Active Suspensions

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Outline

- Motivations & Goals
- Non-linear 1D Model
- Control Problem Formulation
- Linearisation
- Linear Model Investigation
- Control Architecture
- Control Design
- Performance Evaluation

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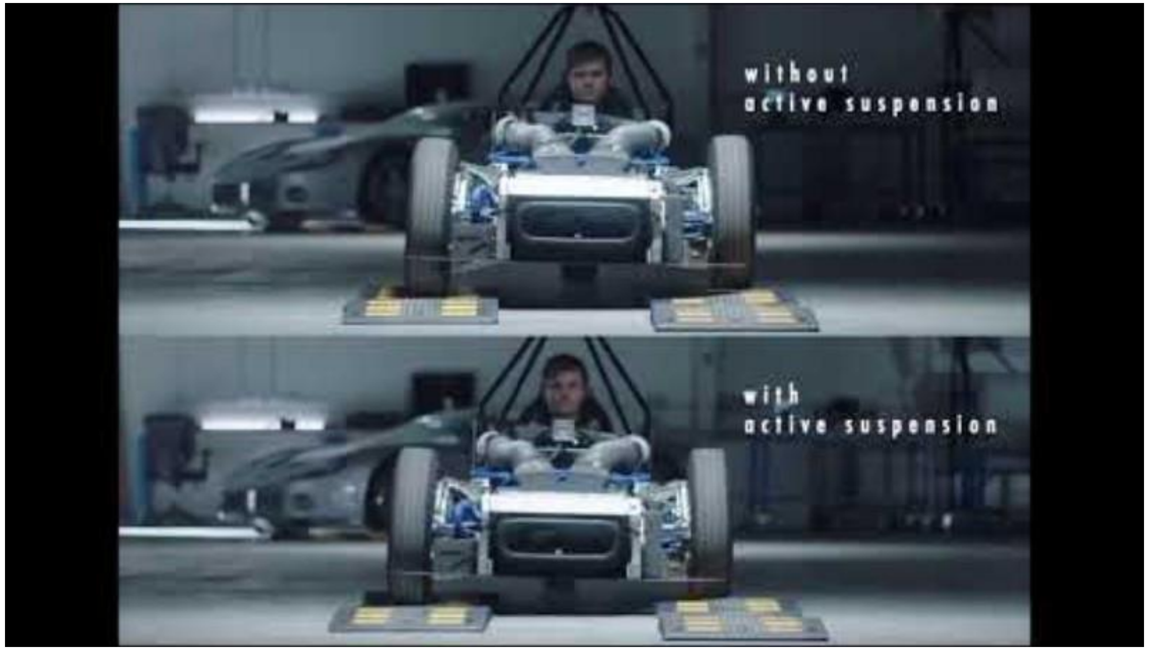
Motivations & Goals

Improve the ride quality and change the setup.



Motivations & Goals

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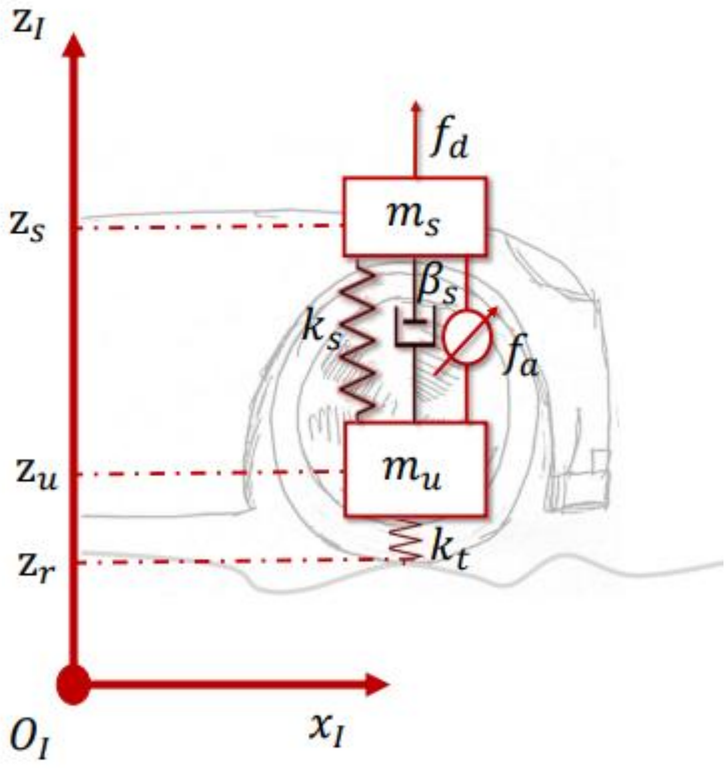
Non-linear 1D Model



Sprung mass $$m_s \ddot{z}_s = -m_s g + f_s + f_d$$

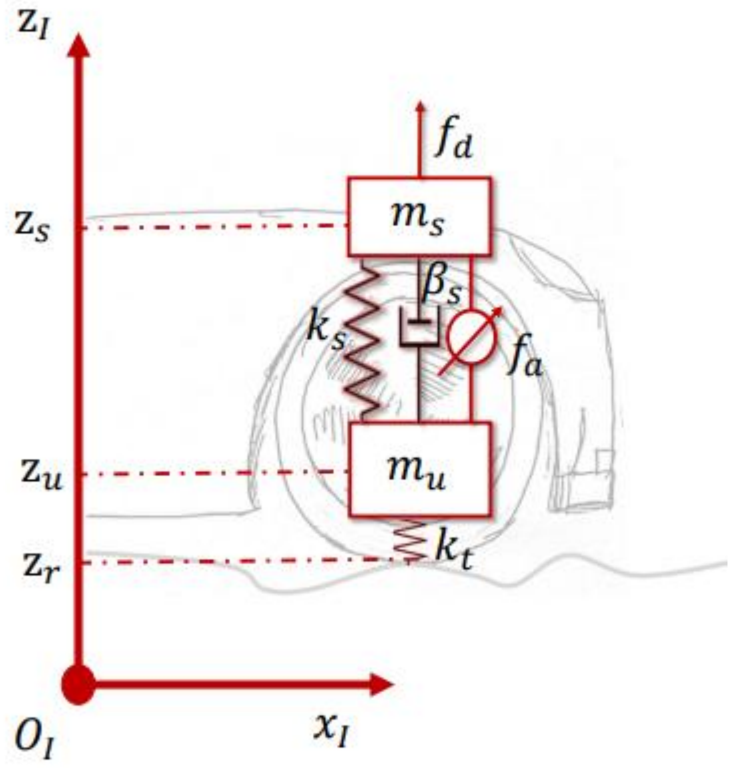
Unsprung mass $$m_u \ddot{z}_u = -m_u g - f_s + f_t(z_u - z_r)$$

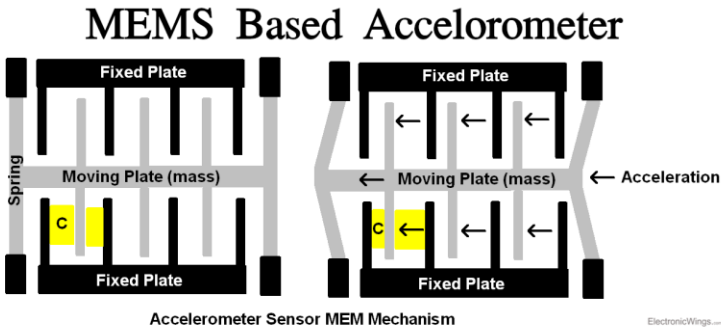
Tire $$f_t(z_u - z_r) = \begin{cases} 0 & z_u - z_r > \ell_t \\ -k_t(z_u - z_r - \ell_t) & z_u - z_r \leq \ell_t \end{cases}$$

Suspension $$f_s := -k_s(z_s - z_u - \ell_s) - \beta_s(\dot{z}_s - \dot{z}_u) + f_a$$

Aerodynamics (downforce) $$f_d = \frac{1}{2}\rho S v^2 C_z$$

Non-linear 1D Model





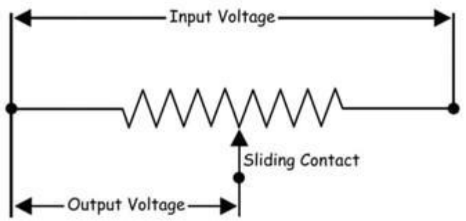
Sensors

Accelerometer (z)

$$y_a = -g - \frac{k_s}{m_s}(z_s - z_u - \ell_s) - \frac{\beta_s}{m_s}(\dot{z}_s - \dot{z}_u) + \frac{f_a + f_d}{m_s} + \nu_a$$

Potentiometer

$$y_p = z_s - z_u + \nu_p$$



Regulated Output

$$e = y_p - r$$

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Problem Formulation

let $\dot{z}_s := v_s$ and $\dot{z}_u := v_u$

$$
\begin{aligned}
\dot{z}_s &= v_s \\
m_s \dot{v}_s &= -m_s g - k_s(z_s - z_u - \ell_s) - \beta_s(v_s - v_u) + f_a + f_d \\
\dot{z}_u &= v_u \\
m_u \dot{v}_u &= -m_u g + k_s(z_s - z_u - \ell_s) + \beta_s(v_s - v_u) - f_a + f_t(z_u - z_r)
\end{aligned}
$$

Problem Formulation

Define

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} z_s - z_u \\ v_s - v_u \\ z_u - z_r \\ v_u - \dot{z}_r \end{bmatrix}$$

let $\dot{z}_s := v_s$ and $\dot{z}_u := v_u$

$$\begin{aligned} \dot{z}_s &= v_s \\ m_s \dot{v}_s &= -m_s g - k_s(z_s - z_u - \ell_s) - \beta_s(v_s - v_u) + f_a + f_d \\ \dot{z}_u &= v_u \\ m_u \dot{v}_u &= -m_u g + k_s(z_s - z_u - \ell_s) + \beta_s(v_s - v_u) - f_a + f_t(z_u - z_r) \end{aligned}$$

