习题 9.3 — 9.5

12. (1) $f'_x = 1 + (y-1)\cdot\dfrac{\frac{1}{y}}{1+(x/y)^2} = 1 + \dfrac{y(y-1)}{x^2+y^2}$

$f'_x(0,1) = 1 + \dfrac{y(y-1)}{x^2+y^2}\Big|_{\substack{x=0\\y=1}} = 1$

$f'_y = \arctan\dfrac{x}{y} + (y-1)\cdot\dfrac{1}{1+(x/y)^2}\cdot\left(-\dfrac{x}{y^2}\right) = \arctan\dfrac{x}{y} - \dfrac{x(y-1)}{x^2+y^2}$

$f'_y(0,1) = \arctan\dfrac{x}{y} - \dfrac{x(y-1)}{x^2+y^2}\Big|_{\substack{x=0\\y=1}} = 0$

13. (2) $\dfrac{\partial z}{\partial x} = \left(\dfrac{\partial}{\partial x}e^{\frac{y}{x}}\right)\cdot(x+y) + e^{\frac{y}{x}}\cdot 1 = \left(1 - \dfrac{y(x+y)}{x^2}\right)e^{\frac{y}{x}}$

$\dfrac{\partial z}{\partial y} = e^{\frac{y}{x}}\cdot\dfrac{1}{x}(x+y) + e^{\frac{y}{x}} = e^{\frac{y}{x}}\left(2+\dfrac{y}{x}\right)$

(3) $\dfrac{\partial z}{\partial x} = \dfrac{1}{2x+\sqrt{x^2+y^2}}\cdot\dfrac{\partial}{\partial x}\left(2x+\sqrt{x^2+y^2}\right) = \dfrac{2+\frac{x}{\sqrt{x^2+y^2}}}{2x+\sqrt{x^2+y^2}} = \dfrac{2\sqrt{x^2+y^2}+x}{2x\sqrt{x^2+y^2}+x^2+y^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{1}{2x+\sqrt{x^2+y^2}}\cdot\dfrac{2y}{2\sqrt{x^2+y^2}} = \dfrac{y}{2x\sqrt{x^2+y^2}+x^2+y^2}$

$(x-2)\cdots(x-100) + (x-1)[(x-2)\cdots(x-100)]'$

$\parallel$

15. (1) $f'_x = (y-1)(y-2)\cdots(y-100)\left[\dfrac{d}{dx}(x-1)(x-2)\cdots(x-100)\right]$

$f'_x(1,0) = 100!\cdot(1-2)(1-3)\cdots(1-100) = -99!\cdot 100!$

$f''_{xy} = \left[\dfrac{d}{dx}(x-1)(x-2)\cdots(x-100)\right]\left[\dfrac{d}{dy}(y-1)(y-2)\cdots(y-100)\right]$

$= \left[(-1)^{99}\cdot 99!\right]^2 = (99!)^2$

18. $\varphi(x) \triangleq f(x,0) = \begin{cases} x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f_x'(0,0) = \lim_{\Delta x \to 0} \frac{\varphi(0+\Delta x) - \varphi(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

$\psi(y) \triangleq f(0,y) = \begin{cases} 1, & y \neq 0 \\ 0, & y = 0 \end{cases}$

$f_y'(0,0) = \lim_{\Delta y \to 0} \frac{\psi(0+\Delta y) - \psi(0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1-0}{\Delta y} = \infty$

$\therefore f_x'(0,0)$ 存在且为 1；$f_y'(0,0)$ 不存在。

20. 令 $u = \frac{y}{x}$，则 $\frac{\partial f(\frac{y}{x})}{\partial x} = \frac{df(u)}{du} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot (-\frac{y}{x^2})$

$\frac{\partial f(\frac{y}{x})}{\partial y} = \frac{df(u)}{du} \cdot \frac{\partial u}{\partial y} = f'(u) \cdot \frac{1}{x}$

$\therefore \frac{\partial z}{\partial x} = y + f(u) + x\frac{\partial f(u)}{\partial x} = y + f(u) - \frac{y}{x} f'(u)$

$\frac{\partial z}{\partial y} = x + x\frac{\partial f(u)}{\partial y} = x + f'(u)$

$\therefore LHS = xy + xf(u) - yf'(u) + xy + yf'(u) = 2xy + f(\frac{y}{x}) = xy + z = RHS$。

(left hand side) 指题中等号左边　　(right hand side)

23. $\frac{\partial z}{\partial x} = e^{-(\frac{1}{x}+\frac{1}{y})} \cdot [\frac{\partial}{\partial x}(-\frac{1}{x}-\frac{1}{y})] = e^{-(\frac{1}{x}+\frac{1}{y})} \cdot \frac{1}{x^2}$ 同理 $\frac{\partial z}{\partial y} = e^{-(\frac{1}{x}+\frac{1}{y})} \cdot \frac{1}{y^2}$

$\therefore LHS = e^{-(\frac{1}{x}+\frac{1}{y})} + e^{-(\frac{1}{x}+\frac{1}{y})} = 2z = RHS$。

31. (1) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2+v^2}} \cdot \sin y + \frac{v}{\sqrt{u^2+v^2}} \cdot ye^{xy} = \frac{x\sin^2 y + ye^{2xy}}{\sqrt{x^2\sin^2 y + e^{2xy}}}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{u}{\sqrt{u^2+v^2}} \cdot x\cos y + \frac{v}{\sqrt{u^2+v^2}} \cdot xe^{xy} = \frac{x^2 \sin y \cos y + xe^{2xy}}{\sqrt{x^2\sin^2 y + e^{2xy}}}$

