Project 2

Archie Mead

The Idea

This method interpolates between points by drawing arcs of *circles* that connect groups of three of them. The subsets of points (x_{2i}, y_{2i}), (x_{2i+1}, y_{2i+1}), (x_{2i+2}, y_{2i+2}) each have a unique circle (or line) that intersects all three of them. We define the function:

$$S_{2i}(t) = \begin{bmatrix} a_{2i} + r_{2i}\cos(\theta_{2i} + \phi_{2i}t) \\ b_{2i} + r_{2i}\sin(\theta_{2i} + \phi_{2i}t) \end{bmatrix}$$

Where (a_i, b_i) is the center of this circle, r_i is the radius, and θ_i, ϕ_i are angles such that $S_{2i}(0) = \langle x_{2i}, y_{2i}\rangle$ and $S_{2i}(1) = \langle x_{2i+2}, y_{2i+2}\rangle$. For some value $t_{2i+1} \in [0, 1]$, we have $S_{2i}(t_{2i+1}) = \langle x_{2i+1}, y_{2i+1}\rangle$.

If the three points happen to be collinear, we instead use a line:

$$S_{2i}(t) = \begin{bmatrix} (1-t)x_{2i} + tx_{2i+2} \\ (1-t)y_{2i} + ty_{2i+2} \end{bmatrix}$$

If there is an even number of points, we also use a line to connect the last two.

This interpolation connects all points. There is usually a corner at every other point. If the figure of points is rotates, the interpolation will consistently rotate with it. The points need not be ordered left-to-right, and the path can bend in any direction.

The Computation

Let each $\vec{x}_i = \langle x_i, y_i\rangle$, and each $\vec{a}_{2i} = \langle a_{2i}, b_{2i}\rangle$.

The circle center $\vec{a}_{2i}$ must be equidistant from $\vec{x}_{2i}, \vec{x}_{2i+1}, \vec{x}_{2i+2}$.

$$||\vec{a}_{2i} - \vec{x}_{2i}||^2 = ||\vec{a}_{2i} - \vec{x}_{2i+1}||^2 = ||\vec{a}_{2i} - \vec{x}_{2i+2}||^2$$

$$||\vec{a}_{2i}||^2 - 2\vec{a}_{2i} \cdot \vec{x}_{2i} + ||\vec{x}_{2i}||^2 =$$
$$||\vec{a}_{2i}||^2 - 2\vec{a}_{2i} \cdot \vec{x}_{2i+1} + ||\vec{x}_{2i+1}||^2 =$$
$$||\vec{a}_{2i}||^2 - 2\vec{a}_{2i} \cdot \vec{x}_{2i+2} + ||\vec{x}_{2i+2}||^2$$

Use Δ notation to refer to forward differences. For example, $\Delta x_{2i} = x_{2i+1} - x_{2i}$.

$$2\vec{a}_{2i} \cdot \Delta\vec{x}_{2i} = \Delta(||\vec{x}_{2i}||^2)$$
$$2\vec{a}_{2i} \cdot \Delta\vec{x}_{2i+1} = \Delta(||\vec{x}_{2i+1}||^2)$$

$$2\begin{bmatrix} \Delta x_{2i} & \Delta y_{2i} \\ \Delta x_{2i+1} & \Delta y_{2i+1} \end{bmatrix} \begin{bmatrix} a_{2i} \\ b_{2i} \end{bmatrix} = \begin{bmatrix} \Delta(||\vec{x}_{2i}||^2) \\ \Delta(||\vec{x}_{2i+1}||^2) \end{bmatrix}$$

If the determinant of the matrix on the left is zero, the points are collinear, and we use the other formula.

Otherwise, once $\vec{a}_{2i}$ is calculated, we get:

$$r_{2i} = ||\vec{x}_{2i} - \vec{a}_{2i}||$$

To work with the angles, we use the a$tan2(x, y)$ function, which determines the angle needed to draw segment in the direction $\langle x, y\rangle$. (This matches $\arctan(y/x)$ when $x > 0$, but is off by $\pm\pi$ otherwise.)

$$\theta_{2i} = \text{atan2}(y_{2i} - b_{2i}, x_{2i} - a_{2i})$$

We want to make is so that $S_{2i}(t)$ moves in the correct direction to pass through $\vec{x}_{2i}, \vec{x}_{2i+1}, \vec{x}_{2i+2}$ in that order. We set:

$$\alpha = \text{atan2}(y_{2i+1} - b_{2i}, x_{2i+1} - a_{2i}) - \theta_{2i}$$
$$\beta = \text{atan2}(y_{2i+2} - b_{2i}, x_{2i+2} - a_{2i}) - \theta_{2i} - \alpha$$

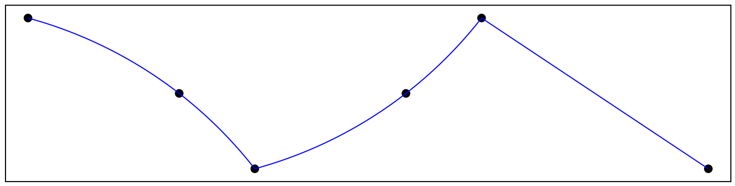
We want α, β to be the radians travels from each point to the next. The should be in the same direction, and with $\alpha + \beta \in [-2\pi, 2\pi]$.

