Sample Solutions on HW7 (*45 exercises in total*)

Sec. 6.5 10, 16, 26, 32, 46, 50, 54, 61

10(a) $C(6+12-1, 12) = C(17, 12) = 6188$

10(b) $C(6+36-1, 36) = C(41, 36) = 749{,}398$

10(c) If we first pick the two of each kind, then we have picked 2 • 6 = 12 croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part (a), namely $C(6+12-1, 12) = C(17, 12) = 6188$

10(d) We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1, 21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1, 24) = C(29, 24) = 118755$ ways to pick
24 croissants without restriction, there must be 118755 – 65780 = 52,975 ways to choose two dozen croissants with no more than two broccoli.

10(e) Eight croissants are specified, so this problem is the same as choosing 24 – 8 =
16 croissants without restriction, which can be done in $C(6+16-1, 16) = C(21, 16)$ =20,349 ways.

10(f) First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are 24 – 9 = 15 left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1, 15) = C(20, 15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1, 11) = C(16, 11)$ =4368 choices. Therefore the number of ways to make the selection without violating the restriction is 15504 – 4368 = 11,136

16(a) We require each $x_i \geqslant 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x_1' + x_2' + x_3' + x_4' + x_5 + x_6' = 17$ with each x_i a nonnegative integer. The number of solutions is therefore $C(6+17-1,17) = C(22, 17)$ =26,334.

16(b) The restriction use up 22 of the total, leaving a free total of 7. Therefore the answer is $C(6+7-1, 7) = C(12, 7)$ =792.

16(c) The number of solutions without restriction is $C(6+29-1, 29) = C(34, 29)$ = 278,526. The number of solution violating the restriction by having $x_1 \geqslant 6$ is $C(6+23-1, 23) = C(28, 23)$ = 98,280. Therefore the answer is 278256 – 98280 = 179,976.

16(d) The number of solutions with x2>=9 (as required) but without the restriction on x1 is C(6+20-1, 20) = C(25, 20) = 53130. The number of solutions violating the additional restriction by having x1≥8 is C(6+12-1, 12) = C(17, 12) = 6188. Therefore

the answer is 53130 – 6188 =46,942.

26 We can model this problem by letting x_i be the i^{th} digit of the number for i = 1,2,3,4,5,6 and asking for the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$, where each x_i is between 0 and 8, inclusive, except that one of them equals 9. First, there are 6 ways to decide which of the digits is 9. Without loss of generality assume that $x_6 = 9$. Then the number of ways to choose the remaining digits is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$ (note that the restriction that each $x_i \leqslant 8$ was moot, since the sum was only 4). By Theorem 2 there are $C(5+4-1, 4) = C(8, 4) = 70$ solutions. Therefore the answer is $6 \cdot 70 = 420$.

32 We can treat the 3 consecutive A's as one letter. Thus we have 6 letters, of which 2 are the same (the two R's), so by Theorem 3 the answer is 6! / 2! = 360.

46 We follow the hint. There are 5 bars (chosen books), and therefore there 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars, …, after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be places in 6 slots. The number of ways to do this is therefore $C(6+3-1, 3)$ = C(8,3) = 56.

50 This is actually a problem about partitions of sets. Let us cal the set of 5 objects {a,b,c,d,e}. We want to partition this set into three pairwise disjoint subsets (some possibly empty). We count in a fairly ad hoc way. First, we could put all five objects into one subset (i.e., all five objects go into one box, with the other two boxes empty). Second, we could put four of the objects into one subset and one into another, such as {a,b,c,d} together with {e}. There are **5** ways to do this, since each of the five objects can be the singleton. Third, we could put three of the objects into one set (box) and two into another; there are $C(5,2)$ = **10** ways to do this, since there are that many ways to choose which objects are to be the doubleton. Similarly, there are **10** ways to distribute the elements so that three go into one set and one each into the other two sets (for example, {a,b,c}. {d}, and {e}). Finally, we could put two items into one set, two into another, and one into the third (for example, {a,b}, {c,d}, and {e}). Here we need to choose the singleton (5 ways), and then we need to choose one of the 3 ways to separate the remaining four elements into pairs; this gives a total of 15 partitions. In all we have 41 different partitions.

