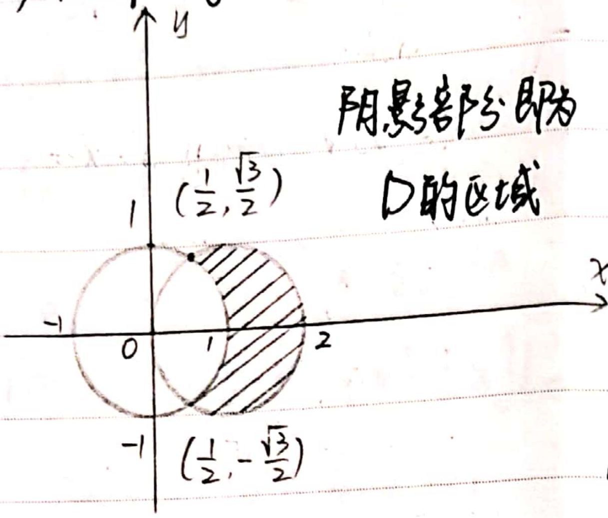
习题10.2, 10.3, 10.5 @hj

12(2) $D=\{(x,y)\mid 1\le x^2+y^2\le 2x\}$



阴影部分即为D的区域

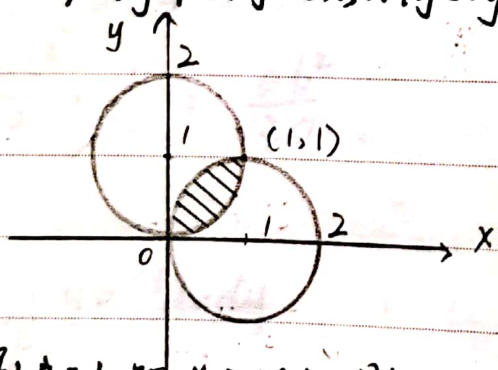
令 $x=r\cos\theta,\ y=r\sin\theta$

$\therefore D'=\{(r,\theta)\mid 1\le r^2\le 2r\cos\theta\}$

$\therefore \iint\limits_D f(x,y)\,dxdy=\iint\limits_{D'} f(r\cos\theta, r\sin\theta)\,r\,drd\theta$

$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}d\theta\int_1^{2\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$

12(3) $D=\{(x,y)\mid x^2+y^2\le 2x,\ x^2+y^2\le 2y\}$



阴影部分即为D的区域

令 $x=r\cos\theta,\ y=r\sin\theta$

$\therefore D'=\{(r,\theta)\mid r^2\le 2r\cos\theta,\ r^2\le 2r\sin\theta\}$

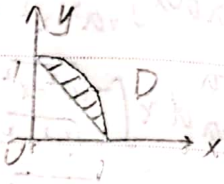
$\therefore \iint\limits_D f(x,y)\,dxdy=\iint\limits_{D'} f(r\cos\theta, r\sin\theta)\,r\,drd\theta$

$=\int_0^{\frac{\pi}{4}}d\theta\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$

$+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$

13(1). $\int_0^1 dx\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$



$D=\{(x,y)\mid 1-x\le y\le\sqrt{1-x^2}\}$

$D'=\{(r,\theta)\mid 1\le r(\sin\theta+\cos\theta),\ r^2\le 1\}$

$\therefore$ 原式 $=$

$\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{\sqrt2}{2}\csc(\theta+\frac{\pi}{4})}^{1} f(r\cos\theta, r\sin\theta)\,r\,dr$

$=\int_{\frac{\sqrt2}{2}}^{1} r\,dr\int_{\frac{\pi}{4}-\arccos\frac{\sqrt2}{2r}}^{\frac{\pi}{4}+\arccos\frac{\sqrt2}{2r}} f(r\cos\theta, r\sin\theta)\,d\theta$

或 $\int_{\frac{\sqrt2}{2}}^{1} dr\int_{\arcsin\frac{1}{\sqrt2 r}-\frac{\pi}{4}}^{\frac{3\pi}{4}-\arcsin\frac{1}{\sqrt2 r}} f(r\cos\theta, r\sin\theta)\,r\,d\theta$

