Mzuzu University

MZUZU UNIVERSITY

Centre for Open and Distance Learning

**Mathematics Module 1**

# *Precalculus*

Nephtale B. Mumba

Published by Mzuzu University,

Centre for Open and Distance Learning,

Private Bag 201,

Luwinga

Mzuzu 2

Editors:

Typesetters:

Designers:

Proofreaders:



ISBN 000–00000–00–00–0
First edition published 2020

Printed by Centre for Open and Distance Learning

## Acknowledgments

The author wishes to thank all members of Mathematics Department for offering sound and professional advice in the development of this module. Past, present and future students of this course also deserve my sincerely gratitude because without them there could not have been notes that led to the construction of this module.

# Dedication

*To all students who have ever felt different.*

# Contents

# Introduction

The course builds upon secondary school mathematics to create a bridge aimed at easing transition from secondary school to college mathematics. Broadly, the course is aimed at reinforcing students background in algebra and trigonometry, preparing them for subsequent courses such as calculus. The course emphasis on algebraic and trigonometric functions, equips students with theoretical knowledge and skills in functions, their graphs, relationships between functions and problem solving. The gained knowledge consequently affords students essential building blocks and prerequisites for calculus and courses that follow calculus.
The material in this module provides a suitable background for mathematics and statistics content for students intending to do more mathematics in years three and four. It also provides adequate material for those students that may not have the chance to do more base in higher levels in their course of study at undergraduate level. This module is a self–contained resource, consisting of explanatory text, activities, examples and exercises. The textual material is presented in such a way that you yourself become involved in the development of ideas. Throughout the module, you have activities to introduce a concept or summarise a textual material.

The questions in the examples and exercises are are mostly based on real problems. As such they are a crucial part of the learning process. Work through them with an open mind.

UNIT 1

# Fundamentals

## 1.1. Real Numbers

Real numbers are used in everyday life to describe quantities such as age, weight, price, population etc. A **real number** is any number that can be expressed in decimal form. Real numbers can be represented as points on a number line as shown below:

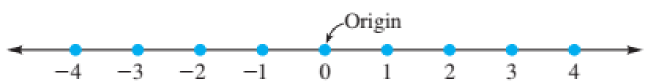



FIGURE 1.1.

The point associated with the number zero is referred to as the **origin**. Each real number can be identified with exactly one point on the line and with each point on the line, we identify exactly one real number. The fundamental fact here is that there is a **one-to-one correspondence**.

Real numbers consist of **natual numbers**, **integers**, **rational numbers** and **irrational numbers**.

**Natural numbers** are just ordinary counting numbers: 1, 2, 3 and so on. **Integers** are just the natural numbers along with their negatives and zero. Examples are -2, 0, 1, 2. **Rational numbers** are ratios of two integers. For example, $\frac{2}{5}$ is a rational number. It can also be shown that a number is rational if and only if its decimal expansion terminates. **Irrational numbers** are the numbers which are not rational.

The decimal expansion of irrational numbers does not terminate. For example, $\pi$ is irrational.

The following flowchart shows the sets of real numbers.

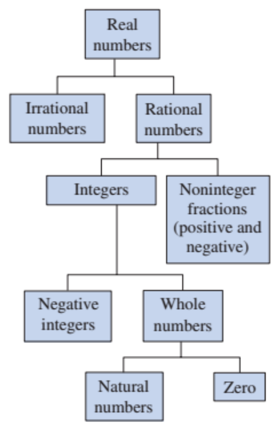



FIGURE 1.2. Subsets of Real Numbers

This is also shown below:

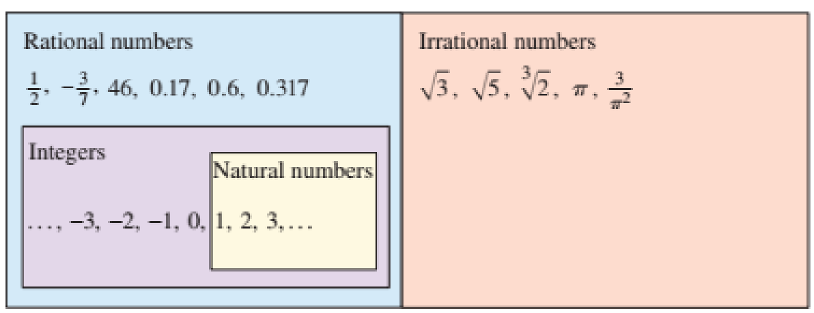
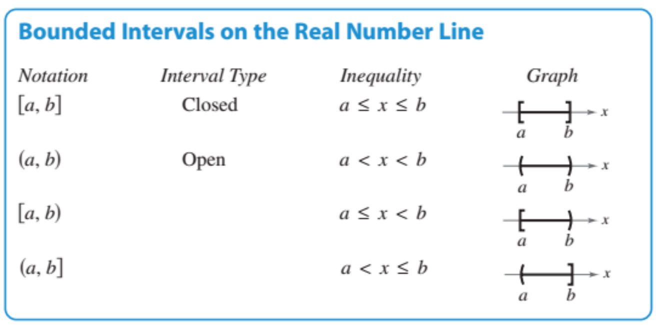



FIGURE 1.3. Subsets of Real Numbers

**1.1.1. Ordering of Real Numbers.** One important property of real numbers is that they are ordered. If $a$ and $b$ are real numbers, $a$ is less than $b$ if $b - a$ is positive. The order of $a$ and $b$ is denoted by the inequality $a < b$. This relationship can also be described by saying that $b$ is greater than $a$ and writing $b > a$. The inequality $a \leq b$ means that

$a$ is less than or equal to $b$, and the inequality $b \geq a$ means that $b$ is greater than or equal to $a$. The symbols $<$, $>$, $\leq$, and $\geq$ are inequality symbols.

Inequalities can be used to describe subsets of real numbers called **intervals**. Intervals can be **bounded** or **unbounded**. In the figure below, we show bounded intervals. The real numbers $a$ and $b$ are the endpoints of each interval. An interval is **closed** if the endpoints are included in the interval, and it is open if the endpoints are not included in the interval.



**Bounded Intervals on the Real Number Line**

| *Notation* | *Interval Type* | *Inequality* | *Graph* |
|---|---|---|---|
| $[a, b]$ | Closed | $a \leq x \leq b$ | |
| $(a, b)$ | Open | $a < x < b$ | |
| $[a, b)$ | | $a \leq x < b$ | |
| $(a, b]$ | | $a < x \leq b$ | |

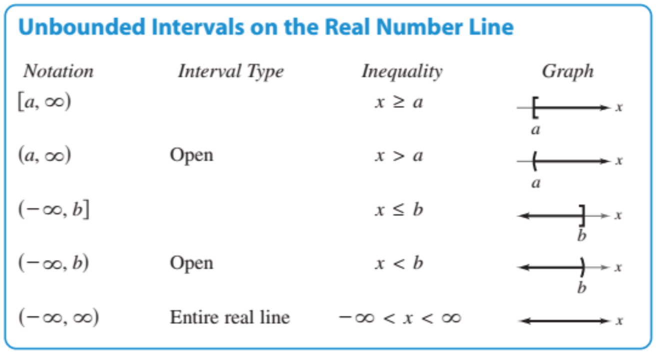
FIGURE 1.4. Intervals

**Example 1.1.** (a) $-2 < x < 3$ *consists of all real numbers between* $-2$ *and* $3$ *but* $-2$ *and* $3$ *are not included.*

(b) $-2 \leq x \leq 3$ *consists of all real numbers between* $-2$ *and* $3$*, and* $-2$ *and* $3$ *are included.*

(c) $-2 \leq x < 3$ *consists of all real numbers from* $-2$ *to* $3$ *but only* $3$ *is not included.*

Below, we show unbounded intervals. **Positive infinity**($\infty$) and **negative infinity**($-\infty$) do not represent real numbers. They are symbols convenient to describe the unboundedness of an interval.



