习题10.4 @ hj

16(2)

$\int_2^3\int_0^\pi\int_0^1 zx\sin(xy)\,dz\,dy\,dx$

$=\int_2^3\int_0^\pi x\sin(xy)\times\frac{1}{2}\,dy\,dx$

$=\int_2^3\int_0^\pi \frac{1}{2}\sin(xy)\,d(xy)\,dx$

$=\frac{1}{2}\int_2^3 (-\cos xy)\Big|_0^\pi\,dx$

$=\frac{1}{2}\int_2^3(1-\cos\pi x)\,dx$

$=\frac{-1}{2}\int_2^3(\cos\pi x-1)\,dx$

$=\frac{-1}{2}\left(\frac{1}{\pi}\sin\pi x-x\right)\Big|_2^3$

$=\frac{1}{2}$

16(5) $\iiint_V(x^2+y^2)\,dx\,dy\,dz$ 投影法

$=\underbrace{\iint_{D_1}\left(\int_{4(x^2+y^2)}^{16(x^2+y^2)}dz\right)(x^2+y^2)\,dx\,dy}_{①}+②$

其中 $D_1=\{(x,y)\mid x^2+y^2\le 4\}$

$①=\iint_{D_1}12(x^2+y^2)^2\,dx\,dy$

令 $x=r\cos\theta,\ y=r\sin\theta$

$\therefore D_1'=\{(r,\theta)\mid 0\le r\le 2,\ 0\le\theta\le 2\pi\}$

$\therefore ①=\iint_{D_1'}12(r^2)^2\cdot r\,dr\,d\theta$

$=12\int_0^{2\pi}d\theta\int_0^2 r^5\,dr$

$=2\int_0^{2\pi}d\theta\cdot r^6\Big|_0^2$

$=256\pi$

$②=\iint_{D_2}\left(\int_{4(x^2+y^2)}^{64}dz\right)(x^2+y^2)\,dx\,dy$

$D_2=\{(x,y)\mid 4\le x^2+y^2\le 16\}$

$D_2'=\{(r,\theta)\mid 2\le r\le 4,\ 0\le\theta\le 2\pi\}$

$②=\iint_{D_2'}[64-4(x^2+y^2)](x^2+y^2)\,dx\,dy$

$=\int_0^{2\pi}d\theta\int_2^4(64-4r^2)r^3\,dr$

$=8\pi\times\left(4r^4-\frac{1}{6}r^6\right)\Big|_2^4$

$=2304\pi$

$\therefore\iiint_V(x^2+y^2)\,dx\,dy\,dz=①+②$

$=2560\pi$

或 $\int_0^{64}dz\iint_{\frac{z}{16}\le x^2+y^2\le\frac{z}{4}}(x^2+y^2)\,dx\,dy$ (这是截面法)

$=\int_0^{64}dz\int_0^{2\pi}d\theta\int_{\frac{\sqrt z}{4}}^{\frac{\sqrt z}{2}}r^2\cdot r\,dr=2560\pi$

17(1) $\int_0^1dx\int_0^{1-x}dy\int_0^{x+y}f\,dz$

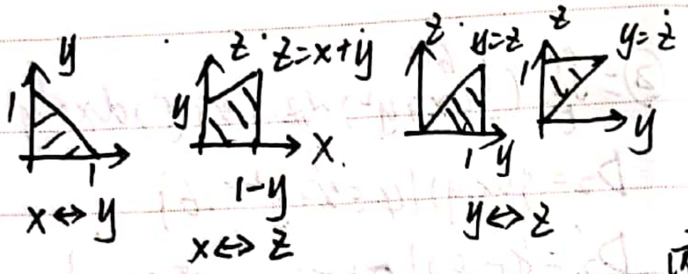
$\xlongequal{交换x,y}\int_0^1dy\int_0^{1-y}dx\int_0^{x+y}f\,dz$

$\xlongequal{交换x,z}\int_0^1dy\int_0^y dz\int_0^{1-y}f\,dx+\int_0^1dy\int_y^1dz\int_{z-y}^{1-y}f\,dx$

