuitive Explanation

Let $\mathbf{X} := \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$. Therefore, we can conceive $\mathbf{Y} := \mathbf{A}\,\mathbf{X}$ as a **space deformation**. Indeed, $\mathbf{A}$ transforms the unitary volume $\mathbf{X}$ into the volume $\mathbf{Y}$.



Let us denote with $V_{\mathbf{X}}$ and $V_{\mathbf{Y}}$ the volumes of $\mathbf{X}$ and $\mathbf{Y}$. Then, we have

$$V_{\mathbf{Y}} = |\det(\mathbf{A})| V_{\mathbf{X}}$$

Remark

Therefore, $\det(\mathbf{A}) = 0$ implies that $\mathbf{A}$ squeezes $\mathbf{X}$ into $\mathbf{Y}$ of a lower dimension (e.g., from a 3D volume to a surface, a line, or a point).

Eigenvalues

- On the one hand, we saw that the BIBS-stability is linked to $\lim_{t\to\infty} \|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| = 0$
- On the other hand, we can interpret $\exp(\mathbf{A}(t - t_0))$ as a time-varying matrix transforming $\mathbf{x}(t_0)$ into $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\,\mathbf{x}(t_0)$
- Moreover, $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))$ represents a solution to $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$

Eigenvalues

- On the one hand, we saw that the BIBS-stability is linked to $\lim_{t\to\infty} \|\boldsymbol{\chi_0}(t, \mathbf{x}(t_0))\| = 0$
- On the other hand, we can interpret $\exp(\mathbf{A}(t - t_0))$ as a time-varying matrix transforming $\mathbf{x}(t_0)$ into $\boldsymbol{\chi_0}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\, \mathbf{x}(t_0)$
- Moreover, $\boldsymbol{\chi_0}(t, \mathbf{x}(t_0))$ represents a solution to $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$

Intuitive Definition - Eigenvalues and Eigenvectors

Let $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ be an LTI system, with $\mathbf{x} \in \mathbb{R}^n$. Then, we look for special **non-trivial** vectors $\mathbf{x} \in \mathbb{C}^n$, called **eigenvectors**, and constants $\lambda \in \mathbb{C}$, called **eigenvalues**, such that

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$$

Roughly, the eigenvectors are special vectors which do not change directions over time because $\dot{\mathbf{x}} = \lambda\,\mathbf{x}$. The eigenvectors changes over time only their length. Intuitively, for $\lambda \in \mathbb{R}$, if $\lambda > 0$ the length increases, if $\lambda = 0$ the length remains constant, and if $\lambda < 0$ the length decreases. Does this remember you $\lim_{t\to\infty} \|\boldsymbol{\chi_0}(t, \mathbf{x}(t_0))\| = 0$?

Eigenvalues

Definition - Eigenvalues

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix and $\mathbf{I}$ be the identity matrix of dimension n. Then, the eigenvalues of $\mathbf{A}$ are the roots of

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = 0$$

Intuitively, we defined the eigenvectors and eigenvalues as solutions to $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$ and therefore to

$$(\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

Then, by finding $\lambda : \det(\mathbf{A} - \lambda\,\mathbf{I}) = 0$, we let $\mathbf{A} - \lambda\,\mathbf{I}$ to transform some non-trivial vectors into the null vector. As a consequence, the eigenvectors $\mathbf{x}$ solutions to $(\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{x} = \mathbf{0}$ are non trivial!

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Modes

- We found a connection between the sign of the real part of the eigenvalues of $\mathbf{A}$ and $\lim_{t\to\infty} \|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| 0$, where $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))$ is interpreted as a solution to $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$
- We intuitively concluded that "$\dot{\mathbf{x}}$ takes a direction which decreases the norm of $\mathbf{x}$" if the real part of the eigenvalue λ is negative

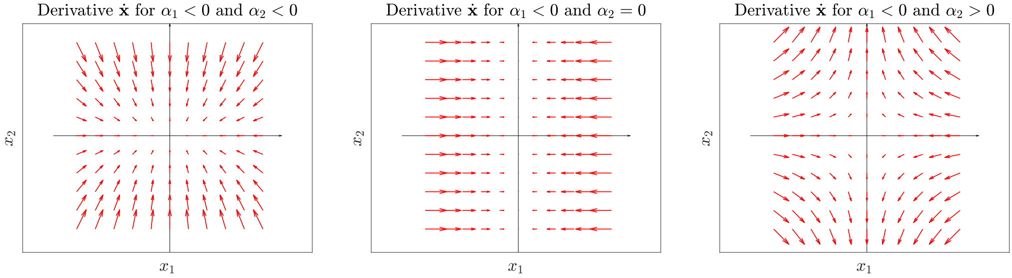
Modes

Example 1: real eigenvalues

Let us take $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and consider the matrix

