Sample Solutions on HW8 (21 exercises in total)

Sec. 8.4 6(d,f), 10 (c, d, e), 16, 24(a), 30, 34, 43

6(d) The power series for the function e^x is $\sum_{n=0}^{\infty} x^n/n!$. That is almost what we have here; the difference is that the denominator is $(n+1)!$ instead of n!. So we have $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!}$ by a change of variable. This last sum is $e^x - 1$ (only the first term is missing), so our answer is $(e^x - 1)/x$.

6(f) By Table 1,

$$\sum_{n=0}^{\infty} C(10, n+1)x^n = \sum_{n=1}^{\infty} C(10, n)x^{n-1}$$
$$= \frac{1}{x}\sum_{n=1}^{\infty} C(10, n)x^n = \frac{1}{x}((1+x)^{10} - 1)$$

10(c) If we factor out as high a power of x from each factor as we can, then we can write this as $x^7(1 + x^2 + x^3)(1 + x)(1 + x + x^2 + x^3 + \cdots)$, and so we seek the coefficient of x^2 in $(1 + x^2 + x^3)(1+ x)(1+ x+ x^2 + x^3+\cdots)$. We could do this by brute force, but let's try it more analytically. We write our expression in closed form as

$$\frac{(1+x^2+x^3)(1+x)}{1-x} = \frac{1+x+x^2+higherorderterms}{1-x}$$

$$= \frac{1}{1-x} + x\cdot\frac{1}{1-x} + x^2\cdot\frac{1}{1-x} + \text{irrelevant terms}$$

The coefficient of x^2 in this power series comes either from the coefficient of x^2 in the first term in the final expression displayed above, or from the coefficient of x^1 in the second factor of the second term of that expression, or from the coefficient of x^0 in the second factor of the third term. Each of these coefficients is 1, so our answer is 3.

10(d) The easiest approach here is simply to note that there are only two combinations of terms that will give us an x^9 term in the product: $x \bullet x^8$ and $x^7 \bullet x^2$. So the answer is 2.

10(e) The highest power of x appearing in this expression when multiplied out is x^6. Therefore the answer is 0.

16 The factors in the generating function for choosing the egg and plain bagels are both $x^2 + x^3 + x^4 +\ldots$ The factor for choosing the salty bagels is $x^2 + x^3$. Therefore the generating function for this problem is $(x^2 + x^3 + x^4 +\ldots)^2 (x^2 + x^3)$. We want to find the coefficient of x12, since we want 12 bagels. This is equivalent to finding the coefficient of x6 in (1 + x + x2 +…)2 (1 + x) This function is (1+x)/(1-x)2, so we want the coefficient of x6 in 1/(1-x)2, which is 7, plus the coefficient of x5 in 1/(1-x)2, which is 6. Thus the answer is 13.

24(a) The restriction on x_1 gives us the factor $x^3 + x^4 + x^5 + \cdots$. The restriction on x_2 gives us the factor $x + x^2 + x^3 + x^4 + x^5$. The restriction on x_3 gives us the factor $1+ x + x^2+ x^3+ x^4$. And the restriction on x_4 gives us the factor $x + x^2 + x^3 + \cdots$. Thus the answer is the product of these:

$$(x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots)$$

We can use algebra to rewrite this in closed form as $x^5(1+x+x^2+x^3+x^4)^2/ (1-x)^2$.

30(a) Multiplication distributes over addition, even when we are talking about infinite sums, so the generating function is just $2G(x)$.

30(b) What used to be the coefficient of x0 is now the coefficient of x1, and similarly for the other terms. The way that happened is that the whole series got multiplied by x. Therefore the generating function for this series is xG9x). In symbols,

$$a_0x + a_1x^2 + a_2x^3 + \cdots = x(a_0 + a_1x + a_2x^2 + \cdots) = xG(x)$$

30(c) The terms involving a_0 and a_1 are missing, $G(x) - a_0 - a_1x = a_2x^2 + a_3x^3 + \cdots$. Here, however, we want a_2 to be the coefficient of x^4, not x^2 (and similarly for the other powers), so we must throw in an extra factor. Thus the answer is $x^2(G(x)-a_0-a_1x)$.

30(d) This is just like part(c), except that we slide the powers down, Thus the answer is $(G(x)-a_0-a_1x)/x^2$.

30(e) Following the hint, we differentiate $G(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain $G'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}$. By a change of variable this becomes $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + 2a_2x + 3a_3x^2 + \cdots$, which is the generating function for precisely the sequence we are given. Thus G'(x) is the generating function for this sequence.

30(f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x) \cdot G(x)$.

34 Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by changing the name of the variable from k to k+1). Thus

$$G(x) - 3xG(x) = {}_{k=0} a_k x^k - {}_{k=1} 3a_{k-1}x^k = a_0 + {}_{k=1}(a_k - 3a_{k-1})x^k$$

$$= 1 + \sum_{k=1}^{\infty} 4^{k-1}x^k = 1 + x\sum_{k=1}^{\infty} 4^{k-1}x^{k-1} = 1 + x\sum_{k=0}^{\infty} 4^k x^k = 1 + x\frac{1}{1-4x} = \frac{1-3x}{1-4x}$$

Thus G(x)(1-3x)=(1-3x)/(1-4x), so G(x)=1/(1-4x). Therefore ak = 4k, from Table 1.

43 Following the hint, we note that $(1+x)^{m+n} = (1+x)^m (1+x)^n$. Then applying the Binomial Theorem, we have

$$\sum_{r=0}^{m+n} C(m+n,r)x^r = \sum_{r=0}^{m} C(m,r)x^r \cdot \sum_{r=0}^{n} C(n,r)x^r = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} C(m,r-k)C(n,k)\right)x^r$$

by Theorem 1. Comparing coefficients gives us the desired identity.

Sec. 8.5 7, 10, 18

7 We need to use the formula:

$$|P \cup F \cup C| = |P| + |F| + |C| - |P \cap F| - |P \cap C| - |F \cap C| + |P \cap F \cap C|$$

where, for example, P is the set of students who have taken a course in Pascal. Thus we have $|P \cup F \cup C|$ = 1876 + 999 + 345 – 876 – 290 – 231 + 189 = 2012. Therefore, since there are 2504 students altogether, we know that 2504 – 2012 = **492** have taken none of these courses.

10 $100 - \lfloor 100/5 \rfloor - \lfloor 100/7 \rfloor + \lfloor 100/(5 \times 7) \rfloor = 100 - 20 - 14 + 2 = 68$

18 There are C(10, 1) + C(10, 2) + · · · + C(10, 10) = 2^{10} – C(10, 0) = 1023 terms on the right-hand side of the equation.

Sec. 8.6 6, 11, 16

6 Square-free numbers are those not divisible by the square of a prime. We count them as follows:

$$99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61$$

11 Here is one approach. Let us ignore temporarily the stipulation about the most difficult job being assigned to be the best employee (We assume that this language uniquely specifies a job and an employee). Then we are looking for the number of onto functions from the set of 7 jobs to the set of 4 employees. By Theorem 1 there are $4^7 - C(4,1)3^7 + C(4,2)2^7 - C(4,1)1^7$ = 8400 such functions. Now by symmetry, in exactly one fourth of those assignments should the most difficult job be given to the best employee, as opposed to one of the other three employees. Therefore the answer is 8400 / 4 = **2100**.

16 There are $n!$ ways to make the first assignment. We can think of this first seating as assigning student n to a chair we will label n. Then the next seating must be a derangement with respect to this numbering, so there are D_n second seating possible. Therefore the answer is $n!D_n$.