Consultatie PS

Ex 1

60% nu folosesc nici F nici T

20% folosesc F

30% folosesc T

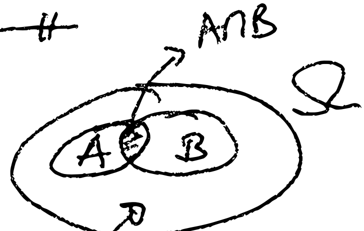
$1 + P + A + E \geq \frac{1}{2} E$

$\geq \frac{1}{2} (2)$

a) Prob ca elevii să folosească F sau T

b) — F și T

a)



$A \to F$

$B \to T$

elevi care nu folosesc nici F nici T

$\mathbb{P}(A \cup B) = ?$

$\mathbb{P}(A \cap B) = ?$

$\mathbb{P}(A) = 0.2$

$\mathbb{P}(B) = 0.3$

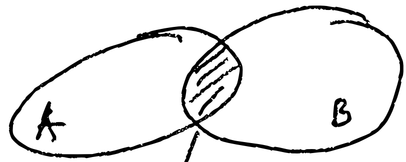
$\mathbb{P}((A \cup B)^c) = 0.6$

$\mathbb{P}(A \cup B) = 1 - \mathbb{P}((A \cup B)^c)$

$= 0.4$

$\overset{0.4}{\mathbb{P}(A \cup B)} = \overset{0.2}{\mathbb{P}(A)} + \overset{0.3}{\mathbb{P}(B)} - \underline{\overset{0.1}{\mathbb{P}(A \cap B)}}$

c) Care este prob. elevilor care folosesc o singură platformă?



"o singură platformă" $= A \triangle B = \boxed{(A\cup B)\setminus(A\cap B)}$

$$= (A\setminus B)\cup(B\setminus A)$$

$$P(A\triangle B) = P(A\setminus B) + P(B\setminus A)$$

$$P(A\setminus B) = P(\text{folosesc doar } F)$$

$$= P(A\setminus(A\cap B)) = P(A) - P(A\cap B)$$

$$= 0.2 - 0.1 = 0.1$$

$$P(B\setminus A) = P(B) - P(A\cap B) = 0.3 - 0.1 = 0.2$$

$$P(A\triangle B) = 0.1 + 0.2 = 0.3$$

$$= P(A\cup B) - P(A\cap B) = 0.4 - 0.1 = 0.3$$

Ex2 Urnă cu $\underset{24}{\overset{\shortparallel}{a}}$ bile albe și $\underset{9}{\overset{\shortparallel}{n}}$ bile negre

Extragem fără întoarcere 3 bile și vrem să știi care este prob. ca bilele extrase să fie în ordinea 1) alb, alb, negru?

2) alb, negru, alb?

3) 2 din cele 3 bile să fie albe?

Sol: A_i - even. ca la a i-a extragere să avem o bilă albă

1) a, a, n

$$\mathbb{P}(A_1 \cap A_2 \cap A_3^c) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 / A_1) \cdot \mathbb{P}(A_3^c / A_1 \cap A_2)$$

formula produs

$$= \frac{a}{a+n} \times \frac{a-1}{a-1+n} \times \frac{n}{a-2+n}$$

2) a, n, a $\longrightarrow \mathbb{P}(A_1 \cap A_2^c \cap A_3)$

3) 2 din cele 3 să fie albe

$$\underbrace{(A_1 \cap A_2 \cap A_3^c)}_{B_1} \cup \underbrace{(A_1 \cap A_2^c \cap A_3)}_{B_2} \cup \underbrace{(A_1^c \cap A_2 \cap A_3)}_{B_3}$$

„2 din cele 3 să fie albe" $= B_1 \cup B_2 \cup B_3$

unde $B_i \cap B_j = \emptyset$, $i \neq j$

$$P(B_1 \cup B_2 \cup B_3) = P(B_1) + P(B_2) + P(B_3)$$

„cel puțin 2 din cele 3 să fie albe" =

$$= (A_1 \cap A_2 \cap A_3) \cup B_1 \cup B_2 \cup B_3$$

<u>Extindem modelul:</u>

a - albe (24)

n - negre (9)

extragem o bilă, notăm culoarea și introducem bila în urnă împreună cu d bile de aceeași culoare
(5)

