ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

State-Feedback Stabiliser

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Motivations and Goals

Motivation

- On the one hand, we know that the boundedness of signals (goal G1)) is linked to $\mathbf{A}$ being Hurwitz
- On the other hand, the dynamics of the open-loop plant is described by the eigenvalues of $\mathbf{A}$

Therefore, changing the eigenvalues of $\mathbf{A}$ we can

- Stabilise a non-BIBS-stable open-loop plant
- Improve/modify the behaviour of a BIBS-stable open-loop plant

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2. **How** can we change the **modifiable** eigenvalues of $\mathbf{A}$?

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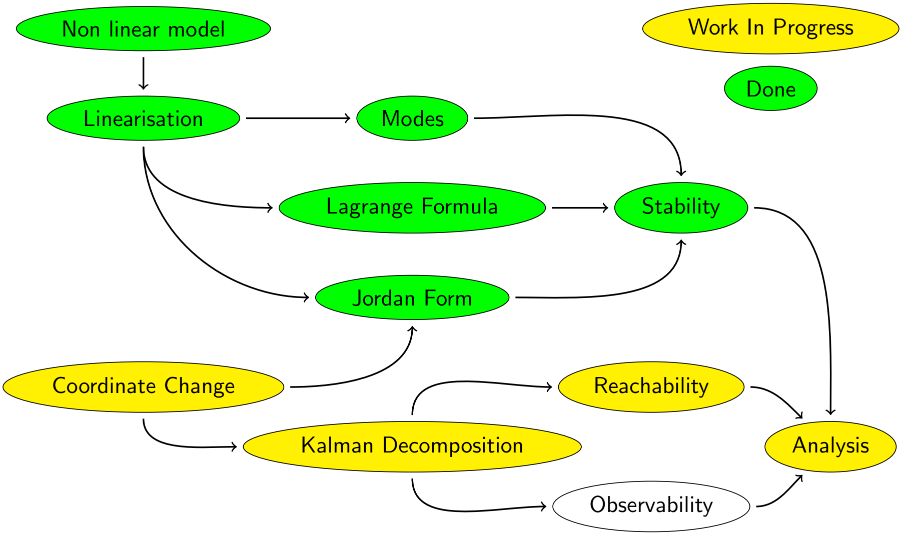
1. **Which** eigenvalues of $\mathbf{A}$ can we change?
2. **How** can we change the **modifiable** eigenvalues of $\mathbf{A}$?

Goal

Answer these questions!

Where are we?

Regarding the course contents ...



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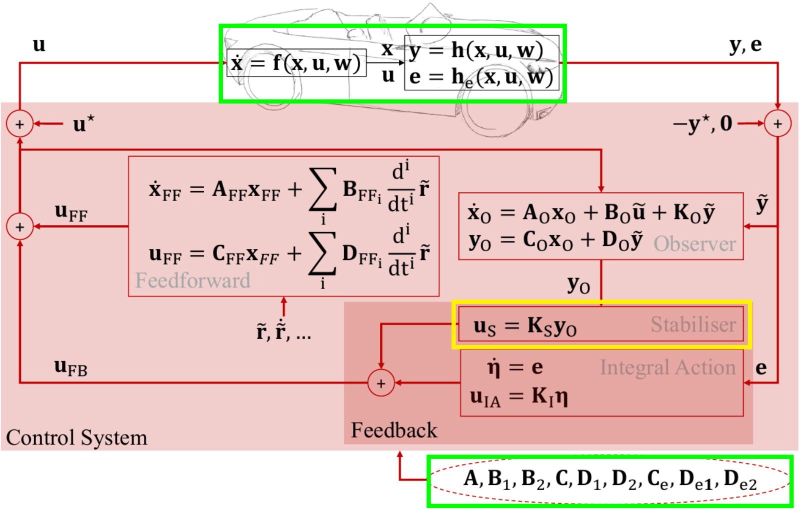

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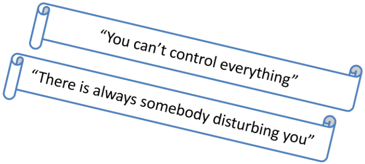
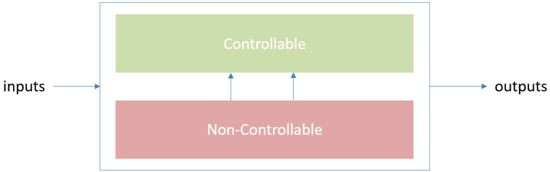
Where are we?