$$\mathbf{A} := \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$

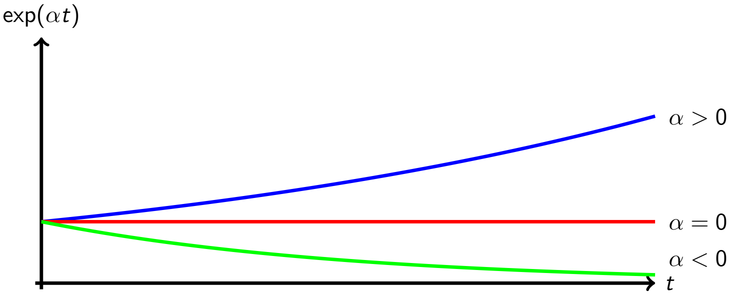
whose eigenvalues are $\lambda_1 = \alpha_1$ and $\lambda_2 = \alpha_2$. Then, $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ is a vector obtained by scaling $\mathbf{x}$ by factors α_1 and α_2 along the first and the second direction respectively.



Modes

Moreover, assume $t_0 = 0$ (to simplify the notation), and compute

$$\chi_{\mathbf{0}}(t, \mathbf{x}(0)) = \exp(\mathbf{A}\, t)\, \mathbf{x}(0) = \exp\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} t\right) \mathbf{x}(0) = \begin{bmatrix} \exp(\alpha_1 t) & 0 \\ 0 & \exp(\alpha_2 t) \end{bmatrix} \mathbf{x}(0)$$



Modes

Example 2: real eigenvalues

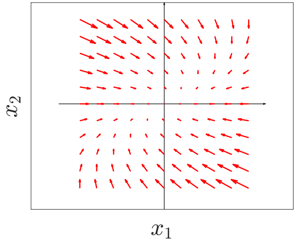
Let us take $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, $\alpha \in \mathbb{R}$ and consider the matrix

$$\mathbf{A} := \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{N}:=} = \alpha\,\mathbf{I} + \mathbf{N}$$

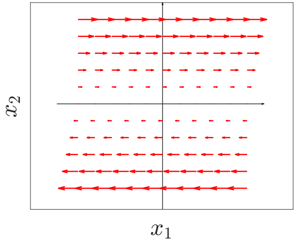
whose eigenvalues are $\lambda_{1,2} = \alpha$. Then, $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} = (\alpha\,\mathbf{I} + \mathbf{N})\,\mathbf{x}$ is a vector obtained by

- scaling $\mathbf{x}$ by the factor α (opposite direction if $\alpha < 0$)
- adding to $\alpha\,\mathbf{x}$ the vector col$(x_2, 0)$, where x_2 represents the second element of $\mathbf{x}$

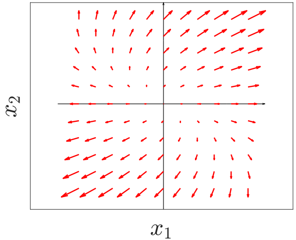
Derivative $\dot{\mathbf{x}}$ for $\alpha < 0$



Derivative $\dot{\mathbf{x}}$ for $\alpha = 0$



Derivative $\dot{\mathbf{x}}$ for $\alpha > 0$



Modes

Moreover, assume $t_0 = 0$ (to simplify the notation), and compute

$$\chi_{\mathbf{0}}(t, \mathbf{x}(0)) = \exp(\mathbf{A}\, t)\, \mathbf{x}(0) = \exp\left(\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} t \right) \mathbf{x}(0) = \exp(\alpha t) \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}(0)$$