4) a doua bilă extrasă să fie neagră

5) prima bilă să fie neagră știind că a doua este neagră

4) $P(A_2^c) = P(A_2^c | A_1) P(A_1) + P(A_2^c | A_1^c) P(A_1^c)$

formula prob. totale

$$= \frac{a}{a+n} \times \frac{n}{a+n+d} + \frac{n}{a+n} \times \frac{n+d}{a+n+d} = \frac{n}{a+n}$$

Putem arăta că (presupunând că repetăm procesul la infinit)

$$\mathbb{P}(A_k^c) = \frac{n}{a+n}$$

inducție

$$\mathbb{P}(A_3^c) = \frac{n}{a+n}$$

5) $$\mathbb{P}(A_1^c / A_2^c) = \frac{\mathbb{P}(A_2^c / A_1^c) \cdot \mathbb{P}(A_1^c)}{\mathbb{P}(A_2^c)}$$

$$\frac{\frac{n}{a+n} \times \frac{n+d}{a+n+d}}{\frac{n}{a+n}} = \frac{n+d}{a+n+d}$$

6) $$\mathbb{P}(A_1^c / A_2^c \cap A_3^c \cap \dots \cap A_{k+1}^c) = ? \simeq 1$$

$$\frac{n+kd}{a+n+kd} \xrightarrow[k \to \infty]{} 1$$

$$\frac{\mathbb{P}(A_1^c \cap A_2^c \cap \dots \cap A_{k+1}^c)}{\mathbb{P}(A_2^c \cap \dots \cap A_{k+1}^c)}$$

$$\mathbb{P}(E) = \mathbb{P}(E \cap A) + \mathbb{P}(E \cap A^c)$$

$$\mathbb{P}(\underbrace{A_2^c \cap \dots \cap A_{k+1}^c}_{E}) = \mathbb{P}(A_1^c \cap \dots \cap A_{k+1}^c) + \mathbb{P}(A_1 \cap A_2^c \cap \dots A_{k+1}^c)$$

$$\mathbb{P}(A_1^c \cap A_2^c \cap \dots \cap A_{k+1}^c) = \mathbb{P}(A_1^c) \times \mathbb{P}(A_2^c \mid A_1^c) \times \mathbb{P}(A_3^c \mid A_1^c \cap A_2^c) \times \dots \times \mathbb{P}(A_{k+1}^c \mid A_1^c \cap \dots \cap A_k^c)$$

$$\frac{n}{a+n} \times \frac{n+d}{a+n+d} \times \frac{n+2d}{a+n+2d} \times \dots \times \frac{n+kd}{a+n+kd}$$

(Ex 3) Două zaruri și efectuăm aruncări succesive (independente)

Vrem să găsim prob. ev. ca suma 5 să apară înaintea sumei 7

Sol: Notăm cu A_i - suma celor 2 zaruri la aruncarea i este 5

B_i - $\underbrace{\qquad\qquad}$ este 7

fie E_n evenimentul prin care în primele $n-1$ aruncări nu a apărut nici 5 nici 7 dar în a n-a aruncare a apărut 5

$$E_n = (A_1^c \cap B_1^c) \cap (A_2^c \cap B_2^c) \dots \cap (A_{n-1}^c \cap B_{n-1}^c) \cap A_n$$

$A = \{$ suma 5 apare înaintea sumei 7 $\}$

$= E_1 \cup E_2 \cup E_3 \cup \dots$

$= \bigcup_{i=1}^{\infty} E_i \qquad E_i \cap E_j = \emptyset,\ i \neq j$

$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \underline{\mathbb{P}(E_i)}$

$\mathbb{P}(E_n) = \mathbb{P}\left((A_1^c \cap B_1^c) \cap \dots (A_{n-1}^c \cap B_{n-1}^c) \cap A_n\right)$

$\overset{\text{indep}}{=} \underline{\mathbb{P}(A_1^c \cap B_1^c)} \times \dots \times \mathbb{P}(A_{n-1}^c \cap B_{n-1}^c) \times \mathbb{P}(A_n)$

