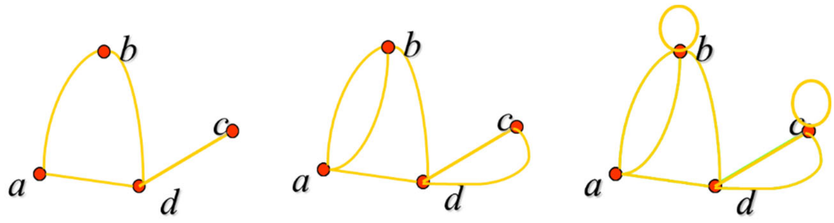
CHAPTER 10 Graphs

10.1 Graphs and Graph Models

【Definition 1】 A *graph* G=(V,E) consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

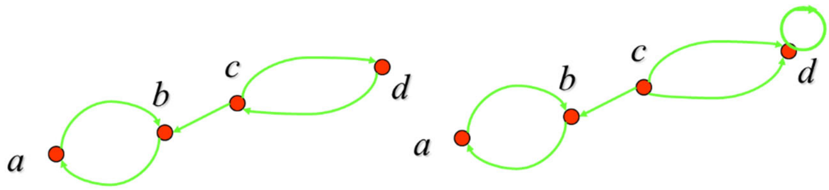
- *Simple* graph: A graph in which each edge connects two different vertices *and* where no two edges connect the same pair of vertices.
- *Multigraph:* Graphs that may have multiple edges connecting the same vertices.
- *Pseudograph:* Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices.



【Definition 2】A *directed graph* (or *digraph*) *(V, E)* consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

Types of digraphs:

- *Simple directed graph:* a directed graph has no loops and has no multiple directed edges.
- *directed multigraph:* a directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex.



10.2 Graph Terminology and Special Types of Graphs

1. Basic Terminology

Undirected Graphs G=(V, E)

- *Vertex, edge*
- Two vertices, u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G, if {u, v} is an edge of G.
- An edge e connecting u and v is called *incident with vertices u and v*, or is said to connect u and v.
- The vertices u and v are called *endpoints* of edge {u, v}.
- *loop*
- The *degree of a vertex* in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex

Notation: deg(v)

- If deg(v) = 0, v is called *isolated*.
- If deg(v) = 1, v is called *pendant*.

【 Theorem 1】 The Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with e edges. Then

$$\sum_{v \in V} deg(v) = 2e$$

The sum of the degrees over all the vertices is twice the number of edges

Note:

This applies even if multiple edges and loops are present.

【 Theorem 2】 An undirected graph has an even number of vertices of odd degree.

Directed Graphs $\boldsymbol{G=(V, E)}$

- Let (u, v) be an edge in G. Then u is an *initial vertex* and is *adjacent to* v and v is a *terminal vertex* and is *adjacent from* u.
- The *in degree* of a vertex v, denoted $\deg^-(v)$ is the number of edges which terminate at v.
- Similarly, the *out degree* of v, denoted $\deg^+(v)$, is the number of edges which initiate at v.
- *underlying undirected graph*: The undirected graph that results from ignoring directions of edges

【 Theorem 3】 Let $G = (V, E)$ be a graph with direct edges. Then

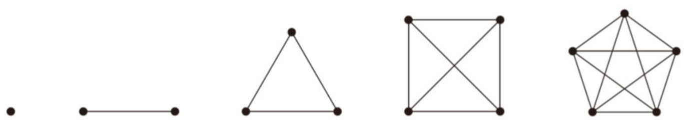
$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$$

2. Some Special Simple Graphs

(1) Complete Graphs - K_n: the simple graph with

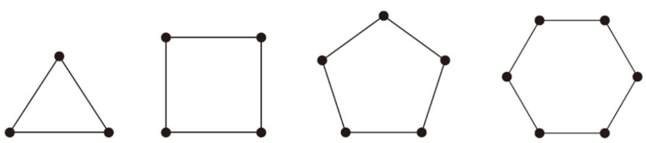
- n vertices
- exactly one edge between every pair of distinct vertices.

The graphs K_n for n=1,2,3,4,5.



K_1 K_2 K_3 K_4 K_5

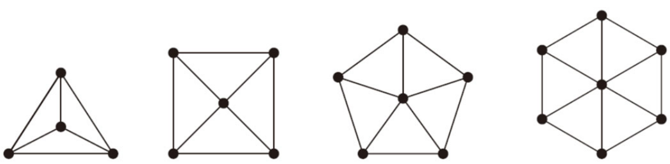
(2) Cycles C_n (n>2)



C_3 C_4 C_5 C_6

(3) Wheels W_n (n>2)

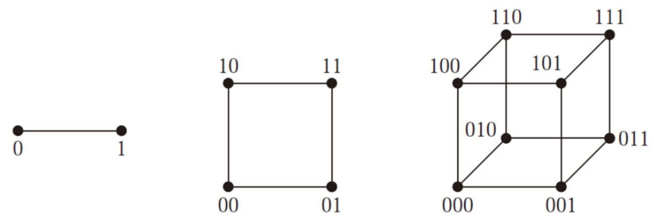
Add one additional vertex to the cycle C_n and add an edge from each vertex in C_n to the new vertex to produce W_n.