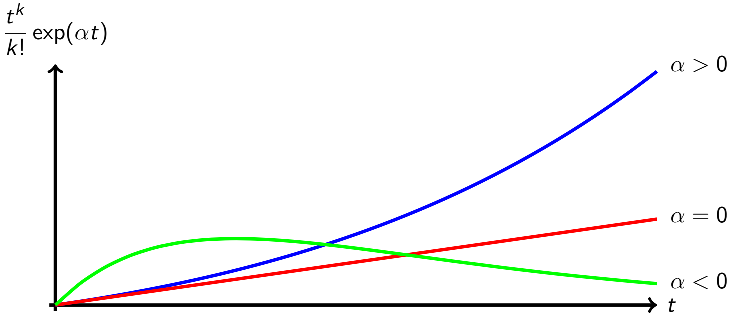


Figure: These plots are valid for any $k \geq 1$

Modes

Example 3: complex eigenvalues

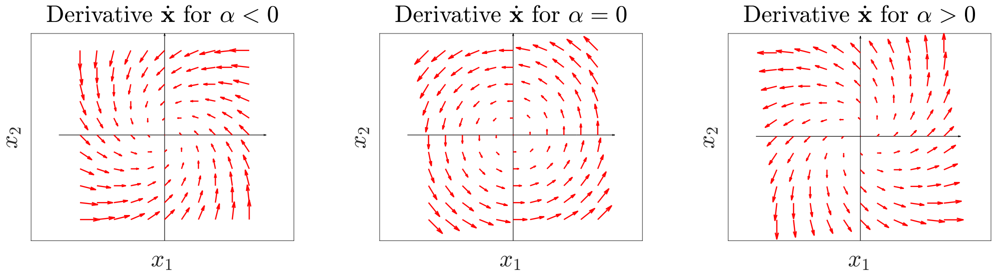
Let us take $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, $\alpha \in \mathbb{R}$, $\beta \geq 0$, and consider the following matrix

$$\mathbf{A} := \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} + \beta \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{R}:=} = \alpha\,\mathbf{I} + \beta\mathbf{R}$$

whose eigenvalues are $\lambda_1 = \alpha + \mathrm{i}\,\beta$ and $\lambda_2 = \alpha - \mathrm{i}\,\beta$.
Now, $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} = (\alpha\,\mathbf{I} + \beta\mathbf{R})\,\mathbf{x}$ where

- $\mathbf{x}' := \mathbf{R}\,\mathbf{x}$ is otrhogonal to $\mathbf{x}$ ($\mathbf{R}$ rotates $\mathbf{x}$ by $\pi/2$)
- $\beta\,\mathbf{x}'$ scales $\mathbf{x}'$ by a factor β
- $\alpha\,\mathbf{x}$ scales $\mathbf{x}$ (opposite direction for $\alpha < 0$) by a factor α

Modes



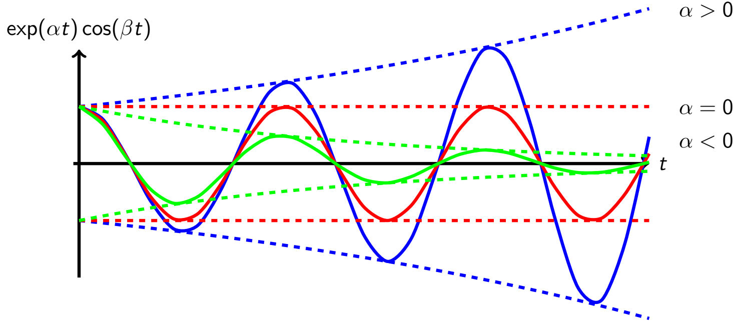
Take-home message

We have complex eigenvalues (and eigenvectors) when $\mathbf{A}$ induces rigid rotations.

Modes

Moreover, assume $t_0 = 0$ (to simplify the notation), and compute

$$\chi_{\mathbf{0}}(t, \mathbf{x}(0)) = \exp(\mathbf{A}\, t)\, \mathbf{x}(0) = \exp\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} t\right) \mathbf{x}(0) = \exp(\alpha t) \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} \mathbf{x}(0)$$



Modes

- $\texttt{Real}(\lambda) < 0 \implies \lim_{t\to\infty} \exp(\mathbf{A}\, t) = \mathbf{0}$ for the previous examples
- In general, $\mathbf{A}$ is not in any of the previous forms
- Fortunately, there exist linear functions transforming $\mathbf{A}$ in the so-called **Jordan Canonical Form**, *i.e.*, a combination of the previous forms

$$\begin{bmatrix} \alpha_1 & 0 & \cdots \\ 0 & \alpha_2 & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \alpha & 1 & 0 & \dots \\ 0 & \alpha & 1 & \ddots \\ 0 & 0 & \alpha & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \alpha\,\mathbf{I} + \beta\mathbf{R} & \mathbf{I} & \\ 0 & \alpha\,\mathbf{I} + \beta\mathbf{R} & \ddots \\ & \ddots & \ddots \end{bmatrix}$$