**Unbounded Intervals on the Real Number Line**

| *Notation* | *Interval Type* | *Inequality* | *Graph* |
|---|---|---|---|
| $[a, \infty)$ | | $x \geq a$ | |
| $(a, \infty)$ | Open | $x > a$ | |
| $(-\infty, b]$ | | $x \leq b$ | |
| $(-\infty, b)$ | Open | $x < b$ | |
| $(-\infty, \infty)$ | Entire real line | $-\infty < x < \infty$ | |

FIGURE 1.5. Unbounded Intervals

**Example 1.2.** *Using inequality notation to describe each of the following:*

(a) *$c$ is at most 2,*

***Solution:***

$c \leq 2$.

(b) *$m$ is at least $-3$,*

***Solution:***

$m \geq -3$.

(c) *All $x$ in the interval $(-3, 5]$,*

***Solution:***

$-3 < x \leq 5$.

The **Law of Trichotomy** states that for any two real numbers $a$ and $b$, precisely one of three relationships is possible:

$$a = b, \quad a < b, \quad a > b.$$

**Activity 1.1.** (1) *Use inequality notation to describe the following:*

(a) $[-2, 5)$

(b) $(-\infty, 0)$

(2) *Find the indicated set if*

$$A = \{x | x \geq -2\} \quad B = \{x | x < 4\} \quad C = \{x | -1 < x \leq 5\}$$

(a) $B \cup C$
(b) $A \cap C$
(c) $A \cap B$

**1.1.2. Properties of Real Numbers.** For any real numbers $a$, $b$ and $c$, the following properties hold:

(1) Commutative Property

$$a + b = b + a,$$

$$ab = ba.$$

(2) Associative Property

$$(a + b) + c = a + (b + c)$$

$$(ab)c = a(bc)$$

(3) Distributive Property

$$a(b + c) = ab + ac$$

$$(b + c)a = ab + ac$$

**1.1.3. Addition and Subtraction of Real Numbers.** The number 0 is called the additive identity because $a + 0 = a$ for every real number $a$. Every real number $a$ has a negative $-a$ that satisfies $a + (-a) = 0$.

By definition, for real numbers $a$ and $b$

$$a - b = a + (-b).$$

**Properties of Negatives**

(1) $(-1)a = -a$

(2) $-(-a) = a$

(3) $(-a)b = a(-b) = -(ab)$

(4) $(-a)(-b) = ab$

(5) $-(a + b) = -a - b$

(6) $-(a - b) = -a + b = b - a$

**Example 1.3.** *Let $x$, $y$, and $z$ be real numbers*

(a) $-(x + z) = -x - z$

(b) $-(x + y - z) = -x - y - (-z) = -x - y + z$

**1.1.4. Multiplication and Division of Real Numbers.** The number 1 is called the **multiplicative identity** because for any real number $a$, $a \cdot 1 = a$.

Every real number $a$ has an inverse $\frac{1}{a}$ that satisfies $a \cdot (1/a) = 1$. Division, by definition, for any real numbers $a$ and $b$ such that $b \neq 0$, is given by

$$a \div b = a \cdot \frac{1}{b} = \frac{a}{b}.$$

**Properties**

(1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(2) $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

(3) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

(4) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

(5) $\frac{ac}{bc} = \frac{a}{b}$

(6) If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

**Example 1.4.** *Evaluate* $\frac{5}{36} + \frac{7}{120}$.

***Solution:***

*We first find the least common denominator (LCD). We first express the numbers 36 and 120 as products of prime powers of their prime factors. Thus,* $36 = 2^2 \cdot 3^2$ *and* $120 = 2^3 \cdot 3 \cdot 5$*. The LCD is the product of greatest prime powers. Hence,* $LCD = 2^3 \cdot 3^2 \cdot 5$.

*Then*

$$\frac{5}{36} + \frac{7}{120} = \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3} = \frac{50}{360} + \frac{21}{360} = \frac{71}{360}.$$

**Activity 1.2.** (1) *Evaluate*

(a) $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

(b) $\frac{2-\frac{3}{4}}{\frac{1}{2}-\frac{1}{3}}$

(2) *Simplify*

(a) $\frac{2x}{3} - \frac{x}{4} \div 2$

(b) $\frac{\frac{4}{3}(-6y)}{3(x+y)}$

## 1.2. Exponents and Radicals

**1.2.1. Exponents.** If $a$ is a real number and $n$ is a positive integer, then the $n$th power of $a$ is

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n\ factors}$$

The number $a$ is called the **base**, and $n$ the **exponent**.

**Example 1.5.** (a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$,

(b) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$,

(c) $(-3)^4 = (-3)(-3)(-3)(-3) = 81.$

**Laws of Exponents**

(1) $a^m a^n = a^{m+n}$

(2) $\frac{a^m}{a^n} = a^{m-n}$

(3) $(a^m)^n = a^{mn}$

(4) $(ab)^n = a^n b^n$

(5) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

**Example 1.6.** *Simplify*

(a) $\frac{c^9}{c^5} = c^{9-5} = c^4$

(b) $\left(\frac{x}{2}\right)^5 = \frac{x^5}{2^5} = \frac{x^5}{32}$

(c) $(2a^3b^2)(3ab^4)^3 = (2a^3b^2)(3^3a^3b^{4\cdot 3}) = 54a^{3+3}b^{2+12} = 54a^6b^{14}$

(d) $\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^7$

$$\begin{aligned}\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^7 &= \frac{x^3}{y^3}\frac{y^{14}x^7}{z^7}\\ &= \frac{x^{10}y^{14}}{y^3z^7}\\ &= \frac{x^{10}y^{11}}{z^7}\end{aligned}$$

**Activity 1.3.** *Simplify*

(a) $\left(\frac{4}{y}\right)^3\left(\frac{3}{y}\right)^4$

(b) $\frac{12(x+y)^3}{9x+9y}$

(c) $\left(\frac{x^4z^2}{4y^5}\right)\left(\frac{2x^3y^2}{z^3}\right)^2$

(d) $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

**1.2.2. Radicals.** We know that $\sqrt{a} = b$ means $b^2 = a$ and $b \geq a$. It is clear to see that $\sqrt{a}$ works only when $a \geq 0$.

**Definition 1.1.** *If $n$ is any positive integer, then the principal $n$th root of $a$ is defined as follows*

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.$$

*If $n$ is even, we must have $a \geq 0$ and $b \geq 0$.*

**Example 1.7.** (a) $\sqrt[4]{81} = 3$ *since* $3^4 = 81$ *and* $3 \geq 0$.

(b) $\sqrt[3]{-8} = -2$ *since* $(-2)^3 = -8$.

