Sample Solutions on HW9 (*33 exercises in total*)

Sec. 9.1 7(a,c,h), 26, 32, 47, 51

7(a) This relation is **not reflexive** since it is not the case that $1 \neq 1$, for instance. It is **symmetric**: if $x \neq y$, then of course $y \neq x$. It is **not antisymmetric**, since, for instance, $1 \neq 2$ and also $2 \neq 1$. It is **not transitive**, since $1 \neq 2$ and $2 \neq 1$, for instance, but it is not the case that $1 \neq 1$

7(c) This relation is **not reflexive**, since (1,1) is not included, for instance. It is **symmetric**; the equation $x = y - 1$ is equivalent to the equation $y = x + 1$, which is the same as the equation $x = y+1$ with the roles of x and y reversed. (A more formal proof of symmetry would be by cases. If x is related to y then either $x = y+1$ or $x = y - 1$. In the former case, $y = x - 1$, so y is related to x; in the latter case $y = x+1$, so y is related to x.) It is **not antisymmetric**, since, for instance, both (1,2) and (2,1) are in the relation. It is **not transitive**, since, for instance, although both (1,2) and (2,1) are in the relation, (1,1) is not.

7(h) This relation is **not reflexive**, since, for instance $17 < 17^2$. It is **not symmetric**, since although $289 \geq 17^2$, it is not the case that $17 \geq 289^2$. To see whether it is antisymmetric, we assume that both (x,y) and (y,x) are in the relation. Then $x \geq y^2$ and $y \geq x^2$. Since both sides of the second inequality are nonnegative, we can square both sides to get $y^2 \geq x^4$. Combining this with the first inequality, we have $x \geq x^4$, which is equivalent to $x - x^4 \geq 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$. The last factor is always positive, so we can divide the original inequality by it to obtain the equivalent inequality $x(1-x) \geq 0$. Now if $x \geq 1$ or $x < 0$, then the factors have different signs, so the inequality does not hold. Thus the only solutions are x =0 and x =1. The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x \geq y^2$ and $y \geq x^2$ is when x = y; this means that the relation is **antisymmetric**. It is **transitive**. Suppose $x \geq y^2$ and $y \geq z^2$. Again the second inequality implies that both sides are nonnegative, so we can square both sides to obtain $y^2 \geq z^4$. Combing these inequalities gives $x \geq z^4$. Now we claim that it is always the case that $z^4 \geq z^2$; if so, then we combine this fact with the last inequality to obtain $x \geq z^2$, so x is related to z. To verify the claim, note that since we are working with integers, it is always the case that $z^2 \geq |z|$ (equality for z =0 and z =1, strict inequality for other z). Squaring both sides gives the desired inequality.

26(a) $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(a,b) \mid a > b\}$

26(b) $\overline{R} = \{(a,b) \mid (a,b) \notin R\} = \{(a,b) \mid a \geq b\}$

32 Since $(1,2) \in R$ and $(2,1) \in S$, we have $(1,1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1,1),(1,2),(2,1),(2,2)\}$

47 Let A be the set with n elements on which the relations are defined.

47(a) To specify a symmetric relation, we need to decide, for each unordered pair $\{a,b\}$ of distinct elements of A, whether to include the pairs (a,b) and (b,a) or leave them out; this can be done in 2 ways for each such unordered pair. Also, for each element $a \in A$, we need to decide whether to include (a,a) or not, again 2 possibilities. We can think of these two parts as one by considering an element to be an unordered pair with repetition allowed. Thus we need to make this 2-fold choice $C(n+1,2)$ times, since there are $C(n+2-1,2)$ ways to choose an unordered pair with repetition allowed. Therefore the answer is $2^{C(n+1,2)} = 2^{n(n+1)/2}$.

47(b) For each unordered pair $\{a,b\}$ of distinct elements of A, we have a 3-way choice – either include (a,b) only, include (b,a) only, or include neither. For each element of A we have a 2-way choice. Therefore the answer is $3^{C(n,2)}2^n = 3^{n(n-1)/2}2^n$.

47(c) As in part (b) we have a 3-way choice for $a \neq b$. There is no choice about including (a,a) in the relation – the definition prohibits it. Therefore the answer is $3^{C(n,2)} = 3^{n(n-1)/2}$.

47(d) For each ordered pair (a,b), with $a \neq b$ (and there $P(n,2)$ such pairs), we can choose to include (a,b) or to leave it out. There is no choice for pairs (a,a). Therefore the answer is $2^{P(n,2)} = 2^{n(n-1)}$.

47(e) This is just like part (a), except that there is no choice about including (a,a). For each unordered pair of distinct elements of A, we can choose to include neither or both of the corresponding ordered pairs. Therefore the answer is $2^{C(n,2)} = 2^{n(n-1)/2}$.

47(f) We have complete freedom with the ordered pairs (a,b) with $a \neq b$, so that part of the choice gives us $2^{P(n,2)}$ possibilities, just as in part (d). For the decision as to whether to include (a,a), two of the 2^n possibilities are prohibited: we cannot include all such pairs, and we cannot leave them all out. Therefore the answer is $2^{P(n,2)}(2^n - 2) = 2^{n^2} - 2^{n^2-n+1}$.

