Chapter 8　Advanced Counting Techniques

8.1　Applications of Recurrence Relations

2. Recurrence Relations

【Definition】 A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \dots, a_{n-1}$, for all integers n with $n \ge n_0$, where n_0 is a nonnegative integers.

$a_n = f(a_0, a_1, a_2, \dots, a_{n-1})$　　$n \ge n_0$

A solution of a recurrence relation is a sequence if its terms satisfy the recurrence relation.

Note:

- Normally, there are infinitely many sequences which satisfy a recurrence relation.
  We distinguish them by the *initial conditions*, the values of $a_0, a_1, a_2, \dots$ to uniquely identify a sequence.
- The *degree* of a recurrence relation
  $a_n = a_{n-1} + a_{n-8}$　　---- a recurrence relation of degree 8

3. Modeling with Recurrence Relations

8.2　Solving Linear Recurrence Relations

Linear homogeneous(齐次) recurrence relation of degree k with constant coefficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$　　where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$

- *Linear*- linear combination of previous terms
- *constant coefficients*- the coefficients of a_is are constants
- *degree k*- a_n is a function of the previous k terms of the sequence
- *Homogeneous*- If we put all the a_i s on the left side of the equation and everything else on the right side, then the right side is 0. Otherwise *nonhomogeneous*.

A sequence satisfying the recurrence relation in the definition is uniquely determined by the recurrence relation and the *k initial conditions*:

a0=C0, a1= C1, … , ak-1=Ck-1

1. Solving Linear Homogeneous Recurrence Relation With Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

The basic approach:

To look for solutions of the form $a_n = r^n$, where r is a constant.

Solution Procedure:

(1) Put all a_is on the left side of the equation $a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0$

(2) Substitute the solution into the equation, factor out the lowest power of r and eliminate it.

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k} = 0$$
$$r^{n-k}(r^k - c_1 r k^{n-1} - c_2 r^{k-2} - \cdots - c_k) = 0$$

(3) We obtain the equivalent equation (*Characteristic equation)*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

(4) Find its k roots $r_1, r_2, \cdots, r_k$ (*Characteristic root)*

These characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

【Theorem 1】Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. Then the Sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n for\ n = 0,1,2,\cdots$, where α_1, α_2 are constants.

Find an explicit formula for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:
(1) Determine the characteristic equation: $r^2 - c_1 r - c_2 = 0$
(2) Find its roots: r_1, r_2
(3) Obtain the general solution: $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$
(4) Determine α_1, α_2:

【Theorem 2】 Let c_1, c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0{}^n + \alpha_2 n r_0{}^n for\ n = 0,1,2,\cdots$, where α_1, α_2 are constants.

【Theorem 3】 Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \cdots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n + \cdots + \alpha_k r_k{}^n$ for $n = 0,1,2,\cdots$ where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.

2. Linear Nonhomogeneous Recurrence Relation With Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

Where c_i(i=1,2,…,k) is real numbers, $F(n)$ is a function not identically zero depending only on n.
the associated *homogeneous* recurrence relation: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$
Note:

- Solutions to nonhomogeneous case is the sum of solutions to associated homogeneous recurrence system and a particular solution to the nonhomogeneous case.

【Theorem 5】 Let $\{a_n{}^{(p)}\}$ be a *particular solution* of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$ Then every solution is of the form $\{a_n{}^{(p)} + a_n{}^{(h)}\}$, where $\{a_n{}^{(h)}\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

【Theorem 6】 Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part *F(n)* of the form

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

If s is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

If s is a root of multiplicity m, a particular solution is of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

8.4 Generating Functions

1. Generating function for a sequence

【Definition 1】 The *generating function* for the sequence $a_0, a_1, a_2, \cdots, a_k, \cdots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

The generating function for a finite sequence of real numbers $a_0, a_1, a_2, \cdots, a_n$ is

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

2. Useful Facts About Power Series

【Theorem 1】Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$\boldsymbol{(1) f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k}$$

$$\boldsymbol{(2) \alpha \cdot f(x) = \sum_{k=0}^{\infty} \alpha \cdot a_k x^k \quad \alpha \in R}$$

$$\boldsymbol{(3) x \cdot f'(x) = \sum_{k=0}^{\infty} k \cdot a_k x^k}$$

$$\boldsymbol{(4) f(\alpha x) = \sum_{k=0}^{\infty} \alpha^k \cdot a_k x^k}$$

$$\boldsymbol{(5) f(x) g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k}$$

The extended binomial coefficient

【Definition 2】 Let *u* be a real number and *k* a nonnegative integer. Then the *extended binomial coefficient* is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if} \quad k > 0 \\ 1 & \text{if} \quad k = 0 \end{cases}$$

The extended Binomial Theorem

【 Theorem 2】Let *x* be a real number with $|x| < 1$ and let *u* be a real number. Then

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

4. Counting with Generating Functions

Generating functions can be used to solve a wide variety of counting problems, such as

- Count the number of combinations from a set when repetition is allowed and additional constraints exist.
- Count the number of permutations

5. Using Generating Functions to Solve Recurrence Relations

The Methods of Solving Recurrence Relations

- Iterative approach
- Use a systematic way to solve an important class of recurrence relations
- Generating functions *Method:*(1) Use the recurrence relation to find the generating function of this sequence; (2) $G(x) \leftrightarrow a_n$

6. Proving Identities via Generating Functions

The method of proving combinatorial identities:

- Use combinatorial proofs
- Use generating functions

8.5 Inclusion-Exclusion

For the union of three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

For the union of *n* finite sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

8.6 Applications of Inclusion-Exclusion

1. An alternative form of inclusion-exclusion

Problems that ask for the number of elements in a set that have none of *n* properties $P_1, P_2, \cdots, P_n$.

Let A_i be the subset containing the elements that have property P_i.

$N(P_1' P_2' \cdots P_n')$ ----The number of elements with none of the properties $P_1, P_2, \cdots, P_n$.

From the inclusion-exclusion principle, we see that

$$N(P_1' P_2' \cdots P_n') = N - |A_1 \cup A_2 \cdots \cup A_n| = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) + \cdots + (-1)^n N(P_1 P_2 \cdots P_n)$$

2. The sieve of Eratosthenes

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of {2, 3, 5, 7} is $\lfloor 100/N \rfloor$, where *N* is the product of the primes in this subset.

3. The number of onto functions

Theorem:

Let *m* and *n* be positive integers with $m \ge n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1} C(n, n-1) \cdot 1^m$$

onto functions from a set with *m* elements to a set with *n* elements.

Applications:

- Assign *m* different jobs to *n* different employees if every employee is assigned at least one job.
- Distribute *m* different toys to *n* different children such that each child gets at least one toy.

4. Derangement

A *derangement* is a permutation of objects that leaves no object in its original position.

Theorem: The number of derangements of a set with *n* elements is

$$D_n = n!\,[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}]$$