14(3) $\iint\limits_D f(x+y)\,dxdy$. $D: |x|+|y|\le 1$

令 $u=x+y,\ v=x-y$ 即 $x=\frac{u+v}{2},\ y=\frac{u-v}{2}$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left\|\begin{matrix}\frac12 & \frac12 \\ \frac12 & -\frac12\end{matrix}\right\|=\frac12$

$D=\{(x,y)\mid |x|+|y|\le 1\}$

$D'=\{(u,v)\mid -1\le u\le 1,\ -1\le v\le 1\}$

$\therefore$ 原式 $=\int_{-1}^{1}du\int_{-1}^{1}f(u)\times\frac12\,dv$

$=\int_{-1}^{1}f(u)\,du$

15(2).

令 $x=r\cos\theta,\ y=r\sin\theta$.

$D=\{(x,y)\mid x^2+y^2\le 1\}$

$D'=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$\therefore$ 原式 $=\iint_{D'}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\theta$

$=\int_0^{2\pi}d\theta\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$

$=\pi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,d(r^2)$

令 $t=\sqrt{\frac{1-r^2}{1+r^2}}\Rightarrow r^2=\frac{1-t^2}{1+t^2}$

$\therefore\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,d(r^2)$

$=\int_1^0 t\,d\left(\frac{1-t^2}{1+t^2}\right)$

$=-4\int_1^0\frac{t^2dt}{(1+t^2)^2}=4\int_0^1\frac{t^2}{(1+t^2)^2}dt$

$\xlongequal{t=\tan u}4\int_0^{\frac{\pi}{4}}\frac{\tan^2u}{\sec^4u}\sec^2u\,du$

$=4\int_0^{\frac{\pi}{4}}\sin^2u\,du$

$=2\int_0^{\frac{\pi}{4}}(1-\cos 2u)du=2\left(u-\frac{1}{2}\sin 2u\right)\Big|_0^{\frac{\pi}{4}}$

$=\frac{\pi}{2}-1$

$\therefore$ 原式 $=\pi\left(\frac{\pi}{2}-1\right)$

15(5)

令 $x=ar\cos\theta,\ y=br\sin\theta$

$D=\{(x,y)\mid\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$

$D'=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\left\|\begin{matrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{matrix}\right\|$

$=abr$

$\therefore$ 原式 $=\iint_{D'}\sqrt{1-r^2}\,abr\,dr\,d\theta$

$=\int_0^{2\pi}d\theta\int_0^1\left(-\frac{1}{2}\right)ab\sqrt{1-r^2}\,d(1-r^2)$

$=-\pi ab\int_0^1\sqrt{1-r^2}\,d(1-r^2)$

$=-\pi ab\,\frac{2}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1$

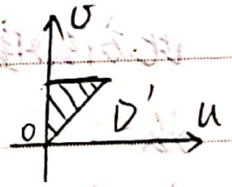
$=\frac{2}{3}\pi ab$

15(8)

令 $u=y,\ v=x+y$ 即 $x=v-u,\ y=u$

$\therefore\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left\|\begin{matrix}-1 & 1\\ 1 & 0\end{matrix}\right\|=1$

D' 是由 $v=u,\ u=0,\ v=1$ 围成的有界区域



$\therefore$ 原式 $=\iint_{D'}e^{\frac{u}{v}}du\,dv$

$=\int_0^1dv\int_0^v e^{\frac{u}{v}}du$

$=\int_0^1 v\,dv\int_0^v e^{\frac{u}{v}}\,d\left(\frac{u}{v}\right)$

$=(e-1)\int_0^1 v\,dv$

$=\frac{e-1}{2}$

15(10)

令 $\begin{cases} a_1x+b_1y+c_1=u & ① \\ a_2x+b_2y+c_2=v & ② \end{cases}$

计算 $\iint_D dxdy$ 即计算曲线围成区域的面积。易知该曲线是一个椭圆,

$S_{椭}=\pi ab$, a为半长轴长, b为半短轴长

令 $f(x,y)=x^2+y^2$, 即求f极值

构造拉格朗日函数 $L(x,y,\lambda)=$

$x^2+y^2+\lambda[(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2-1]$

$$\begin{cases} L'_x=2x+\lambda[2a_1^2x+2a_1b_1y+2a_1c_1+2a_2^2x+2a_2b_2y+2a_2a_3]=0 \\ L'_y=2y+\lambda[2b_1^2y+2a_1b_1x+2b_1c_1+2b_2^2y+2a_2b_2x+2b_2c_2]=0 \\ L'_\lambda=(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2-1=0 \end{cases}$$