W_3 W_4 W_5 W_6

(4) *n-Cubes* Q_n (n>0)

Q_n is the graph with 2^n vertices representing bit strings of length n. An edge exists between two vertices that differ in exactly one bit position.



Q_1 Q_2 Q_3

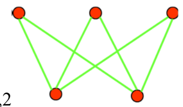
3. Bipartite Graphs

A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

Note:

- There are no edges which connect vertices in V_1 or in V_2.

The *complete bipartite graph* is the simple graph that has its vertex set partitioned into two subsets V_1 and V_2 with m and n vertices, respectively, and *every vertex* in V_1 is connected to *every vertex* in V_2, denoted by $K_{m,n}$, where $m = | V_1 |$ and $n = | V_2 |$.

$K_{3,2}$ 

【Theorem 4】 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Regular graph

- A simply graph is called *regular* if every vertex of this graph has the same degree.
- A *regular graph* is called n-regular if every vertex in this graph has degree n.

4. Some applications of special types of graphs

5. New Graphs from Old

【Definition】 $G= (V, E)$, $H= (W, F)$

- H is a *subgraph* of G if $W \subseteq V, F \subseteq E$.
- subgraph H is a *proper subgraph* of G *if* $H \neq G$.
- H is a *spanning subgraph* of G if $W = V, F \subseteq E$.

The union of G_1 and G_2

The *union* of two simple graphs $G_1 = (V_1 , E_1)$ and $G_2 = (V_2 , E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Notation: $G_1 \cup G_2$

10.3 Representing Graphs and Graph Isomorphism

1. Representing Graphs

Methods for representing graphs:

- Graphs
- Adjacency lists - lists that specify all the vertices that are adjacent to each vertex
- Adjacency matrices
- Incidence matrices

An adjacency list for a simple graph	
vertex	Adjacent vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

An adjacency list for a directed graph	
Initial vertex	terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

2. Adjacency Matrices

A simple graph $G = (V, E)$ with n vertices $(v_1, v_2, \cdots, v_n)$ can be represented by its adjacency matrix, A, with respect to this listing of the vertices, where

- $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,
- $a_{ij} = 0$ otherwise.

Note:

- An adjacency matrix of a graph is based on the ordering chosen for the vertices.
- Adjacency matrices of undirected graphs are always symmetric.

The adjacency matrix of a multigraph or pseudograph

- For the representation of graphs with multiple edges, we use matrices of nonnegative integers.
- The (i, j)th entry of such a matrix equals the number of edges that are associated to $\{v_i, v_j\}$.

The adjacency matrix of a directed graph

Let $G = (V, E)$ be a directed graph with $|V| = n$. Suppose that the vertices of G are listed in an arbitrary order as $v_1, v_2, \ldots, v_n$.

The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when there is an edge from v_i to v_j, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,

- $a_{ij} = 1$ if (v_i, v_j) is an edge of G,
- $a_{ij} = 0$ otherwise.

4. Isomorphism of Graphs

Graphs with the same structure are said to be *isomorphic*.

- Formally, two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a 1-1 and onto function f from V_1 to V_2 such that for all a and b in V_1, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.Such a function f is called an *isomorphism.*
- In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.

Invariants -- things that G_1 and G_2 must have in common to be isomorphic.

Important *invariants in isomorphic graphs*:

- the number of vertices
- the number of edges
- the degrees of corresponding vertices
- if one is bipartite, the other must be
- if one is complete, the other must be
- if one is a wheel, the other must be

- The adjacency matrix of a graph G is the same as the adjacency matrix of another graph H, when rows and columns are labeled to correspond to the images under f of the vertices in G that are the labels of these rows and columns in the M_G

10.4 Connectivity

1. Paths

A *path of length n* in a simple graph is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $\{v_0, v_1\}$, $\{v_1, v_2\}$, ..., $\{v_{n-1}, v_n\}$ are *n* edges in the graph.

- The path is a *circuit* if it begins and ends at the same vertex (length greater than 0).
- A path is *simple* if it does not contain the same edge more than once.

Note:

- There is nothing to prevent traversing an edge back and forth to produce arbitrarily long paths. This is usually not interesting which is why we define a simple path.
- The notation of a path: vertex sequence
- A path of length zero consists of a single vertex.