Regarding the course contents ...

Automatic control theory: a live coach



Can we always reach our goals?



System Oriented Model

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Basic Mathematics Notions

Definition - Vector Space

Let $\mathbb{V}(\mathbb{R}) := \{\mathbf{x} \in \mathbb{R}^n\}$ with $n \in \mathbb{N}$. Then $\mathbb{V}$ is a **vector space** if the following conditions hold true:

1. for any $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{V}$ then $(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$
2. for any $\mathbf{u} \in \mathbb{V}$ the null vector $\mathbf{0}$ is such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$;
3. for any $\mathbf{u}_1 \in \mathbb{V}$ there exists $\mathbf{u_2} \in \mathbb{V}$ such that $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{0} \implies \mathbf{u}_2 = -\mathbf{u}_1$;
4. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$ $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_2 + \mathbf{u}_1$;
5. for any $\mathbf{u} \in \mathbb{V}$ and for any $\alpha, \beta \in \mathbb{R}$ $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$;
6. there exists a neutral element, namely 1, such that $1\mathbf{u} = \mathbf{u}$;
7. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$ and for any $\alpha \in \mathbb{R}$ $\alpha(\mathbf{u}_1 + \mathbf{u}_2) = \alpha\mathbf{u}_1 + \alpha\mathbf{u}_2$;
8. for any $\mathbf{u} \in \mathbb{V}$ and any $\alpha, \beta \in \mathbb{R}$ $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

Basic Mathematics Notions

Definition - Basis

Let $\mathbb{V}(\mathbb{R})$ be an n-dimensional vector space with $n \in \mathbb{N}$. Then, a **basis** of $\mathbb{V}(\mathbb{R})$ is a set of linearly independent vectors

$$\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}, \quad \mathbf{b}_i \in \mathbb{V}(\mathbb{R}), \quad i = 1, \ldots, n$$

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such that $\forall\, \mathbf{v} \in \mathbb{V}(\mathbb{R})\ \exists\ \beta_i \in \mathbb{R},\ i = 1, \ldots, n\ :\ \mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

In other words, any vector $\mathbf{v} \in \mathbb{V}(\mathbb{R})$ can be represented as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_n$. The term β_i means the i-th component of $\mathbf{v}$ on $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

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The bases are not unique!

Let $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two bases of $\mathbb{V}(\mathbb{R})$. Then, for any $\mathbf{v} \in \mathbb{V}(\mathbb{R})$ exist

- $\mathbf{u} := \text{col}(\beta_1, \dots, \beta_n)$ such that $\mathbf{v} = [\mathbf{b}_1, \dots, \mathbf{b}_n]\, \mathbf{u}$
- $\mathbf{w} := \text{col}(\gamma_1, \dots, \gamma_n)$ such that $\mathbf{v} = [\mathbf{c}_1, \dots, \mathbf{c}_n]\, \mathbf{w}$

Basic Mathematics Notions

Definition - Change of Coordinates

Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be two bases of $\mathbb{V}(\mathbb{R})$ and take $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ such that

$$\begin{cases} \mathbf{v} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]\,\mathbf{u} \\ \mathbf{v} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]\,\mathbf{w} \end{cases} \iff [\mathbf{b}_1, \ldots, \mathbf{b}_n]\,\mathbf{u} = [\mathbf{c}_1, \ldots, \mathbf{c}_n]\,\mathbf{w}$$

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Then,

$$\mathbf{T} := \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$$

is called **change of coordinates** and it is such that

$$\mathbf{u} = \mathbf{T}\,\mathbf{w}.$$

Finally,

Definition - Change of Coordinates

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$$\mathbf{u} = \mathbf{T}\,\mathbf{w}.$$

Finally, since the bases are made of linearly independent vectors, coordinate changes are invertible such that

$$\mathbf{w} = \mathbf{T}^{-1}\mathbf{u}.$$

Basic Mathematics Notions

Let $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n \in \mathbb{N}$, be a linear function with $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be two vectors, both defined on the basis $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ and such that

$$\mathbf{y} = \mathbf{A}\,\mathbf{x}. \tag{1}$$

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Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a second basis of $\mathbb{R}^n$ and define $\mathbf{T} := [\mathbf{b}_1 \cdots \mathbf{b}_n]^{-1} [\mathbf{c}_1 \cdots \mathbf{c}_n]$,

$$\chi := \mathbf{T}\mathbf{x}, \text{ and } \boldsymbol{\mu} := \mathbf{T}\mathbf{y}$$