- If α, β have the same sign, and $|\alpha + \beta| \leq 2\pi$, then we leave them as-is.
- If $\alpha, \beta > 0$, but $|\alpha + \beta| > 2\pi$, then we adjust them both by -2π. If instead $\alpha, \beta < 0$, we adjust them by $+2\pi$.
- If α, β have different signs, then we take the one closer to zero and adjust it by $\pm 2\pi$, using the opposite of its current sign (sending it to the other side of zero).

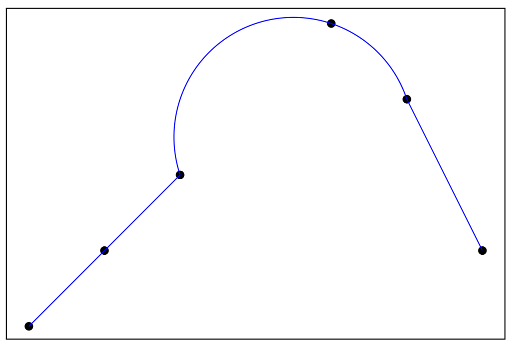
After this step, we set $\phi_{2i} = \alpha + \beta$.

The Results

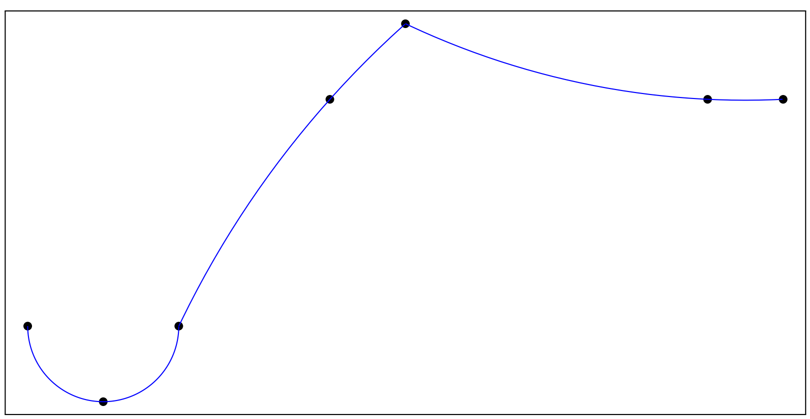
Graph 1:



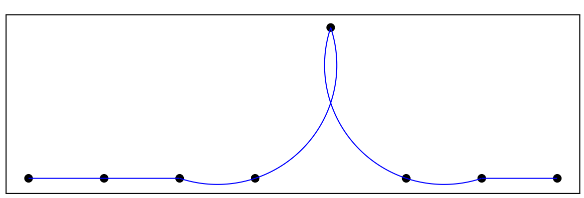
Graph 2:



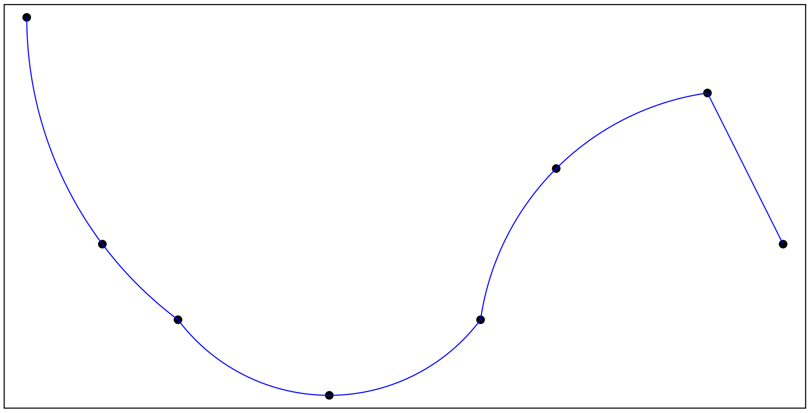
Graph 3:



Graph 4:



Graph 5:



Possible Issues

This method usually makes sharp corners at every other point, which may not be desirable.

Inserting a point anywhere except the right end will change the even/odd designation of every point to its right, which may alter the picture drastically.

Possible Extensions

We believe that it is possible to interpolate arcs that lie tangent on the points they intersect (no corners) if we set each to connect two consecutive points instead of three.

It might also be possible to make the current arcs all tangent if we permit ourselves to use *ellipses* as well as circles, or other conic sections.