54 We are asked for the partitions of 5 into at most 3 parts, notice that we are not required to use all three boxes. We can easily list these partitions explicitly: 5 = 5, 5 = 4 +1, 5 = 3+2, 5 = 3+1+1, and 5 = 2+2+1. Therefore the answer is 5.

61 Without the restriction on site X, we are simply asking for the number of ways to order the ten symbols V, V, W, W, X, X, Y, Y, Z, Z (the ordering will give us the visiting schedule). By Theorem 3 this can be done in $10! / (2!)^5 = 113{,}400$ ways. If the inspector visits site X on consecutive days, then in effect we are ordering nine symbols (including only one X)., where now the X means to visit site X twice in a row. There are $9! / (2!)^4 = 22{,}680$ ways to do this. Therefore the answer is 113,400 - 22,680 = 90,720.

Sec. 6.6 6(f), 7, 9

6(f) The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_7=1$ and $a_8=6$. The least integer to the right of 1 that is greater than 1 is 6. Hence 6 is placed in the 7th position. The integer 1 is then placed in the 8th positions, giving the permutation 23587461

7 We begin with the permutation **1234**. Then we apply Algorithm 1 23 times in succession, giving us the other 23 permutations in lexicographic order: **1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312,** and **4321**. The last permutation is the one entirely in decreasing order.

9 We begin with the first 3-combination, namely **{1,2,3}**. Let us trace through Algorithm 3 to find the next. Note that $n = 5$ and $r = 3$; also $a_1=1$, $a_2=2$, and $a_3=3$. We set i equal to 3 and then decrease i until $a_i \neq 5-3+i$. This inequality is already satisfied for $i = 3$, since $a_3 \neq 5$. At this point we increment a_i by 1 (so that now $a_3 = 4$), and fill the remaining spaces with consecutive integers following a_i (in this case there are no more remaining spaces). Thus our second 3-combination is **{1,2,4}**. The next call to Algorithm 3 works the same way, producing the third 3-combination, namely **{1,2,5}**. To get the fourth 3-combination, we again call Algorithm 3. This time the i that we end up with is 2, since $5 = a_3 = 5-3+3$. Therefore the second element in the list is incremented, namely goes from a 2 to a 3, and the third element is the next larger element after 3, namely 4. Thus this 3-combination is **{1,3,4}**. Another call to the algorithm gives us **{1,3,5}**, and another call gives us **{1,4,5}**. Now when we call the algorithm, we find $i=1$ at the end of the **while** loop, since in this case the last two elements are the two largest elements in the set. Thus a_1 is increased to 2, and the remainder of the list is filled with the next two consecutive integers, giving us **{2,3,4}**. Continuing in this manner, we get the rest of the 3-combinations: **{2,3,5}**, **{2,4,5}**, **{3,4,5}**.

Sec.8.1 8, 10, 26, 29, 32

8:

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

c) We will compute a_3 through a_7 using the recurrence relation:

$$a_3 = a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1$$
$$a_4 = a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3$$
$$a_5 = a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8$$
$$a_6 = a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20$$
$$a_7 = a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

10

First let us solve this problem without using recurrence relations at all. It is clear that the only strings that do not contain the string 01 are those that consist of a string of 1's follows by a string of 0's. The string can consist of anywhere from 0 to n 1's, so the number of such strings is $n+1$. All the rest have at least one occurrence of 01. Therefore the number of bit strings that contain 01 is $2^n - (n+1)$. However, this approach does not meet the instructions of this exercise.

