Sample Solutions on HW11 (*59 exercises in total*)

Sec. 9.6 5, 10, 23(a), (c), 32, 44, 66

5 The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

5(a) The equality relation on any set satisfies all three conditions and is therefore a partial ordering. (It is the smallest partial ordering; reflexivity insures that every partial order contains at least all the pairs (a,a).)

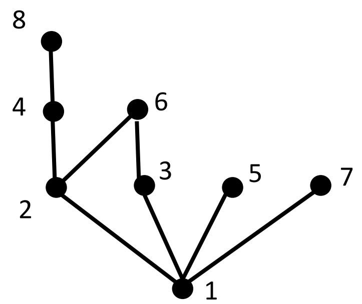
5(b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

5(c) This is a poset, as explained in Example 1 in this section.

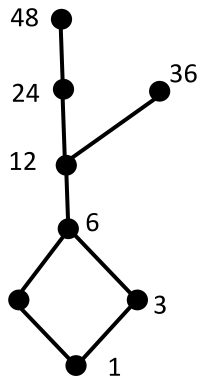
5(d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that 2 does not divide 2. (It also is not antisymmetric and not transitive.)

10 This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

23(a)



23(c)



32(a) The maximal elements are the ones with no other elements above them, namely l and m.

32(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

32(c) There is no greatest element, since neither *l* nor *m* is greater than the other.
32(d) There is no least element, since none of *a*, *b*, and *c* is less than the other two.

32(e) We need to find elements from which we can find downward paths to all of *a*, *b*, and *c*. It is clear that *k*, *l*, and *m* are the elements fitting this description.

32(f) Since *k* is less than both *l* and *m*, it is the least upper bound of *a*, *b*, and *c*.

32(g) No element is less than both *f* and *h*, so there are no lower bounds.

32(h) Since there are no lower bounds, there can be no greatest lower bound.

44 In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.
44(a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

44(b) This is a lattice; in fact it is a linear order, since each element in this list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

44(c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in this list is the smaller number (since here "greater" really means "less"), and the greatest lower bound is the larger of the two numbers.

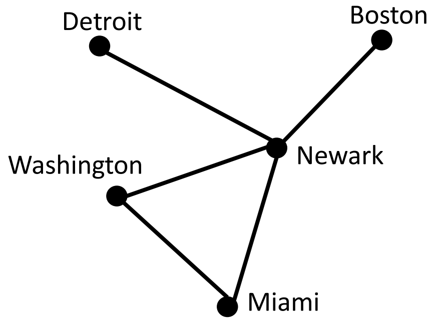
44(d) This is similar to Example 24 of this section, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

64 There are many compatible total orders here. We just need to work from the bottom up. One answer is to take
Foundation < Framing < Roof < Exterior siding < Wiring < Plumbing < Flooring < Wall-board < Exterior painting < Interior painting < Carpeting < Interior fixtures < Exterior fixtures < Completion.

Sec. 10.1 1, 3-9

1 In **part (a)** we have a simple graph, with undirected edges, no loops or multiple edges. In **part (b)** we have a multigraph, since there are multiple edges. In **part (c)** we have the same picture as in part (b) except that there is now a loop at one vertex, thus this is a pesudograph. In **part (d)** we have a directed graph, the directions of the edges telling the directions of the flights. In **part (e)** we have a directed multigraph, since there are parallel edges.