$= \mathbb{P}(A_1^c \cap B_1^c)^{n-1} \cdot \mathbb{P}(A_n)$

$\Omega = \{(i,j) \mid i,j \in \{1, \dots 6\}\} = \{1, \dots 6\}^2$

$\mathbb{P}(\Sigma$ să fie $5) = \mathbb{P}(A_n) = 4/36$

$\mathbb{P}(A_1^c \cap B_1^c) = \frac{26}{36}$

$\mathbb{P}(A) = \sum_{n=1}^{\infty} \left(\frac{26}{36}\right)^{n-1} \cdot (4/36) = \frac{4}{36} \cdot \frac{1}{1 - \frac{26}{36}} = 2/5$

Ex 4

$$X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

Rep. $\underline{5X-2,\ X^3,\ X+X^2}$

Media și var. ↗ dar și $\mathbb{P}(X < 1/8 \mid X \geq -1/8)$

Sol:

$5X-2 \in \{-7, -2, 3\}$

$$5X-2 \sim \begin{pmatrix} -7 & -2 & 3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$\mathbb{E}[5X-2]$
$= 5\,\mathbb{E}[X]-2$
$= 5 \times 0.2 - 2$

$Var(5X-2) = 25\, Var(X)$

$Var(X) = \underline{\mathbb{E}[X^2]} - \mathbb{E}[X]^2 =$

$\mathbb{E}[X^2] = 0.8$ $\quad X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}$

$\mathbb{E}[g(X)] = \sum g(x)\,\mathbb{P}(X=x)$

$= (-1)^2 \times 0.3 + 0^2 \times 0.2 + 1^2 \times 0.5 = 0.8$

$X^3 \in \{-1, 0, 1\}$ $\quad X^3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

$\mathbb{E}(X^3) = \mathbb{E}(X)$, $Var(X^3) = Var(X)$

$$X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix} \qquad X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$$X + X^2 \in \{0, 0, 2\} \qquad \begin{cases} -1 \to -1 + (-1)^2 = 0 \\ 0 \to 0 + 0^2 = 0 \\ 1 \to 1 + 1^2 = 2 \end{cases}$$

$$Y = X + X^2 \sim \begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \end{pmatrix}$$

$$\mathbb{E}[X + X^2] = 1 \quad \text{sau} \quad \mathbb{E}[X + X^2] = \mathbb{E}[X] + \mathbb{E}[X^2]$$

$$\operatorname{Var}(\underbrace{X + X^2}_{Y}) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 1$$

$$Y^2 \sim \begin{pmatrix} 0 & 4 \\ 0.5 & 0.5 \end{pmatrix} \Rightarrow \mathbb{E}[Y^2] = 2$$

$$\mathbb{P}(\underbrace{X \leq 1/8}_{A} \mid \underbrace{X \geq -1/8}_{B}) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$A \cap B = \{X \leq 1/8\} \cap \{X \geq -1/8\} = \{X = 0\}$$

$$B = \{X \geq -1/8\} = \{X = 0\} \cup \{X = 1\}$$

$$= \frac{\mathbb{P}(X = 0)}{\mathbb{P}(X = 0) + \mathbb{P}(X = 1)} = \frac{0.2}{0.7} = 2/7.$$

Ex 5 (X, Y)

$X \backslash Y$	2	4	6	Σ
0	0.1	0.2	0.1	0.4
1	0.1	0.1	0.1	0.3
2	0.1	$\boxed{0.1}$	0	0.2
3	0.05	0	0.05	0.1
Σ	0.35	0.4	0.25	1

$P(X=2, Y=4)$

a) Rep. marginale pt X și Y, $E[Y]$, $Var(Y)$

b) Rep cond. a lui $Y | X=1$

c) Media cond $E[Y|X]$, $Var(Y|X)$

d) Coef. de corelație

Sol:

a) Rep. marginală a lui X (Σ linii)

