

Theory of Computation, Fall 2023

Assignment 7 Solutions

- Q1. For any alphabet Σ , Σ^* is countable. Since a language L is a subset of Σ^* , L must be countable.
- Q2. The power set of $\{1\}^*$ is uncountable, and the set of all Turing machines is countable. Therefore, some subset of $\{1\}^*$ cannot be decided by any Turing machine.
- Q3. Suppose there's some Turing machine M_H that decides H . We show that there is a Turing machine M^* that decides A . Since A is recursively enumerable, there is some Turing machine M_A that semidecides A . That is M_A halts on a string w if and only if $w \in A$. Therefore, to decide whether $w \in A$ or not, it suffices to decide whether M_A halts on w , and this can be done by M_H . More precisely, the following Turing machine M^* decides A .

$M =$ on input w :

1. run M_H on " M_A " " w "
2. return the result of M_H

- Q4. Let $A = \{ \langle M \rangle : \langle M \rangle \text{ is a Turing machine that halts on some input} \}$. Let B be the language in the question. In class, we have proved that A is not recursive. To prove that B is not recursive, it suffices to reduce A to B . Suppose that there is a Turing machine M_B that decides B . Then we can construct the following Turing machine M_A to decide A .

$M_A =$ on input " M ":

1. Construct a Turing machine M_{all} that halts on every input as follows.

$M_{\text{all}} =$ on input x

1. halt
2. run M_B on " M " " M_{all} " " 1 "
3. return the result of M_B

This completes the proof.

- Q5. We can conclude that A is also recursively enumerable. Since B is recursively enumerable, there is a Turing machine M_B that semidecides B . We can construct the following Turing machine M_A .

$M_A =$ on input " x ":

1. compute $f(x)$
2. run M_B on $f(x)$

M_A halts on x if and only if M_B halts on $f(x)$ if and only if $f(x) \in B$ if and only if $x \in A$. So M_A semidecides A .

- Q6. Let f be a reduction from A to B . By definition, for any $x \in \Sigma^*$, $x \in A$ if and only if $f(x) \in B$. In other words, $x \in \bar{A}$ if and only if $f(x) \in \bar{B}$. So f is also a reduction from \bar{A} to \bar{B} .
- Q7. Let M_1 and M_2 be Turing machines that semidecide A and \bar{A} , respectively. Consider the following Turing machine M .

$M =$ on input w :

1. run M_1 and M_2 on w in parallel
2. If M_1 halts, accept w , and if M_2 halts, reject w .

If $w \in A$, M_1 will halt on w , and M will accept w . Otherwise, M_2 will halt on w , and M will reject w . So M decides A . A is recursive.

- Q8. Since $A \leq \bar{A}$, by the conclusion of Q6, we have that $\bar{A} \leq A$. Since A is recursively enumerable, by the conclusion of Q5, \bar{A} is also recursively enumerable. Then by the conclusion of Q7, A is recursive.