Sample Solutions on HW3 (*30 exercises in total*)

Sec. 1.6 12, 14(d), 18, 24, 29, 34(a)

12. Exercise 11 states that $((p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \to r) \to ((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to (q \to r))$

Applying exercise 11, we just need to show that the conclusion r follows

from the five premises

$(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $\neg s$, and q

Step	Reason
1. q	Premise
2. $q \to (u \wedge t)$	Premise
3. $u \wedge t$	Modus ponens, 1,2
4. u	Simplification, 3
5. t	Simplification, 3
6. $u \to p$	Premise
7. p	Modus ponens, 4,6
8. $p \wedge t$	Conjunction, 5,7
9. $(p \wedge t) \to (r \vee s)$	Premise
10. $r \vee s$	Modus ponens, 8,9
11. $\neg s$	Premise
12. r	Disjunctive syllogism,10,11

Q.E.D.

14(d) Let C(x) be “x is in this class,” let F(x) be “x has been to France,” and let L(x) be “x has visited the Louvre.” We are given premises $\exists x(C(x) \wedge F(x))$, $\forall x(F(x) \rightarrow L(x))$, and we want to conclude $\exists x(C(x) \wedge L(x))$. In the following proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(C(x) \wedge F(x))$	Premise
2. $C(y) \wedge F(y)$	EI, 1
3. F(y)	Simplification,2
4. $C(y)$	Simplification,2
5. $\forall x(F(x) \rightarrow L(x))$	Premise
6. $F(y) \rightarrow L(y)$	UI, 5
7. $L(y)$	Modus ponens, 3,6
8. $C(y) \wedge L(y)$	Conjunction, 4,7
9. $\exists x(C(x) \wedge L(x))$	EG, 8

Q.E.D.

18. We know that some *s* exists that makes *S*(*s*, Max) true, but we cannot conclude that Max is one such *s*. Therefore this first step is invalid.

24. Steps 3 and 5 are incorrect; simplification applies to conjunction, not disjunction.

29.

Step	Reason
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	EI, 1
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	UI, 3
5. $Q(c)$	Disjunctive syllogism, 4,2
6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	UI, 6
8. $S(c)$	Disjunctive syllogism, 5,7
9. $\forall x(R(x) \rightarrow \neg S(x))$	Premise
10. $R(c) \rightarrow \neg S(c)$	UI, 9
11. $\neg R(c)$	Modus tollens, 8,10
12. $\exists x \neg R(x)$	EG, 11

Q.E.D.

34(a) Let d be “logic is difficult,” s be “many students like logic,” and e for “mathematics is easy.”. Then the assumptions are $d \vee \neg s$ and $e \rightarrow \neg d$. Note that the first of these is equivalent to

$s \rightarrow d$. This exercise asks whether we can conclude $s \rightarrow \neg e$

Step	Reason
1. $d \vee \neg s$	Premise
2. $s \rightarrow d$	Implication Rule, 1
3. $e \rightarrow \neg d$	Premise
4. $d \rightarrow \neg e$	Contrapositive, 3
5. $s \rightarrow \neg e$	Hypothetical syllogism, 2,4

Q.E.D.

Sec. 1.7 7, 8, 34

7. Assume n is odd. By the definition of an odd integer, it follows that

$n=2k+1$, where k is some integer.

Then $(k+1)^2-k^2 = k^2+2k+1-k^2 = 2k+1 = n$

Therefore, every odd integer can be expressed as the difference of two squares.

8. Let $n = m^2$. If $m = 0$, then $n+2 = 2$, which is not a perfect square, so we can assume that $m \geq 1$. The smallest perfect square greater than n is $(m+1)^2$, and we have $(m+1)^2 = m+2m+1 = n+2m+1 > n+2\cdot 1+1 > n+1$.

Therefore $n+2$ cannot be a perfect square.

34. No. The line of reasoning shows that if $2x^2-1=x$, then we must have $x = 1$ or $x= -1$. These are therefore the only possible solutions, but we have no guarantee that they are solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we must substitute these values back into the original equation to determine whether they do indeed satisfy it.