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 2 & 4 & 6 \\ 0.35 & 0.4 & 0.25 \end{pmatrix} \quad (\Sigma \text{ col})$$

d) Coef. de corelație

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = \sum xy\, P(X=x, Y=y)$$

$= 0\times2\times0.1 + 0\times4\times0.2 + 0\times6\times0.1 +$
$1\times2\times0.1 + 1\times4\times0.1 + 1\times6\times0.1 +$
$2\times2\times0.1 + 2\times4\times0.1 + 2\times6\times0 +$
$3\times2\times0.05 + 3\times4\times0 + 3\times6\times0.05$

e) $Y|X=1$

X\Y	2	4	6	Σ
⋮				
1	0.1/0.3	0.1/0.3	0.1/0.3	0.3
⋮				

$$Y|X=1 \sim \begin{pmatrix} 2 & 4 & 6 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$Y|X=0$

$$\sim \begin{pmatrix} 2 & 4 & 6 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$

$X \mid Y=4$

$X \backslash Y$	2	4	6	Σ
0		0.2		
1		0.1		
2		0.1		
3		0		
Σ		0.4		

$$X \mid Y=4 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1/2 & 1/4 & 1/4 & 0 \end{pmatrix}$$

c) $\mathbb{E}[Y|X] \to$ v.a

$$\mathbb{E}[Y|X=0] = 2\,\mathbb{P}(Y=2|X=0) + 4\,\mathbb{P}(Y=4|X=0) + 6\,\mathbb{P}(Y=6|X=0) = 4$$

$$\mathbb{E}[Y|X=1] = 4$$

$$\mathbb{E}[Y|\underline{X=2}] = 3$$

$$\mathbb{E}[Y|X=3] = 4$$

$$\mathbb{E}[Y|\underline{X=0}] = \underline{a_1}$$
$$\mathbb{E}[Y|\underline{X=1}] = \underline{a_2}$$
$$\mathbb{E}[Y|X=2] = a_4$$
$$\mathbb{E}[Y|X=3] = a_4$$

$$\mathbb{E}[Y|X] \sim \begin{pmatrix} 3 & 4 \\ \mathbb{P}(X=2) & \mathbb{P}(X\neq 2) \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0.2 & 0.8 \end{pmatrix}$$

$\mathrm{Var}(Y|X)$ tot v.a

$$\mathrm{Var}(Y|X=0) = \mathbb{E}[Y^2|X=0] - \mathbb{E}[Y|X=0]^2$$
$$= \left(2^2 \cdot \frac{0.1}{0.4} + 4^2 \cdot \frac{0.2}{0.4} + 6^2 \cdot \frac{0.1}{0.4}\right) - 16 = 2$$

$$Var(Y|X) \sim \begin{pmatrix} 1 & 2 & 2.66 & 4 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{pmatrix}$$

$\mathbb{P}(X=2)$ $\quad$ $\mathbb{P}(X=0)$ $\quad$ $\mathbb{P}(X=1)$ $\quad$ $\mathbb{P}(X=3)$

Arătați că rel. este adev.

$$Var(Y) = \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$$

Ex 6 $\quad c \in (0,1)$

$$f(x) = \begin{cases} 1/x & , x \in [1-c, 1+c] \\ 0 & , \text{altfel} \end{cases}$$

a) Det c a.î. f densitate

b) medie, varianța

Sol: a) f densitate $\Leftrightarrow$ i) $f \geq 0$

(ii) $\int_{\mathbb{R}} f(x)\,dx = 1$

$f \geq 0$ pt că $c \in (0,1)$ și $1-c > 0$

$$\int_{\mathbb{R}} f(x)\,dx = \int \frac{1}{x} \mathbb{1}_{[1-c, 1+c]}(x)\,dx = \int_{1-c}^{1+c} \frac{1}{x}\,dx$$

$$= \ln\left(\frac{1+c}{1-c}\right)$$

$$\ln\left(\frac{1+c}{1-c}\right) = 1 \iff \frac{1+c}{1-c} = e \iff \boxed{c = \frac{e-1}{1+e}}$$

$X \sim f$

$$E[X] = \int x f(x)\,dx = \int \cancel{x}\,\frac{1}{\cancel{x}} \cdot \mathbb{1}_{[1-c,1+c]}(x)\,dx$$

$$= \int_{1-c}^{1+c} dx = 2c = 2 \cdot \frac{e-1}{1+e}$$

$$\operatorname{Var}(X) = \underline{E[X^2]} - E[X]^2$$

$$= \int x^2 f(x)\,dx = \int_{1-c}^{1+c} x^{\cancel{2}} \cdot 1/\cancel{x}\,dx = x^2/2\Big|_{1-c}^{1+c}$$

$$= 2c$$

$$\operatorname{Var}(X) = 2c - (2c)^2 = 2c(1-2c)$$

Def. fct. de rep $F = ?$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad , x \in \mathbb{R}$$

