EE435 - Communications I

Term Project - Fall 2020

Prepared by Anıl Kurt and Ali Bulut Üçüncü

Frequency Modulation

A frequency modulated (FM) signal is written as

$$x(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\right)$$

where A_c is carrier magnitude, f_c is carrier frequency, k_f is frequency sensitivity and $m(t)$ is the message signal. In this project, frequency modulated signals and their spectra will be investigated for narrowband and wideband cases. Experimental findings and analytical expressions will be compared. Also, generation methods will be implemented and the relation with digital modulations will be observed.

Part 1: Discrete Representation of Continuous Signals

In order to work in a simulated environment, we need to transform analog signals into discrete ones, and modify the operations that we implement on analog signals accordingly. Given a signal $x(t)$ that is nonzero only in the interval $0 \leq t < T$, let the discrete signal $x[n]$ be the sampled version of $x(t)$ such that $x[n] = x(nt_s)$ for $n = 0, \dots, N-1$, where t_s is the duration between two samples and $N = \frac{T}{t_s}$. (For this part only, consider x(t) as an arbitrary signal, not the FM signal given above)

a) Find an approximation to $x_I(t) = \int_0^t x(\tau)d\tau$ for a given t in terms of x[n]. (Hint: Riemann sum)
b) Find a way to compute $x_I[n] = x_I(nt_s)$ in MATLAB using *cumsum()* function.
c) Find the relation between CTFT $X(f)$ of $x(t)$ and DFT $X[k]$ of $x[n]$. To do that:
   i. Define the impulse train $z(t) = \sum_{n=-\infty}^{\infty} \delta(t - nt_s)$, whose CTFT $Z(f) = \frac{1}{t_s}\sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{t_s})$. Then define $x_s(t) = x(t)z(t)$. Express $X_s(f)$ in terms of $x(t)$ by using the general CTFT expression. Compare this expression with the expression of DTFT $X(e^{j\omega})$ of $x[n]$. Try to relate $X_s(f)$ and $X(e^{j\omega})$.
   ii. Find $X_s(f)$ in terms of $X(f)$ using the property of multiplication in time. State the condition to be able to obtain $x(t)$ from $x_s(t)$ back again.
   iii. Relate $X(e^{j\omega})$ and $X(f)$ assuming the condition is satisfied. Considering an arbitrary signal to satisfy the condition, state the interval in ω axis with maximum width that is meaningful for $X(f)$. Indicate the corresponding interval of f. Establish the relation between ω and f in that interval. Let F be the width of the f interval, namely the maximum width of frequency spectrum $X(f)$ that we can observe using $X(e^{j\omega})$. What is F in terms of t_s?

iv. Let $X[k]$ be the N_f point DFT of x[n] for $k = 0, \dots, N_f - 1$ and $N_f \geq N$. $X[k]$ is the sampled version of $X(e^{j\omega})$. Properly express $X[k]$ for $k = 0, \dots, N_f - 1$ in terms of $X(e^{j\omega})$ using only the meaningful ω interval stated in part iii. Rearrange $X[k]$ to obtain $X_{sorted}[k]$ for $k = 0, \dots, N_f - 1$ such that for ascending values of k, $X_{sorted}[k]$ is equal to $X(e^{j\omega_k})$ for ascending values of ω_k.

v. Show that $X_{sorted}[k] = \frac{1}{t_s} X(kf_s - \frac{F}{2})$ for $k = 0, \dots, N_f - 1$ and for some f_s and F. Find f_s and F in terms of t_s and T. Which of them is dependent of N_f?

vi. Find a way to compute $X_{sorted}[k]$, which is established in previous parts, in MATLAB using *fft()* and *fftshift()* functions (DFT definition might differ in different sources. Go to MATLAB reference page for *fft()* function, and check the expression at the bottom of the page. If it is different from the definition that you use (except the numeration of indices), modify your answer accordingly)

d) The signals that are not time-limited may have impulsive CTFT. How do we observe impulses in the theoretic CTFT of a signal with our discrete equivalent framework derived in part c? Note that, we observe only a limited time interval $0 \leq t < T$ of a possibly time-unlimited signal. Consider which operation in time might be equivalent to observing the signal with limited observation time, and its effect in frequency spectrum. How does it affect impulses in CTFT?