It is good to note that $\sqrt{4^2} = \sqrt{16} = 4$ and $\sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$. So $\sqrt{a^2} = a$ is not always true. In general, it is true when $a \geq 0$. Actually, for any even root $n$, $\sqrt[n]{a^n} = |a|$.

**Properties of $n$th roots**

(1) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

(2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(3) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

(4) $\sqrt[n]{a^n} = a$ if $n$ is odd

(5) $\sqrt[n]{a^n} = |a|$ if $n$ is even

**Example 1.8.** *Simplify*

(a) $\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\sqrt[3]{x} = x\sqrt[3]{x}$

(b) $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4} = 3\sqrt[4]{(x^2)^4}|y| = 3x^2|y|$

(c) $\sqrt{32} + \sqrt{200}$

$$\begin{aligned}\sqrt{32} + \sqrt{200} &= \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2} \\ &= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2} \\ &= 4\sqrt{2} + 10\sqrt{2} \\ &= 14\sqrt{2}\end{aligned}$$

**Activity 1.4.** *Simplify*

(a) $\frac{\sqrt[3]{8x^2}}{\sqrt{x}}$

(b) $\sqrt[3]{y\sqrt{y}}$

(c) $\sqrt[5]{x^3y^2}\ \sqrt[10]{x^4y^{16}}$

(d) $\sqrt[3]{\frac{54x^2y^4}{2x^5y}}$

### 1.2.3. Rational Exponents.

**Definition 1.2.** *For any rational exponent $\frac{m}{n}$ in lowest terms where $m$ and $n$ are integers and $n > 0$, we define*

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

*or equivalently*

$$a^{m/n} = \sqrt[n]{a^m}.$$

*If $n$ is even, then we require that $a \geq 0$.*

**Example 1.9.** (a) $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

(b) $125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$

(c) $\frac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$

(d) $(2a^3b^4)^{\frac{3}{2}} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2} = (\sqrt{2})^3a^{3(3/2)}b^{4(3/2)} = 2\sqrt{2}a^{9/2}b^6$

(e) $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 = \left(\frac{2^3(x^{3/4})^3}{(y^{1/3})^3}\right) \cdot (y^4x^{1/2}) = \frac{8x^{9/4}}{y} \cdot y^4x^{1/2} = 8x^{1/4}y^3$

**Activity 1.5.** *Perform the operations and simplify.*

(a) $\frac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}}$

(b) $\frac{5^{-1/2} \cdot 5x^{5/2}}{(5x)^{3/2}}$

(c) $\left(\frac{a^{1/6}b^{-3}}{x^{-1}y}\right)^3 \left(\frac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$

(d) $\frac{(9st)^{3/2}}{27s^3t^{-4}} \cdot \left(\frac{3s^{-2}}{4t^{1/3}}\right)^{-1}$

**1.2.4. Rationalizing the Denominator.** It is often useful to eliminate the radical in the denominator by multiplying both the numerator and denominator by an appropriate expression. The process is called **rationalizing the denominator**.

For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}.$$

If the denominator is of the form $\sqrt[n]{a^m}$, then

$$\sqrt[n]{a^m} \cdot \sqrt[n]{a^{n-m}} = \sqrt[n]{a^{n-m+m}} = \sqrt[n]{a^n} = a.$$

**Example 1.10.** (a) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

(b) $\sqrt[7]{\frac{1}{a^2}} = \frac{1}{\sqrt[7]{a^2}} = \frac{1}{\sqrt[7]{a^2}} \cdot \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}} = \frac{\sqrt[7]{a^5}}{a}$

**Activity 1.6.** *Rationalize the denominator:*

(a) $\frac{a}{\sqrt[3]{a^2}}$

(b) $\frac{1}{c^{3/7}}$

(c) $\frac{2}{5-\sqrt{3}}$

(d) $\frac{3}{\sqrt{5}+\sqrt{6}}$

## 1.3. Absolute Value and Distance

**Definition 1.3.** *The* ***absolute value*** *of a real number $a$ is the distance from $a$ to 0 on the real number line, denoted by $|a|$.*

Distance is always positive or zero. In general, $|a| \geq 0$. If $a$ is a real number, then the absolute value of $a$ is

$$|a| = \begin{cases} a, & if\ a \geq 0, \\ -a, & if\ a < 0. \end{cases}$$

**Example 1.11.** (a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3 - \pi| = -(3 - \pi) = \pi - 3$ *since* $3 < \pi \implies 3 - \pi < 0$.

**Properties of Absolute Values**

(1) $|a| \geq 0$

(2) $|a| = |-a|$

(3) $|ab| = |a||b|$

(4) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

**Definition 1.4.** *If $a$ and $b$ are real numbers, then the distance between the points $a$ and $b$ on the real line is*

$$d(a, b) = |b - a|.$$

**Activity 1.7.** *Evaluate each expression*

(a) $||-6| - |-4||$

(b) $|\sqrt{5} - 5|$

(c) $\left|\frac{-6}{24}\right|$

(d) $\left|\frac{7-12}{12-7}\right|$

### 1.4. Rectangular Coordinates

Two perpendicular lines that intersect at zero on each line are drawn. The horizontal line has positive vlues on the right and an arrow points that side. The vertical line has positive values upwards of the intersection. The horizontal line is the $x$**-axis** and the vertical line is the $y$**-axis**. The point of intersection is known as the **Origin** $O$. The axes divide the plane into 4 quadrants as shown below:
A point on the $xy$-plane is located by a unique pair of numbers $(a, b)$ where $a$ is the $x$-coordinate and $b$ is the $y$-coordinate.

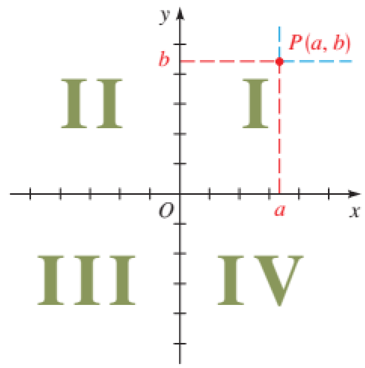



FIGURE 1.6.

### 1.4.1. Graphing Regions in the Coordinate Plane.

**Example 1.12.** (a) $\{(x,y)|x \geq 0\}$

***Solution:***

*This region consists of all points whose $x$-coordinates are positive or 0.*

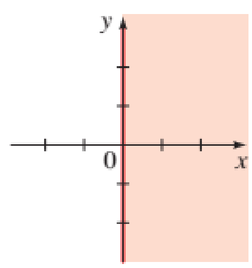



FIGURE 1.7.

(b) $\{(x,y)||y| < 1\}$

***Solution:***

*The region consists of all the points whose $y$-coordinates lie between $-1$ and $1$.*

**Activity 1.8.** *Sketch the regions given by each set.*

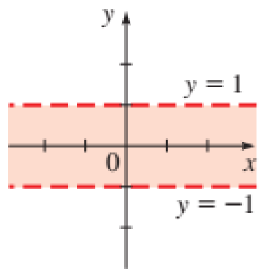



FIGURE 1.8.