51 We need to show two things. First, we need to show that if a relation R is symmetric, then $R = R^{-1}$, which means we must show that $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$. To do this, let $(a,b) \in R$. Since R is symmetric, this implies that $(b,a) \in R$. But since R^{-1} consists of all pairs (a,b) such that $(b,a) \in R$, this means that $(a,b) \in R^{-1}$. Thus we have shown that $R \subseteq R^{-1}$. Next let $(a,b) \in R^{-1}$. By definition this means that $(b,a) \in R$. Since R is symmetric, this implies that $(a,b) \in R$ as well. Thus we have shown that $R^{-1} \subseteq R$.

Second, we need to show that $R = R^{-1}$ implies that R is symmetric. To this end we let $(a,b) \in R$ and try to show that (b,a) is also necessarily an element of R. Since $(a,b) \in R$, the definition tells us that $(b,a) \in R^{-1}$. But since we are under the hypothesis that $R = R^{-1}$, this tells us that $(b,a) \in R$, exactly as desired.

Sec. 9.3 13,14,31

13(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

13(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

13(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

14(a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

14(b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

14(c) The matrix is the Boolean product $M_{R1} \odot M_{R2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

14(d) The matrix is the Boolean product $M_{R1} \odot M_{R1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

14(e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

31 Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in antiparallel pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; and transitive if all paths of length 2 (a pair of edges (x,y) and (y,z)) are accompanied by the corresponding path of length 1 (the edge (x,z)).

The relation drawn in Exercise 23 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge (a,b) is present but not the edge (b,a). It is not antisymmetric, since both edges (b,c) and (c,b) are present. It is not transitive, since the path (b,c), (c,b) from b to b is not accompanied by the edge (b,b).

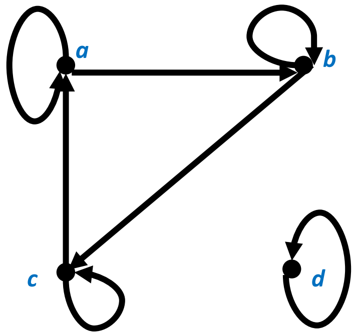
The relation drawn in Exercise 24 is reflexive and not irreflexive since there is a loop at each vertex. It is not symmetric, since, for instance, the edge (b,a) is present but not the edge (a,b). It is antisymmetric, since there are no pairs of antiparallel edges. It is transitive, since the only nontrivial path of length 2 is *bac*, and the edge (b,c) is present.

The relation drawn in Exercise 25 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge (b,a) is present but not the edge (a,b). It is antisymmetric, since there are no pairs of antiparallel edges. It is not transitive, since edges (a,c) and (c,d) are present, but not (a,d).

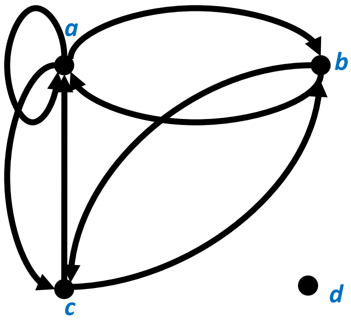
Sec. 9.4 2, 6, 9(6), 11(6), 20, 28(a), 29

2 When we add all the pairs (x,x) to the given relation we have all of $Z \times Z$; in other words, we have the relation that always holds.

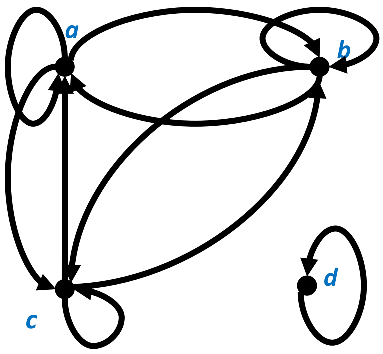
6 We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



9(6) We form the symmetric closure by taking the given directed graph and appending an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there); in other words, we append the edge (b, a) whenever we see the edge (a, b).



11(6) We are asked for the symmetric and reflexive closure of the given relation. We form it by taking the given directed graph and appending (1) a loop at each vertex at which there is not already a loop and (2) an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there).



20(a) The pair (a,b) is in R2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b – in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.

20(b) The pair (a,b) is in R3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b – in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.

20(c) The pair (a,b) is in R^* precisely when it is possible to fly from a to b.

28(a) We compute the matrices Wi for i = 0, 1, 2, 3, 4, 5, and then W5 is the answer.

$$W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = W_5$$

29(a) We need to include at least the transitive closure, which we can compute by Algorithm 1 or Algorithm 2 to be $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. All we need in addition is the pair (2,2) in order to make the relation reflexive. Note that the result is still transitive (the addition of a pair (a,a) cannot make a transitive relation no longer transitive), so our answer is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

29(b) The symmetric closure of the original relation is represented by $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

We need at least the transitive closure of this relation, namely $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Since it is also symmetric, we are done. Note that it would not have been correct to find first the transitive closure of the original matrix and then make it symmetric, since the pair (2,2) would be missing. What is going on here is that the transitive closure of a symmetric relation is still symmetric, but the symmetric closure of a transitive relation might not be transitive.

29(c) Since the answer to part (b) was already reflexive, it must be the answer to this part as well.