Definition - Algebraic Multiplicty

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, with $n \in \; N$, be a matrix. Then, direct computations show that there exist $p \leq n$, $a_i \in \mathbb{N}$, and $\lambda_i \in \mathbb{C}^n$, for $i = 1, \dots, p$, such that

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{a_i} = 0, \qquad \sum_{i=1}^{p} a_i = n$$

where

- p denotes the number of eigenvalues
- λ_i represents the ith eigenvalue
- a_i is called algebraic multiplicity of λ_i

Definition - Kernel of a Matrix

Let $\mathbf{A} \in \mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}$. Then, we define $\ker(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^m \ : \ \mathbf{0} = \mathbf{A}\mathbf{x}\}$

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Definition - Geometric Multiplicty

Let $\lambda_i \in \mathbb{C}$ be an eigenvalue of $\mathbf{A} \in \mathbb{R}^{n\times n}$, with $n \in \mathbb{N}$, and let $a_i \in \mathbb{N}$ be the algebraic multiplicity of λ_i. Then, the geometric multiplicity associated to λ_i is $g_i \in \mathbb{N}$, with $1 \leq g_i \leq a_i$. It represents the dimension of $\ker(\mathbf{A} - \lambda_i \mathbf{I})$.

Modes

Building the Jordan Canonical Form

Definition - Kernel of a Matrix

Let $\mathbf{A} \in \mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}$. Then, we define $\ker(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^m \ : \ \mathbf{0} = \mathbf{A}\mathbf{x}\}$

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Remark

Since the eigenvectors are the solutions to $(\mathbf{A} - \lambda_i \mathbf{I})\, \mathbf{x} = \mathbf{0}$, the number of eigenvectors associated to the eigenvalue λ_i is equal to g_i. It is worth remembering that $1 \leq g_i \leq a_i$, which implies $\sum_{i=1}^{p} g_i \leq n$. Therefore, the overall number of eigenvectors may be less than the dimension n of the state space.

Definition - Generalised Eigenvectors

The generalised eigenvectors are the solutions to $(\mathbf{A}-\lambda\,\mathbf{I})^q\,\mathbf{x}=\mathbf{0}$, with $(\mathbf{A}-\lambda\,\mathbf{I})^{q-1}\,\mathbf{x}\neq\mathbf{0}$, where $q\in\mathbb{N}$ is called length of the chain of generalised eigenvectors.

Idea

In the case $\sum_{i=1}^{p} g_i < n$, we exploit the generalised eigenvector to find n linearly independent vectors.

Modes

Building the Jordan Canonical Form

Definition - Generalised Eigenvectors

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Idea

In the case $\sum_{i=1}^{p} g_i < n$, we exploit the generalised eigenvector to find n linearly independent vectors.

Generalised Eigenvectors Computation

Denote with $\mathbf{v}_{i,j,1}$, with $j=1,\dots,g_i$, the jth eigenvector associated with λ_i. Then, for all $j=1,\dots,g_i$ and $i=1,\dots,p$ we compute

$$
\begin{aligned}
(\mathbf{A}-\lambda_i\,\mathbf{I})\mathbf{v}_{i,j,1} &= \mathbf{0}\\
(\mathbf{A}-\lambda_i\,\mathbf{I})\mathbf{v}_{i,j,k+1} &= \mathbf{v}_{i,j,k} \quad 1\le k\le q_{i,j}
\end{aligned}
$$

and we stop at $q_{i,j}\in\mathbb{N}$ such that $\mathbf{v}_{i,j,q_{i,j}+1}$ is linearly dependent on $\{\mathbf{v}_{i,j,k}\}_{k=1,\dots,q_{i,j}}$

Modes

Building the Jordan Canonical Form

The eigenvalues and generalised eigenvectors are organised as

$$\begin{cases} \lambda_1 \\ \vdots \\ \lambda_i & \begin{cases} \mathbf{v}_{i,1,1} & \cdots & \mathbf{v}_{i,1,q_{i,1}} \\ \vdots \\ \mathbf{v}_{i,g_i,1} & \cdots & \mathbf{v}_{i,g_i,q_{i,g_i}} \end{cases} \\ \vdots \\ \lambda_p \end{cases} \qquad \text{such that } \sum_{i=1}^{p} \sum_{j=1}^{g_i} q_{i,j} = n$$

where

- $p \in \mathbb{N}$ denotes the number of eigenvalues
- $a_i \in \mathbb{N}$ represents the algebraic multiplicity of the eigenvalue λ_i
- $g_i \in \mathbb{N}$ is the geometric multiplicity of the eigenvalue λ_i
- $q_{i,j} \in \mathbb{N}$ is called length of the chain of generalised eigenvectors associated to λ_i and $\mathbf{x}_{i,j,1}$
- $\mathbf{v}_{i,j,k} \in \mathbb{C}^n$ are the generalised eigenvectors associated with λ_i and $g_{i,j}$