(a) $\{(x, y)|y < 2\}$

(b) $\{(x, y)||x| < 3\}$

### 1.4.2. Distance and Midpoint Formulas.

**Definition 1.5.** *The **distance** between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is*

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

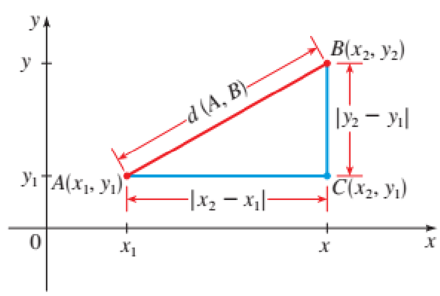



FIGURE 1.9.

**Example 1.13.** *Which of the points $P(1, -2)$ and $Q(8, 9)$ is closer to the point $A(5, 3)$?*

***Solution:***

*We compute the distances $d(P, A)$ and $d(Q, A)$ and see which one is* *smaller.*

$$d(P,A) = \sqrt{(5-1)^2 + (3-(-2))^2} = \sqrt{4^2+5^2} = \sqrt{41}$$

$$d(Q,A) = \sqrt{(5-8)^2 + (3-9)^2} = \sqrt{(-3)^2+(-6)^2} = \sqrt{45}$$

*So, $d(P,A) < d(Q,A)$, then $P$ is closer to $A$.*

**Definition 1.6.** *The **midpoint** of the line segment from $A(x_1,y_1)$ to $B(x_2,y_2)$ is*

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

**Example 1.14.** *Find the midpoint of the line segment from $P(1,-2)$ to $A(5,8)$.*

***Solution:***

*The midpoint is*

$$\left(\frac{1+5}{2}, \frac{-2+8}{2}\right) = \left(\frac{6}{2}, \frac{6}{2}\right) = (3,3).$$

**Example 1.15.** *Show that the quadrilateral with vertices $P(1,2)$, $Q(4,4)$, $R(5,9)$ and $S(2,7)$ is a parallelogram by proving that its two diagonals bisect each other.*

***Solution:***

*We look at the midpoints of $PR$ and $SQ$ as shown in the figure below. If the midpoints are equal, then the diagonals bisect each other. Midpoint of $PR = \left(\frac{1+5}{2}, \frac{2+9}{2}\right) = (3, 11/2)$.*

*Midpoint of $SQ = \left(\frac{4+2}{2}, \frac{4+7}{2}\right) = (3, 11/2)$. The midpoints are equal. Therefore, the diagonals bisect each other and the quadrilateral is a parallelogram.*

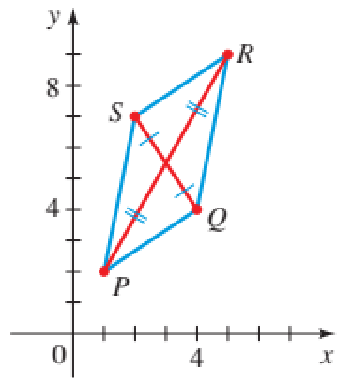



FIGURE 1.10.

**Activity 1.9.** (1) *Draw the rectangle with vertices* $A(1, 3)$, $B(5, 3)$, $C(1, -3)$ *and* $D(5, -3)$ *on a coordinate plane. Find the area of the rectangle.*

(2) *Show that the triangle with vertices* $A(0, 2)$, $B(-3, -1)$ *and* $(-4, 3)$ *is isosceles.*

(3) *The point* $M$ *in the figure below is the midpoint of the line segment* $AB$. *Show that* $M$ *is equidistant from the vertices of triangle* $ABC$.

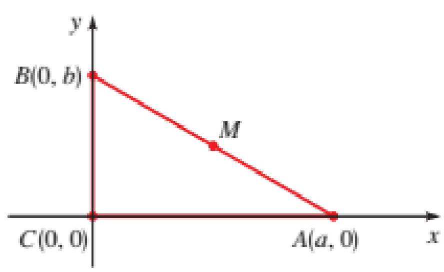



FIGURE 1.11.

**1.4.3. Graphs of Equations in Two Variables.** The graph of an equation in $x$ and $y$ is the set of all points $(x, y)$ in the coordinate plane that satisfy the equation.

**Example 1.16.** *Sketch the graph of the equation $2x - y = 3$.*

***Solution:***

*Firstly, we make $y$ the subject,*

$$y = 2x - 3.$$

*We then come up with a table of values as below:*

| $x$ | $y = 2x - 3$ | $(x, y)$ |
|---|---|---|
| $-1$ | $-5$ | $(-1, -5)$ |
| $0$ | $-3$ | $(0, -3)$ |
| $1$ | $-1$ | $(1, -1)$ |
| $2$ | $1$ | $(2, 1)$ |
| $3$ | $3$ | $(3, 3)$ |
| $4$ | $5$ | $(4, 5)$ |

FIGURE 1.12.

*The graph is as follows:*

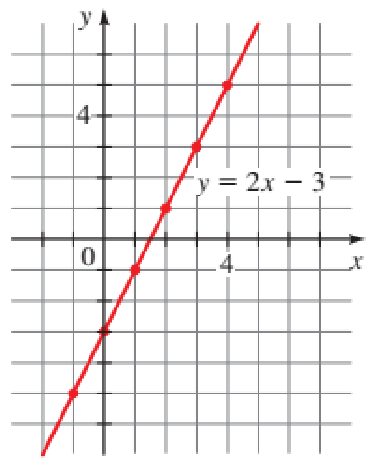



FIGURE 1.13.

**Definition 1.7.** *The $x$-**intercept** is the $x$-coordinate of a point where a graph intersects the $x$-axis. The $y$-intercept is the $y$-coordinate pf a point where a graph intersects the $y$-axis. The value of $y$ at the $x$-intercept is 0 and the value of $x$ at the $y$-intercept is 0.*

**Example 1.17.** *Find the $x$- and $y$-intercepts of the graph of the equation $y = x^2 - 2$.*

***Solution:***

*To find the $x$-intercepts, we set $y = 0$,*

$$0 = x^2 - 2$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

*Therefore, the $x$-intercepts are $x = \sqrt{2}$ and $x = -\sqrt{2}$.*

*The $y$-intercept is where $x = 0$. Thus $y = 0 - 2 = -2$ is the $y$-intercept.*

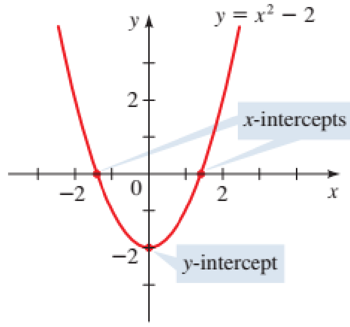



FIGURE 1.14.

**Activity 1.10.** (1) *Determine the $x$ and $y$ intercepts in the following:*

(a) $y = x^2 + 3x + 2$

(b) $y = 6x^3 + 9x^2 + x$

(c) $y = 11x - 2x^2 - x^3$

(2) *Graph the equations.*

(a) $3x + 2y = 6$

(b) $y = x^3 - 3x + 1$

## 1.5. Equation of a Circle

An equation of the circle with center $(h, k)$ and radius $r$ is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is called the standard form of the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation becomes

$$x^2 + y^2 = r^2.$$

**Example 1.18.** *Graph each equation.*

(a) $x^2 + y^2 = 25$.