Problem Formulation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{m_s + m_u}{m_s m_u} k_s(x_1 - \ell_s) - \frac{m_s + m_u}{m_s m_u} \beta_s x_2 + \frac{m_s + m_u}{m_s m_u} f_a + \frac{1}{m_s} f_d - \frac{1}{m_u} f_t(x_3)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -g + \frac{k_s}{m_u}(x_1 - \ell_s) + \frac{\beta_s}{m_u} x_2 - \frac{f_a}{m_u} + \frac{1}{m_u} f_t(x_3) - \ddot{z}_r$$

Define

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} z_s - z_u \\ v_s - v_u \\ z_u - z_r \\ v_u - \dot{z}_r \end{bmatrix}$$

let $\dot{z}_s := v_s$ and $\dot{z}_u := v_u$

$$\begin{aligned} \dot{z}_s &= v_s \\ m_s \dot{v}_s &= -m_s g - k_s(z_s - z_u - \ell_s) - \beta_s(v_s - v_u) + f_a + f_d \\ \dot{z}_u &= v_u \\ m_u \dot{v}_u &= -m_u g + k_s(z_s - z_u - \ell_s) + \beta_s(v_s - v_u) - f_a + f_t(z_u - z_r) \end{aligned}$$

Problem Formulation

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\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{m_s + m_u}{m_s m_u} k_s (x_1 - \ell_s) - \frac{m_s + m_u}{m_s m_u} \beta_s x_2 + \frac{m_s + m_u}{m_s m_u} f_a + \frac{1}{m_s} f_d - \frac{1}{m_u} f_t(x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -g + \frac{k_s}{m_u}(x_1 - \ell_s) + \frac{\beta_s}{m_u} x_2 - \frac{f_a}{m_u} + \frac{1}{m_u} f_t(x_3) - \ddot{z}_r
\end{aligned}
$$

$$
\begin{aligned}
y_p &= x_1 + \nu_p \\
y_a &= -g - \frac{k_s}{m_s}(x_1 - \ell_s) - \frac{\beta_s}{m_s} x_2 + \frac{f_a + f_d}{m_s} + \nu_a
\end{aligned}
$$

$$
e = x_1 + \nu_p - r.
$$

Problem Formulation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{m_s+m_u}{m_s m_u} k_s(x_1-\ell_s) - \frac{m_s+m_u}{m_s m_u}\beta_s x_2 + \frac{m_s+m_u}{m_s m_u} f_a + \frac{1}{m_s} f_d - \frac{1}{m_u} f_t(x_3) \\
\dot{x}_3 &= x_4 \\
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\end{aligned}
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$$
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\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x} &= \mathrm{col}(x_1, x_2, x_3, x_4), \quad u = f_a, \\
\mathbf{d} &= \mathrm{col}(\ddot{z}_r,\ f_d), \\
\boldsymbol{\nu} &= \mathrm{col}(\nu_p, \nu_a), \\
\mathbf{w} &= \mathrm{col}(\mathbf{d}, \boldsymbol{\nu}, r),
\end{aligned}
$$

$$
e = x_1 + \nu_p - r.
$$

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, u, \mathbf{w}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\
\mathbf{y} &= \mathbf{h}(\mathbf{x}, u, \mathbf{w}) \\
e &= h_e(\mathbf{x}, u, \mathbf{w}),
\end{aligned}
$$

Problem Formulation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{m_s+m_u}{m_s m_u} k_s(x_1-\ell_s) - \frac{m_s+m_u}{m_s m_u}\beta_s x_2 + \frac{m_s+m_u}{m_s m_u} f_a + \frac{1}{m_s} f_d - \frac{1}{m_u} f_t(x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -g + \frac{k_s}{m_u}(x_1-\ell_s) + \frac{\beta_s}{m_u} x_2 - \frac{f_a}{m_u} + \frac{1}{m_u} f_t(x_3) - \ddot{z}_r
\end{aligned}
$$

$$
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y_p &= x_1 + \nu_p \\
y_a &= -g - \frac{k_s}{m_s}(x_1-\ell_s) - \frac{\beta_s}{m_s} x_2 + \frac{f_a+f_d}{m_s} + \nu_a
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\boldsymbol{\nu} &= \mathrm{col}(\nu_p, \nu_a), \\
\mathbf{w} &= \mathrm{col}(\mathbf{d}, \boldsymbol{\nu}, r),
\end{aligned}
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e = x_1 + \nu_p - r.
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\mathbf{y} &= \mathbf{h}(\mathbf{x}, u, \mathbf{w}) \\
e &= h_e(\mathbf{x}, u, \mathbf{w}),
\end{aligned}
$$

Control Goals

1) Keep all the signals bounded
2) Asymptotically steer the regulated output to zero assuming constant disturbances.

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Linearisation

assume $u_0 = 0,\ \mathbf{d}_0,\ \boldsymbol{\nu}_0 = \mathbf{0}, \mathbf{x}_0 := \mathrm{col}(x_{1_0}, x_{2_0}, x_{3_0}, x_{4_0}), \mathbf{y}_0 := \mathrm{col}(y_{p0}, y_{a0})$, and $r = r_0$

Impose $\dot{\mathbf{x}} = \mathbf{0}$

$$
\begin{aligned}
0 &= x_{2_0} \\
0 &= -\frac{m_s + m_u}{m_s m_u} k_s (x_{1_0} - \ell_s) + \frac{k_t}{m_u}(x_{3_0} - \ell_t) \\
0 &= x_{4_0} \\
0 &= -g + \frac{k_s}{m_u}(x_{1_0} - \ell_s) - \frac{k_t}{m_u}(x_{3_0} - \ell_t) \\
y_{p0} &= x_{1_0} \\
y_{a0} &= -g - \frac{k_s}{m_s}(x_{1_0} - \ell_s) \\
r_0 &= x_{1_0},
\end{aligned}
$$

$$
\begin{bmatrix} x_{1_0} \\ x_{2_0} \\ x_{3_0} \\ x_{4_0} \end{bmatrix} = \begin{bmatrix} \ell_s - g\dfrac{m_s}{k_s} \\ 0 \\ \ell_t - g\dfrac{m_s + m_u}{k_t} \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} y_{p0} \\ y_{a0} \end{bmatrix} = \begin{bmatrix} \ell_s - g\dfrac{m_s}{k_s} \\ 0 \end{bmatrix}
$$

$$
r_0 = \ell_s - g\frac{m_s}{k_s}.
$$

Linearisation

Define the errors to the equilibrium point as

$$\tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}_0,\ \tilde{u} := u - u_0,$$
$$\tilde{\mathbf{d}} := \mathbf{d} - \mathbf{d}_0,\ \tilde{\boldsymbol{\nu}} = \boldsymbol{\nu} - \boldsymbol{\nu}_0,$$
$$\tilde{r} := r - r_0,\ \tilde{\mathbf{w}} = \mathrm{col}(\tilde{d}, \tilde{\boldsymbol{\nu}}, \tilde{\mathbf{r}}),$$
$$\tilde{\mathbf{y}} := \mathbf{y} - \mathbf{y}_0$$
$$\tilde{e} := e - 0$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s+m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & m_s^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To obtain

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{u} + \mathbf{B}_2\tilde{\mathbf{w}} \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0$$
$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1\tilde{u} + \mathbf{D}_2\tilde{\mathbf{w}}$$
$$\tilde{e} = \mathbf{C}_e\tilde{\mathbf{x}} + \mathbf{D}_{e2}\tilde{\mathbf{w}}.$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 \end{bmatrix},\ \mathbf{C}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 \\ m_s^{-1} \end{bmatrix},\ \mathbf{D}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & m_s^{-1} & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D}_{e2} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

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Linear Model Investigation

Let us study **A**

How many eigenvalues?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \dfrac{m_s + m_u}{m_s m_u} & -\beta_s \dfrac{m_s + m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Linear Model Investigation

Let us study **A**

How many eigenvalues?

Numerically …

```
K>> eig(A)

ans =

 -11.4354 +60.8968i
 -11.4354 -60.8968i
  -1.6757 + 7.5142i
  -1.6757 - 7.5142i
```

```
A =

   1.0e+03 *

         0    0.0010         0         0
   -0.4196   -0.0262    3.5556         0
         0         0         0    0.0010
    0.3556    0.0222   -3.5556         0
```

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        0    0.0010         0         0
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        0         0         0    0.0010
   0.3556    0.0222   -3.5556         0
```

Let $\lambda_{1,2} = -\alpha_1 \pm i\beta_1$ and $\lambda_{3,4} = -\alpha_2 \pm i\beta_2$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = e^{-\alpha_1 t} \begin{bmatrix} \cos(\beta_1 t) & \sin(\beta_1 t) \\ -\sin(\beta_1 t) & \cos(\beta_1 t) \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

$$\begin{bmatrix} z_3(t) \\ z_4(t) \end{bmatrix} = e^{-\alpha_2 t} \begin{bmatrix} \cos(\beta_2 t) & \sin(\beta_2 t) \\ -\sin(\beta_2 t) & \cos(\beta_2 t) \end{bmatrix} \begin{bmatrix} z_3(0) \\ z_4(0) \end{bmatrix}$$

Linear Model Investigation

Let us study **A**

$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$

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```

$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$