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$$\boldsymbol{\chi} := \mathbf{T}\mathbf{x}, \text{ and } \boldsymbol{\mu} := \mathbf{T}\mathbf{y}$$

Then,

$$\begin{aligned} \mathbf{y} &= \mathbf{A}\,\mathbf{x} \\ \text{pre-multiply both sides by } \mathbf{T} \quad \implies \mathbf{T}\mathbf{y} &= \mathbf{T}\,\mathbf{A}\,\mathbf{x} \\ \text{exploit } \mathbf{x} = \mathbf{T}^{-1}\boldsymbol{\chi} \quad \implies \quad \boldsymbol{\mu} &= \bar{\mathbf{A}}\boldsymbol{\chi} \end{aligned}$$

where $\bar{\mathbf{A}} := \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1}$ represents $\mathbf{A}$ described on $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. This result was implicitly exploited to build the Jordan canonical form ...

Basic Mathematics Notions

Example (Jordan Canonical Form)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, and $[\mathbf{c}_1 \cdots \mathbf{c}_n] = \mathbf{I}$. Then, take $[\mathbf{b}_1 \cdots \mathbf{b}_n] = \mathbf{V}$, where $\mathbf{V}$ is column composition of the generalised eigenvectors of $\mathbf{A}$. Therefore $\mathbf{T} = \mathbf{V}^{-1}$ leads to $\bar{\mathbf{A}} = \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V} = \mathbf{J}$.

Definition - Orthogonal Complement

Let $\mathbb{S}(\mathbb{R}) \subseteq \mathbb{V}(\mathbb{R})$ be a p-dimensional subspace of $\mathbb{V}(\mathbb{R})$. Then, the **orthogonal complement** of $\mathbb{S}(\mathbb{R})$, denoted with $[\mathbb{S}(\mathbb{R})]^{\perp}$, is defined as a subspace of $\mathbb{V}(\mathbb{R})$ such that

$$\mathbb{V}(\mathbb{R}) = \mathbb{S}(\mathbb{R}) \oplus [\mathbb{S}(\mathbb{R})]^{\perp}$$

where the direct sum $\oplus$ means that $\mathbb{S}(\mathbb{R}) \cup [\mathbb{S}(\mathbb{R})]^{\perp} = \mathbb{V}(\mathbb{R})$ and $\mathbb{S}(\mathbb{R}) \cap [\mathbb{S}(\mathbb{R})]^{\perp} = \{0\}$

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Remark

Let $\mathbb{S}(\mathbb{R}) \subseteq \mathbb{V}(\mathbb{R})$ be a p-dimensional subspace of $\mathbb{V}(\mathbb{R})$ and let

- $\{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ be a basis for $\mathbb{S}(\mathbb{R})$
- $\{\mathbf{s}_1^\star, \ldots, \mathbf{s}_{n-p}^\star\}$ be a basis for $[\mathbb{S}(\mathbb{R})]^\perp$

then, the set $\{\mathbf{s}_1, \ldots, \mathbf{s}_p, \mathbf{s}_1^\star, \ldots, \mathbf{s}_{n-p}^\star\}$ represents a basis for $\mathbb{V}(\mathbb{R})$.

Basic Mathematics Notions

Definition - Kernel of a Matrix

Let $\mathbf{R} \in \mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}$. Then, we define $\ker(\mathbf{R}) := \{\mathbf{x} \in \mathbb{R}^m \,:\, \mathbf{0} = \mathbf{R}\,\mathbf{x}\}$

The dimension of $\ker(\mathbf{R})$ is <u>m minus the number of linearly independent rows</u>

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Definition - Span of a Matrix

Let $\mathbf{R} \in \mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}$. Then, we define $\text{span}(\mathbf{R}) := \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = \mathbf{R}\,\mathbf{x}, \mathbf{x} \in \mathbb{R}^m\}$

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Span/Kernel Property

The kernel and the span are such that $[\operatorname{span}(\mathbf{R})]^{\perp} = \ker(\mathbf{R}^{\top})$. As a consequence