***Solution:***

*We can rewrite the equation as* $(x-0)^2 + (y-0)^2 = 5^2$*. So the radius is* $5$ *and the center is* $(0,0)$.

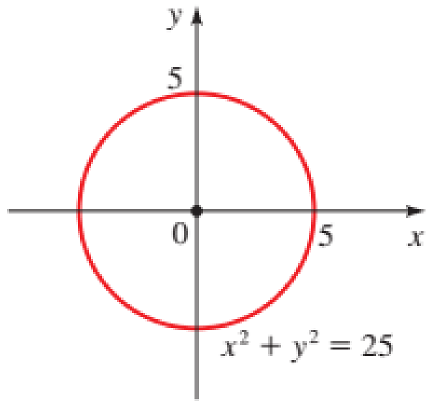



FIGURE 1.15.

(b) $(x-2)^2 + (y+1)^2 = 25$.

***Solution:***

*The center is* $(2,-1)$ *and radius 5.*

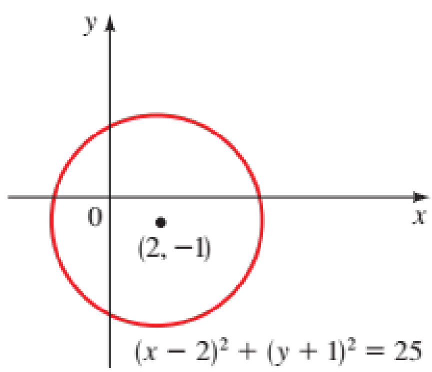



FIGURE 1.16.

**Example 1.19.** (a) *Find an equation of a circle with radius 3 and center* $(2,-5)$.

***Solution:***

*We have* $r = 3$*, center is* $(2, -5)$*, thus* $h = 2$ *and* $k = -5$*. The equation is*

$$(x-2)^2 + (y+5)^2 = 9.$$

(b) *Find an equation of the circle that has the points* $P(1, 8)$ *and* $Q(5, -6)$ *as the endpoints of a diameter.*

***Solution:***

*The midpoint of the diameter* $PQ$ *is the center of the circle.*

$$\left(\frac{1+5}{2}, \frac{8-6}{2}\right) = (3, 1).$$

*The radius* $r$ *is the distance from* $P$ *to the center. Thus, by the distance formula*

$$r^2 = (3-1)^2 + (1-8)^2 = 2^2 + (-7)^2 = 53.$$

*The equation is therefore*

$$(x-3)^2 + (y-1)^2 = 53.$$

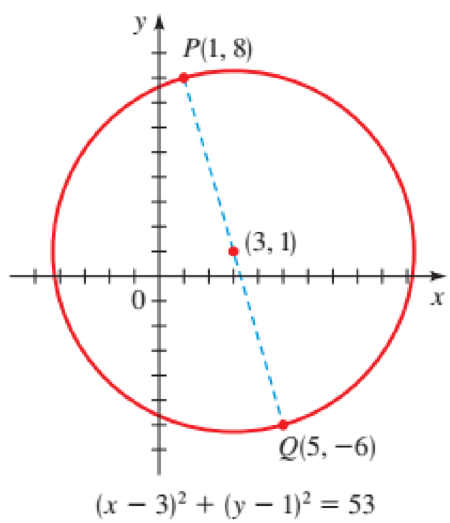



FIGURE 1.17.

**Example 1.20.** *Show that the equation* $x^2 + y^2 + 2x - 6y + 7 = 0$ *represents a circle, and find the center and radius of the circle.*

***Solution:***

$$\begin{aligned}
x^2 + y^2 + 2x - 6y + 7 &= 0 \\
(x^2 + 2x) + (y^2 - 6y) &= -7 \\
(x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 &= -7 \\
(x^2 + 2x + 1) + (y^2 - 6y + 9) &= -7 + 1 + 9 \\
(x + 1)^2 + (y - 3)^2 &= 3
\end{aligned}$$

*Therefore, the center is* $(-1, 3)$ *and the radius* $r = \sqrt{3}$.

**Activity 1.11.** (1) *Find an equation of the circle that satisfies the given conditions:*

(a) *Center* $(2, -1)$*; radius 3.*

(b) *Center* $(-1, 5)$*; passes through* $(-4, -6)$.

(c) *Endpoints of a diameter are* $P(-1, 3)$ *and* $Q(7, -5)$.

(d) *Center* $(7, -3)$*; tangent to the* $x$*-axis.*

(2) *Find the equation of the circle shown below:*

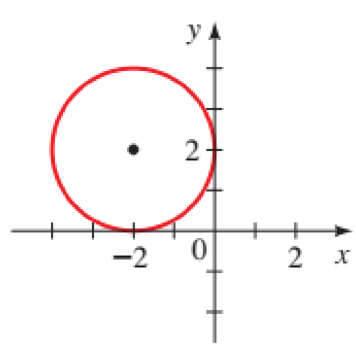



FIGURE 1.18.

(3) *Show that the equation represents a circle, find the center and radius of the circle.*

(a) $x^2 + y^2 - 4x + 10y + 13 = 0$;

(b) $2x^2 + 2y^2 - 3x = 0$.

(c) $x^2 + y^2 + \frac{1}{2}x + 2y = -\frac{1}{16}$

(d) $3x^2 + 3y^2 + 6x - y = 0$

## 1.6. Symmetry

Given a graph, if points on the graph on the left of a line are reflections of points on the other side of the line, then we say the graph is symmetric with respect to that line. We will discuss three aspects of symmetry.

**1.6.1. Symmetry with Respect to the $x$-axis.** If a point $(x, y)$ is on the graph, then the point $(x, -y)$ is also on the graph. In other words, the graph is unchanged when reflected on the $x$-axis. We can test for this symmetry by observing the equation when $y$ is replaced with $-y$ (equation is unchanged).

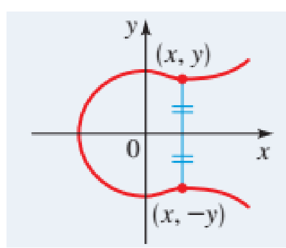



FIGURE 1.19.

**1.6.2. Symmetry with Respect to the $y$-axis.** If a point $(x, y)$ is on the graph, then $(-x, y)$ is also on the graph. In other words, the graph is unchanged when reflected on the $y$-axis. We can test for this symmetry by observing the equation when $x$ is replaced by $-x$.

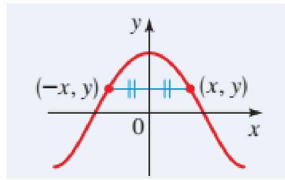



FIGURE 1.20.

**1.6.3. Symmetry with Respect to the Origin.** If a point $(x, y)$ is on the graph, then $(-x, -y)$ is also on the graph. In other words, the graph is unchanged when reflected on the origin. We can test for this symmetry by observing that the equation is unchanged if $x$ and $y$ are replaced by $-x$ and $-y$, respectively.

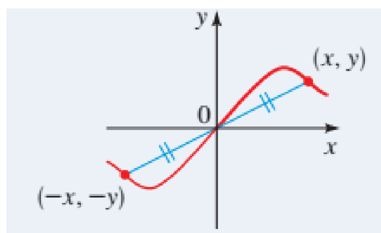



FIGURE 1.21.