```
V =

    0.0021    0.0114   -0.0279   -0.1251
   -0.7162         0    0.9865         0
   -0.0027   -0.0109    0.0035   -0.0127
    0.6965   -0.0411    0.0893    0.0477
```

Let $\lambda_{1,2} = -\alpha_1 \pm i\beta_1$ and $\lambda_{3,4} = -\alpha_2 \pm i\beta_2$

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```
V =
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.0021 & 0.0114 & -0.0279 & -0.1251 \\ -0.7162 & 0 & 0.9865 & 0 \\ -0.0027 & -0.0109 & 0.0035 & -0.0127 \\ 0.6965 & -0.0411 & 0.0893 & 0.0477 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_4 \end{bmatrix}$$

Let $\lambda_{1,2} = -\alpha_1 \pm i\beta_1$ and $\lambda_{3,4} = -\alpha_2 \pm i\beta_2$

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$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$

V =

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.0021 & 0.0114 & -0.0279 & -0.1251 \\ -0.7162 & 0 & 0.9865 & 0 \\ -0.0027 & -0.0109 & 0.0035 & -0.0127 \\ 0.6965 & -0.0411 & 0.0893 & 0.0477 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.2816 & 6.8250 & 16.7559 & 75.1375 \\ 42.0622 & 0 & 57.9378 & 0 \\ 9.1364 & 36.6087 & 11.7927 & 42.4622 \\ 79.6353 & 4.7001 & 10.2131 & 5.4516 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_4 \end{bmatrix}$$

Normalised (by row)

$$\bar{V}(i,:) = \frac{|V(i,:)|}{\sum_j |V(i,j)|}$$

Linear Model Investigation

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How many eigenvalues?

Numerically …

$$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$$

V =

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.0021 & 0.0114 & -0.0279 & -0.1251 \\ -0.7162 & 0 & 0.9865 & 0 \\ -0.0027 & -0.0109 & 0.0035 & -0.0127 \\ 0.6965 & -0.0411 & 0.0893 & 0.0477 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.2816 & 6.8250 & 16.7559 & 75.1375 \\ 42.0622 & 0 & 57.9378 & 0 \\ 9.1364 & 36.6087 & 11.7927 & 42.4622 \\ 79.6353 & 4.7001 & 10.2131 & 5.4516 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_4 \end{bmatrix}$$

Normalised (by row)

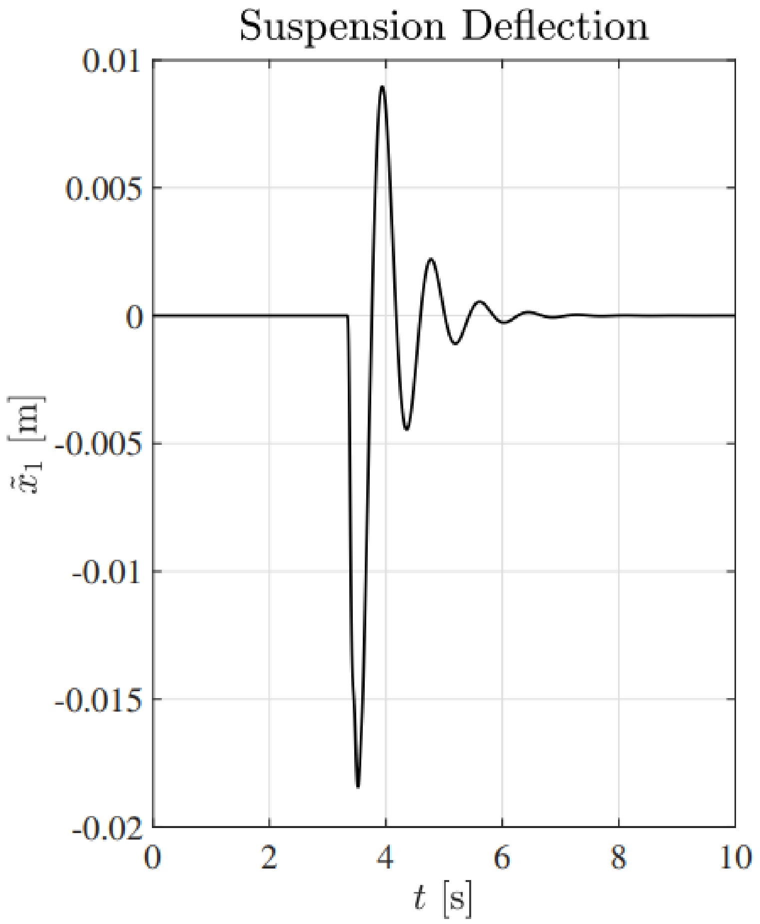
$$\bar{V}(i,:) = \frac{|V(i,:)|}{\sum_j |V(i,j)|}$$

Linear Model Investigation

Let us study **A**

How many eigenvalues?

Numerically …



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Linear Model Investigation

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Normalised (by row)

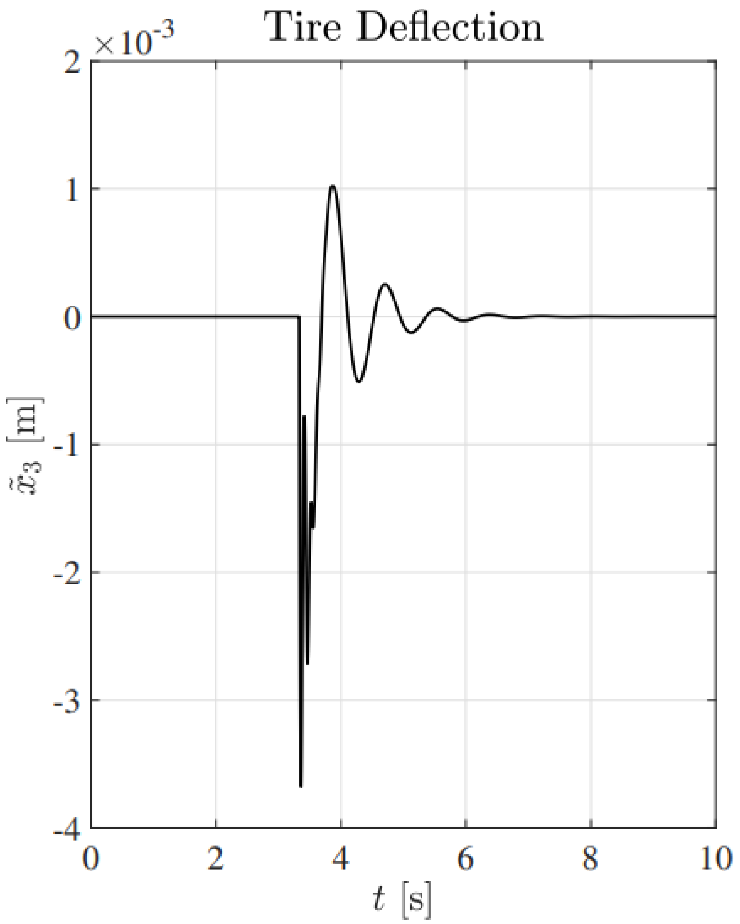
$$\bar{V}(i,:) = \frac{|V(i,:)|}{\sum_j |V(i,j)|}$$

Linear Model Investigation

Let us study **A**

How many eigenvalues?

Numerically …



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Normalised (by row)

$$\bar{V}(i,:) = \frac{|V(i,:)|}{\sum_j |V(i,j)|}$$

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \dfrac{m_s + m_u}{m_s m_u} & -\beta_s \dfrac{m_s + m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s + m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix}$$

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\frac{m_s+m_u}{m_s m_u} & -\beta_s\frac{m_s+m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

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$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \frac{m_s+m_u}{m_s m_u} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix}$$

$$\mathbf{R} = \left[\ \mathbf{B}_1 \quad \cdots \quad \mathbf{A}^3\mathbf{B}_1\ \right]$$

$$= \frac{1}{m_u}\begin{bmatrix} 0 & 0 & -1 & \frac{\bar{m}\beta_s}{m_u} \\ \bar{m} & -\frac{\bar{m}^2\beta_s}{m_u} & \frac{\bar{m}^3{\beta_s}^2-k_t m_u}{{m_u}^2} & \frac{\bar{m}\left(2k_t\beta_s m_u+k_s{m_u}^2-\bar{m}^3{\beta_s}^3\right)}{{m_u}^3} \\ 0 & -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u-\bar{m}^2{\beta_s}^2}{{m_u}^2} \\ -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u-\bar{m}^2{\beta_s}^2}{{m_u}^2} & \frac{\bar{m}^3{\beta_s}^3-k_t\beta_s m_u(1+\bar{m})}{{m_u}^3} - \frac{k_s}{m_s} \end{bmatrix}$$

Fully reachable

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\frac{m_s+m_u}{m_s m_u} & -\beta_s\frac{m_s+m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \frac{m_s+m_u}{m_s m_u} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix}$$

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Fully reachable, on the paper … but numerically

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \dfrac{m_s + m_u}{m_s m_u} & -\beta_s \dfrac{m_s + m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s + m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix}$$

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Fully reachable, on the paper … but numerically