In this part, a convention is established for representing signals in discrete domain and equivalent operations of integration and CTFT are derived, to work with simulation tools. In order for you to continue the project properly, the correct convention is given below. Your answers to this part will be evaluated based on the derivation of the results given. Throughout the project, please use the convention below.

- Any time signal $x(t)$ is represented in simulation environment by $x[n]$ of N samples, representing $x(t)$ for $t = 0, t_s, 2t_s, \dots, T - t_s$.
- Any frequency signal $X(f)$ is represented by $X[k]$ of N_f samples, representing $X(f)$ for $f = -\frac{F}{2}, \dots, -f_s, 0, f_s, \dots, \frac{F}{2} - f_s$ where $f_s = \frac{N}{N_f}\frac{1}{T}$ and $F = \frac{1}{t_s}$.

Part 2: MATLAB Code

Assume $A_C = 1$ and write a single MATLAB code that

- Generates t and f for $T = 1$ seconds, $t_s = 10^{-6}$ seconds, $N_f = 5 \times 10^6$,
- Generates $m(t)$, $M(f)$, $x(t)$, $X(f)$, and $S(f) = \frac{1}{T}\,|X(f)|^2$ for $f_c = 20kHz$, and for **any given** $m(t)$ and k_f,
- Calculates B_{exp}, the experimental effective two-sided bandwidth that covers at least %98 of the signal power, to be measured from $S(f)$. (You can assume that the band is centered around f_c. Then search for the B_{exp} value for which %98 of the signal power is in $f_c - \frac{B_{exp}}{2} < |f| < f_c + \frac{B_{exp}}{2}$ interval. Noting that the power in this interval is a monotonic increasing function of B_{exp}, you can use a basic search algorithm. Check : “One Dimensional Search”)

Throughout the project, we will only use some options for the message signal $m(t)$, so you can add them into your code and select one among them when needed:

- Single tone cosine wave with frequency f_m, ($m(t) = \cos(2\pi f_m t)$),

- Triangle wave with period $\frac{1}{f_m}$ which takes values between -1 and +1 (Hint: Use *sawtooth(<.>,0.5)*),
- Square wave with period $\frac{1}{f_m}$ which takes values between -1 and +1 (Hint: Use *square(<.>)*),
- Two tone cosine wave with frequencies $\frac{f_m}{2}$ and f_m, $\left(m(t) = \cos\left(2\pi\frac{f_m}{2}t\right) + \cos(2\pi f_m t)\right)$.

Submit your code.

Be careful: Even if the message signals are just given, your code should be able to process **any given** message signal $m(t)$, so **do not** calculate any message-specific detail about the modulated signal $x(t)$ beforehand. For a given $m(t)$, all of the calculations should be done by the code.

Note: If your computer cannot process with the above settings due to insufficient memory etc., firstly try $N_f = 10^6$. If the problem persists, try $t_s = 2 \times 10^{-6}$ in addition. If neither can help, contact the teaching assistant.

<u>Part 3: FM Signal and Spectrum</u>

Run your code for single tone cosine wave message signal with the parameters

a) $f_m = 1\ kHz$ and $k_f = 0.1\ kHz/V$
b) $f_m = 1\ kHz$ and $k_f = 1\ kHz/V$
c) $f_m = 1\ kHz$ and $k_f = 10\ kHz/V$
d) $f_m = 2\ kHz$ and $k_f = 1\ kHz/V$
e) $f_m = 2\ kHz$ and $k_f = 2\ kHz/V$

- For each case :
  - Calculate β and effective bandwidth B_c. Compare B_c and B_{exp}.
  - In a single figure of 4 subplots (2×2):
    - Plot $m(t)$ and $x(t)$ from $t = 0$ to $t = \frac{2}{f_m}$.
    - Plot $|M(f)|$ from $f = -2f_m$ to $f = 2f_m$.
    - Plot $|X(f)|$ from $f = f_c - \max(B_c, B_{exp})$ to $f = f_c + \max(B_c, B_{exp})$.

    Titles should include the parameters, and numbers in x axis should be relatable (should not be discrete time instances n or k). Axes should have labels.
- Only for the first and the second cases, compare magnitudes and positions of the impulses in the plot of $X(f)$ with the corresponding analytical expressions.
- Make comparisons among those which have the same f_m and those which have the same β.