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Span/Kernel Property

The kernel and the span are such that $[\mathrm{span}(\mathbf{R})]^\perp = \ker(\mathbf{R}^\top)$. As a consequence

$$\mathbb{V}(\mathbb{R}) = \mathbb{S}(\mathbb{R}) \oplus [\mathbb{S}(\mathbb{R})]^\perp \implies \mathbb{R}^n = \mathrm{span}(\mathbf{R}) \oplus [\mathrm{span}(\mathbf{R})]^\perp = \mathrm{span}(\mathbf{R}) \oplus \ker(\mathbf{R}^\top)$$

Let $p \leq n$ be the dimension of $\mathrm{span}(\mathbf{R})$, $\{\mathbf{r}_1, \cdots, \mathbf{r}_p\}$ be a basis for $\mathrm{span}(\mathbf{R})$, and $\{\mathbf{r}_1^\star, \cdots, \mathbf{r}_{n-p}^\star\}$ be a basis for $\ker(\mathbf{R}^\top)$. Then

$$\{\mathbf{r}_1, \cdots, \mathbf{r}_p, \mathbf{r}_1^\star, \cdots, \mathbf{r}_{n-p}^\star\} \text{ is a basis for } \mathbb{R}^n$$

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Reachability Kalman Decomposition

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With $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^p$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, and $\mathbf{B}_1 \in \mathbb{R}^{n \times p}$, consider

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} \;\text{ with }\; \mathbf{x}(0) = \mathbf{0} \implies \dot{\mathbf{x}}(0) = \mathbf{B}_1\,\mathbf{u}(0)$$

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Therefore,

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Therefore, $\mathbf{B}_1$ represents a basis for describing $\dot{\mathbf{x}}(0)$ at $\mathbf{x}(0) = \mathbf{0}$.

Remark

At $\mathbf{x}(0) = \mathbf{0}$, the derivative $\dot{\mathbf{x}}(0)$ is a linear combination of the columns of $\mathbf{B}_1$

Reachability Kalman Decomposition

Compute now the second derivative of $\mathbf{x}$

$$\ddot{\mathbf{x}} = \frac{d}{dt}\dot{\mathbf{x}}$$

Reachability Kalman Decomposition

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$$
\begin{aligned}
& \ddot{\mathbf{x}} = \frac{d}{dt}\dot{\mathbf{x}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} \quad & \Longrightarrow = \frac{d}{dt}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u}) \\
\text{distributive law and } \mathbf{A} \text{ and } \mathbf{B}_1 \text{ constant} \quad & \Longrightarrow = \mathbf{A}\,\dot{\mathbf{x}} + \mathbf{B}_1\,\dot{\mathbf{u}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} \quad & \Longrightarrow = \mathbf{A}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u}) + \mathbf{B}_1\,\dot{\mathbf{u}} \\
\text{distributive law} \quad & \Longrightarrow = \mathbf{A}^2\,\mathbf{x} + \mathbf{A}\,\mathbf{B}_1\,\mathbf{u} + \mathbf{B}_1\,\dot{\mathbf{u}}
\end{aligned}
$$

and use the latter with $\mathbf{x}(0) = \mathbf{0}$ to evaluate

Reachability Kalman Decomposition

Compute now the second derivative of $\mathbf{x}$

$$
\begin{aligned}
&& \ddot{\mathbf{x}} &= \frac{d}{dt}\dot{\mathbf{x}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} & \implies & &= \frac{d}{dt}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u}) \\
\text{distributive law and } \mathbf{A} \text{ and } \mathbf{B}_1 \text{ constant} & \implies & &= \mathbf{A}\,\dot{\mathbf{x}} + \mathbf{B}_1\,\dot{\mathbf{u}} \\
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\end{aligned}
$$

and use the latter with $\mathbf{x}(0) = \mathbf{0}$ to evaluate

$$
\ddot{\mathbf{x}}(0) := \mathbf{A}\,\mathbf{B}_1\,\mathbf{u}(0) + \mathbf{B}_1\,\dot{\mathbf{u}}(0) = [\mathbf{B}_1 \;\; \mathbf{A}\,\mathbf{B}_1] \begin{bmatrix} \dot{\mathbf{u}}(0) \\ \mathbf{u}(0) \end{bmatrix}
$$