$\xlongequal{交换y,z}\int_0^1dz\int_z^1dy\int_0^{1-y}f\,dx+\int_0^1dz\int_0^z dy\int_{z-y}^{1-y}f\,dx$

$\therefore$ 综上: $\int_0^1dx\int_0^{1-x}dy\int_0^{x+y}f\,dz$

$=\int_0^1dz\int_z^1dy\int_0^{1-y}f(x,y,z)\,dx+\int_0^1dz\int_0^z dy\int_{z-y}^{1-y}f(x,y,z)\,dx$



18(3) $\iiint\limits_V f(z)\,dV$

$= \int_{-1}^{1}\pi(1-z^2)f(z)\,dz$

$= \pi\int_{-1}^{1}(1-z^2)f(z)\,dz$

19(3)

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$\therefore \begin{cases} r^2 = 2ar\cos\theta \\ r^2 = az \\ z = 0 \end{cases}$

$\therefore V' = \{(r,\theta,z) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2a\cos\theta,$
$0 \le z \le \frac{r^2}{a}\}$

$\therefore V = \iiint\limits_{V'} r\,d\theta\,dr\,dz$

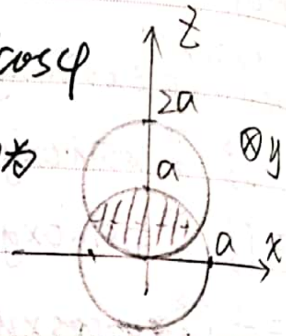
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} dr \int_0^{\frac{r^2}{a}} r\,dz$

20(3) 令 $\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

则 $\Rightarrow \begin{cases} \rho^2 \le a^2 \\ \rho^2 \le 2a\rho\cos\varphi \end{cases}$

两个球的公共界面为

$\begin{cases} x^2 + y^2 = \frac{3}{4}a^2 \\ z = \frac{a}{2} \end{cases}$



$\therefore V = \iiint\limits_{V'} \rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$

$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} d\varphi \int_0^{a} \rho^2\sin\varphi\,d\rho$

$+ \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2a\cos\varphi} \rho^2\sin\varphi\,d\rho$

20(5)

令 $\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

$\therefore \rho^6 = 3\rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta$

$\Rightarrow \rho = \sqrt[3]{3\sin^2\varphi\cos\varphi\sin\theta\cos\theta}$

$x \ge 0 \quad y \ge 0 \quad z \ge 0$

$\therefore \varphi \in [0, \frac{\pi}{2}],\ \theta \in [0, \frac{\pi}{2}]$

$\therefore V = \iiint\limits_{V'} \rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$

$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt[3]{3\sin^2\varphi\cos\varphi\sin\theta\cos\theta}} \rho^2\sin\varphi\,d\rho$

21(1).

x(1)

$$J=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$=a_{11}a_{22}-a_{12}a_{21}$$

21(4)

$$J=\frac{\partial(x,y,z)}{\partial(\theta,\varphi,r)}=\begin{vmatrix}-ar\sin\varphi\sin\theta & ar\cos\theta\cos\varphi & a\sin\varphi\cos\theta\\ br\sin\varphi\cos\theta & br\cos\varphi\sin\theta & b\sin\varphi\sin\theta\\ 0 & -cr\sin\varphi & c\cos\varphi\end{vmatrix}$$

$$=abcr^2\begin{vmatrix}-\sin\varphi\sin\theta & \cos\theta\cos\varphi & \sin\varphi\cos\theta\\ \sin\varphi\cos\theta & \cos\varphi\sin\theta & \sin\varphi\sin\theta\\ 0 & -\sin\varphi & \cos\varphi\end{vmatrix}$$

$$=abcr^2\sin\varphi\begin{vmatrix}-\sin\theta & \cos\theta\cos\varphi & \sin\varphi\cos\theta\\ \cos\theta & \cos\varphi\sin\theta & \sin\varphi\sin\theta\\ 0 & -\sin\varphi & \cos\varphi\end{vmatrix}$$

$$\triangleq abcr^2\sin\varphi\times D$$

$$D=-\sin\theta\times\sin\theta(\cos^2\varphi+\sin^2\varphi)-\cos\theta\times\cos\theta(\cos^2\varphi+\sin^2\varphi)=-1$$