**Example 1.21.** (a) *Test the equation $x = y^2$ for symmetry and sketch the graph.*

***Solution:***

*The equation is unchanged if $y$ is replaced by $-y$,*

$$x = (-y)^2 = y^2.$$

*We plot points for $y > 0$ and reflect the graph in the $x$-axis.*

| $y$ | $x = y^2$ | $(x, y)$ |
|---|---|---|
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 2 | 4 | (4, 2) |
| 3 | 9 | (9, 3) |

FIGURE 1.22.

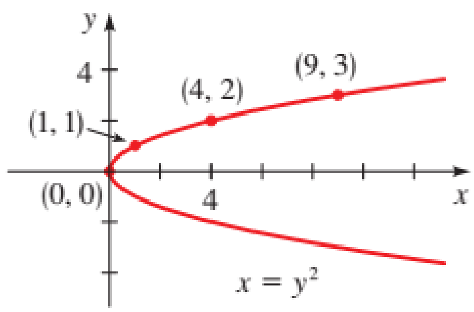



FIGURE 1.23.

(b) *Test the equation $y = x^3 - 9x$ for symmetry and sketch its graph.*

***Solution:***

*Replacing $x$ by $-x$ and $y$ by $-y$ in the equation, we get*

$$-y = (-x)^3 - 9(-x)$$

$$-y = -x^3 + 9x$$

$$y = x^3 - 9x$$

*The equation is unchanged. The graph is symmetric with respect to the origin. We sketch it by plotting points for $x > 0$ and then reflect it about the origin.*

| $x$ | $y = x^3 - 9x$ | $(x, y)$ |
|---|---|---|
| 0 | 0 | $(0, 0)$ |
| 1 | $-8$ | $(1, -8)$ |
| 1.5 | $-10.125$ | $(1.5, -10.125)$ |
| 2 | $-10$ | $(2, -10)$ |
| 2.5 | $-6.875$ | $(2.5, -6.875)$ |
| 3 | 0 | $(3, 0)$ |
| 4 | 28 | $(4, 28)$ |

FIGURE 1.24.

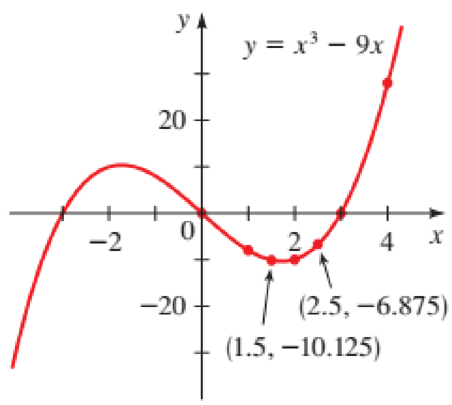



FIGURE 1.25.

**Activity 1.12.** (1) *Test the equation for symmetry.*

(a) $y = x^4 + x^2$

(b) $y = x^3 + 10x$

(c) $y = x^2 + |x|$

(d) $x^4y^4 + xy = 1$

(2) *Complete the graph using the indicated symmetry.*

(a) *Symmetry with respect to the $y$-axis.*

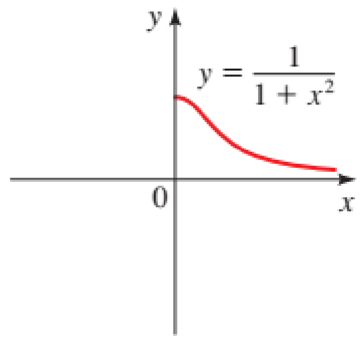



FIGURE 1.26.

(b) *Symmetry with respect to the origin.*

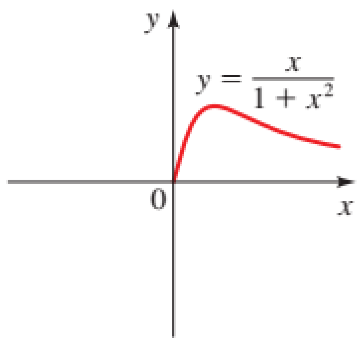



FIGURE 1.27.

## 1.7. Inequalities

There are some problems in algebra that lead to inequalities. An inequality looks just like an equation except that in the place of the equal sign is one of the symbols $<$, $>$, $\leq$ or $\geq$, e.g., $4x + 7 \leq 19$.
Solving an inequality that contains a variable means to find all values that make the inequality true. For example:

| $x$ | $4x + 7 \leq 19$ |
|---|---|
| 1 | $11 \leq 19$ ✓ |
| 2 | $15 \leq 19$ ✓ |
| 3 | $19 \leq 19$ ✓ |
| 4 | $23 \leq 19$ ✗ |
| 5 | $27 \leq 19$ ✗ |

FIGURE 1.28.

In the table, it can be seen that values of $x$ greater than 3 do not satisfy the inequality. We can use the notation for intervals or the number line to represent a solution set.

### Rules for Inequalities

(1) $A \leq B \iff A + C \leq B + C$
(2) $A \leq B \iff A - C \leq B - C$
(3) If $C > 0$, then $A \leq B \iff CA \leq CB$
(4) If $C < 0$, then $A \leq B \iff CA \geq CB$
(5) If $A > 0$ and $B > 0$, then $A \leq B \iff \frac{1}{A} \geq \frac{1}{B}$
(6) If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$

**Example 1.22.** (a) *Solve the inequality* $3x < 9x + 4$ *and sketch the solution set.*

***Solution:***

$$3x < 9x + 4$$

$$3x - 9x < 9x + 4 - 9x$$

$$-6x < 4$$

$$\left(-\frac{1}{6}\right) > \left(-\frac{1}{6}\right)(4)$$

$$x > -\frac{2}{3}$$

*The solution set is* $(-2/3, \infty)$. *We can also represent the solution on the number line as follows:*


FIGURE 1.29.

(b) *Solve the inequality* $4 \leq 3x - 2 < 13$.

***Solution:***

$$4 \leq 3x - 2 < 13$$

$$6 \leq 3x < 15$$

$$2 \leq x < 5$$

*The solution set is* $[2, 5)$. *We can also represent the solution set on the number line:*


FIGURE 1.30.

**Activity 1.13.** *Solve the inequality. Express the solution using interval notation and graph the solution set.*

(a) $3x + 11 < 5$

(b) $5 - 3x \leq -16$

(c) $2(7x - 3) \leq 12x + 16$

(d) $-3 \leq 3x + 7 \leq \frac{1}{2}$

(e) $-\frac{1}{2} \leq \frac{4-3x}{5} \leq \frac{1}{4}$

## 1.8. Nonlinear Inequalities

We look at how to find solutions for inequalities which are not linear in this section.

**1.8.1. Sign of a Product or Quotient.** If a product or a quotient has an even number of negative factors, then its value is positive. Otherwise, if it has an odd number of negative factors, then its value is negative.
For example, to solve $x^2 - 5x \leq -6$, we first move all terms to the left hand side and factor to get

$$(x - 2)(x - 3) \leq 0.$$

Since this product is supposed to be less than 0, we must determine where the product is negative. Thus, the sign of the product depends on the sign of the factors.