```
R =

   1.0e+03 *

         0    0.0000   -0.0007   -0.0720
    0.0000   -0.0007   -0.0720    4.2479
         0   -0.0000    0.0006    0.0731
   -0.0000    0.0006    0.0731   -3.9160
```

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\frac{m_s+m_u}{m_s m_u} & -\beta_s\frac{m_s+m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \frac{m_s+m_u}{m_s m_u} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{R} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^3\mathbf{B}_1 \end{bmatrix}$$

$$= \frac{1}{m_u}\begin{bmatrix} 0 & 0 & -1 & \frac{\bar{m}\beta_s}{m_u} \\ \bar{m} & -\frac{\bar{m}^2\beta_s}{m_u} & \frac{\bar{m}^3\beta_s{}^2 - k_t m_u}{m_u{}^2} & \frac{\bar{m}\left(2k_t\beta_s m_u + k_s m_u{}^2 - \bar{m}^3\beta_s{}^3\right)}{m_u{}^3} \\ 0 & -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u - \bar{m}^2\beta_s{}^2}{m_u{}^2} \\ -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u - \bar{m}^2\beta_s{}^2}{m_u{}^2} & \frac{\bar{m}^3\beta_s{}^3 - k_t\beta_s m_u(1+\bar{m})}{m_u{}^3} - \frac{k_s}{m_s} \end{bmatrix}$$

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```

Linear Model Investigation

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Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s+m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^3\mathbf{B}_1 \end{bmatrix}$$

$$= \frac{1}{m_u}\begin{bmatrix} 0 & 0 & -1 & \frac{\bar{m}\beta_s}{m_u} \\ \bar{m} & -\frac{\bar{m}^2\beta_s}{m_u} & \frac{\bar{m}^3\beta_s{}^2-k_t m_u}{m_u{}^2} & \frac{\bar{m}\left(2k_t\beta_s m_u+k_s m_u{}^2-\bar{m}^3\beta_s{}^3\right)}{m_u{}^3} \\ 0 & -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u-\bar{m}^2\beta_s{}^2}{m_u{}^2} \\ -1 & \frac{\bar{m}\beta_s}{m_u} & \frac{k_t m_u-\bar{m}^2\beta_s{}^2}{m_u{}^2} & \frac{\bar{m}^3\beta_s{}^3-k_t\beta_s m_u(1+\bar{m})}{m_u{}^3}-\frac{k_s}{m_s} \end{bmatrix}$$

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    0.0000   -0.0007   -0.0720    4.2479
         0   -0.0000    0.0006    0.0731
   -0.0000    0.0006    0.0731   -3.9160
```

$$\bar{R}(:,i) = \frac{R(:,i)}{|R(:,i)|}$$

Normalised (by column)

```
         0    0.0291   -0.0067   -0.0125
    0.7629   -0.7623   -0.7018    0.7351
         0   -0.0246    0.0057    0.0126
   -0.6465    0.6461    0.7123   -0.6777
```

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{R} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^3\mathbf{B}_1 \end{bmatrix} \qquad \bar{m} = (m_s + m_u)/m_s$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s+m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix}$$

$$= \frac{1}{m_u}\begin{bmatrix} 0 & 0 & -1 & \dfrac{\bar{m}\beta_s}{m_u} \\ \bar{m} & -\dfrac{\bar{m}^2\beta_s}{m_u} & \dfrac{\bar{m}^3\beta_s{}^2 - k_t m_u}{m_u{}^2} & \dfrac{\bar{m}\left(2k_t\beta_s m_u + k_s m_u{}^2 - \bar{m}^3\beta_s{}^3\right)}{m_u{}^3} \\ 0 & -1 & \dfrac{\bar{m}\beta_s}{m_u} & \dfrac{k_t m_u - \bar{m}^2\beta_s{}^2}{m_u{}^2} \\ -1 & \dfrac{\bar{m}\beta_s}{m_u} & \dfrac{k_t m_u - \bar{m}^2\beta_s{}^2}{m_u{}^2} & \dfrac{\bar{m}^3\beta_s{}^3 - k_t\beta_s m_u(1+\bar{m})}{m_u{}^3} - \dfrac{k_s}{m_s} \end{bmatrix}$$

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```

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Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

$$\mathbf{R} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{A}^3\mathbf{B}_1 \end{bmatrix} \qquad \bar{m} = (m_s + m_u)/m_s$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{m_s+m_u}{m_s m_u} \\ 0 \\ -\dfrac{1}{m_u} \end{bmatrix}$$

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```
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```

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Normalised (by column)

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Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

Reachability of $(\mathbf{A}, \mathbf{B}_1)$

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Fully reachable, on the paper … but numerically

```
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Linear Model Investigation

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \frac{m_s + m_u}{m_s m_u} & -\beta_s \frac{m_s + m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \dfrac{m_s + m_u}{m_s m_u} & -\beta_s \dfrac{m_s + m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \vdots \\ \mathbf{CA}^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{k_s\,\bar{m}}{m_u} & -\dfrac{\beta_s\,\bar{m}}{m_u} & \dfrac{k_t}{m_u} & 0 \\ \dfrac{\beta_s\,k_s\,\bar{m}^2}{{m_u}^2} & \dfrac{{\beta_s}^2\,\bar{m}^2}{{m_u}^2} - \dfrac{k_s\,\bar{m}}{m_u} & -\dfrac{\beta_s\,k_t\,\bar{m}}{{m_u}^2} & \dfrac{k_t}{m_u} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \dfrac{m_s + m_u}{m_s m_u} & -\beta_s \dfrac{m_s + m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \vdots \\ \mathbf{C}\mathbf{A}^3 \end{bmatrix}$$

Fully observable, on the paper …

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{k_s\,\bar{m}}{m_u} & -\dfrac{\beta_s\,\bar{m}}{m_u} & \dfrac{k_t}{m_u} & 0 \\ \dfrac{\beta_s\,k_s\,\bar{m}^2}{{m_u}^2} & \dfrac{{\beta_s}^2\,\bar{m}^2}{{m_u}^2} - \dfrac{k_s\,\bar{m}}{m_u} & -\dfrac{\beta_s\,k_t\,\bar{m}}{{m_u}^2} & \dfrac{k_t}{m_u} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Linear Model Investigation

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \vdots \\ \mathbf{CA}^3 \end{bmatrix}$$

Fully observable, on the paper … but numerically

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{k_s\,\bar{m}}{m_u} & -\frac{\beta_s\,\bar{m}}{m_u} & \frac{k_t}{m_u} & 0 \\ \frac{\beta_s\,k_s\,\bar{m}^2}{{m_u}^2} & \frac{{\beta_s}^2\,\bar{m}^2}{{m_u}^2} - \frac{k_s\,\bar{m}}{m_u} & -\frac{\beta_s\,k_t\,\bar{m}}{{m_u}^2} & \frac{k_t}{m_u} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\frac{m_s+m_u}{m_s m_u} & -\beta_s\frac{m_s+m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

```
1.0e+07 *

 0.0000         0         0         0
-0.0000   -0.0000         0         0
      0    0.0000         0         0
 0.0002    0.0000   -0.0014         0
-0.0000   -0.0000    0.0004         0
-0.0017    0.0001    0.0145   -0.0014
 0.0011    0.0000   -0.0093    0.0004
-0.5311   -0.0349    5.2723    0.0145
```

Linear Model Investigation

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \vdots \\ \mathbf{CA}^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{k_s\,\bar{m}}{m_u} & -\frac{\beta_s\,\bar{m}}{m_u} & \frac{k_t}{m_u} & 0 \\ \frac{\beta_s\,k_s\,\bar{m}^2}{{m_u}^2} & \frac{{\beta_s}^2\,\bar{m}^2}{{m_u}^2} - \frac{k_s\,\bar{m}}{m_u} & -\frac{\beta_s\,k_t\,\bar{m}}{{m_u}^2} & \frac{k_t}{m_u} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Fully observable, on the paper … but numerically

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\frac{m_s+m_u}{m_s m_u} & -\beta_s\frac{m_s+m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

```
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0.0000	0	0	0
-0.0000	-0.0000	0	0
0	0.0000	0	0
0.0002	0.0000	-0.0014	0
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Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

Observability of $(\mathbf{A}, \mathbf{C})$

```
O =

    1.0000         0         0         0
   -0.9981   -0.0624         0         0
         0    1.0000         0         0
    0.1172    0.0029   -0.9931         0
   -0.1172   -0.0073    0.9931         0
   -0.1166    0.0041    0.9884   -0.0967
    0.1171    0.0029   -0.9924    0.0378
   -0.1002   -0.0066    0.9949    0.0027
```

Normalised (by rows)

$$\bar{O}(i,:) = \frac{O(i,:)}{|O(i,:)|}$$

```
1.0e+07 *

    0.0000         0         0         0
   -0.0000   -0.0000         0         0
         0    0.0000         0         0
    0.0002    0.0000   -0.0014         0
   -0.0000   -0.0000    0.0004         0
   -0.0017    0.0001    0.0145   -0.0014
    0.0011    0.0000   -0.0093    0.0004
   -0.5311   -0.0349    5.2723    0.0145
```

Linear Model Investigation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s \frac{m_s + m_u}{m_s m_u} & -\beta_s \frac{m_s + m_u}{m_s m_u} & \frac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{\beta_s}{m_u} & -\frac{k_t}{m_u} & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{\beta_s}{m_s} & 0 & 0 \end{bmatrix}$$

Observability of $(\mathbf{A}, \mathbf{C})$

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1.0000	0	0	0
-0.9981	-0.0624	0	0
0	1.0000	0	0
0.1172	0.0029	-0.9931	0
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-0.0000	-0.0000	0.0004	0
-0.0017	0.0001	0.0145	-0.0014
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-0.5311	-0.0349	5.2723	0.0145

Linear Model Investigation

Observability of $(\mathbf{A}, \mathbf{C})$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_s\dfrac{m_s+m_u}{m_s m_u} & -\beta_s\dfrac{m_s+m_u}{m_s m_u} & \dfrac{k_t}{m_u} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{m_u} & \dfrac{\beta_s}{m_u} & -\dfrac{k_t}{m_u} & 0 \end{bmatrix}$$

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O =
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1.0000	0	0	0
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0.1172	0.0029	-0.9931	0
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```
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-0.0000	-0.0000	0.0004	0
-0.0017	0.0001	0.0145	-0.0014
0.0011	0.0000	-0.0093	0.0004
-0.5311	-0.0349	5.2723	0.0145

