## Chapter 6 Counting

### 6.1 The Basic of Counting

**1. Basic Counting Principles**

**1) The Sum Rule**

**2) The Product Rule**

**2. The Inclusion-Exclusion Principle (Subtraction Rule)**

If a task can be done in either $n1$ ways or $n2$ ways, then the number of ways to do the task is $n1 + n2$ minus the number of ways to do the task that are common to the two different ways.

$|A1 \cup A2| = |A1| + |A2| - |A1 \cap A2|$.

**3. Tree Diagrams**

### 6.2 The Pigeonhole Principle

**1.The Pigeonhole Principle :** If $k$ is a positive integer and $k$+1 or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

If f is a function from A to B, where A and B are finite sets with $|A| > |B|$, then there are elements $a_1$, $a_2$ in A($a_1 \neq a_2$) such that $f(a_1) = f(a_2)$

***Corollary* 1** *A* function *f* from a set with *k*+1 or more elements to a set with *k* elements is not one-to-one.

**2. The Generalized Pigeonhole Principle:** If *N* objects are placed into *k* boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

$f: A \to B$, If $\left\lceil \frac{|A|}{|B|} \right\rceil = i$, then there must exist elements $a_1, a_2, \cdots, a_i \in A$ such that $f(a_1) = f(a_2) = \cdots = f(a_i) = b \in B$

### 6.3 Permutations and Combinations

**1. Permutations**

【Definition】A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects. An ***r-permutation*** is an ordered arrangement of *r* elements of a set.

Notation: P(n, r)

**【Theorem 1】** P(n, r)=n(n-1)(n-2)…(n-r+1) $= \frac{n!}{(n-r)!}$

$B = \{b_1, b_2, \ldots, b_r\}$, $A = \{a_1, a_2, \cdots, a_n\}$, $f: B \to A$

1) *f* is an injection from *B* to *A* ⇔ an *r*-permutation of the set *A*

2) the number of injections from *B* to *A* ⇔ *P*(*n*, *r*)

**2. Combinations**

【Definition】An ***r-combination*** of elements of a set is an unordered selection of r elements from the set.

Notation: $\boldsymbol{C(n,r)} = \binom{\boldsymbol{n}}{\boldsymbol{r}}$ ←Binomial coefficient

**【Theorem 2】** $\boldsymbol{C(n,r)} = \frac{\boldsymbol{n!}}{\boldsymbol{r!(n-r)!}} = n(n-1)(n-2)\ldots\frac{n-r+1}{r!}$

**【Corollary 1】** ***combination corollary*** Let *n* and *r* be nonnegative integers with *r* ≤ *n*. Then

$C\ (n, r) = C\ (n, n-r)$

**3. Combinatorial Proofs**

【**Definition**】 A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

**6.4 Binomial Coefficients**

【 **Theorem 1** 】 ***The Binomial Theorem*** Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then $(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$

【 **Corollary 1** 】 Let $n$ be a nonnegative integer. Then $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

【 **Corollary 2** 】 Let $n$ be a positive integer. Then $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

【 **Corollary 3** 】 Let $n$ be a nonnegative integer. Then $\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n$

【 **Theorem 2** 】 ***PASCAL'S Identity*** Let $n$ and $k$ be positive integers with $k \le n$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

【 **Theorem 3** 】 ***Vandermonde's Identity*** Let $m$, $n$ and $r$ be nonnegative integers with $r$ not exceeding either $m$ or $n$. Then $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$

【 **Corollary 4** 】 If $n$ is a nonnegative integer. Then $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$

【 **Theorem 4** 】 Let $n$ and $r$ be nonnegative integer with $r \le n$. Then $\binom{n+1}{r+1} = \sum_{j=r}^{n}\binom{j}{r}$

**6.5 Generalized Permutations and Combinations**

**2. Permutations with Repetition**

【 **Theorem 1** 】 The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$.

**3. Permutations with Indistinguishable Objects**

$A = \{ n_1 \bullet a_1, n_2 \bullet a_2, \ldots, n_k \bullet a_k \}$, where $n_1+n_2+\ldots+n_k = n$

【**Theorem 2** 】 The number of different permutations of $n$ objects, where there are $n_1$ indistinguishable objects of type1, …, and $n_k$ indistinguishable objects of type $k$, is $n!/(n_1!n_2!\ldots n_k!)$

【 **Theorem 3** 】 The number of $r$- Circle permutations of a set of $n$ objects is $P(n, r)/r$ .