a) Let a_n be the number of bit strings of length n that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n-1$ that contains 01, and there are a_{n-1} of these. Alternatively, for any k from 1 to $n-1$, we could start with k 0's, follow this by a 1, and then follow this by any $n-k-1$ bits. For each such k there are 2^{n-k-1} such strings, since the final bits are free. Therefore the number of such strings is $2^0 + 2^1 + 2^2 + \cdots + 2^{n-2}$, which equals $2^{n-1} - 1$. Thus our recurrence relation is $a_n = a_{n-1} + 2^{n-1} - 1$. It is valid for all $n \geq 2$.

b) The initial conditions are $a_0 = a_1 = 0$, since no string of length less than 2 can have 01 in it.

c) We will compute a_2 through a_7 using the recurrence relation:

$$a_2 = a_1 + 2^1 - 1 = 0 + 2 - 1 = 1$$
$$a_3 = a_2 + 2^2 - 1 = 1 + 4 - 1 = 4$$
$$a_4 = a_3 + 2^3 - 1 = 4 + 8 - 1 = 11$$
$$a_5 = a_4 + 2^4 - 1 = 11 + 16 - 1 = 26$$
$$a_6 = a_5 + 2^5 - 1 = 26 + 32 - 1 = 57$$
$$a_7 = a_6 + 2^6 - 1 = 57 + 64 - 1 = 120$$

Thus there are 120 bit strings of length 7 containing 01. Note that this agrees with our nonrecursive analysis, since $2^7 - (7+1) = 120$.

26 Let a_n be the number of coverings.

(a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the leftmost $n-1$ columns, and this can be done in a_{n-1} ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

(b) Clearly $a_1 = 1$ and $a_2 = 2$

(c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584. . ., so the answer to this part is 2584

29 If the codomain has only one element, then there is only one function (namely the function that takes each element of the domain to the unique element of the codomain). Therefore when ***n***=1 we have ***S(m,n)=S(m,1)=1***, the initial condition we are asked to verify. Now assume that ***m>=n>1***, and we want to count ***S***(***m,n***), the number of functions from a domain with ***m*** elements onto a codomain with ***n*** elements. The form of the recurrence relation we are supposed to verify suggests that what we want to do is to look at the non-onto functions. There are ***n^m*** functions from the ***m***-set to the ***n***-set altogether (by the product rule, since we need to choose an element from the ***n***-set, which can be done in ***n*** ways, a total of ***m*** times). Therefore we must show that there are $\sum_{k=1}^{n-1} C(n,k)S(m,k)$ functions from the domain to the codomain that are not onto. First we use the sum rule and break this count down into the disjoint cases determined by the number of elements – let us call it ***k*** – in the range of the function. Since we want the function not to be onto, ***k*** can have any value from ***1*** to ***n-1***, but ***k*** cannot equal ***n***. Once we have specified ***k***, in order to specify a function we need to first specify the actual range, and this can be done in ***C(n,k)*** ways, namely choosing the subset of ***k*** elements from the codomain that are to constitute the range; and second choose an onto function from the domain to this set of ***k*** elements. This latter task can be done in ***S(m,k)*** ways, since (and here is the key recursive point) we are defining ***S(m,k)*** to be precisely this number. Therefore by the product rule there are ***C(n,k)S(m,k)*** different functions with our original domain to our original codomain. Note that this two-dimensional recurrence relation can be used to computer ***S(m,n)*** for any desired positive integers ***m*** and ***n***. Using it is much easier than trying to list all onto functions.