$\therefore J=-abcr^2\sin\varphi$

取$|J|=\left|\frac{\partial(x,y,z)}{\partial(\theta,\varphi,r)}\right|=abcr^2\sin\varphi$

22(2)

$$\iiint_V xy^2z^3\,dxdydz$$

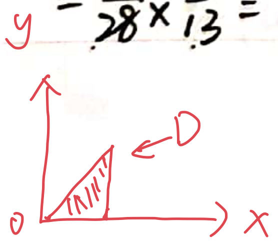
$$=\int_0^1dx\int_0^x dy\int_0^{xy}xy^2z^3\,dz$$

$$=\int_0^1dx\int_0^x\frac{1}{4}x^4y^4\times xy^2\,dy$$

$$=\frac{1}{4}\int_0^1x^5dx\int_0^x y^6dy$$

$$=\frac{1}{28}\int_0^1x^{12}dx$$

$$=\frac{1}{28}\times\frac{1}{13}=\frac{1}{364}.$$



投影法.

$$\iint_D dxdy\int_0^{xy}xy^2z^3\,dz$$

22(6)

作广义球面变换

令 $\begin{cases}x=3\rho\sin\varphi\cos\theta\\ y=2\rho\sin\varphi\sin\theta\\ z=\rho\cos\varphi\end{cases}$

则原式$=$

$$\iiint_{V'}[\rho^2+2\rho^2\sin^2\varphi\cos\theta\sin\theta+2\rho^2\sin\varphi\cos\varphi\cos\theta+2\rho^2\sin\varphi\cos\varphi\sin\theta]\times36\times6\rho^2\sin\varphi\,d\theta d\varphi d\rho$$

含 $\sin\varphi\cos\varphi$ 或 $\cos\theta\sin\theta$ 之项积分之值为0.(由周期性和对称性可知)

$$\therefore 上式=\iiint_{V'}216\rho^4\sin\varphi\,d\theta d\varphi d\rho$$

$$=216\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1\rho^4\sin\varphi\,d\rho$$

$$=\frac{216}{5}\times2\pi\int_0^{\pi}\sin\varphi\,d\varphi$$

$$=\frac{864}{5}\pi$$

(由于ρ的积分上下限与φ、θ无关，φ的积分上下限与θ无关，所以θ和φ的积分可以拎出来提前积)

22(8)

作球坐标变换

$$则原式=\iiint_{V'}\rho^2\cos^2\varphi\cdot\rho^2\sin\varphi\,d\theta d\varphi d\rho$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{3}}d\varphi\int_0^1\rho^4\cos^2\varphi\sin\varphi\,d\rho \quad ①$$

$$+\int_0^{2\pi}d\theta\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}d\varphi\int_0^{2\cos\varphi}\rho^4\cos^2\varphi\sin\varphi\,d\rho \quad ②$$

本题图同20(3)(a=1)

① $= \frac{2\pi}{5}\int_0^{\frac{\pi}{3}} \cos^2\varphi \sin\varphi \, d\varphi$

$= -\frac{2\pi}{5}\int_0^{\frac{\pi}{3}} \cos^2\varphi \, d(\cos\varphi)$

$= -\frac{2\pi}{5} \times \frac{1}{3}\cos^3\varphi \Big|_0^{\frac{\pi}{3}}$

$= \frac{7\pi}{60}$

② $= \frac{2\pi}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi \cdot 32\cos^5\varphi \, d\varphi$

$= \frac{64\pi}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} -\cos^7\varphi \, d(\cos\varphi)$

$= -\frac{64\pi}{5} \times \frac{1}{8}\cos^8\varphi \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{64\pi}{5\times 8\times 256} = \frac{\pi}{160}$

∴原式=①+②= $\frac{59}{480}\pi$