69、求在指定点的泰勒展开式. 参考例 9.6.11

(1) $f(x,y)=xy^2$ 在点 $P(2,1)$ 处(二阶)

解: $\frac{\partial f}{\partial x}=y^2$, $\frac{\partial f}{\partial y}=2xy$

$\frac{\partial^2 f}{\partial x^2}=0$, $\frac{\partial^2 f}{\partial x\partial y}=2y$, $\frac{\partial^2 f}{\partial y^2}=2x$

$\frac{\partial^3 f}{\partial x^3}=0$, $\frac{\partial^3 f}{\partial x^2\partial y}=0$, $\frac{\partial^3 f}{\partial x\partial y^2}=2$, $\frac{\partial^3 f}{\partial y^3}=0$

∴ 在 $P(2,1)$ 处,

$\frac{\partial f}{\partial x}=1$, $\frac{\partial f}{\partial y}=4$, $\frac{\partial^2 f}{\partial x^2}=0$, $\frac{\partial^2 f}{\partial x\partial y}=2$, $\frac{\partial^2 f}{\partial y^2}=4$

$$\therefore f(x,y)=f(2,1)+\left[(x-2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]f(2,1)+\frac{1}{2!}\left[(x-2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]^2 f(2,1)+R_2$$

$$=2+(x-2)+4(y-1)+2(x-2)(y-1)+2(y-1)^2+R_2$$

其中 $R_2=\frac{1}{3!}\left[(x-2)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]^3 f(2+\theta(x-2),1+\theta(y-1))$

$$=\frac{1}{3!}C_3^2(x-2)(y-1)^2\cdot 2$$

$$=(x-2)(y-1)^2 \quad (0<\theta<1)$$

(3) $f(x,y)=\sin(x^2+y^2)$ 在点 $P(0,0)$ 处（二阶）.

解：$\frac{\partial f}{\partial x}=2x\cos(x^2+y^2)$， $\frac{\partial f}{\partial y}=2y\cos(x^2+y^2)$

$$\frac{\partial f}{\partial x^2}=2\cos(x^2+y^2)-4x^2\sin(x^2+y^2)$$

$$\frac{\partial^2 f}{\partial x\partial y}=-4xy\sin(x^2+y^2)$$

$$\frac{\partial^2 f}{\partial y^2}=2\cos(x^2+y^2)-4y^2\sin(x^2+y^2)$$

$$\frac{\partial^3 f}{\partial x^3}=-4x\sin(x^2+y^2)-8x\sin(x^2+y^2)-8x^3\cos(x^2+y^2)$$

$$=-12x\sin(x^2+y^2)-8x^3\cos(x^2+y^2)$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=-4y\sin(x^2+y^2)-8x^2y\cos(x^2+y^2)$$

$$\frac{\partial^3 f}{\partial x\partial y^2}=-4x\sin(x^2+y^2)-8xy^2\cos(x^2+y^2)$$

$$\frac{\partial^3 f}{\partial y^3}=-4y\sin(x^2+y^2)-8y\sin(x^2+y^2)-8y^3\cos(x^2+y^2)$$

$$=-12y\sin(x^2+y^2)-8x^3\cos(x^2+y^2)$$

∴ 在 $P(0,0)$ 处.

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$， $\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial y^2}=2$， $\frac{\partial^2 f}{\partial x\partial y}=0$

$$f(x,y) = f(0,0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f(0,0)$$

$$+R_2$$

$$= x^2+y^2+R_2$$

其中$R_2 = \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$

$$= \frac{1}{3!}\Big[x^3\left(-12\theta x\sin[(\theta x)^2+(\theta y)^2] - 8(\theta x)^3\cos[(\theta x)^2+(\theta y)^2]\right)$$

$$+3x^2y\left(-4(\theta y)\sin[(\theta x)^2+(\theta y)^2] - 8(\theta x)^2(\theta y)\cos[(\theta x)^2+(\theta y)^2]\right)$$

$$+3xy^2\left(-4(\theta x)\sin[(\theta x)^2+(\theta y)^2] - 8(\theta x)(\theta y)^2\cos[(\theta x)^2+(\theta y)^2]\right)$$

$$+y^3\left(-12(\theta y)\sin[(\theta x)^2+(\theta y)^2] - 8(\theta y)^3\cos[(\theta x)^2+(\theta y)^2]\right)$$