Outline

- Motivations & Goals
- Non-linear 1D Model
- Control Problem Formulation
- Linearisation
- Linear Model Investigation
- Control Architecture
- Control Design
- Performance Evaluation

Control System Architecture

- Open-loop BIBS (we want to modify the eig.s)
- Const. disturbance/reference
- No time-varying reference
- Non-fully measurable state

Control System Architecture

- Open-loop BIBS (we want to modify the eig.s) $\Rightarrow$
- Const. disturbance/reference
- No time-varying reference
- Non-fully measurable state

- Stabiliser

Control System Architecture

- Open-loop BIBS (we want to modify the eig.s)
- Const. disturbance/reference
- No time-varying reference
- Non-fully measurable state

- Stabiliser
- Integral Action

Control System Architecture

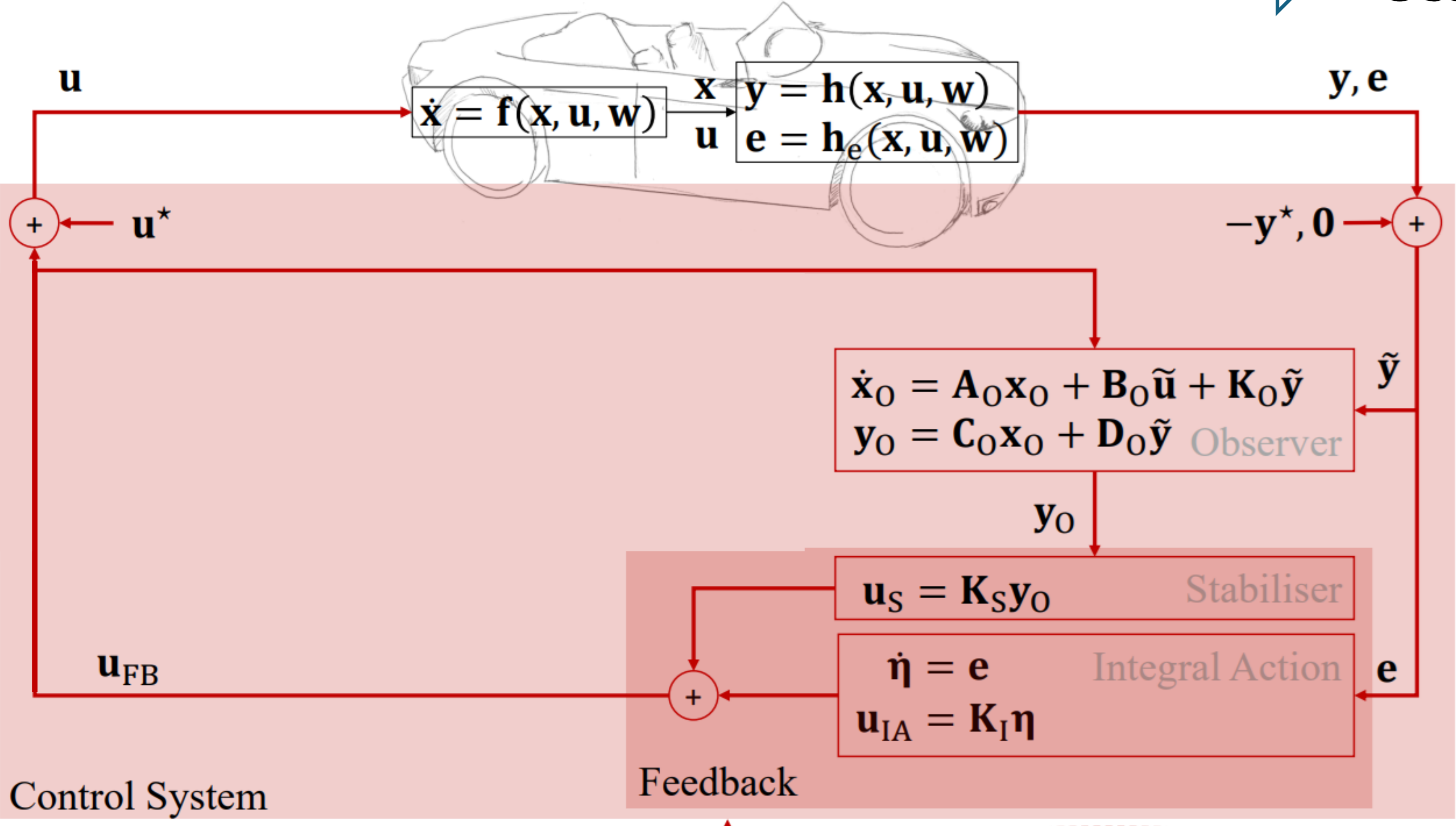
- Open-loop BIBS (we want to modify the eig.s) ⇒ Stabiliser
- Const. disturbance/reference ⇒ Integral Action
- No time-varying reference ⇒ No Feed-Forward
- Non-fully measurable state

Control System Architecture

- Open-loop BIBS (we want to modify the eig.s) $\Rightarrow$ Stabiliser
- Const. disturbance/reference $\Rightarrow$ Integral Action
- No time-varying reference $\Rightarrow$ No Feed-Forward
- Non-fully measurable state $\Rightarrow$ Observer

Control System Architecture

- Open-loop BIBS (we want to modify the eig.s) → Stabiliser
- Const. disturbance/reference → Integral Action
- No time-varying reference → No Feed-Forward
- Non-fully measurable state → Observer



Control System Architecture

Let the plant be
$$\dot{\mathbf{x}}_e = \mathbf{A}_e \mathbf{x}_e + \mathbf{B}_e \tilde{u}$$
$$\epsilon = \mathbf{C}_\epsilon \mathbf{x}_e + \mathbf{D}_\epsilon \tilde{u}$$

where

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B}_e = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{C}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

We stabilise it through $u_S = \mathbf{K}_e \mathbf{x}_O$, with $\mathbf{K}_e \coloneqq [\mathbf{K}_S \quad k_I]$, such that $\mathbf{A}_e + \mathbf{B}_e \mathbf{K}_e$ is Hurwitz.

The integral action is computed as
$$\dot{\eta} = \tilde{e}$$
$$u_{IA} = k_I \eta$$

Finally, $\tilde{u} = u_S + u_{IA}$.

The observer takes the form $\dot{\mathbf{x}}_O = (\mathbf{A} + \mathbf{K}_O \mathbf{C}) \mathbf{x}_O + \mathbf{B}_1 \tilde{\mathbf{u}} + \mathbf{K}_O \tilde{\mathbf{y}}$

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Control Design

Let the plant be $$\dot{\mathbf{x}}_e = \mathbf{A}_e \mathbf{x}_e + \mathbf{B}_e\, \tilde{u}$$
$$\boldsymbol{\epsilon} = \mathbf{C}_\epsilon \mathbf{x}_e + \mathbf{D}_\epsilon\, \tilde{u}$$

to which we associate the cost function $$J = \int_{t_0}^{\infty} \boldsymbol{\epsilon}^\top \mathbf{Q} \boldsymbol{\epsilon} + R\, \tilde{u}^2\, dt$$

With $\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix}$, $\quad \mathbf{B}_e = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$ $\quad \alpha \geq 0,\ R > 0,$ and $\mathbf{Q} = \mathbf{Q}^\top \succeq 0$

Take

1) Comfort $\quad \epsilon_1 = \begin{bmatrix} -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 & 0 \end{bmatrix} \mathbf{x}_e + m_s^{-1} u$

2) Race $\quad \epsilon_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}_e$

3) Off-road $\quad \epsilon_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}_e.$

Control Design

Impose $\boldsymbol{\epsilon} = \mathrm{col}(\epsilon_1, \epsilon_2, x_2, \epsilon_3, x_4, \eta)$

$$\mathbf{C}_\epsilon = \begin{bmatrix} -\dfrac{k_s}{m_s} & -\dfrac{\beta_s}{m_s} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{D}_\epsilon = \begin{bmatrix} m_s^{-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{Q} := 1/6\,\mathtt{diag}(|\epsilon_{1_{\max}}|^{-2}, \cdots, |\epsilon_{6_{\max}}|^{-2})$$

Take

1) Comfort $|\epsilon_{1_{\max}}| \ll |\epsilon_{2_{\max}}|, |\epsilon_{3_{\max}}|$

2) Race $|\epsilon_{2_{\max}}| \ll |\epsilon_{1_{\max}}|, |\epsilon_{3_{\max}}|$

3) Off-road $|\epsilon_{3_{\max}}| \ll |\epsilon_{1_{\max}}|, |\epsilon_{2_{\max}}|$

Remember $\bar{R} = R + \mathbf{D}_\epsilon^\top \mathbf{Q} \mathbf{D}_\epsilon$

$$\bar{R} = R + (1/3) m_s^{-2} |\epsilon_{1_{\max}}|^{-2}$$

Control Design

As for the observer

$$\mathbf{Q}_d := \mathtt{diag}(q_1, q_2, r_1, r_2, 0)$$

$$\mathbf{R}_d = \mathbf{0}$$

Where we assume

$$\mathrm{E}[\ddot{z}_r(t)\ddot{z}_r(\tau)] = q_1\delta(t-\tau) \quad \mathrm{E}[f_d(t)f_d(\tau)] = q_2\delta(t-\tau)$$
$$\mathrm{E}[\nu_p(t)\nu_p(\tau)] = r_1\delta(t-\tau) \quad \mathrm{E}[\nu_a(t)\nu_a(\tau)] = r_2\delta(t-\tau)$$