Therefore,

Reachability Kalman Decomposition

Compute now the second derivative of $\mathbf{x}$

$$
\begin{aligned}
\ddot{\mathbf{x}} &= \frac{d}{dt}\dot{\mathbf{x}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} \implies &= \frac{d}{dt}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u}) \\
\text{distributive law and } \mathbf{A} \text{ and } \mathbf{B}_1 \text{ constant} \implies &= \mathbf{A}\,\dot{\mathbf{x}} + \mathbf{B}_1\,\dot{\mathbf{u}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} \implies &= \mathbf{A}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u}) + \mathbf{B}_1\,\dot{\mathbf{u}} \\
\text{distributive law} \implies &= \mathbf{A}^2\,\mathbf{x} + \mathbf{A}\,\mathbf{B}_1\,\mathbf{u} + \mathbf{B}_1\,\dot{\mathbf{u}}
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and use the latter with $\mathbf{x}(0) = \mathbf{0}$ to evaluate

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$$

Therefore, $[\mathbf{B}_1 \;\; \mathbf{A}\,\mathbf{B}_1]$ represents a basis for describing $\ddot{\mathbf{x}}(0)$ at $\mathbf{x}(0) = \mathbf{0}$.

Reachability Kalman Decomposition

Generically, the kth derivative of $\mathbf{x}$, evaluated at $t = 0$ with $\mathbf{x}(0) = \mathbf{0}$, is

$$\left.\frac{d^k}{dt^k}\mathbf{x}(t)\right|_{t=0} = \underbrace{\begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^{k-1}\mathbf{B}_1 \end{bmatrix}}_{\text{basis for } \left.\frac{d^k}{dt^k}\mathbf{x}(t)\right|_{t=0}} \begin{bmatrix} \left.\frac{d^{k-1}}{dt^{k-1}}\mathbf{u}(t)\right|_{t=0} \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

Remark

$$\text{span}\left(\begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^{k-1}\mathbf{B}_1 \end{bmatrix}\right) \subseteq \text{span}\left(\begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^{k}\mathbf{B}_1 \end{bmatrix}\right) \quad \forall\, k = 1, \ldots, n-1$$

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Definition - Reachability Matrix

We define the **reachability matrix**

$$\mathbf{R} := \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 \end{bmatrix}$$

as the basis for $\left.\frac{d^n}{dt^n}\mathbf{x}(t)\right|_{t=0}$ (where n is the system's dimension)

Reachability Kalman Decomposition

Some thoughts ...

- The span of $\mathbf{R}$ represents the set of all possible directions $\mathbf{x}$ can take (in terms of its nth time derivative)

Reachability Kalman Decomposition

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Definition - Reachability Subspace

We define the **reachability subspace** as $\text{span}(\mathbf{R})$. (note that it is a vector space)

Definition - Reachabile Couple

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_1 \in \mathbb{R}^{n\times m}$ and their associated reachability matrix $\text{span}(\mathbf{R})$. Then, we say the couple $(\mathbf{A}, \mathbf{B}_1)$ is **reachable** if $\text{span}(\mathbf{R}) = \mathbb{R}^n$, *i.e.*, if $\text{rank}(\mathbf{R}) = n$

Reachability Kalman Decomposition

Therefore, we use the kernel/span property (hereafter recalled)

$$\mathbb{R}^n = \text{span}(\mathbf{R}) \oplus \ker(\mathbf{R}^\top)$$

to define the change of coordinate $\mathbf{T}^{-1} := \begin{bmatrix} \mathbf{r}_1 & \cdots & \mathbf{r}_p & \mathbf{r}_1^\star & \cdots & \mathbf{r}_{n-p}^\star \end{bmatrix}$ where

- $p \leq n$ is the dimension of span($\mathbf{R}$)
- $\{\mathbf{r}_1, \cdots, \mathbf{r}_p\}$ is a basis for span($\mathbf{R}$)
- $\{\mathbf{r}_1^\star, \cdots, \mathbf{r}_{n-p}^\star\}$ is a basis for $\ker(\mathbf{R}^\top)$