Path in directed graph

A *path of length n* in a directed graph is a sequence of vertices $v_0, v_1, \ldots v_n$ such that (v_0, v_1) , (v_1, v_2) , ..., (v_{n-1}, v_n) are *n* directed edges in the graph.

- *Circuit or cycle :* the path begins and ends with the same vertex.
- *Simple path:* the path does not contain the same edge more than once.

2. Counting paths between vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

【 Theorem 2】 The *number of different paths of length r* from v_i to v_j is equal to the $(i, j)th$ entry of A^r, where A is the adjacency matrix representing the graph consisting of vertices $v_1, v_2, \ldots v_n$.

Note: This is the standard power of A, not the Boolean product.

3. Connectedness in undirected graphs

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

【 Theorem 1】 There is a simple path between every pair of distinct vertices of a connected undirected graph.

The maximally connected subgraphs of *G* are called the *connected components* or just the *components*.

A vertex is a *cut vertex* (or *articulation point*), if removing it and all edges incident with it results in more connected components than in the original graph.

Similarly if removal of an edge creates more components the edge is called a *cut edge* or *bridge*.

4. Connectedness in directed graphs

A directed graph is *strongly connected* if there is a path from *a* to *b* and from *b* to *a* for all vertices *a* and *b* in the graph.

The graph is *weakly connected* if the underlying undirected graph is connected.

Note:

- By the definition, any strongly connected directed graph is also weakly connected.

For directed graph, the maximal strongly connected subgraphs are called the *strongly connected components* or just the *strong components*.

5. Paths and Isomorphism

Idea:

（1）Some other invariants

- The number and size of connected components
- Path
- Two graphs are isomorphic only if they have simple circuits of the same length.
- Two graphs are isomorphic only if they contain paths that go through vertices so that the corresponding vertices in the two graphs have the same degree.

（2）We can also use paths to find mapping that are potential isomorphism.

10.5 Euler and Hamilton Paths

Euler Paths

1. Konigsberg Seven Bridge Problem

Terminologies:

- *Euler Path* An Euler path is a simple path containing every edge of *G*.
- *Euler Circuit* An Euler circuit is a simple circuit containing every edge of *G*.
- *Euler Graph* A graph contains an Euler circuit.

2. Necessary and sufficient conditions for Euler circuit and paths

【 Theorem 1】 A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

【 Theorem 2】 A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

3. Euler circuit and paths in directed graphs

A directed multigraph having no isolated vertices has an Euler circuit if and only if

- the graph is weakly connected
- the in-degree and out-degree of each vertex are equal.

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if

- the graph is weakly connected
- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree.

4. Applications

Hamilton paths and circuit

A *Hamilton path* in a graph *G* is a path which visits every vertex in *G* exactly once.

A *Hamilton circuit* (or *Hamilton cycle*) is a cycle which visits every vertex exactly once, *except for the first vertex*, which is also visited at the end of the cycle.

If a connected graph *G* has a Hamilton circuit, then *G* is called a *Hamilton graph*.

1. The sufficient condition for the existence of Hamilton path and Hamilton circuit

【 Theorem 3】 DIRAC'THEOREM

If *G* is a simple graph with *n* vertices with $n>=3$ such that the degree of every vertex in *G* is at least $n/2$, then *G* has a Hamilton circuit.

【 Theorem 4】 ORE'THEOREM

If *G* is a simple graph with *n* vertices with $n>=3$ such that $\deg(u)+\deg(v) >= n$ for every pair of nonadjacent vertices *u* and *v* in *G*, then *G* has a Hamilton circuit.

2. The necessary condition for Hamilton path and Hamilton circuit

For undirected graph:

The necessary condition for the existence of Hamilton path:

- *G* is connected;

- There are at most two vertices which degree are less than 2.

The necessary condition for the existence of Hamilton circuit:

- The degree of each vertex is larger than 1.

Some properties:

- If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
- When a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit , can be removed from consideration.

Another important necessary condition

- For any nonempty subset S of set V, the number of connected components in G-S <=$|S|$.

Note:

(1) G-S is a subgraph of G

(2) Suppose that C is a H circuit of G. For any nonempty subset S of set V, the number of connected components in C-S <=$|S|$.

(3) the number of connected components in G-S <= the number of connected components in C-S

10.6 Shortest Path Problems

1. Introduction

- *Weighted graph* $G = (V, E, W)$

  We can assign weights to the edges of graphs.
- *The length of a path in a weighted graph*

  the sum of the weights of the edges of this path.
- *Shortest Path Problems*

  $G = (V, E, W)$ *is a weighted graph, where* $w(x, y)$ *is the weight of edge (x , y) (if* $(x, y) \notin E, w(x, y) = \infty$*).* $a, z \in V$*，find the shortest path between a and z.*

2. A shortest path algorithm

Dijkstra's Algorithm (undirected graph with positive weights)

Dijkstra's algorithm proceeds by forming a distinguished set of vertices iteratively. Let S_k denote this set of vertices after k iterations of labeling procedure.