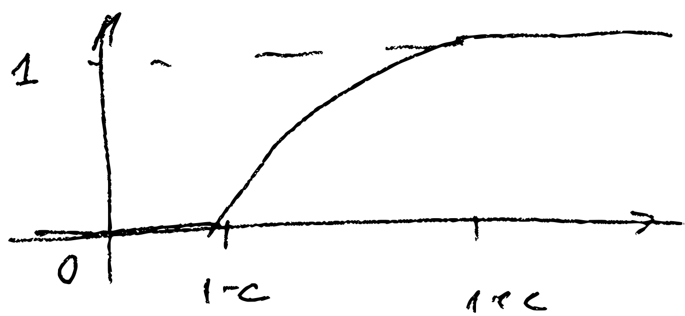
$$F(x) = \int_{-\infty}^{x} \frac{1}{t} \mathbb{1}_{[1-c,1+c]}(t)\, dt$$

Dacă $x < 1-c$ atunci $F(x) = 0$

$x > 1+c$ atunci $F(x) = 1$

$x \in [1-c, 1+c]$

$$F(x) = \int_{1-c}^{x} \frac{1}{t}\, dt = \ln\left(\frac{x}{1-c}\right)$$



Ex 7 (X,Y)

$$f_{(X,Y)}(x,y) = \begin{cases} k(x+y+2), & x \in [0,1], \\ & y \in [0,2] \\ 0, & \text{altfel} \end{cases}$$

a) $k = ?$

b) densitățile marginale

c) densitățile condiționate

a) $f_{(X,Y)}$ densitate $\Rightarrow f_{(X,Y)} \geq 0 \Rightarrow k \geq 0$

$$\iint f_{(X,Y)}(x,y)\,dx\,dy = 1$$

$$\Leftrightarrow \int_0^1\int_0^2 k(x+y+1)\,\underline{dy}\,\underline{dx} = \int_0^2\int_0^1 k(x+y+1)\,dx\,dy$$

$$\Leftrightarrow \int_0^2 k\left(\frac{x^2}{2}+yx+x\right)\Big|_0^1 dy = 1$$

$$\Leftrightarrow \int_0^2 k\left(y+\frac{3}{2}\right) dy = 1$$

$$\Leftrightarrow k\left(\frac{y^2}{2}+\frac{3y}{2}\right)\Big|_0^2 = 1$$

$$\Leftrightarrow k\left(\frac{4}{2}+\frac{6}{2}\right) = 1 \Rightarrow \boxed{k = 1/5}$$

b) Densitățile marginale

$$f_X(x) = \int f_{(X,Y)}(x,y)\,dy = \int \frac{1}{5}(x+y+1)\,\mathbb{1}_{(0,1)}(x)\cdot\mathbb{1}_{(0,2)}(y)\,dy$$

$$= \int_0^2 \frac{1}{5}(x+y+1)\,dy \cdot \mathbb{1}_{(0,1)}(x) = \frac{2x+4}{5}\,\mathbb{1}_{(0,1)}(x)$$

$$f_Y(y) = \int f_{(X,Y)}(x,y)\,dx = \int_0^1 \frac{x+y+1}{5}\,dx \cdot \mathbb{1}_{[0,2]}(y)$$

$$= \frac{2y+3}{10}\,\mathbb{1}_{[0,2]}(y)$$

Sunt X şi Y indep? $f_{(X,Y)}(x,y) \neq f_X(x) f_Y(y)$

nu sunt

c) Densitatile cond.