**Guidelines for Solving Inequalities**

(1) Move all factors to one side.

(2) Factor.

(3) Find the intervals. Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers

(4) Make a table or diagram with intervals as columns and factors as rows. Include another row for the product of the factors.

Use test values in the intervals to determine if a factor is positive or negative in the interval.

(5) Use the last row to determine whether the product is positive or negative in the intervals. In other words, find which intervals satisfy the inequality.

**Example 1.23.** (a) *Solve the inequality* $x^2 \leq 5x - 6$. ***Solution:***

– *Move all terms to one side:*

$$x^2 - 5x + 6 \leq 0$$

– *Factor:*

$$(x-2)(x-3) \leq 0$$

– *Find the intervals: The zeros are* $x = 2$ *and* $x = 3$.

$$(-\infty, 2), (2, 3), (3, \infty)$$

– *Make a table:*

| **Interval** | $(-\infty, 2)$ | $(2, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| **Sign of** $x - 2$ | $-$ | $+$ | $+$ |
| **Sign of** $x - 3$ | $-$ | $-$ | $+$ |
| **Sign of** $(x-2)(x-3)$ | $+$ | $-$ | $+$ |

FIGURE 1.31.

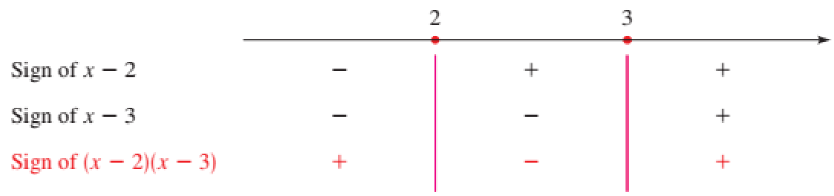



FIGURE 1.32.

– *Solve:* $(x-2)(x-3)$ *is negative on the interval* $(2,3)$. *Thus, the solution of the inequality* $(x-2)(x-3) \leq 0$ *is*

$$x|2 \leq x \leq 3 = [2,3].$$

(b) *Solve the inequality* $x(x-1)^2(x-3) < 0$.

***Solution:***

*The terms are already on one side and the expression already factored. So the zeros are* $x = 0$, $x = 1$ *and* $x = 3$. *So the intervals are* $(-\infty, 0), (0,1), (1,3), (3,\infty)$. *Thus,*

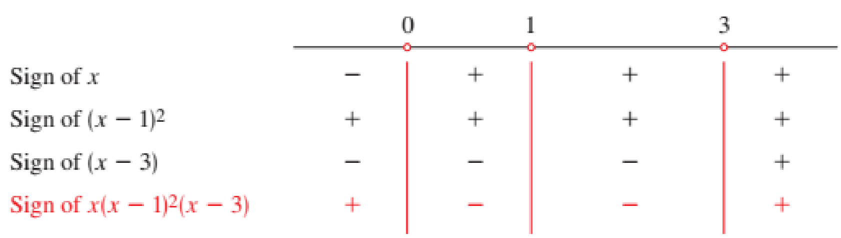



FIGURE 1.33.

*From the diagram above, we see that* $x(x-1)^2(x-3) < 0$ *for* $x$ *in the intervals* $(0,1)$ *and* $(1,3)$. *The solution set is the union of these two intervals:*

$$(0,1) \cup (1,3).$$

*We can also represent the solution on a number line as follows:*

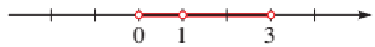



FIGURE 1.34.

(c) *Solve the inequality* $\frac{1+x}{1-x} \geq 1$. ***Solution:***

*We first move all terms to the left hand side:*

$$\frac{1+x}{1-x} \geq 1$$

$$\frac{1+x}{1-x} - 1 \geq 0$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \geq 0$$

$$\frac{1+x-1+x}{1-x} \geq 0$$

$$\frac{2x}{1-x} \geq 0$$

*The intervals are* $(-\infty, 0), (0, 1), (1, \infty)$. *Thus,*

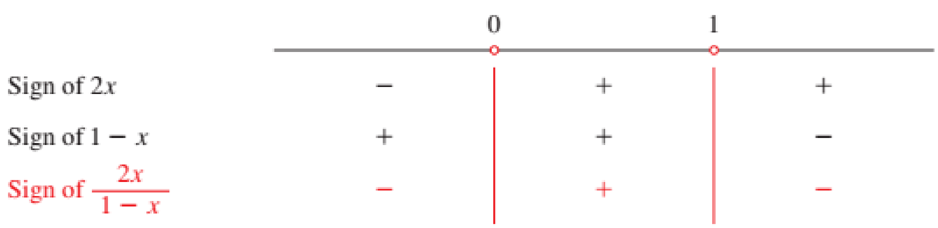



FIGURE 1.35.

*We find that the interval where* $\frac{2x}{1-x} \geq 0$ *is* $[0, 1)$.

**Activity 1.14.** *Solve the nonlinear inequality. Express the solution in interval notation and graph the solution set.*

(a) $x^2 < x + 2$

(b) $(x-2)^2(x-3)(x+1) \leq 0$

(c) $\frac{3}{x-1} - \frac{4}{x} \geq 1$

(d) $\frac{x+2}{x+3} < \frac{x-1}{x-2}$

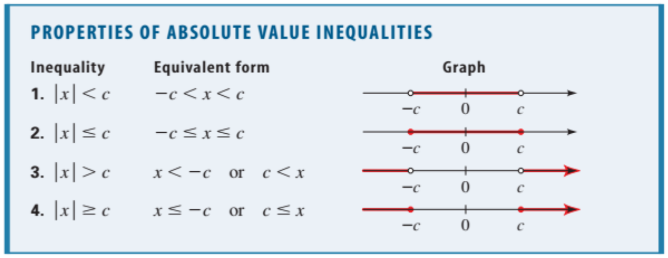



FIGURE 1.36.

### 1.8.2. Absolute Value Inequalities.

**Example 1.24.** *Solve the inequality:*

(a) $|x-5| < 2$, ***Solution:***

*This inequality is equivalent to*

$$-2 < x - 5 < 2$$

$$3 < x < 7$$

*The solution set is* $(3, 7)$.

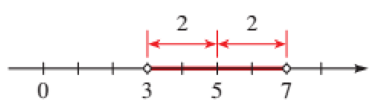



FIGURE 1.37.

(b) $|3x+2| \ge 4$, ***Solution:***

$$3x + 2 \ge 4 \quad or \quad 3x + 2 \le -4$$

$$3x \ge 2 \qquad 3x \le -6$$

$$x \ge \frac{2}{3} \qquad x \le -2$$

*The solution set is* $\{x | x \le -2 \quad or \quad x \ge \frac{2}{3}\} = (-\infty, -2] \cup [2/3, \infty)$.

**Activity 1.15.** *Solve the nonlinear inequality. Express the solution in interval notation and graph the solution set.*

(a) $|5x - 2| < 6$

(b) $\left|\frac{x-2}{3}\right| < 2$

(c) $8 - |2x - 1| \geq 6$

(d) $\left|\frac{x+1}{2}\right| \geq 4$

UNIT 2

# Sets and Functions

## 2.1. Introduction

A certain amount of mathematical maturity is necessary to find and study applications of abstract algebra. A basic knowledge of set theory, equivalence relations, and functions is a must. Even more important is the ability to read and understand mathematical proofs. In this chapter we will outline the background needed for a course in abstract algebra.