**4. Combinations with Repetition**

【 **Theorem 4** 】 There are $C\ (n-1+r, r)$ $r$-combination from a set with $n$ elements when repetition of elements is allowed.

**5. Distributing objects into boxes**

**1) Distinguishable objects and distinguishable boxes**

【 **Theorem 5** 】 The number of ways to distribute $n$ distinguishable objects into $k$ distinguishable boxes so that $n_i$ objects are place into box $i$, $i$=1,2,…,$k$, equals $n!/(n_1!n_2!\ldots n_k!)$

**2) Distinguishable objects and indistinguishable boxes**

counting the ways to place $n$ distinguishable objects into $k$ indistinguishable boxes

*S(n, j):* Stirling numbers of the second kind

- the number of ways to distribute $n$ distinguishable objects into $j$ indistinguishable boxes so that **no boxes is empty**

(1) $S(r, 1)=S(r, r)=1$ (r≥1)

(2) $S(r, 2)=2^{r-1}-1$

(3) $S(r, r-1)=C(r,2)$

(4) $S(r+1, n)=S(r,n-1)+nS(r, n)$

(5) $S(n,j) = (\frac{\sum_{i=0}^{j-1}(-1)^i C_j^i (j-i)^n)}{j!}$

the number of ways to place n distinguishable objects into k indistinguishable boxes (exist empty boxes) $\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} ((\sum_{i=0}^{j-1}(-1)^i C_j^i (j-i)^n)/j!)$

**3) Indistinguishable objects and distinguishable boxes**

-counting the ways to distribute $n$ indistinguishable objects into $k$ distinguishable boxes

-Same as counting the number of n-combinations for a set with k elements when repetitions are allowed.

**4) Indistinguishable objects and indistinguishable boxes**

-counting the ways to distribute indistinguishable objects into indistinguishable boxes

## 6.6 Generating Permutations and Combinations

### 1. Generating Permutations

***The lexicographic ordering* of the set of permutations of {1, 2, …, *n*}**

The permutation $a_1a_2\ldots a_n$ precedes the permutation of $b_1b_2\ldots b_n$, if for some $k$, with $1\leq k \leq n$, $a_1 = b_1$, $a_2 = b_2$, …, $a_{k-1} = b_{k-1}$, and $a_k < b_k$

- Given permutation $a_1a_2\ldots a_n$, find the next larger permutation in increasing order:
  (1) Find the integers $a_j, a_{j+1} with\ a_j < a_{j+1} and\ a_{j+1} > a_{j+2} > \cdots > a_n$
  (2) Put in the $j$th position the least integer among $a_{j+1}, a_{j+2}, \cdots, a_n$ that is greater than $a_j$
  (3) List in increasing order the rest of the integers $a_j, a_{j+1}, \cdots, a_n$
  **124653 → 125346**

### 2. Generating Combinations

***Problem 1:***

Generate all combinations of the elements of a finite set.

*Solution:*

- A combination is just a subset. ⇒ We need to list all subsets of the finite set.
- Use bit strings of length $n$ to represent a subset of a set with $n$ elements. ⇒ We need to list all bit strings of length $n$.
- The $2^n$ bit strings can be listed in order of their increasing size as integers in their binary expansions.

Algorithm of producing all bit strings

- Start with the bit string 000…00, with $n$ zeros.
- Then, successively find the next larger expansion until the bit string 111…11 is obtained.

The method to find the next larger binary expansion:

Locate the first position from the right that is not a 1, then changing all the 1s of the right of this position to 0s and making this first 0 a 1. **1000110011 → 1000110100**

***Problem 2:***

Generate all *r*-combinations of the set {1, 2, …, *n*}

The algorithm for generating the *r*-combination of the set {1, 2, …, *n*}

(1) $S_1 = \{1, 2, \ldots, r\}$

(2) If $S_i = \{a_1, a_2, \cdots, a_r\}, 1 \leq i \leq C_n^r - 1$has found, then the next combination can be obtained using the following rules.

First, locate the last element$a_i$in the sequence such that$a_i \neq n - r + i$. Then replace$a_i$ with$a_i + 1$ and$a_j$ with$a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$.