For the message signals below, run your code and submit only the aforementioned figure of 4 subplots. (Change limits for $|M(f)|$ as from $f = -5f_m$ to $f = 5f_m$)

f) Triangle wave with period 1 milliseconds and $k_f = 10\ kHz/V$
g) Square wave with period 1 milliseconds and $k_f = 10\ kHz/V$

Part 4: Wideband FM

For $k_f = 10\ kHz/V$ and $f_m = 100\ Hz$ (or period of 0.01 seconds), run your code for the message signals

a) Single tone cosine wave ($m(t) = \cos(2\pi f_m t)$),
b) Triangle wave with period $\frac{1}{f_m}$ which takes values between -1 and +1,
c) Square wave with period $\frac{1}{f_m}$ which takes values between -1 and +1,

For each case, in a single figure of 3 subplots (3×1):

- Plot $k_f m(t)$ from $t = 0$ to $t = \frac{2}{f_m}$.
- Plot $S(f)$ from $f = f_c - \max(B_c, B_{exp})$ to $f = f_c + \max(B_c, B_{exp})$.
- Plot histogram of $k_f m(t)$.

Compare $S(f)$ and the histogram for each case (Hint: Woodward's theorem). Be sure that you understand the relation between $k_f m(t)$ and its histogram.

Part 5: Armstrong's Method for FM Generation

a) Rewrite the FM signal expression using the narrowband FM assumption. ($\cos(\theta) \cong 1, \sin(\theta) \cong \theta$ for small θ)
b) Run your code for single-tone message signal with $f_m = 1\ kHz$ and $\beta = 0.2$ to obtain FM signal $x(t)$ at $f_c = 20\ kHz$ (without using any NBFM approximation). This is the direct generation of the FM signal. In addition, obtain the FM signal $x_{armstrong}(t)$ with the same settings (single-tone message signal with $f_m = 1\ kHz$ and $\beta = 0.2$) by Armstrong's method, which first generates a narrowband FM (NBFM) signal and then changes it to a wideband signal. Let the narrowband signal has $\beta_{NBFM} = 0.1$ and use a doubler that outputs $b(t) = 2\left(a(t)^2 - \frac{1}{2}\right)$ for input $a(t)$. Implement the procedure in MATLAB and plot resulting FM signals $x(t)$ and $x_{armstrong}(t)$ from $t = 0$ to $t = \frac{2}{f_m}$ on a single figure (on top of each other). Compare the two signals.
c) Make necessary changes and repeat part b) for $\beta = 12.8$ and $\beta_{NBFM} = 0.1$.

Part 6: Digital Modulation

In communication systems, information is delivered from one end to another by means of modulation techniques. This information might be a continuous signal $m(t)$, or might be a sequence of symbols a_k from some finite set of symbols. Modulation is specified as analog in the former case and digital in the second case.

Digital modulation techniques have specific expressions and analysis methods. However, the transmitted signal is still a continuous signal. Therefore, they can also be expressed in the form of analog modulation techniques. That means, digital modulations are some specific cases of analog modulations. We will investigate the relation between analog and digital modulation techniques.

The sequence a_k might take values from a set of numbers with different cardinalities. Consider that a_k is a binary information sequence, that is, it can take only two values. Let it randomly take the values -1 and +1 with equal probabilities, that is,

$$a_k = \begin{cases} -1, & with\ probability\ 0.5 \\ +1, & with\ probability\ 0.5 \end{cases}, \qquad k = 0,1,\dots,N_s - 1$$

According to the value of a_k, we can modulate amplitude, frequency or phase of a carrier signal (or combinations of these) for a time interval specified for kth instance of the sequence. We can express that generally as

$$x(t) = \begin{cases} u_1(t - kT_s), & if\ a_k = -1 \\ u_2(t - kT_s), & if\ a_k = +1 \end{cases}, \qquad for\ kT_s \le t < (k+1)T_s$$

or

$$x(t) = \begin{cases} x_1(t), & if\ a_k = -1 \\ x_2(t), & if\ a_k = +1 \end{cases}, \qquad for\ kT_s \le t < (k+1)T_s$$

according to common practice. The signals $u_1(t)$ and $u_2(t)$ are nonzero only for $0 \le t < T_s$ for some signaling interval T_s but $x_1(t)$ and $x_2(t)$ are nonzero for $0 \le t < T$. These signals are chosen according to the modulation type.

