Sample Solutions on HW10 (*14 exercises in total*)

Sec. 9.5 3, 10, 16, 36(b), 39, 41

3(a) This is an equivalence relation, one of the general form that two things are considered equivalent if they have the same "something" (see Exercise 9 for a formalization of this idea). In this case the "something" is the value at 1.

3(b) This is not an equivalence relation because it is not transitive. Let $f(x) = 0$, $g(x) = x$, and $h(x) = 1$ for all $x \in Z$. Then f is related to g since $f(0) = g(0)$, and g is related to h since $g(1) = h(1)$, but f is not related to h since they have no values in common. By inspection we see that this relation is reflexive and symmetric.

3(c) This relation has none of the three properties required for an equivalence relation. It is not reflexive, since $f(x) - f(x) = 0 \neq 1$. It is not symmetric, since if $f(x) - g(x) = 1$, then $g(x) - f(x) = -1 \neq 1$. It is not transitive, since if $f(x) - g(x) = 1$ and $g(x) - h(x) = 1$, then $f(x) - h(x) = 2 \neq 1$.

3(d) This is an equivalence relation. Two functions are related if they differ by a constant. It is clearly reflexive (the constant is 0). It is symmetric, since if $f(x) - g(x) = C$, then $g(x) - f(x) = -C$. It is transitive, since if $f(x) - g(x) = C_1$ and $g(x) - h(x) = C_2$, then $f(x) - h(x) = C_3$, where $C_3 = C_1 + C_2$.

3(e) This relation is not reflexive, since there are lots of functions f (for instance, $f(x) = x$) that do not have the property that $f(0) = f(1)$. It is symmetric by inspection (the roles of f and g are the same). It is not transitive. For instance, let $f(0) = g(1) = h(0) = 7$, and let $f(1) = g(0) = h(1) = 3$; fill in the remaining values arbitrarily. Then f and g are related, as are g and h, but f is not related to h since $7 \neq 3$.

10 The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

16 This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since clearly ad = bc if and only if a/b = c/d.
If we want an explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then ad = bc, which also means that cb = da, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then ad = bc and cf = de. Multiplying these equations gives acdf = bcde, and since all these numbers are nonzero, we have af = be, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

36(b) The equivalence class of 4 is the set of all integers congruent to 4, modulo m. $\{4 + 3n \mid n \in Z\} = \{\ldots,-2,1,4,7,\ldots\}$

39(a) We observed in the solution to Exercise 15 that (a,b) is equivalent to (c,d) if $a - b = c - d$. Thus because $1 - 2 = -1$, we have $[(1,2)]=\{(a,b)|a - b = -1\}=\{(1,2), (2,3), (3,4), (4,5), (5,6),\cdots\}$.

39(b) Since the equivalence class of (a,b) is entirely determined by the integer $a - b$, which can be negative, positive, or zero, we can interpret the equivalences classes as being the integers. This is a standard way to define the integers once we have defined the whole numbers.

41 The sets in a partition must be nonempty, pairwise disjoint, and have as there union all of the underlying set.
41(a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

41(b) This is a partition.

41(c) This is a partition.

41(d) This is not a partition, since none of the sets includes the element 3.