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} = \frac{\frac{x+y+1}{5}\,\mathbb{1}_{[0,1]}(x)\,\mathbb{1}_{[0,2]}(y)}{\frac{2y+3}{10}\,\mathbb{1}_{[0,2]}(y)}$$

$$= \frac{2(x+y+1)}{2y+3}\,\mathbb{1}_{[0,1]}(x)\,\mathbb{1}_{[0,2]}(y)$$

Dacă $y \in [0,2]$ at $f_{X|Y}(x|y) = \begin{cases} \frac{2(x+y+1)}{2y+3}\,\mathbb{1}_{[0,1]}(x) \\ 0, \text{ altfel} \end{cases}$

$$f_{Y|X}(y|x) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$$

$$\mathbb{E}[X|Y=y] = \int x \, f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 \frac{2x(x+y+1)}{2y+3}\, dx \quad , y \in (0,1)$$

$$= g(y)$$

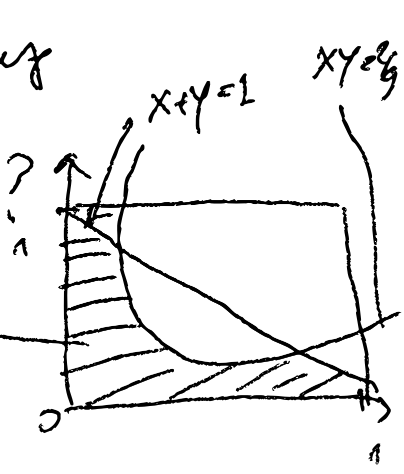
$$\mathbb{E}[X|Y] = g(Y)$$

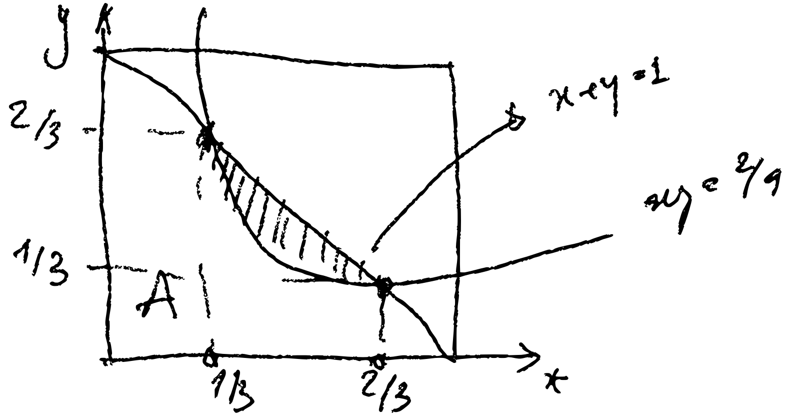
Ex 8 Să se determine prob. ca suma a 2 nr. alese la întâmplare din $[0,1]$ să nu depășească valoarea 1 iar produsul lor să nu depășească $2/9$.

Sol: $X, Y \sim \mathcal{U}[0,1]$ indep

$$\mathbb{P}(X+Y \le 1 \ , XY \le 2/9) = ?$$

$A = \{(x,y) \in [0,1]^2 \mid x+y \le 1, xy \le 2/9\}$ aria reg. hașurate





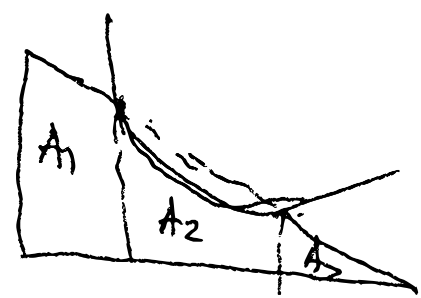
$$\mathbb{P}(X+Y\le 1,\ XY\le 2/9) = \mathbb{P}((X,Y)\in A) = \iint_A f_{(X,Y)}(x,y)\,dx\,dy$$

$$f_{(X,Y)}(x,y) = f_X(x)\, f_Y(y) \quad (\text{din indep})$$

$$= \mathbb{1}_{(0,1)}(x)\cdot \mathbb{1}_{(0,1)}(y)$$

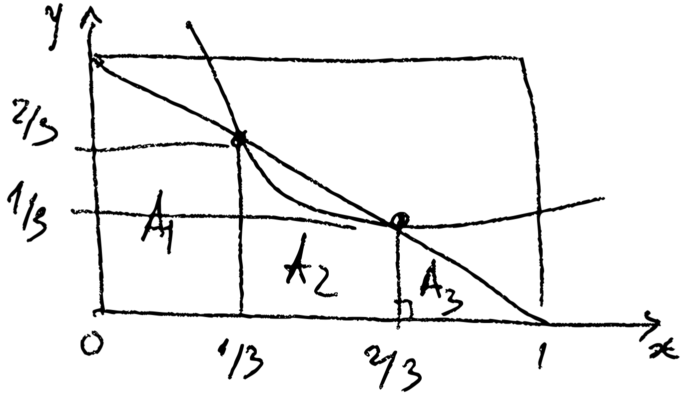
$$= \iint_A \mathbb{1}_{(0,1)}(x)\, \mathbb{1}_{(0,1)}(y)\, dy = \text{Aria}(A)$$