- Step 0: Label a with 0 and other with ∞, i.e. $L_0(a)=0$, and $L_0(v)=\infty$ and $S_0=\phi$.
- Step 1: The set S_1 is formed from S_0 by adding a vertex u not in S_0 with the smallest label. Once u is added to S_1 , we update the labels of all vertices not in S_1 , so that $L_1(v)$, the label of the vertex v at the *1st* stage, is the length of the shortest path from a to v that containing vertices only in S_1 .
- Step k: The set S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with the smallest label. Once u is added to S_k , we update the labels of all vertices not in S_k , so that $L_k(v)$, the label of the vertex v at the kth stage, is the length of the shortest path from a to v that containing vertices only in S_k .

【 Theorem 1】 Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

【 Theorem 2】 Dijkstra's algorithm uses $O(n^2)$operations (additions and comparisons) to find the length of the shortest path between two vertices in a connected simple undirected weighted graph.

3. The Traveling Salesman Problem

Find a Hamilton circuit with minimum total weight in the complete graph.

10.7 Planar Graphs

【Definition】A graph is called *planar* if it can be drawn in the plane without any edges crossing . Such a drawing is called a *planar representation* of the graph.

Note:

- To prove that a graph is planar amounts to redrawing the edges in a way that no edges will cross. May need to move vertices around and the edges may have to be drawn in a very indirect fashion.

1. Euler's Formula

Region -- A region is a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.

- Bounded region
- Unbounded region

Note: There is one unbounded region in a planar graph.

【 Theorem 1】 Euler's formula

Let G be a *connected planar simple* graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r=e-v+2$.

Note: The Euler's formula is a *necessary condition*.

【Definition】 Suppose R is a region of a connected planar simple graph, the number of the edges on the boundary of R is called the *Degree of R* .

Notation: Deg(R)

【 Corollary 1 】 If G is a connected planar simple graph with e edges and v vertices where $v\geq3$, then $e\leq3v-6$

Note:

- The equality holds if and only if every region has exactly three edges.
- For unconnected planar simple graph, $e \leq 3v-6$ also holds.

  Since for a component, $e_i \leq 3v_i - 6$

$$e = \sum e_i \leq \sum (3v_i - 6) < 3\sum v_i - 6 = 3v - 6$$

【 Corollary 2 】 If G is a connected planar simple graph, then G has *a* vertex of degree not exceeding five.

【 Corollary 3 】 If a connected planar simple graph has e edges and v vertices with $v\geq3$ and no circuits of length 3,then $e \leq 2v-4$.

Generally, if every region of a connected planar simple graph has at least k edges, then

$$e \leq \frac{(v-2)k}{k-2}$$

2. KURATOWSKI'S THEOREM

Elementary subdivision

- If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$

Homeomorphic

- The graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

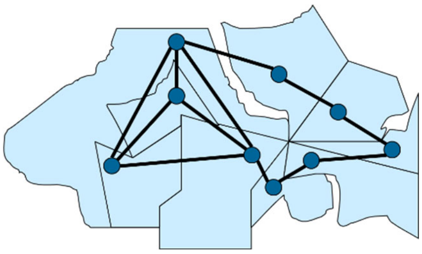
【 Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

10.8 Graph Coloring

The problem of coloring a map, can be reduced to a graph-theoretic problem.

Each map in the plane can be represented by a graph, namely *the dual graph of the map*.

- Each region of the map is represented by a vertex.
- An edge connects two vertices if the regions represented by these vertices have a common border.
- Two regions that touch at only one point are not considered adjacent.



Coloring regions of a map is equivalent to coloring vertices of its dual graph.

coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The chromatic number of a graph

- is the least number of colors needed for a coloring of this graph, denoted by $x(G)$

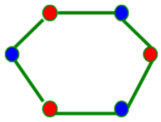
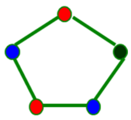
【 Theorem 1】 The Four Color Theorem

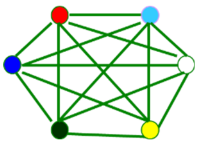
The chromatic number of a planar graph is no greater than four.

1. The chromatic numbers of some simple graphs

(1) The graph G is a path containing no circuit. $x(G) = 2$ 

(2) C_n $\begin{cases} x(G) = 2 & \textbf{if } \boldsymbol{n} \textbf{ is even} \\ x(G) = 3 & \textbf{if } \boldsymbol{n} \textbf{ is odd} \end{cases}$

(3) K_n, $x(G) = n$ 

(4) A simple graph with a chromatic number of 2 is bipartite.

A connected bipartite graph has a chromatic number of 2.

2. Applications of graph colorings