32

We let a_n be the number of moves required for this puzzle.

a) In order to move the bottom disk off peg 1, we must have transferred the other $n-1$ disks to peg 3 (since we must move the bottom disk to peg 2); this will require a_{n-1} steps. Then we can move the bottom disk to peg 2 (one more step). Our goal, though, was to move it to peg 3, so now we must move the other $n-1$ disks from peg 3 back to peg 1, leaving the bottom disk quietly resting on peg 2. By symmetry, this again takes a_{n-1} steps. One more step lets us move the bottom disk from peg 2 to peg 3. Now it takes a_{n-1} steps to move the remaining disks from peg 1 to peg 3. So our recurrence relation is $a_n = 3a_{n-1} + 2$. The initial condition is of course that $a_0 = 0$.

b) Computing the first few values, we find that $a_1 = 2$, $a_2 = 8$, $a_3 = 26$, and $a_4 = 80$. It appears that $a_n = 3^n - 1$. This is easily verified by induction: The base case is $a_0 = 3^0 - 1 = 1 - 1 = 0$, and $3a_{n-1} + 2 = 3 \cdot (3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1 = a_n$.

c) The only choice in distributing the disks is which peg each disk goes on, since the order of the disks on a given peg is fixed. Since there are three choices for each disk, the answer is 3^n.

d) The puzzle involves $1 + a_n = 3^n$ arrangements of disks during its solution—the initial arrangement and the arrangement after each move. None of these arrangements can repeat a previous arrangement, since if it did so, there would have been no point in making the moves between the two occurrences of the same arrangement. Therefore these 3^n arrangements are all distinct. We saw in part (c) that there are exactly 3n arrangements, so every arrangement was used.

Sec. 8.2 2, 4(g), 20, 30, 35

2(a) linear, homogeneous, with constant coefficients, degree 2

2(b) linear with constant coefficients but not homogeneous

2(c) not linear

2(d) linear, homogeneous, with constant coefficients, degree 3

2(e) linear and homogeneous, but not with constant

coefficients **2(f)** linear with constant coefficients, but not

homogeneous

2(g) linear, homogeneous, with constant coefficients, degree 7

4(g) $r^2 + 4r - 5 = 0$ r = -5, 1

$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$

$2 = \alpha_1 + \alpha_2$, $8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1$ $\alpha_2 = 3$

$a_n = -(-5)^n + 3$

20 This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, which factors as $(r^2 - 4)^2$, which then further factors into $(r\text{-}2)^2(r\text{+}2)^2$. The roots are 2 and -2, each with multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3(-2)^n + \alpha_4 n \cdot (-2)^n$

30(a) The associated homogeneous recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 + 5r + 6 = 0$, find that $r = -2$ and $r = -3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c \cdot 4^n$. We plug this into our recurrence relation and obtain $c \cdot 4^n = -5\, c \cdot 4^{n-1} - 6c \cdot 4^{n-2} + 42 \cdot 4^n$. We divide through by 4^{n-2}, obtaining $16c = -20c - 6c + 42 \cdot 16$, whence with a little simple algebra $c = 16$. Therefore the particular solution we seek is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$

30(b) We plug the initial conditions into our solution from part (a) to solve for α and β. So the solution is $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$

35 The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 3a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 3 = 0$, find that $r = 1$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice, we want to look for a function of the form $a_n = c \cdot 2^n + n(dn + e) = c \cdot 2^n + dn^2 + en$. We plug this into our recurrence relation an $c \cdot 2^n + dn^2 + en = 4c \cdot 2^{n-1} + 4d(n-1)^2 + 4e(n-1) - 3c \cdot 2^{n-2} - 3d(n-2)^2 - 3e \cdot (n-2) + 2^n + n + 3$ .A lot of messy algebra transforms this into the following equation, where we group by function of n: $2^{n-2}(-c-4) + n^2 \cdot 0 + n(-4d - 1) + (8d - 2e - 3) = 0$. The coefficients must therefore all be 0, whence $c = -4$, $d = -1/4$, and $e = -5/2$. Therefore the particular solution we see is $a_n^{(p)} = -4 \cdot 2^n - n^2/4 - 5n/2$. So the general solution is $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + \alpha + \beta 3^n$. We solve this system of equations to obtain $\alpha = 1/8$ and $\beta = 39/8$. So the final solution is $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$. As a check of our work, we can compute a_2 both from the recurrence and from the solution, and we find that $a_2 = 22$ both ways.