$A = A_1 + A_2 + A_3$



$$\begin{cases} x+y=1 \\ xy = 2/9 \end{cases} \iff \begin{cases} x = 1-y \\ (1-y)y = 2/9 \end{cases} \iff y_{1/2} = \frac{1\pm\sqrt{\frac{1}{9}}}{2}$$

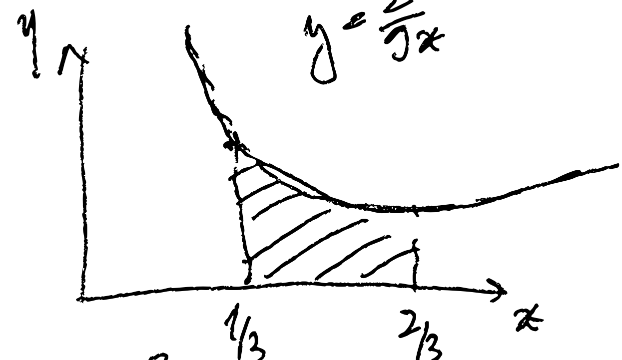
$$y_{1/2} \begin{cases} \to 1/3 \\ \to 2/3 \end{cases} \implies x_{1/2} \begin{cases} \to 2/3 \\ \to 1/3 \end{cases}$$



$A_1 = \frac{1}{3} \cdot (1 + 2/3) \cdot \frac{1}{2}$

$= 5/18$

$A_3 = \frac{1/3 \cdot 1/3}{2} = \frac{1}{18}$



$$A_2 = \int_{1/3}^{2/3} \frac{2}{9x}\,dx = \frac{2}{9}\ln 2$$

$$\boxed{A = 1/3 + \frac{2}{9}\ln 2}$$

Ex 10 $\quad X, Y \sim N(0,1)$ indep

$\rho(\min(X,Y), \max(X,Y)) = ?$

$$M = \max(X,Y)$$
$$L = \min(X,Y)$$

$$\boxed{\rho(M,L) = \frac{1}{\pi - 1}}$$

$$\begin{cases} M + L = X + Y \\ M - L = |X - Y| \end{cases}$$

$$E[M] + E[L] = 0$$
$$E[M] - E[L] = a$$
$$\Downarrow \quad E[M] = ? \qquad E[L] = ?$$

$$\rho(M,L) = \frac{Cov(M,L)}{\sqrt{Var(M)}\sqrt{Var(L)}}$$

$$Cov(M,L) = E[ML] - E[M]\cdot E[L]$$

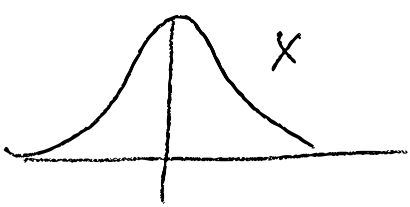
$$E[ML] = E[XY] \underset{\text{indep}}{=} E[X]E[Y] = 0$$

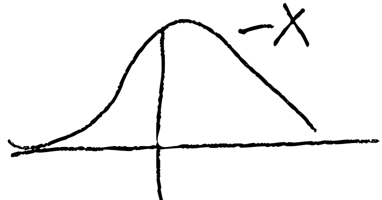
$$Var(M+L) = Var(X+Y) = Var(X) + Var(Y) = 2$$

$$\underbrace{Var(M) + Var(L)} + 2Cov(M,L)$$

$$\underline{Var(M) = Var(L)}$$

$$\max(\underbrace{X,Y}) = -\min(-X,-Y)$$





(X,Y) și $(-X,-Y)$ sunt rep la fel