(a)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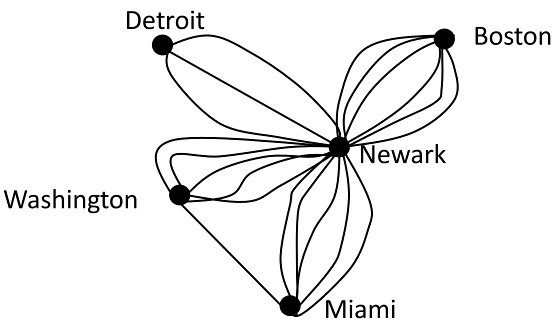
(b)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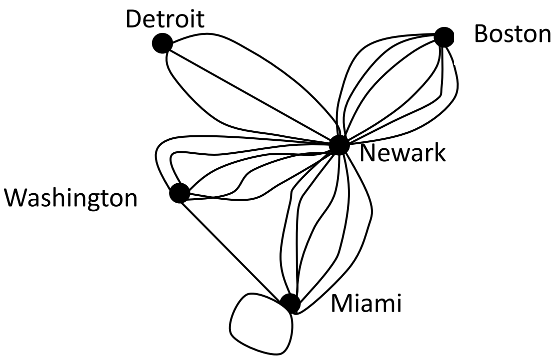
(c)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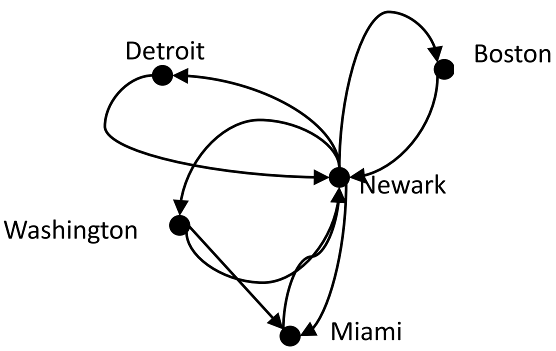
(d)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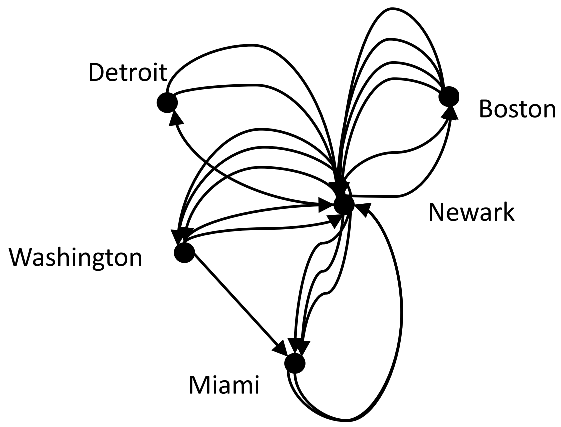
(e)



3 This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

4 This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

5 This is a pseudograph; the edges are undirected, but there are loops and parallel edges.

6 This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

7 This is a directed multigraph; the edges are directed, and there are no parallel edges, but there are loops.

8 This is a directed multigraph; the edges are directed, and there are parallel edges.

9 This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

Sec.10.2 5, 24, 25, 42(b,f,h), 53, 60

5 By Theorem 2 in this section, the number of vertices of odd degree must be even. Hence there cannot be a graph with 15 vertices of odd degree 5. (We assume that the problem was meant to imply that the graph contained only these 15 vertices.)

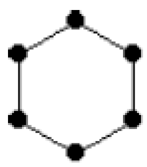
24 This is a complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

25 We can show that this graph is not bipartite by looking at a triangle, in this case the triangle formed by vertices b, d, and e. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.
(An alternative way to look at this is given by Theorem 4 in this section. Because of the existence of a triangle, it is impossible to color the three vertices of the triangle in two colors so that any two adjacent vertices are colored differently.)

42(a) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

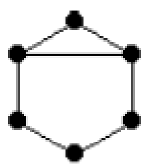
42(b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

42(c) A 6-cycle is such a graph.

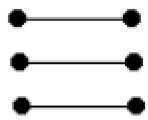


42(d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

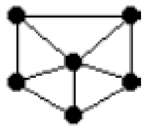
42(e) A 6-cycle with one of its diagonals added is such a graph.



42(f) A graph consisting of three edges with no common vertices is such a graph.



42(g) The 5-wheel is such a graph.



42(h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.