Modes

Building the Jordan Canonical Form

We use

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_{i,j,1} = \mathbf{0} \iff \mathbf{A}\,\mathbf{v}_{i,j,1} = \lambda_i \mathbf{v}_{i,j,1}$$

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_{i,j,k+1} = \mathbf{v}_{i,j,k} \iff \mathbf{A}\,\mathbf{v}_{i,j,k+1} = \mathbf{v}_{i,j,k} + \lambda_i \mathbf{v}_{i,j,k+1} \qquad 1 \leq k \leq q_{i,j}$$

Modes

Building the Jordan Canonical Form

We use

$$
\begin{aligned}
(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_{i,j,1} = \mathbf{0} \iff \mathbf{A}\,\mathbf{v}_{i,j,1} = \lambda_i \mathbf{v}_{i,j,1} \\
(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_{i,j,k+1} = \mathbf{v}_{i,j,k} \iff \mathbf{A}\,\mathbf{v}_{i,j,k+1} = \mathbf{v}_{i,j,k} + \lambda_i \mathbf{v}_{i,j,k+1} \qquad 1 \leq k \leq q_{i,j}
\end{aligned}
$$

to build

$$
\mathbf{A} \underbrace{\begin{bmatrix} \mathbf{v}_{i,j,1} & \dots & \mathbf{v}_{i,j,q_{i,j}} \end{bmatrix}}_{\mathbf{V}_{i,j} :=} = \underbrace{\begin{bmatrix} \mathbf{v}_{i,j,1} & \dots & \mathbf{v}_{i,j,q_{i,j}} \end{bmatrix}}_{\mathbf{V}_{i,j} =} \underbrace{\left(\lambda_i \mathbf{I}_{q_{i,j}} + \underbrace{\begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}}_{\mathbf{N}_{q_{i,j}} :=} \right)}_{\mathbf{J}_{i,j} :=}
$$

from which $\mathbf{A}\,\mathbf{V}_{i,j} = \mathbf{V}_{i,j}\mathbf{J}_{i,j}$.

Remark $\lambda \in \mathbb{C} \implies \mathbf{V}_{i,j} \in \mathbb{C}^{n \times q_{i,j}}$ and $\mathbf{J}_{i,j} \in \mathbb{C}^{q_{i,j} \times q_{i,j}}$

Modes

Building the Jordan Canonical Form

Moreover, we exploit $\mathbf{A}\,\mathbf{V}_{i,j} = \mathbf{V}_{i,j}\mathbf{J}_{i,j}$ for all $j = 1, \ldots, g_i$ to build

$$\mathbf{A}\underbrace{\begin{bmatrix} \mathbf{V}_{i,1} & \ldots & \mathbf{V}_{i,g_i} \end{bmatrix}}_{\mathbf{V}_i :=} = \underbrace{\begin{bmatrix} \mathbf{V}_{i,1} & \ldots & \mathbf{V}_{i,g_i} \end{bmatrix}}_{\mathbf{V}_i =}\underbrace{\mathtt{blkdiag}(\mathbf{J}_{i,1}, \ldots, \mathbf{J}_{i,g_i})}_{\mathbf{J}_i :=}$$

Modes

Building the Jordan Canonical Form

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and $\mathbf{A}\,\mathbf{V}_i = \mathbf{V}_i\mathbf{J}_i$ for $i = 1, \ldots, p$ to create

$$\mathbf{A}\underbrace{\begin{bmatrix} \mathbf{V}_1 & \ldots & \mathbf{V}_p \end{bmatrix}}_{\mathbf{V} :=} = \underbrace{\begin{bmatrix} \mathbf{V}_1 & \ldots & \mathbf{V}_p \end{bmatrix}}_{\mathbf{V} =}\underbrace{\mathtt{blkdiag}(\mathbf{J}_1, \ldots, \mathbf{J}_p)}_{\mathbf{J} :=}$$

from which

$$\mathbf{A}\,\mathbf{V} = \mathbf{V}\mathbf{J} \iff \mathbf{J} = \mathbf{V}^{-1}\mathbf{A}\,\mathbf{V}$$

where $\mathbf{J}$ takes the name of Jordan matrix

Remark if $\lambda_i \in \mathbb{C}$ for some $i \in \{1, \ldots, p\}$, then $\mathbf{V} \in \mathbb{C}^{n\times n}$ and $\mathbf{J} \in \mathbb{C}^{n\times n}$