Then,

Reachability Kalman Decomposition

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Then, define $\mathbf{z} := \mathbf{T}\,\mathbf{x}$ and compute

$$\begin{aligned}
\mathbf{T} \text{ constant} \quad &\Longrightarrow \quad \dot{\mathbf{z}} = \mathbf{T}\dot{\mathbf{x}} \\
\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} + \mathbf{B}_2\,\mathbf{w} \quad &\Longrightarrow \quad \dot{\mathbf{z}} = \mathbf{T}(\mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} + \mathbf{B}_2\,\mathbf{w}) \\
\text{distributive law} \quad &\Longrightarrow \quad \dot{\mathbf{z}} = \mathbf{T}\,\mathbf{A}\,\mathbf{x} + \mathbf{T}\,\mathbf{B}_1\,\mathbf{u} + \mathbf{T}\,\mathbf{B}_2\,\mathbf{w} \\
\mathbf{x} = \mathbf{T}^{-1}\mathbf{z} \quad &\Longrightarrow \quad \dot{\mathbf{z}} = \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\,\mathbf{B}_1\,\mathbf{u}\,\mathbf{T}\,\mathbf{B}_2\,\mathbf{w} \\
\bar{\mathbf{A}} := \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1},\ \bar{\mathbf{B}}_1 := \mathbf{T}\,\mathbf{B}_1,\ \bar{\mathbf{B}}_2 := \mathbf{T}\,\mathbf{B}_2 \quad &\Longrightarrow \quad \dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}_1\,\mathbf{u} + \bar{\mathbf{B}}_2\,\mathbf{w}
\end{aligned}$$

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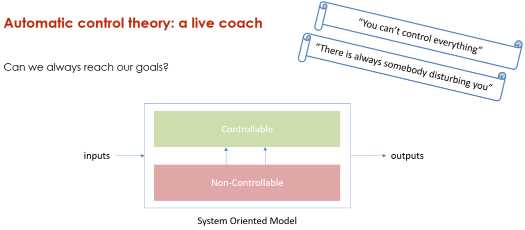
Reachability Kalman Decomposition

Let us investigate $\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}_1\,\mathbf{u} + \bar{\mathbf{B}}_2\,\mathbf{w}$

- define $\mathbf{z}_\mathrm{R} \in \mathbb{R}^p$ and $\mathbf{z}_\mathrm{NR} \in \mathbb{R}^{n-p}$ such that $\mathbf{z} = \mathrm{col}(\mathbf{z}_\mathrm{R}, \mathbf{z}_\mathrm{NR})$

Then,

$$\begin{array}{ccccccccc} \dot{\mathbf{z}} & = & \bar{\mathbf{A}} & \mathbf{z} & + & \bar{\mathbf{B}}_1 & \mathbf{u} + & \bar{\mathbf{B}}_2 & \mathbf{w} \\ \begin{bmatrix} \dot{\mathbf{z}}_\mathrm{R} \\ \dot{\mathbf{z}}_\mathrm{NR} \end{bmatrix} & = & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} & \begin{bmatrix} \mathbf{z}_\mathrm{R} \\ \mathbf{z}_\mathrm{NR} \end{bmatrix} & + & \begin{bmatrix} \quad \\ \quad \end{bmatrix} & \mathbf{u} + & \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \end{bmatrix} & \mathbf{w} \end{array}$$



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Reachability Kalman Decomposition

Let us investigate $\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}_1\,\mathbf{u} + \bar{\mathbf{B}}_2\,\mathbf{w}$

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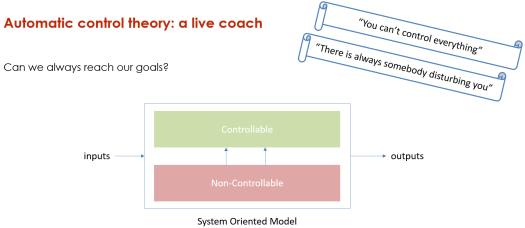

Reachability Kalman Decomposition

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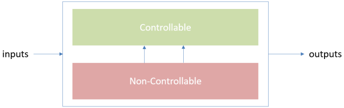
Automatic control theory: a live coach

Can we always reach our goals?

"You can't control everything"

"There is always somebody disturbing you"



System Oriented Model

Reachability Kalman Decomposition

Let us investigate $\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}_1\,\mathbf{u} + \bar{\mathbf{B}}_2\,\mathbf{w}$

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$$
\begin{array}{ccccccccc}
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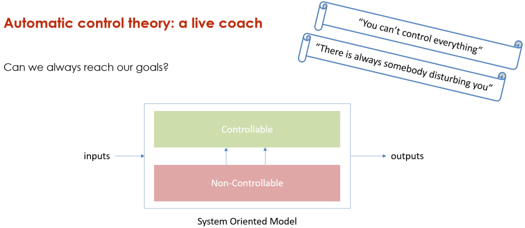
\end{array}
$$