Sec. 1.8 22, 24

22. We follow the hint. The square of every real number is nonnegative, so $(x - 1/x)^2 \geqslant 0$. Multiplying this out and simplifying, we obtain $x^2 - 2 + 1/x^2 \geqslant 0$, so $x^2 + 1/x^2 \geqslant 2$, as desired.

24. Let $x = 1$ and $y = 10$. Then their arithmetic is 5.5 and their quadratic mean is $\sqrt{50.5} \approx 7.11$. Similarly, if $x = 5$ and $y = 8$, then the arithmetic mean is $(5+8)/2 = 6.5$ and the quadratic mean is $\sqrt{(5^2 + 8^2)/2} \approx 6.67$. So we conjecture that the quadratic mean is always greater than or equal to the arithmetic mean. Thus we want to prove that

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2}$$

for all positive real numbers x and y. Doing some algebra, we find that this inequality is equivalent to the true statement that $(x - y)^2 \geqslant 0$:

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2}$$

$$\equiv 2x^2 + 2y^2 \geq x^2 + 2xy + y^2$$

$$\equiv x^2 - 2xy + y^2 \geq 0$$

$$(x + y)^2 \geq 0$$

In fact, our argument also shows that equality holds if and only if $x = y$.

Sec. 2.1 11, 18, 22, 24, 32(a),(c)

11. **(a)** True **(b)** True **(c)** False **(d)** True **(e)** True **(f)** False

18. Since the empty set is a subset of every set, we just need to take a set B that contains ϕ as an element. Thus we can let $A = \phi$ and $B = \{\phi\}$ as the simplest example.

22. The union of all the sets in the power set of a set **X** must be exactly **X**. In other words, we can recover **X** from its power set, uniquely. Therefore the answer is Yes.

24.

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus ϕ is not the power set of any set.

b) This is the power set of {a}

c) This set has three elements. Since 3 is not a power of any integer, this set cannot be the power set of any set.

d) This is the power set of {a,b}.

32.

a) {(a,x,0), (a,x,1), (a,y,0), (a,y,1), (b,x,0), (b,x,1), (b,y,0), (b,y,1), (c,x,0), (c,x,1), (c,y,0), (c,y,1)}

c) {(0,a,x), (0,a,y), (0,b,x), (0,b,y), (0,c,x), (0,c,y), (1,a,x), (1,a,y), (1,b,x), (1,b,y), (1,c, x), (1,c,y)}

Sec. 2.2 17, 48, 57(c)

17.

(a) Suppose $x \in \overline{A \cap B \cap C}$, then $x \notin A \cap B \cap C$. This means

$x \notin A$ or $x \notin B$ or $x \notin C$.

Equivalently, we can say

$x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$

Therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$

Hence $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ (1)

Suppose $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, then $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$

This means $x \notin A$ or $x \notin B$ or $x \notin C$.

So $x \notin A \cap B \cap C$

It follows that $x \in \overline{A \cap B \cap C}$

Hence $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$ (2)

Based on (1) and (2), we can conclude that $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

(b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1

1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

Because columns 5 and 9 are identical, we can conclude that $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

48. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

a) $A_n = \{\cdots,-2,-1,0,1,\cdots,n\}$

b) $A_1 = \{\cdots,-2,-1,0,1\}$

57(c)

Assume the universal set U is the set of 26 lower-case English letters, and the ordering of elements of U has the elements in alphabetical order.

Represent sets A, B, C, and D by bit strings:

A 11 1110 0000 0000 0000 0000 0000

B 01 1100 1000 0000 0100 0101 0000

C 00 1010 0010 0000 1000 0010 0111

D 00 0110 0110 0001 1000 0110 0110

$A \cup D$

= 11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110

0110

= 11 1110 0110 0001 1000 0110 0110

$B \cup C$

= 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010

0111

= 01 1110 1010 0000 1100 0111 0111

$(A \cup D) \cap (B \cup C)$

= 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111

0111

= 01 1110 0010 0000 1000 0110 0110, which represents the set

$\{b, c, d, e, i, o, t, u, x, y\}$