Modes

Building the Jordan Canonical Form

Complex Conjugate - Existence

Assume $\lambda_i \in \mathbb{C}$, for some $i \in \{1, \dots, p\}$, and let $\alpha_i, \beta_i \in \mathbb{R}$ such that $\lambda_i = \alpha_i + \mathtt{i}\,\beta_i$. Then,

$$\exists\ i^\star \in \{1, \dots, p\} : \lambda_{i^\star} = \lambda_i^\star := \alpha_i - \mathtt{i}\,\beta_i \ \ \forall\, i \in \{1, \dots, p\} : \lambda_i \in \mathbb{C}$$

Complex Conjugate - Existence

Assume $\lambda_i \in \mathbb{C}$, for some $i \in \{1, \dots, p\}$, and let $\alpha_i, \beta_i \in \mathbb{R}$ such that $\lambda_i = \alpha_i + \mathrm{i}\, \beta_i$. Then,

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Therefore

- Let $\mathbf{V}_i \in \mathbb{C}^{n \times \sum_{j=1}^{g_i} q_{i,j}}$ be associated with $\lambda_i \in \mathbb{C}$
- Use $\mathbf{V}_i = \left[\begin{array}{ccc} \mathbf{V}_{i,1} & \dots & \mathbf{V}_{i,g_i} \end{array}\right]$ and $\mathbf{V}_{i,j} = \left[\begin{array}{ccc} \mathbf{v}_{i,j,1} & \dots & \mathbf{v}_{i,j,q_{i,j}} \end{array}\right]$, $j = 1, \dots, g_i$
- Let $\mathbf{a}_{i,j,k}$, $\mathbf{b}_{i,j,k} \in \mathbb{R}^n$ such that for $j = 1, \dots, g_i$ and $k = 1, \dots, q_{i,j}$ we have $\mathbf{v}_{i,j,k} = \mathbf{a}_{i,j,k} + \mathrm{i}\, \mathbf{b}_{i,j,k}$
- Let $\mathbf{V}_{i^\star} \in \mathbb{C}^{n \times \sum_{j=1}^{g_{i^\star}} q_{i^\star,j}}$ be associated with $\lambda_{i^\star} \in \mathbb{C}$. Then we have

$$\mathbf{V}_{i^\star} = \left[\begin{array}{ccc} \mathbf{V}_{i^\star,1} & \dots & \mathbf{V}_{i^\star,g_{i^\star}} \end{array}\right]$$
$$\mathbf{V}_{i^\star,j} = \left[\begin{array}{ccc} \mathbf{v}_{i^\star,j,1} & \dots & \mathbf{v}_{i^\star,j,q_{i^\star,j}} \end{array}\right]$$
$$\mathbf{v}_{i^\star,j,k} = \mathbf{v}_{i,j,k}^\star := \mathbf{a}_{i,j,k} - \mathrm{i}\, \mathbf{b}_{i,j,k}$$

Modes

Building the Jordan Canonical Form

If $\lambda_i \in \mathbb{C}$ we can rearrange $\left[\begin{array}{cc} \mathbf{V}_i & \mathbf{V}_{i^\star} \end{array}\right]$ and $\texttt{blkdiag}(\mathbf{J}_i, \mathbf{J}_{i^\star})$ to equivalent real forms $\bar{\mathbf{V}}_i$ and $\bar{\mathbf{J}}_i$.