There are several digital modulation techniques. For example, frequency shift keying (FSK) modulates the frequency, amplitude shift keying (ASK) modulates the amplitude, phase shift keying (PSK) modulates the phase, quadrature-amplitude modulation (QAM) modulates both the phase and the amplitude. The cardinality of the set of symbols or some other features are often mentioned in the name of the technique, resulting in new definitions such as binary PSK (BPSK), quadrature PSK (QPSK), offset QPSK (OQPSK), differential PSK (DPSK), 16-QAM, 64-QAM, minimum FSK (MFSK or MSK), Gaussian FSK (GFSK) and continuous-phase FSK (CPFSK).

For binary FSK, we can specify $x_1(t)$ and $x_2(t)$ as $\cos(2\pi f_1 t)$ and $\cos(2\pi f_2 t)$, respectively, for $0 \le t < T$. Therefore, we can write

$$x(t) = \begin{cases} \cos(2\pi f_1 t), & if\ a_k = -1 \\ \cos(2\pi f_2 t), & if\ a_k = +1 \end{cases}, \qquad for\ kT_s \le t < (k+1)T_s$$

which means that there are two running oscillators and we switch between them according to the symbol to be transmitted. The instantaneous frequency of the modulated signal $x(t)$ is

$$f_i(t) = \begin{cases} f_1, & if\ a_k = -1 \\ f_2, & if\ a_k = +1 \end{cases}, \qquad for\ kT_s \le t < (k+1)T_s$$

a) For $f_1 = 18\ kHz$, $f_2 = 22\ kHz$ and $T_s = 250\ \mu s$ and general convention specified in parts 1 and 2 ($T = 1$ seconds, $t_s = 10^{-6}$ seconds, $N_f = 5 \times 10^6$, etc.), implement FSK as an FM signal. Review the FM expression given before and find A_c, f_c, k_f and properly express $m(t)$ in terms of a_k so that the instantaneous frequency satisfies the equation above. $m(t)$ should be zero mean and take values between -1 and +1.

b) Generate a_k sequence for $N_s = 4000$. Then, generate corresponding $x(t)$ for the message signal $m(t)$. Repeat this procedure 1000 times to generate 1000 different a_k sequences and modulated $x(t)$ signals (ensemble of a_k sequences and $x(t)$ signals).

i. One definition of PSD is $S(f) = \frac{1}{T} E\{|X(f)|^2\}$. Calculate the PSD empirically. (In case of a memory problem, you do not have to keep all the ensemble at the same time in the memory to calculate $S(f)$). According to the weak law of large numbers, the average of independent realizations of $|X(f)|^2$ should converge to $E\{|X(f)|^2\}$.

ii. In a figure of 3 subplots (3×1), plot
   - Only one realization of $m(t)$ from $t = 0$ to $t = 1.5\ ms$ (In the plot, several transitions should be observed. If not, shift the time axis to a more appropriate interval of 1.5 ms),
   - Modulated signal $x(t)$ corresponding to plotted message signal $m(t)$ for the same time interval,
   - $S(f)$ from $f = 8\ kHz$ to $f = 32\ kHz$ in dB scale.

iii. Measure the experimental BW B_{exp}. Is it equal to $f_2 - f_1$? If not, what might be the reason?

QPSK is the digital modulation in which the phase is modulated. Now, assume that the cardinality of the set of symbols is four and a_k takes values of 0, 1, 2, 3 with equal probabilities. We can write the modulated signal as

$$x(t) = \cos\left(2\pi f_c t + a_k \frac{\pi}{2}\right), \qquad for\ kT_s \leq t < (k+1)T_s$$

As the phase is modulated, QPSK can be expressed better with phase modulation (PM), rather than FM.

c) Using the PM expression

$$x(t) = A_c \cos\big(2\pi f_c t + k_p m(t)\big),$$

choose a value for the constant k_p and properly express $m(t)$ in terms of a_k.