此方法难算,不继续往下算了

法二:

由上面①、②式可得

$$\begin{cases} x=\dfrac{b_1(v-c_2)-b_2(u-c_1)}{a_1b_2-a_2b_1} \\ y=\dfrac{a_1(v-c_2)-a_2(u-c_1)}{a_1b_2-a_2b_1} \end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{1}{(a_1b_2-a_2b_1)^2}\left\|\begin{matrix}-b_2 & b_1\\ -a_2 & a_1\end{matrix}\right\|$$

$$=\frac{1}{|a_2b_1-a_1b_2|}$$

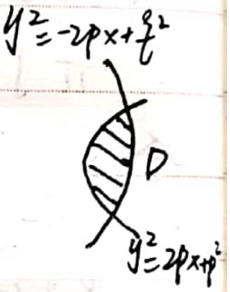
或利用 $\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$\therefore D'=\{(u,v)\mid u^2+v^2\le 1\}$

$$\therefore 原式=\iint_{D'}\frac{1}{|a_2b_1-a_1b_2|}dudv=\frac{\pi}{|a_2b_1-a_1b_2|}$$

23(1)

$A=\iint_D dxdy$



$\begin{cases} y^2=-2qx+q^2 \\ y^2=2px+p^2 \end{cases} \Rightarrow \begin{cases} x=\dfrac{q-p}{2} \\ y=\pm\sqrt{pq} \end{cases}$

$$\therefore A=\int_{-\sqrt{pq}}^{\sqrt{pq}}dy\int_{\frac{y^2-p^2}{2p}}^{\frac{y^2-q^2}{-2q}}dx$$

$$=-\int_{-\sqrt{pq}}^{\sqrt{pq}}\left(\frac{y^2-q^2}{2q}+\frac{y^2-p^2}{2p}\right)dy$$

$$=-\int_{-\sqrt{pq}}^{\sqrt{pq}}\frac{(p+q)}{2}\left(\frac{y^2}{pq}-1\right)dy$$

$$=-\frac{p+q}{2pq}\left(\frac{1}{3}y^3-pqy\right)\Big|_{-\sqrt{pq}}^{\sqrt{pq}}$$

$$=\frac{2}{3}(p+q)\sqrt{pq}$$

23(2)

令 $x=r\cos\theta$, $y=r\sin\theta$

$\therefore r^4=2a^2r^2(\cos^2\theta-\sin^2\theta)$

$= 2a^2r^2\cos2\theta$

当x≥0时有 $r=\sqrt{2}a\sqrt{\cos2\theta}$

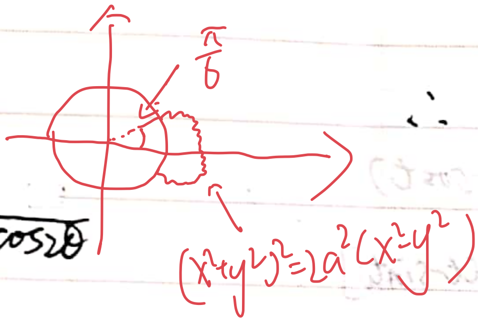
$(\theta\in[-\frac{\pi}{4},\frac{\pi}{4}])$

考虑到整个图形关于y轴对称，故只需计算一、二象限面积。

$x^2+y^2\ge a^2$

$\Rightarrow r\ge a$

令 $\begin{cases} r=a \\ r=\sqrt{2}a\sqrt{\cos2\theta} \end{cases}$



$\Rightarrow \theta=\pm\frac{\pi}{6}$

$\therefore \frac{A}{2}=\iint_D dxdy$

$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta\int_a^{\sqrt{2}a\sqrt{\cos2\theta}} r\,dr$

$=\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(2a^2\cos2\theta-a^2)d\theta$

$=\frac{1}{2}(a^2\sin2\theta-a^2\theta)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$