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Performance Evaluation

Settings

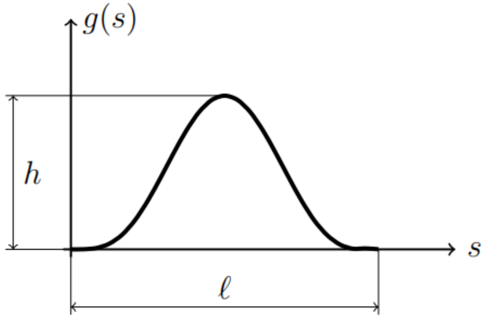
Symbol	Unit	Value		
		Comfort	Race	Off-road
ϵ_1	$\mathrm{m/s^2}$	$0.1g$	$10g$	$10g$
$\epsilon_{2_{\max}}$	m	10^4	10^{-1}	10^4
$\epsilon_{3_{\max}}$	m/s	10^4	10^4	$5 \cdot 10^{-3}$
$\epsilon_{4_{\max}}$	m	10^2	$2.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$
$\epsilon_{5_{\max}}$	m/s	10^{-1}	10^{-2}	$5 \cdot 10^{-3}$
$\epsilon_{6_{\max}}$	m s	10^{-1}	10^{-2}	10^{-2}
$u_{\max}$	N	10^3	10^3	10^3
r_1	$\mathrm{m^2}$	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	$\mathrm{m^2/s^4}$	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	$\mathrm{m^2/s^4}$	1	1	1
q_2	$\mathrm{N^2}$	1	1	1

Performance Evaluation

Comfort

Bump

$$v(t) = v_0 = 30 \text{ km/h}, \; h = 10 \text{ cm}$$



$$\ell = 2 \text{ m}.$$

Symbol	Unit	Value		
		Comfort	Race	Off-road
ϵ_1	m/s^2	$0.1g$	$10g$	$10g$
$\epsilon_{2\max}$	m	10^4	10^{-1}	10^4
$\epsilon_{3\max}$	m/s	10^4	10^4	$5 \cdot 10^{-3}$
$\epsilon_{4\max}$	m	10^2	$2.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$
$\epsilon_{5\max}$	m/s	10^{-1}	10^{-2}	$5 \cdot 10^{-3}$
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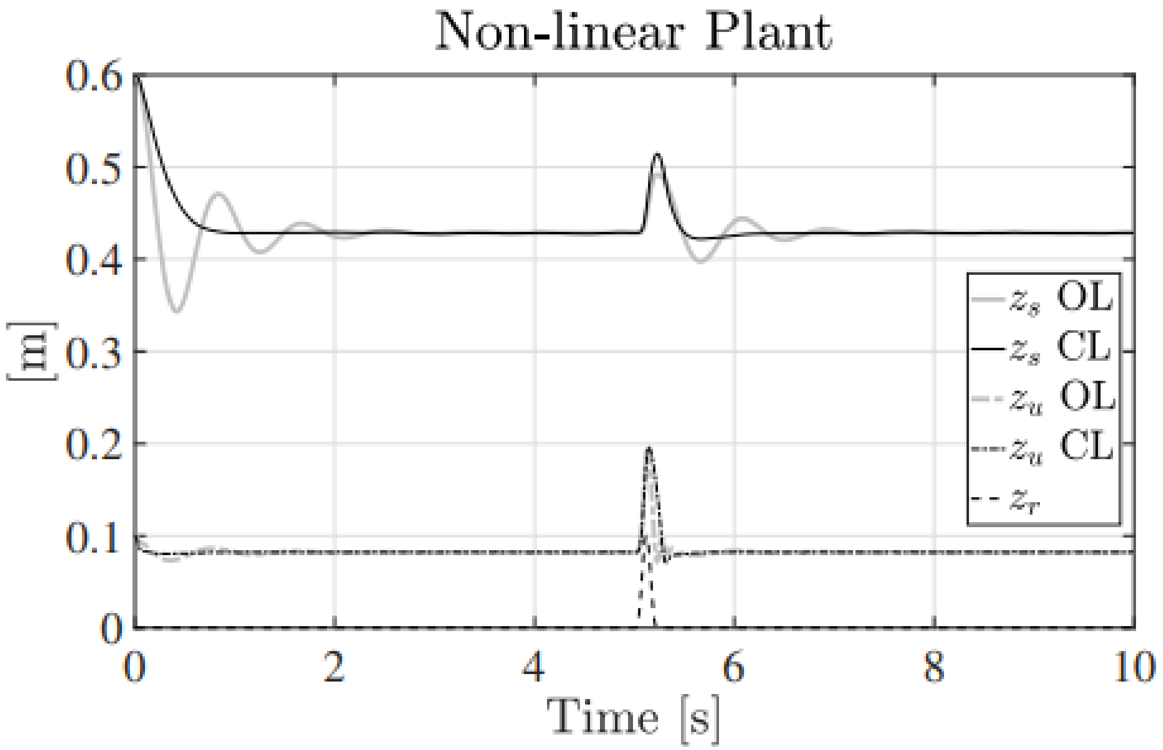
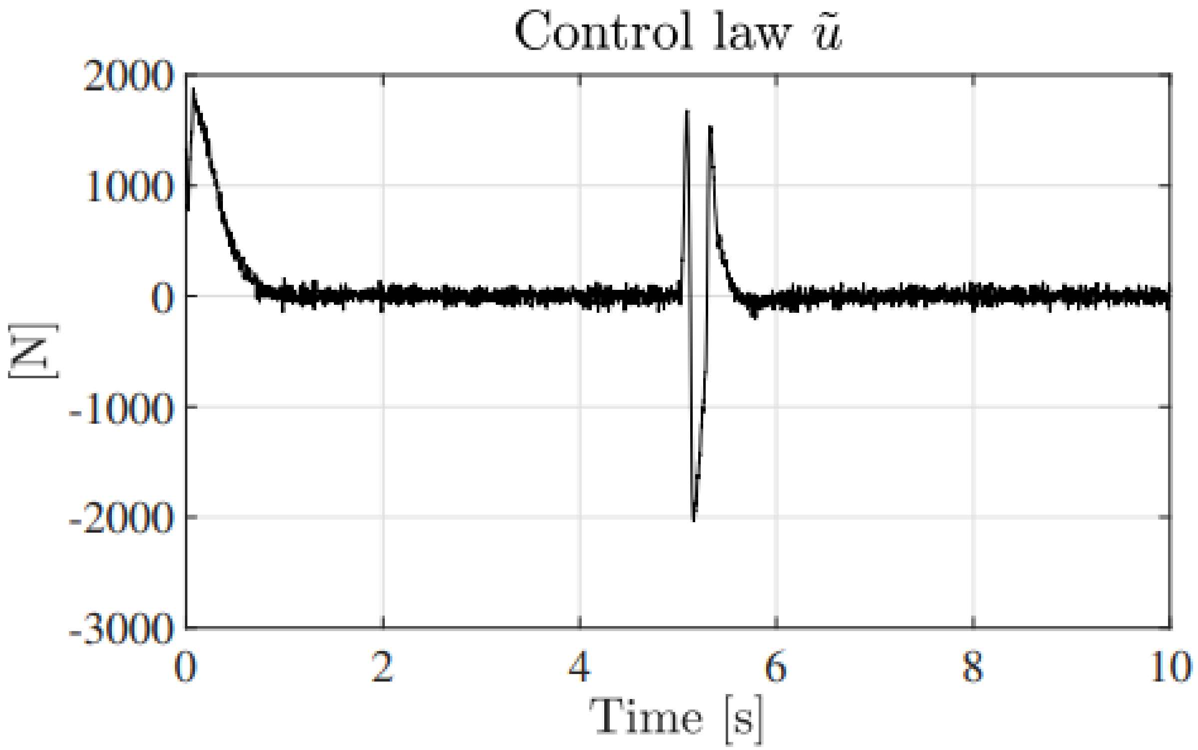
$u_{\max}$	N	10^3	10^3	10^3
r_1	m^2	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	m^2/s^4	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	m^2/s^4	1	1	1
q_2	N^2	1	1	1





Performance Evaluation

Race

Target = suspension length (piecewise constant)