Modes

Building the Jordan Canonical Form

If $\lambda_i \in \mathbb{C}$ we can rearrange $\begin{bmatrix} \mathbf{V}_i & \mathbf{V}_{i^\star} \end{bmatrix}$ and $\texttt{blkdiag}(\mathbf{J}_i, \mathbf{J}_{i^\star})$ to equivalent real forms $\bar{\mathbf{V}}_i$ and $\bar{\mathbf{J}}_i$. In detail, let

$$
\begin{aligned}
\bar{\mathbf{V}}_{i,j} &:= \begin{bmatrix} \mathbf{a}_{i,j,1} & \mathbf{b}_{i,j,1} & \mathbf{a}_{i,j,2} & \mathbf{b}_{i,j,2} & \dots & \mathbf{a}_{i,j,q_{i,j}} & \mathbf{b}_{i,j,q_{i,j}} \end{bmatrix} \\
\bar{\mathbf{J}}_{i,j} &:= \begin{bmatrix} \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \mathbf{I} & \dots & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} \end{bmatrix} \\
\bar{\mathbf{V}}_i &:= \begin{bmatrix} \bar{\mathbf{V}}_{i,1} & \dots & \bar{\mathbf{V}}_{i,g_i} \end{bmatrix} \\
\bar{\mathbf{J}}_i &:= \texttt{blkdiag}(\bar{\mathbf{J}}_{i,1}, \dots, \bar{\mathbf{J}}_{i,g_i})
\end{aligned}
$$

Modes

Building the Jordan Canonical Form

If $\lambda_i \in \mathbb{C}$ we can rearrange $\begin{bmatrix} \mathbf{V}_i & \mathbf{V}_{i^\star} \end{bmatrix}$ and $\texttt{blkdiag}(\mathbf{J}_i, \mathbf{J}_{i^\star})$ to equivalent real forms $\bar{\mathbf{V}}_i$ and $\bar{\mathbf{J}}_i$. In detail, let

$$\bar{\mathbf{V}}_{i,j} := \begin{bmatrix} \mathbf{a}_{i,j,1} & \mathbf{b}_{i,j,1} & \mathbf{a}_{i,j,2} & \mathbf{b}_{i,j,2} & \dots & \mathbf{a}_{i,j,q_{i,j}} & \mathbf{b}_{i,j,q_{i,j}} \end{bmatrix}$$

$$\bar{\mathbf{J}}_{i,j} := \begin{bmatrix} \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \mathbf{I} & \dots & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} \end{bmatrix}$$

$$\bar{\mathbf{V}}_i := \begin{bmatrix} \bar{\mathbf{V}}_{i,1} & \dots & \bar{\mathbf{V}}_{i,g_i} \end{bmatrix}$$

$$\bar{\mathbf{J}}_i := \texttt{blkdiag}(\bar{\mathbf{J}}_{i,1}, \dots, \bar{\mathbf{J}}_{i,g_i})$$

then, $\mathbf{A} \begin{bmatrix} \mathbf{V}_i & \mathbf{V}_{i^\star} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_i & \mathbf{V}_{i^\star} \end{bmatrix} \texttt{blkdiag}(\mathbf{J}_i, \mathbf{J}_{i^\star}) \iff \mathbf{A}\bar{\mathbf{V}}_i = \bar{\mathbf{V}}_i \bar{\mathbf{J}}_i$.

Jordan Canonical Form

Consider $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ and let $\mathbf{V}$ be the (real form) matrix collecting all the generalised eigenvectors of $\mathbf{A}$. Define $\mathbf{z} := \mathbf{V}^{-1}\,\mathbf{x}$ and $\mathbf{J} := \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}$. Then, $\mathbf{x} = \mathbf{V}\mathbf{z}$ and

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{x} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}\mathbf{z} = \mathbf{J}\mathbf{z}$$

where $\dot{\mathbf{z}} = \mathbf{J}\mathbf{z}$ is said to be in the Jordan Canonical Form

Modes

Building the Jordan Canonical Form

Jordan Canonical Form

Consider $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ and let $\mathbf{V}$ be the (real form) matrix collecting all the generalised eigenvectors of $\mathbf{A}$. Define $\mathbf{z} := \mathbf{V}^{-1}\,\mathbf{x}$ and $\mathbf{J} := \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}$. Then, $\mathbf{x} = \mathbf{V}\mathbf{z}$ and

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{x} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}\mathbf{z} = \mathbf{J}\mathbf{z}$$

where $\dot{\mathbf{z}} = \mathbf{J}\mathbf{z}$ is said to be in the Jordan Canonical Form

Remark

The matrix $\mathbf{J}$ is a block-diagonal matrix whose blocks are in the form of the elementary matrices

$$\begin{bmatrix} \alpha_1 & 0 & \cdots \\ 0 & \alpha_2 & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \alpha & 1 & 0 & \cdots \\ 0 & \alpha & 1 & \ddots \\ 0 & 0 & \alpha & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \alpha\,\mathbf{I} + \beta\mathbf{R} & \mathbf{I} & \cdots \\ 0 & \alpha\,\mathbf{I} + \beta\mathbf{R} & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

Remember, $\mathbf{J}$ has the same eigenvalues of $\mathbf{A}$!