53(a) The complete graph K_n is regular for all values of $n \geqslant 1$, since the degree of each vertex is n-1.

53(b) The degree of each vertex of C_n is 2 for all n which C_n is defined, namely $n \geqslant 3$, so C_n is regular for all these values of n.

53(c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the rim is 3. Therefore W_n is regular if and only if n = 3. Of course W_3 is the same as K_4.

53(d) The cube Q_n is regular for all values of $n \geqslant 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

60 The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n-1)/2$ edges, we want to solve $n(n-1)/2 = 28$. Thus $n = 8$.

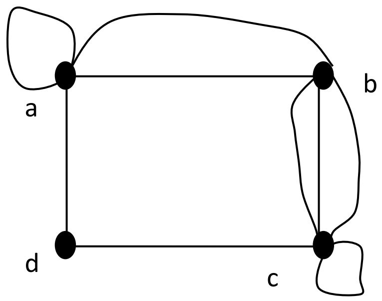
Sec. 10.3 8, 15, 17, 34-37

8 $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ with respect to the ordering of vertices *a, b, c, d, e*.

15 In this graph there are loops, which are represented by entries on the diagonal.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$ with respect to the ordering of vertices *a, b, c, d*.

17 Because of the numbers larger than 1, we need multiple edges in this graph.



34 These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_3$.

35 These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$, $f(u_5) = v_4$.

36 These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

37 These graphs are isomorphic, since each is the 7-cycle (this is just like Ex. 35).

Sec. 10.4 27(e), 28, 29, 62

27(e) There are two approaches here.
One is to use matrix multiplication on the adjacency matrix of this directed graph (by Theorem 2), which is $$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$ with respect to the ordering

of vertices a, b, c, d, e. Thus we just need to compute A^6, and look at the (1,5)th entry to determine the number of paths from a to e of length 6, which is 5.

The other method is to examine the number of possibilities for a path of length 6. Since the only way to get to e is from b, we are asking for the number of paths of length 5 from a to b. We can go around the square (a,b,e,d,a,b), or else we can jog over to either b or d and back twice – there being 4 ways to choose where to do the jogging. Therefore there are 5 paths in all.

28 We show this by induction on n. For $n=1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geqslant 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degree is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

29 The definition given here makes it clear that u and v are related if and only if they are in the same component – in other words $f(u) = f(v)$ where $f(x)$ is the component in which x lies. Therefore this is an equivalence relation.

62 The adjacency matrix of G_1 is as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute A^2 and A^3, obtaining

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 & 1 & 2 & 1 \\ 5 & 5 & 4 & 6 & 1 & 6 & 1 \\ 2 & 2 & 6 & 2 & 3 & 5 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 6 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

Already every off-diagonal entry in A^3 is nonzero, so we know that there is a path of length 3 between every pair of distinct vertices in this graph. Therefore the graph G_1 is connected.

On the other hand, the adjacency matrix of G_2 is as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We compute A^2 through A^5, obtaining the following matrices:

$$A^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & 5 \\ 0 & 0 & 0 & 5 & 6 & 5 \\ 0 & 0 & 0 & 5 & 5 & 6 \end{bmatrix} \qquad A^5 = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 11 & 11 \\ 0 & 0 & 0 & 11 & 10 & 11 \\ 0 & 0 & 0 & 11 & 11 & 10 \end{bmatrix}$$

If we compute the sum $A + A^2 + A^3 + A^4 + A^5$ we obtain

$$\begin{bmatrix} 6 & 7 & 7 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 21 & 21 \\ 0 & 0 & 0 & 21 & 20 & 21 \\ 0 & 0 & 0 & 21 & 21 & 20 \end{bmatrix}.$$

There is a 0 in the (1,4) position, telling us that there is no path of length at most 5 from vertex a to vertex d. Since the graph only has six vertices, this tells us that there is no path at all from a to d. Thus the fact that there was a 0 as an off-diagonal entry in the sum told us that the graph was not connected.