MSc DS - Semester: 3

20XD33 - LINEAR ALGEBRA – Model Exam

1. a-i) Let A be a 3×3 invertible matrix. What will be the rank of $A^T A^{-1}$? Justify your answer.

   a-ii) Consider the matrix $A(x) = \begin{bmatrix} x^2+2 & x \\ 3 & 1 \end{bmatrix}$. What are the values of $x \in \mathbb{R}$ for which rank of $A(x)$ is 2?

   b) Solve the linear system

$$\begin{aligned} 5x_1 + 2x_2 - 8x_3 &= 36 \\ -8x_1 + x_2 + 4x_3 &= 28 \\ x_1 - 13x_2 - x_3 &= 12 \end{aligned}$$

   using Gauss-Seidel iteration method. Assume initial values as $x = -6.\ y = 0$ and $z = -5$.

   c) Determine the values of $k \in \mathbb{R}$ for which the system

$$\begin{aligned} 2x + 2y \quad &= 2 \\ ky + z &= 1 \\ x + 2y + kz &= 2 \end{aligned}$$

   has no solutions, exactly one solution, or infinitely many solutions.

2. a-i) Verify that the set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \text{ and } k(1, y) = (1, ky)$$

   is a vector space over $\mathbb{R}$ or not?

   a-ii) Prove that if a nonempty subset S of a vector space $V(F)$ is linear independent, then $0 \notin S$, where 0 is a additive identity of $V(F)$.

   b) Verify that the set of all 3×3 skew symmetric matrices is a subspace of $M_3(\mathbb{R})$? If it is a subspace, find its basis and dimension?

   c) Prove that any n-dimensional vector space over a field F is isomorphic to $V_n(F)$ over F.

3. a-i) Prove that $\{(2,-2,1), (2,1,-2), (1,2,2)\}$ is an orthogonal subset of the inner product space $\mathbb{R}^3$.

   a-ii) State projection theorem. Also give an example of projection theorem.

   b) State and prove generalized Pythagoras theorem.

   c) Apply Gram Schmidt orthogonalization process to transform the basis vectors $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ where $v_1 = (1,-1,1)$, $v_2 = (1,0,1)$ and $v_3 = (1,1,2)$ into an orthonormal basis.

4. a-i) Explain compressed sparse row representation format of a sparse matrix.

a-ii) Find the matrix corresponding to the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(a,b,c) = (3a + c, -2a + b, a + 2b + 4c)$$

w.r.t the basis $\{(1,0,1), (-1,2,1), (2,1,1)\}$ as a basis for domain and standard basis for codomain.

b) Prove that the function $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(a,b,c) = (a - b, 3b, 4a + 5b)$$

is a linear transformation. Also find its range space, null space, rank, nullity and T^{-1}, if exists.

c) Let$V$ and V' be two vector spaces over a field F and $T: V \to V'$ a linear transformation. Prove that $Rank\ of\ T + Nullity\ of\ T = dimension\ of\ V$. Also give an example of a linear transformation whose domain is an infinite dimensional vector space.

5. a-i) If the product of eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 1 & k & 2 \end{bmatrix}$ is 2, find the value of k?

a-ii) Prove that A and A^T have same eigenvalues.

b) Let $v_1, v_2, \dots, v_n$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of an $n \times n$ matrix. Then prove that the set $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

c) Prove that the matrix $A = \begin{bmatrix} -4 & 4 & -8 \\ 4 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix}$ is diagonalizable. Also, find a diagonal matrix that is similar to A.