Modes

- Let $\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0)) := \exp(\mathbf{J}(t - t_0))\mathbf{z}(t_0)$ be the solution to $\dot{\mathbf{z}} = \mathbf{J}\mathbf{z}$
- remember $\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$ for all $t \geq t_0$, from which $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0)) = \mathbf{V}\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0))$

Modes

- Let $\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0)) := \exp(\mathbf{J}(t - t_0))\mathbf{z}(t_0)$ be the solution to $\dot{\mathbf{z}} = \mathbf{J}\mathbf{z}$
- remember $\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$ for all $t \geq t_0$, from which $\chi_{\mathbf{0}}(t, \mathbf{x}(t_0)) = \mathbf{V}\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0))$

Then, finally, we have

$$
\begin{aligned}
\|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| &= \|\mathbf{V}\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0))\| \\
\text{1) Matrix product inequality} \implies &\leq \|\mathbf{V}\|\|\zeta_{\mathbf{0}}(t, \mathbf{z}(t_0))\| \\
\text{2) } \zeta_{\mathbf{0}}(t, \mathbf{z}(t_0)) = \exp(\mathbf{J}(t - t_0))\mathbf{z}(t_0) \implies &= \|\mathbf{V}\|\|\exp(\mathbf{J}(t - t_0))\mathbf{z}(t_0)\| \\
\text{3) Matrix product inequality} \implies &\leq \|\mathbf{V}\|\|\exp(\mathbf{J}(t - t_0))\|\|\mathbf{z}(t_0)\| \\
\text{4) } \mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \implies &= \|\mathbf{V}\|\|\exp(\mathbf{J}(t - t_0))\|\|\mathbf{V}^{-1}\mathbf{x}(t_0)\| \\
\text{5) Matrix product inequality} \implies &\leq \|\mathbf{V}\|\|\exp(\mathbf{J}(t - t_0))\|\|\mathbf{V}^{-1}\|\|\mathbf{x}(t_0)\|
\end{aligned}
$$

from which

$$
\|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| \leq \|\mathbf{V}\|\|\exp(\mathbf{J}(t - t_0))\|\|\mathbf{V}^{-1}\|\|\mathbf{x}(t_0)\|
$$

Modes

To conclude

$$\texttt{Real}(\lambda) < 0 \implies \lim_{t\to\infty} \exp(\mathbf{J}(t - t_0)) = 0$$

$$\begin{aligned}
\text{previous slide's result} \implies \lim_{t\to\infty} \|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| &\leq \lim_{t\to\infty} \|\mathbf{V}\| \| \exp(\mathbf{J}(t - t_0))\| \|\mathbf{V}^{-1}\| \| \mathbf{x}(t_0)\| \\
&= \|\mathbf{V}\| \lim_{t\to\infty} \| \exp(\mathbf{J}(t - t_0))\| \|\mathbf{V}^{-1}\| \| \mathbf{x}(t_0)\| \\
&= \mathbf{0}
\end{aligned}$$

Definition - Hurwitz Matrices

A matrix **A** is said Hurwitz if the real part of its eigenvalues is negative

Modes

To conclude

$$\texttt{Real}(\lambda) < 0 \implies \lim_{t\to\infty} \exp(\mathbf{J}(t-t_0)) = 0$$

$$\text{previous slide's result} \implies \lim_{t\to\infty} \|\chi_{\mathbf{0}}(t, \mathbf{x}(t_0))\| \leq \lim_{t\to\infty} \|\mathbf{V}\| \|\exp(\mathbf{J}(t-t_0))\| \|\mathbf{V}^{-1}\| \|\mathbf{x}(t_0)\|$$
$$= \|\mathbf{V}\| \lim_{t\to\infty} \|\exp(\mathbf{J}(t-t_0))\| \|\mathbf{V}^{-1}\| \|\mathbf{x}(t_0)\|$$
$$= \mathbf{0}$$

Definition - Hurwitz Matrices

A matrix $\mathbf{A}$ is said Hurwitz if the real part of its eigenvalues is negative

Theorem

The LTI system $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{w}$ is BIBS-stable if and only if $\mathbf{A}$ is Hurwitz.

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Nicola Mimmo

Department of Electrical, Electronics and Information Engineering "G. Marconi"
University of Bologna

nicola.mimmo2@unibo.it