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\end{array}
$$

Automatic control theory: a live coach

"You can't control everything"

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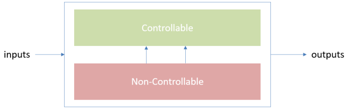

System Oriented Model

Reachability Kalman Decomposition

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$$

Automatic control theory: a live coach

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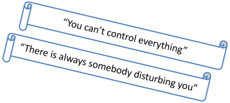

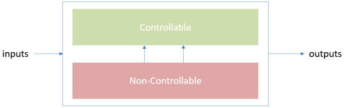

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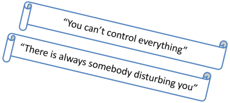

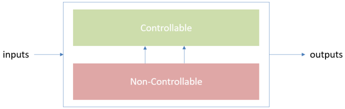

System Oriented Model

The couple $(\bar{\mathbf{A}}_{11}, \bar{\mathbf{B}}_{11})$ is reachable!

$$
\boxed{\dot{\mathbf{z}}_\mathrm{R} = \bar{\mathbf{A}}_{11}\mathbf{z}_\mathrm{R} + \bar{\mathbf{A}}_{12}\mathbf{z}_\mathrm{NR} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w}}
$$

$$
\boxed{\dot{\mathbf{z}}_\mathrm{NR} = \bar{\mathbf{A}}_{22}\mathbf{z}_\mathrm{NR} + \bar{\mathbf{B}}_{22}\mathbf{w}}
$$

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State-Feedback Stabiliser

Theorem

Let $(\mathbf{A}, \mathbf{B})$ be a reachable couple with $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$, $n, m \in \mathbb{N}$. Then, there exists $\mathbf{K}_\mathrm{R} \in \mathbb{R}^{m \times n}$ such that $\mathbf{A} + \mathbf{B}\,\mathbf{K}_\mathrm{R}$ is Hurwitz

Roughly, we can modify all the eigenvalues of $\mathbf{A}$ if $(\mathbf{A}, \mathbf{B})$ is reachable

State-Feedback Stabiliser

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Finally, we are ready to answer the question "**Which** eigenvalues of $\mathbf{A}$ can we change?"

State-Feedback Stabiliser

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Roughly, we can modify all the eigenvalues of $\mathbf{A}$ if $(\mathbf{A}, \mathbf{B})$ is reachable

Finally, we are ready to answer the question "**Which** eigenvalues of $\mathbf{A}$ can we change?"

Answer

$(\bar{\mathbf{A}}_{11}, \bar{\mathbf{B}}_{11})$ reachable $\implies$ we can only modify the eigenvalues of the reachable part!

State-Feedback Stabiliser

But also, we can answer the question "**How** can we change the **modifiable** eigenvalues of $\mathbf{A}$?"

State-Feedback Stabiliser

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Answer

Inspired by Theorem, we define $\mathbf{u}_\mathrm{S} := \mathbf{K}_\mathrm{R}\mathbf{z}_\mathrm{R} + \mathbf{K}_\mathrm{NR}\mathbf{z}_\mathrm{NR}$ such that $\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11}\mathbf{K}_\mathrm{R}$ is Hurwitz. Then,

$$\left.\begin{aligned} \dot{\mathbf{z}}_\mathrm{NR} &= \bar{\mathbf{A}}_{22}\mathbf{z}_\mathrm{NR} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \dot{\mathbf{z}}_\mathrm{R} &= \bar{\mathbf{A}}_{11}\mathbf{z}_\mathrm{R} + \bar{\mathbf{A}}_{12}\mathbf{z}_\mathrm{NR} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w} \\ \mathbf{u} &= \mathbf{u}_\mathrm{S} \end{aligned}\right\} \implies$$

State-Feedback Stabiliser

But also, we can answer the question "**How** can we change the **modifiable** eigenvalues of $\mathbf{A}$?"