Symbol	Unit	Value		
		Comfort	Race	Off-road
ϵ_1	m/s^2	$0.1g$	$10g$	$10g$
$\epsilon_{2_{\max}}$	m	10^4	10^{-1}	10^4
$\epsilon_{3_{\max}}$	m/s	10^4	10^4	$5 \cdot 10^{-3}$
$\epsilon_{4_{\max}}$	m	10^2	$2.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$
$\epsilon_{5_{\max}}$	m/s	10^{-1}	10^{-2}	$5 \cdot 10^{-3}$
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$u_{\max}$	N	10^3	10^3	10^3
r_1	m^2	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	m^2/s^4	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	m^2/s^4	1	1	1
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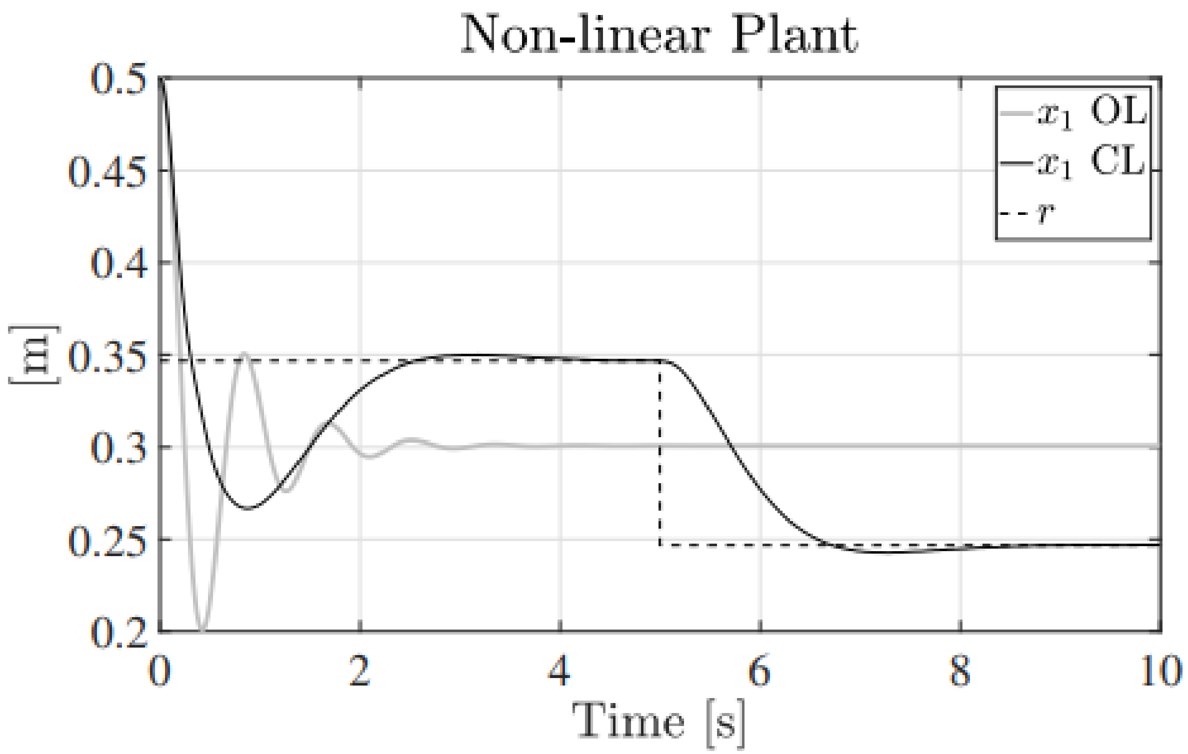

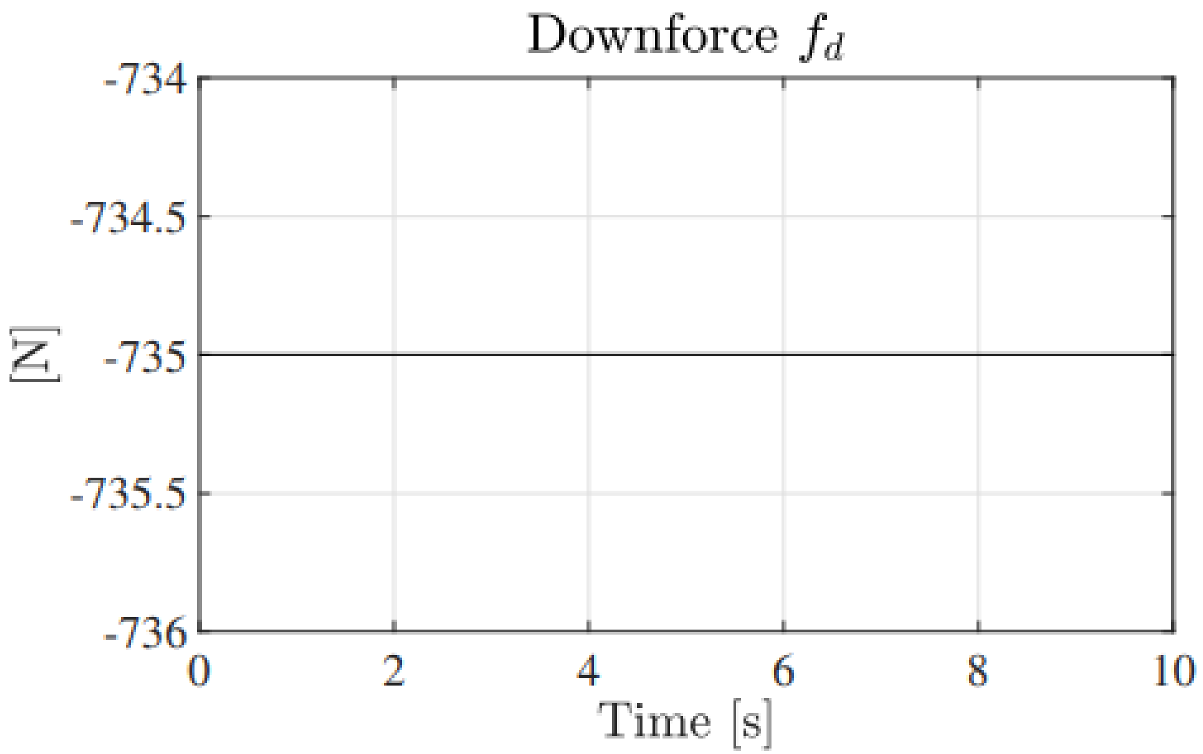

Performance Evaluation

Race

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ϵ_1	m/s^2	$0.1g$	$10g$	$10g$
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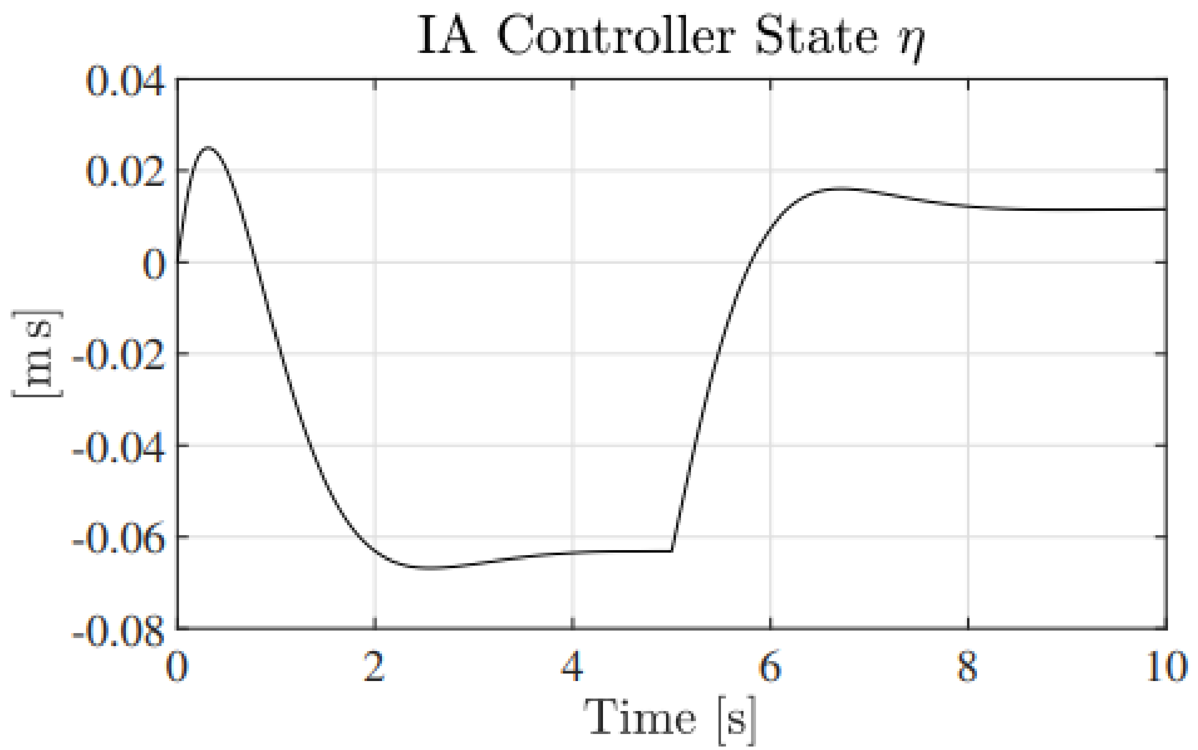
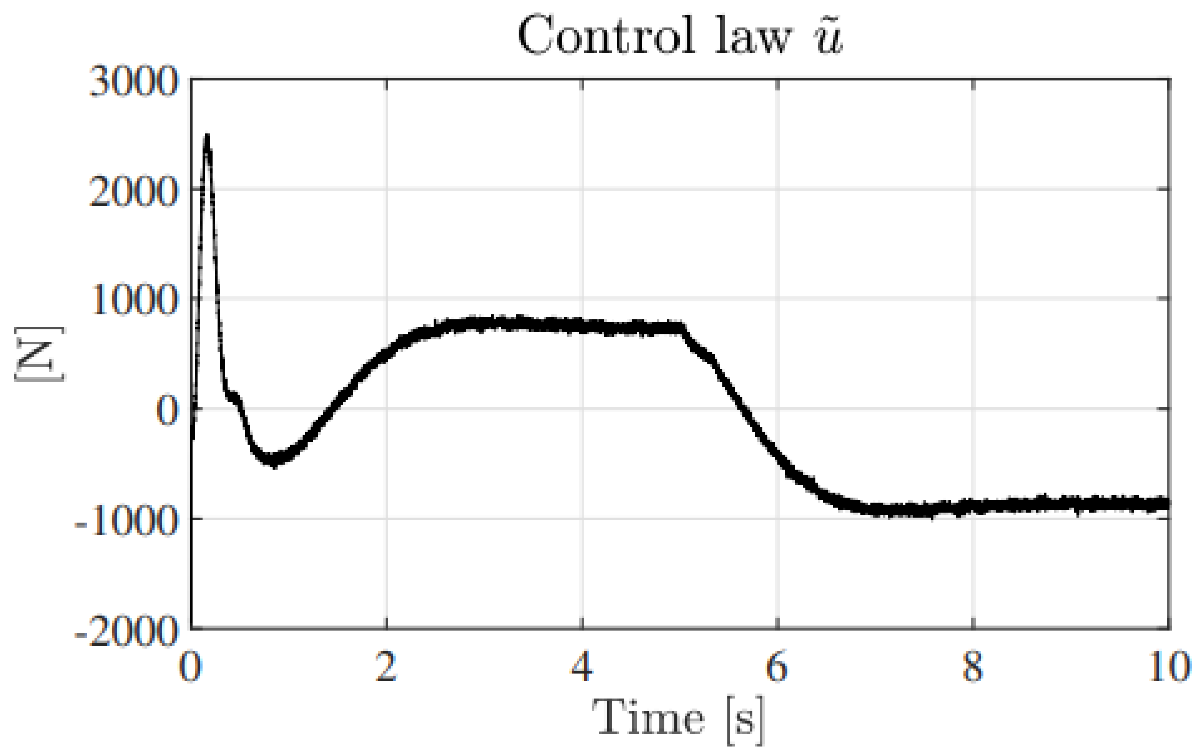
$\epsilon_{6_{max}}$	m s	10^{-1}	10^{-2}	10^{-2}
u_{max}	N	10^3	10^3	10^3
r_1	m^2	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	m^2/s^4	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	m^2/s^4	1	1	1
q_2	N^2	1	1	1





