Theory of Computation, Fall 2023
Assignment 4 Solutions

Q1. $L = \emptyset$

Q2. $S \to e|0S0|1S1$

Q3. The PDA is as follows.

$$P = (K, \Sigma, \Gamma, \Delta, s, F)$$

$$\begin{array}{rcl} K & = & \{s, f\} \\ \Sigma & = & \{0, 1\} \\ \Gamma & = & \Sigma \\ F & = & \{f\} \end{array}$$

and Δ contains the following transitions.

(q, a, β)	(p, γ)
(s, a, e)	(s, a)
(s, b, e)	(s, b)
(s, e, e)	(f, e)
(f, a, a)	(f, e)
(f, b, b)	(f, e)

Q4. Consider the PDA $P = (K, \Sigma, \Gamma, \Delta, s, F)$ where

$$\begin{aligned} K &= \{q_1\}, \\ \Sigma &= \{0, 1\}, \\ \Gamma &= \{1\}, \\ s &= q_1, \\ F &= \{q_1\}, \end{aligned}$$

and Δ contains the following transitions.

(q, a, β)	(p, γ)
$(q_1, 0, 1)$	(q_1, e)
$(q_1, 1, e)$	$(q_1, 1)$
$(q_1, e, 1)$	(q_1, e)

Whenever it reads a 1, it pushes a 1 onto the stack. Whenever it reads a 0, it must pop a 1. After consuming all the input symbols, it pops all the 1's remaining in the stack. If some prefix of the input has more 0's than 1's, the PDA will fail to accept the input because it does not have enough 1's to be popped.

Q5. As the PDA reads the input, we use the stack as a unary counter to record the value of $A-2B$, where A and B are the number of a's and b's that have been read so far. After all the input symbols are consumed, if the stack is empty (i.e., $A = 2B$), the PDA will accept the input.

We use the symbol $+$ to denote $+1$, and $-$ to denote -1. The PDA $P = (K, \Sigma, \Gamma, \Delta, s, F)$ is as follows.

$$
\begin{aligned}
K &= \{q\}, \\
\Sigma &= \{a, b\}, \\
\Gamma &= \{+, -\}, \\
s &= q, \\
F &= \{q\},
\end{aligned}
$$

and Δ contains the following transitions.

(q, a, β)	(p, γ)
(q_1, a, e)	$(q_1, +)$
$(q_1, a, -)$	(q_1, e)
(q_1, b, e)	$(q_1, --)$
$(q_1, b, +)$	$(q_1, -)$
$(q_1, b, ++)$	(q_1, e)

Q6. The PDA first runs M_A, and the stack will be used as a unary counter to record the number of symbols it read. When it reaches any final state of M_A, it will non-deterministically switches to M_B. When running M_B, the PDA decreases the unary counter (by popping a 1 from the stack) whenever it reads a symbol. The PDA $P = (K, \Sigma, \Gamma, \Delta, s, F)$ is as follows.

$$
\begin{array}{rcll}
K & = & K_A \cup K_B \\
\Sigma & = & \Sigma_A \cup \Sigma_B \\
\Gamma & = & \{1\} \\
s & = & s_A \\
F & = & F_B \\
\Delta & = & \{((q_A, a, e), (p_A, 1)) \mid ((q_A, a), p_A) \in \Delta_A\} & \cup \\
 & & \{((q_B, b, 1), (p_B, e)) \mid ((q_B, b), p_B) \in \Delta_B\} & \cup \\
 & & \{((f_A, e, e), (s_B, e)) \mid f_A \in F_A\}
\end{array}
$$

One can see that P accepts $A \diamond B = \{xy \mid x \in A, y \in B, |x| = |y|\}$. Therefore, $A \diamond B$ is context-free.