(3) $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{2x}{x^2+y^2} \cdot (\cos t - t\sin t) + \frac{2y}{x^2+y^2}(-\cos t)$

$= \frac{(1-t^2)\sin 2t + t(\cos 2t + 1)}{t^2\cos^2 t + \sin^2 t}$

33. $z=f(u,x+y)\triangleq f(u,v)$ $\quad \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'\cdot e^y+f_2'\cdot 1=e^y f_1'+f_2'$

$$\frac{\partial^2 z}{\partial x\partial y}=e^y f_1'+e^y[f_{11}''\cdot xe^y+f_{12}''\cdot 1]+f_{21}''\cdot xe^y+f_{22}''\cdot 1=e^y f_1'+xe^{2y}f_{11}''+\cancel{e^y f_{12}''}+e^y(x+1)f_{12}''+f_{22}''$$

34. $\frac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot\sin y$ $\quad \frac{\partial^2 z}{\partial x\partial y}=2x(f_{11}''(-2y)+f_{12}''x\cos y)+f_2'\cos y+\sin y(f_{21}''(-2y)+f_{22}''x\cos y)=-4xyf_{11}''+2(x^2\cos y-y\sin y)f_{12}''+x\sin y\cos y f_{22}''+\cos y f_2'$

38. $z=z(x,y)\triangleq f(u,v)$ $\quad \frac{\partial z}{\partial x}=f_1'+f_2'$ $\quad \frac{\partial z}{\partial y}=-2f_1'+3f_2'$

$$\frac{\partial^2 z}{\partial x^2}=f_{11}''\cdot 1+f_{12}''\cdot 1+f_{21}''\cdot 1+f_{22}''\cdot 1=f_{11}''+2f_{12}''+f_{22}''$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}''\cdot(-2)+f_{12}''\cdot 3+f_{21}''\cdot(-2)+f_{22}''\cdot 3=-2f_{11}''+f_{12}''+3f_{22}''$$

$$\frac{\partial^2 z}{\partial y^2}=-2[f_{11}''\cdot(-2)+f_{12}''\cdot 3]+3[f_{21}''\cdot(-2)+f_{22}''\cdot 3]=4f_{11}''-6f_{12}''+9f_{22}''$$

代入 $6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$ 得 $f_{12}''=0$，即 $\frac{\partial^2 z}{\partial u\partial v}=0$。

43. 等式两边同时对 x 求导，得 $\frac{1}{2}\left[\frac{1}{x^2+y^2}(2x+2yy_x')\right]=\frac{xy_x'-y}{x^2+y^2}$，整理得

$x+yy_x'=xy_x'-y$ (*)。 $\therefore \frac{dy}{dx}=\frac{x+y}{x-y}$。

(*)两边同时对 x 求导，得 $1+y'+yy''=xy''$，得 $\frac{d^2y}{dx^2}=\frac{2(x^2+y^2)}{(x-y)^3}$。

46. 在 $g(x^2,e^y,z)=0$ 的两边同时对 x 求导，得 (或者通过隐函数定理算出 $\frac{dz}{dx}$)

$$g_1'\cdot 2x+g_2'\cdot\cos x e^{\sin x}+g_3'\cdot z_x'=0 \Rightarrow z_x'=-\frac{1}{g_3'}(g_1'\cdot 2x+g_2'\cos x e^{\sin x})$$

$$\frac{du}{dx}=f_1'+f_2'\cdot y_x'+f_3'\cdot z_x'=f_1'+f_2'\cos x-\frac{f_3'}{g_3'}(2xg_1'+\cos x e^{\sin x}g_2')$$

49. 对方程组同时求微分，得 $\begin{cases} 2u\,du - dv = -dx \\ du + 2v\,dv = dy \end{cases}$。

$$du = \frac{\begin{vmatrix} -dx & -1 \\ dy & 2v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & 2v \end{vmatrix}} = \frac{-2v\,dx + dy}{4uv+1} \qquad dv = \frac{\begin{vmatrix} 2u & -dx \\ 1 & dy \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & 2v \end{vmatrix}} = \frac{dx + 2u\,dy}{4uv+1}$$

这里的大F其实是小f

52. 方程两边同时对 x 求导，有 $\cancel{F_1' + F_2'}$ $F_1' \cdot (1 + \frac{z_x'}{y}) + F_2' \cdot (\frac{x z_x' - z}{x^2}) = 0$

$$\Rightarrow (\tfrac{1}{y}F_1' + \tfrac{1}{x}F_2') \cdot z_x' = \frac{z}{x^2}F_2' - F_1'$$

同理有 $F_1' \cdot \frac{y z_y' - z}{y^2} + F_2'(1 + \frac{z_y'}{x}) = 0 \Rightarrow (\frac{1}{x}F_2' + \frac{1}{y}F_1') z_y' = \frac{z}{y^2}F_1' - F_2'$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x \cdot \frac{\frac{z}{x^2}F_2' - F_1'}{\frac{1}{y}F_1' + \frac{1}{x}F_2'} + y \cdot \frac{\frac{z}{y^2}F_1' - F_2'}{\frac{1}{x}F_2' + \frac{1}{y}F_1'} = \frac{(\frac{z}{x}F_2' + \frac{z}{y}F_1') - (xF_1' + yF_2')}{\frac{1}{x}F_2' + \frac{1}{y}F_1'}$$

$= z - xy$。

(或者用隐函数定理直接算出 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$)

记 $F(x,y,z) = f(x + \frac{z}{y}, y + \frac{z}{x}) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x'}{F_z'} = \frac{f_1' + f_2'(-\frac{z}{x^2})}{f_1'\frac{1}{y} + f_2'\frac{1}{x}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y'}{F_z'} = \cdots$$