Answer

Inspired by Theorem, we define $\mathbf{u}_S := \mathbf{K}_R \mathbf{z}_R + \mathbf{K}_{NR} \mathbf{z}_{NR}$ such that $\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11} \mathbf{K}_R$ is Hurwitz. Then,

$$\left.\begin{aligned} \dot{\mathbf{z}}_{NR} &= \bar{\mathbf{A}}_{22} \mathbf{z}_{NR} + \bar{\mathbf{B}}_{22} \mathbf{w} \\ \dot{\mathbf{z}}_R &= \bar{\mathbf{A}}_{11} \mathbf{z}_R + \bar{\mathbf{A}}_{12} \mathbf{z}_{NR} + \bar{\mathbf{B}}_{11} \mathbf{u} + \bar{\mathbf{B}}_{21} \mathbf{w} \\ \mathbf{u} &= \mathbf{u}_S \end{aligned}\right\} \Longrightarrow \begin{aligned} \dot{\mathbf{z}}_{NR} &= \bar{\mathbf{A}}_{22} \mathbf{z}_{NR} + \bar{\mathbf{B}}_{22} \mathbf{w} \\ \dot{\mathbf{z}}_R &= (\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11} \mathbf{K}_R) \mathbf{z}_R \\ &\quad + (\bar{\mathbf{A}}_{12} + \bar{\mathbf{B}}_1 \mathbf{K}_{NR}) \mathbf{z}_{NR} + \bar{\mathbf{B}}_{21} \mathbf{w} \end{aligned}$$

$$\boxed{\dot{\mathbf{z}}_R = (\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11} \mathbf{K}_R) \mathbf{z}_R + (\bar{\mathbf{A}}_{12} + \bar{\mathbf{B}}_1 \mathbf{K}_{NR}) \mathbf{z}_{NR} + \bar{\mathbf{B}}_{21} \mathbf{w}}$$

$$\uparrow$$

$$\boxed{\dot{\mathbf{z}}_{NR} = \bar{\mathbf{A}}_{22} \mathbf{z}_{NR} + \bar{\mathbf{B}}_{22} \mathbf{w}}$$

State-Feedback Stabiliser

Is this system BIBS-stable?

$$\boxed{\dot{\mathbf{z}}_{\mathsf{R}} = (\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11}\mathbf{K}_{\mathsf{R}})\mathbf{z}_{\mathsf{R}} + (\bar{\mathbf{A}}_{12} + \bar{\mathbf{B}}_{1}\mathbf{K}_{\mathsf{NR}})\mathbf{z}_{\mathsf{NR}} + \bar{\mathbf{B}}_{21}\mathbf{w}}$$

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State-Feedback Stabiliser

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$$\uparrow$$

$$\boxed{\dot{\mathbf{z}}_{\mathrm{NR}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{NR}} + \bar{\mathbf{B}}_{22}\mathbf{w}}$$

It depends ... on the eigenvalues of $\bar{\mathbf{A}}_{22}$!

State-Feedback Stabiliser

Is this system BIBS-stable?

$$\boxed{\dot{\mathbf{z}}_{\mathrm{R}} = (\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11}\mathbf{K}_{\mathrm{R}})\mathbf{z}_{\mathrm{R}} + (\bar{\mathbf{A}}_{12} + \bar{\mathbf{B}}_{1}\mathbf{K}_{\mathrm{NR}})\mathbf{z}_{\mathrm{NR}} + \bar{\mathbf{B}}_{21}\mathbf{w}}$$

$$\uparrow$$

$$\boxed{\dot{\mathbf{z}}_{\mathrm{NR}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{NR}} + \bar{\mathbf{B}}_{22}\mathbf{w}}$$

It depends ... on the eigenvalues of $\bar{\mathbf{A}}_{22}$!

Definition - Stabilisability

We say that the couple $(\mathbf{A}, \mathbf{B})$ is stabilisable if $\bar{\mathbf{A}}_{22}$ is Hurwitz

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State-Feedback Stabiliser in the original coordinates

$$\mathbf{u}_{\mathrm{S}} := \mathbf{K}_{\mathrm{R}}\mathbf{z}_{\mathrm{R}} + \mathbf{K}_{\mathrm{NR}}\mathbf{z}_{\mathrm{NR}} =$$

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State-Feedback Stabiliser

Fundamental Result

Consider the LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}_1\,\mathbf{u} + \mathbf{B}_2\,\mathbf{w}$$

with the couple $(\mathbf{A}, \mathbf{B}_1)$ stabilisable. Then, there exists $\mathbf{K}_\mathrm{S}$ such that

$$\mathbf{u}_\mathrm{S} = \mathbf{K}_\mathrm{S}\,\mathbf{x}$$

makes the closed-loop system

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}_1\,\mathbf{K}_\mathrm{S})\,\mathbf{x} + \mathbf{B}_2\,\mathbf{w}$$

BIBS-stable

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