Performance Evaluation

Symbol	Unit	Value		
		Comfort	Race	Off-road
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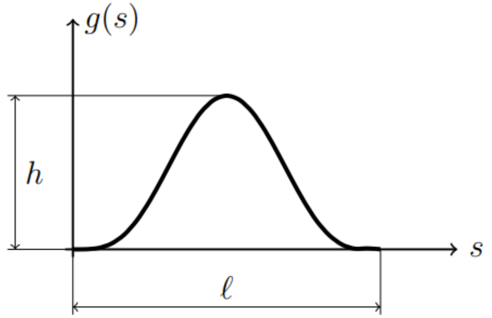
r_1	$\mathrm{m^2}$	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	$\mathrm{m^2/s^4}$	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	$\mathrm{m^2/s^4}$	1	1	1
q_2	$\mathrm{N^2}$	1	1	1

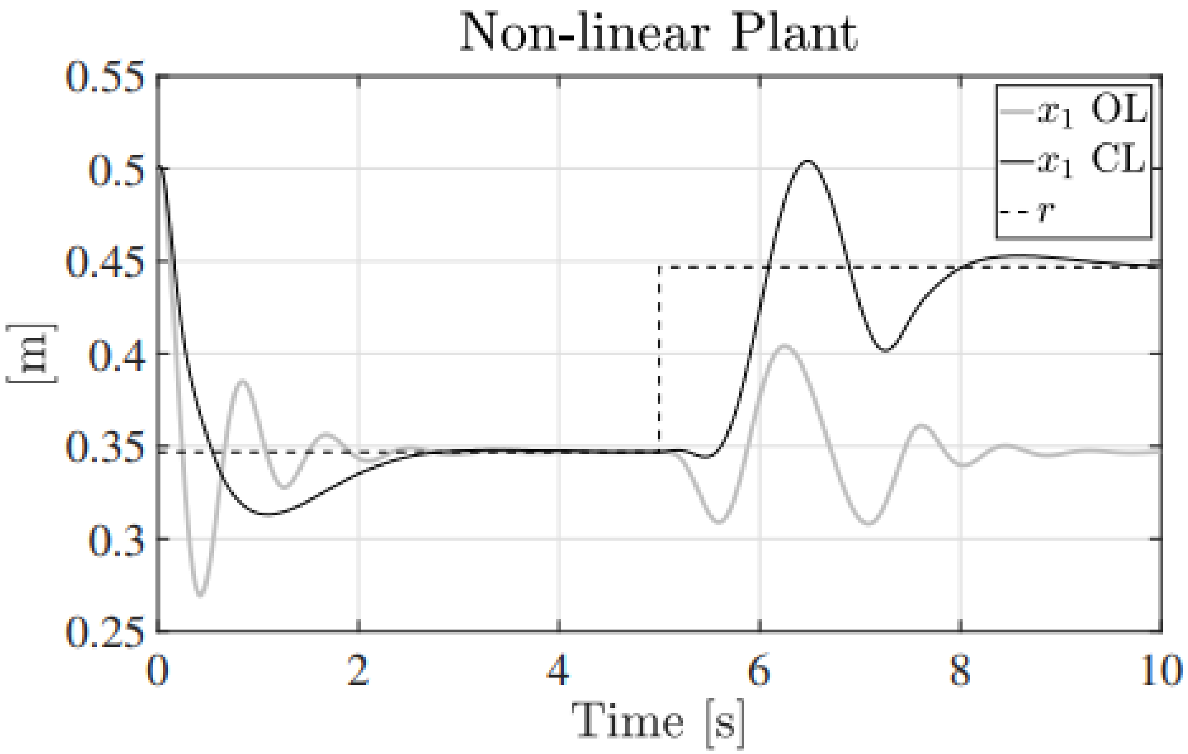
Off-road

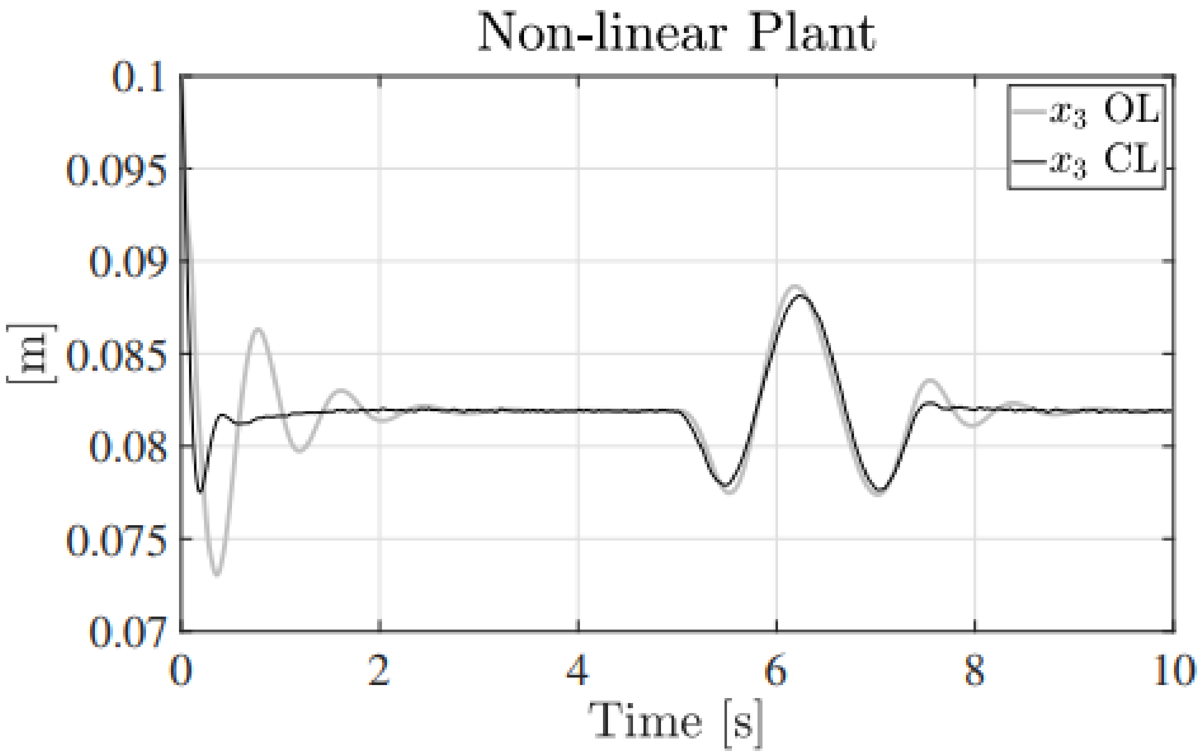
Target = tire deflection
+ longer suspension (const.)

$v(t) = v_0 = 3 \text{ km/h}, \; h = 50 \text{ cm}$



$\ell = 2$ m.



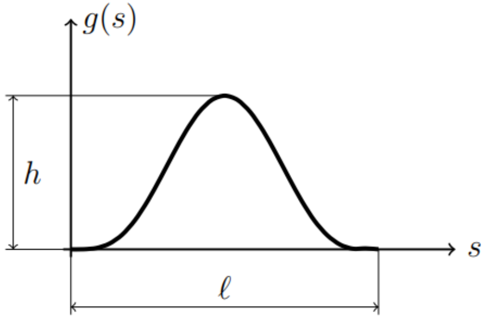


Performance Evaluation

Off-road

Target = tire deflection
+ longer suspension (const.)

$v(t) = v_0 = 3$ km/h, $h = 50$ cm



$\ell = 2$ m.

Symbol	Unit	Value: Comfort	Race	Off-road
ϵ_1	m/s^2	$0.1g$	$10g$	$10g$
$\epsilon_{2_{\max}}$	m	10^4	10^{-1}	10^4
$\epsilon_{3_{\max}}$	m/s	10^4	10^4	$5 \cdot 10^{-3}$
$\epsilon_{4_{\max}}$	m	10^2	$2.5 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$
$\epsilon_{5_{\max}}$	m/s	10^{-1}	10^{-2}	$5 \cdot 10^{-3}$
$\epsilon_{6_{\max}}$	m s	10^{-1}	10^{-2}	10^{-2}
$u_{\max}$	N	10^3	10^3	10^3
r_1	m^2	$(10^{-3})^2$	$(10^{-3})^2$	$(10^{-3})^2$
r_2	m^2/s^4	$(0.05g)^2$	$(0.05g)^2$	$(0.05g)^2$
q_1	m^2/s^4	1	1	1
q_2	N^2	1	1	1