$=\frac{a^2}{2}(\sqrt{3}-\frac{\pi}{3})$

$\therefore A= a^2(\sqrt{3}-\frac{\pi}{3})$

24(1)

$\begin{cases} z^2=x^2+\frac{y^2}{4} \\ 2z=x^2+\frac{y^2}{4} \end{cases} \Rightarrow z=2$

直接求三重积分也可. $\int_0^2 dz\iint_{z^2\le x^2+\frac{y^2}{4}\le 2z} 1\,dxdy$

或 $\iint_{x^2+\frac{y^2}{4}\le 4} dxdy\int_{\frac{1}{2}(x^2+\frac{y^2}{4})}^{\sqrt{x^2+\frac{y^2}{4}}}dz$

$V_{圆锥}=\int_0^2\pi\cdot z\cdot 2z\,dz$

$=\pi\frac{2}{3}z^3\Big|_0^2$

$=\frac{16}{3}\pi$

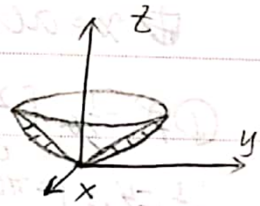
$V_{椭抛体}=\int_0^2\pi\cdot\sqrt{2z}\cdot\sqrt{8z}\,dz$

$=\int_0^2\pi\cdot 4z\,dz$

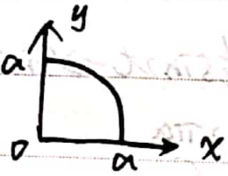
$=8\pi$

$\therefore V=V_{椭抛体}-V_{圆锥}$

$=\frac{8}{3}\pi$



25(2)



$A=\frac{1}{4}\pi a^2$

$\iint_D x\,dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_0^a r^2\cos\theta\,dr$

$=\frac{1}{3}a^3\times\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta=\frac{1}{3}a^3$

$\therefore \bar{x}=\frac{\frac{1}{3}a^3}{\frac{1}{4}\pi a^2}=\frac{4}{3\pi}a$

$\iint_D y\,dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_0^a r^2\sin\theta\,dr$

$=\frac{1}{3}a^3\times\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta=\frac{1}{3}a^3$

$\therefore \bar{y}=\frac{4}{3\pi}a$

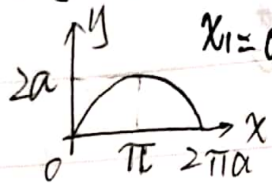
$\therefore$ 形心为 $(\frac{4a}{3\pi},\frac{4a}{3\pi})$

26. 设面密度为μ.

$I_x=\int R^2dm=\mu\iint_D y^2dxdy$

令 $y=0 \Rightarrow t_1=0,\ t_2=2\pi$

$x_1=0,\ x_2=2\pi a$



$\therefore \mu\iint_D y^2\,dxdy$

$=\mu\int_0^{2\pi a}dx\int_0^{y(x)}y^2\,dy$

$=\frac{\mu}{3}\int_0^{2\pi a}y^3(x)\,dx$

由 $x=a(t-\sin t),\ y=a(1-\cos t)$

①式 $=\frac{1}{3}\mu\int_0^{2\pi}a^3(1-\cos t)^3\,d[a(t-\sin t)]$

$=\frac{1}{3}a^3\mu\int_0^{2\pi}(1-\cos t)^4\,dt$

$=\frac{1}{3}a^3\mu\left(\frac{35}{8}x-7\sin x+\frac{7}{4}\sin 2x-\frac{1}{3}\sin 3x\right.$

$\left.+\frac{1}{32}\sin 4x\right)\Big|_0^{2\pi}$

$=\frac{1}{3}a^3\mu\times\frac{35}{4}\pi a$

$=\frac{35}{12}\pi a^4\mu$ (μ为面密度)

$A_{总}=\int_0^{2\pi a}y(x)\,dx$

$=a^2\int_0^{2\pi}(1-\cos t)^2\,dt$

$=\frac{a^2}{4}(6x-8\sin x+\sin 2x)\Big|_0^{2\pi}$

$=3\pi a^2=\frac{m}{\mu}$

$\therefore I_x=\frac{35}{12}\pi a^4\times\frac{m}{3\pi a^2}$

$=\frac{35}{36}ma^2$

(m为薄片质量)