## 2.2. A Short Note on Proofs

Abstract mathematics is different from other sciences. In laboratory sciences such as chemistry and physics, scientists perform experiments to discover new principles and verify theories. Although mathematics is often motivated by physical experimentation or by computer simulations, it is made rigorous through the use of logical arguments. In studying abstract mathematics, we take what is called an axiomatic approach; that is, we take a collection of objects $\mathcal{S}$ and assume some rules about their structure. These rules are called axioms. Using the axioms for $\mathcal{S}$, we wish to derive other information about $\mathcal{S}$ by using logical arguments. We require that our axioms be consistent; that is, they should not contradict one another. We also demand that there not be too many axioms. If a system of axioms is too restrictive, there will be few examples of the mathematical structure.

A statement in logic or mathematics is an assertion that is either true or false. Consider the following examples:

- $3 + 56 - 13 + 8/2$.
- All cats are black.
- $2 + 3 = 5$.
- $2x = 6$ exactly when $x = 4$.
- If $ax^2 + bx + c = 0$ and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $x^3 - 4x^2 + 5x - 6$.

All but the first and last examples are statements, and must be either true or false.

A mathematical proof is nothing more than a convincing argument about the accuracy of a statement. Such an argument should contain enough detail to convince the audience; for instance, we can see that the statement "$2x = 6$ exactly when $x = 4$" is false by evaluating $2 \cdot 4$ and noting that $6 \neq 8$, an argument that would satisfy anyone. Of course, audiences may vary widely: proofs can be addressed to another student, to a professor, or to the reader of a text. If more detail than needed is presented in the proof, then the explanation will be either long-winded or poorly written. If too much detail is omitted, then the proof may not be convincing. Again it is important to keep the audience in mind. High school students require much more detail than do graduate students. A good rule of thumb for an argument in an introductory abstract algebra course is that it should be written to convince one's peers, whether those peers be other students or other readers of the text.

Let us examine different types of statements. A statement could be as simple as "$10/5 = 2$"; however, mathematicians are usually interested in more complex statements such as "If $p$, then $q$," where $p$ and $q$ are both statements. If certain statements are known or assumed to be true, we wish to know what we can say about other statements. Here $p$ is called the hypothesis and $q$ is known as the conclusion. Consider the following statement: If $ax^2 + bx + c = 0$ and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The hypothesis is $ax^2 + bx + c = 0$ and $a \neq 0$; the conclusion is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that the statement says nothing about whether or not the hypothesis is true. However, if this entire statement is true and we can show that $ax^2 + bx + c = 0$ with $a \neq 0$ is true, then the conclusion *must* be true. A proof of this statement might simply be a series of equations:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

If we can prove a statement true, then that statement is called a proposition. A proposition of major importance is called a theorem. Sometimes instead of proving a theorem or proposition all at once, we break the proof down into modules; that is, we prove several supporting propositions, which are called lemmas, and use the results of these propositions to prove the main result. If we can prove a proposition or a theorem, we will often, with very little effort, be able to derive other related propositions called corollaries.

**Cautions and Suggestions**

There are several different strategies for proving propositions. In addition to using different methods of proof, students often make some common mistakes when they are first learning how to prove theorems. To aid students who are studying abstract mathematics for the first time, we list here some of the difficulties that they may encounter and some of the strategies of proof available to them. It is a good idea to keep referring back to this list as a reminder. (Other techniques of proof will become apparent throughout this chapter and the remainder of the text.)

- A theorem cannot be proved by example; however, the standard way to show that a statement is not a theorem is to provide a counterexample.
- Quantifiers are important. Words and phrases such as *only*, *for all*, *for every*, and *for some* possess different meanings.
- Never assume any hypothesis that is not explicitly stated in the theorem. *You cannot take things for granted.*
- Suppose you wish to show that an object *exists* and is *unique*. First show that there actually is such an object. To show that

it is unique, assume that there are two such objects, say $r$ and $s$, and then show that $r = s$.

- Sometimes it is easier to prove the contrapositive of a statement. Proving the statement "If $p$, then $q$" is exactly the same as proving the statement "If not $q$, then not $p$."
- Although it is usually better to find a direct proof of a theorem, this task can sometimes be difficult. It may be easier to assume that the theorem that you are trying to prove is false, and to hope that in the course of your argument you are forced to make some statement that cannot possibly be true.

Remember that one of the main objectives of higher mathematics is proving theorems. Theorems are tools that make new and productive applications of mathematics possible. We use examples to give insight into existing theorems and to foster intuitions as to what new theorems might be true. Applications, examples, and proofs are tightly interconnected—much more so than they may seem at first appearance.

## 2.3. Set Theory

A set is a well-defined collection of objects; that is, it is defined in such a manner that we can determine for any given object $x$ whether or not $x$ belongs to the set. The objects that belong to a set are called its elements or members. We will denote sets by capital letters, such as $A$ or $X$; if $a$ is an element of the set $A$, we write $a \in A$.

A set is usually specified either by listing all of its elements inside a pair of braces or by stating the property that determines whether or not an object $x$ belongs to the set. We might write

$$X = \{x_1, x_2, \ldots, x_n\}$$

for a set containing elements $x_1, x_2, \ldots, x_n$ or

$$X = \{x : x \text{ satisfies } \mathcal{P}\}$$

if each $x$ in $X$ satisfies a certain property $\mathcal{P}$. For example, if $E$ is the set of even positive integers, we can describe $E$ by writing either

$$E = \{2, 4, 6, \ldots\} \quad \text{or} \quad E = \{x : x \text{ is an even integer and } x > 0\}.$$

We write $2 \in E$ when we want to say that 2 is in the set $E$, and $-3 \notin E$ to say that $-3$ is not in the set $E$.

Some of the more important sets that we will consider are the following:

$$\mathbb{N} = \{n : n \text{ is a natural number}\} = \{1, 2, 3, \ldots\};$$

$$\mathbb{Z} = \{n : n \text{ is an integer}\} = \{\ldots, -1, 0, 1, 2, \ldots\};$$

$$\mathbb{Q} = \{r : r \text{ is a rational number}\} = \{p/q : p, q \in \mathbb{Z} \text{ where } q \neq 0\};$$

$$\mathbb{R} = \{x : x \text{ is a real number}\};$$

$$\mathbb{C} = \{z : z \text{ is a complex number}\}.$$

We find various relations between sets and can perform operations on sets. A set $A$ is a subset of $B$, written $A \subset B$ or $B \supset A$, if every element of $A$ is also an element of $B$. For example,

$$\{4, 5, 8\} \subset \{2, 3, 4, 5, 6, 7, 8, 9\}$$

and

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Trivially, every set is a subset of itself. A set $B$ is a proper subset of a set $A$ if $B \subset A$ but $B \neq A$. If $A$ is not a subset of $B$, we write $A \not\subset B$; for example, $\{4, 7, 9\} \not\subset \{2, 4, 5, 8, 9\}$. Two sets are equal, written $A = B$, if we can show that $A \subset B$ and $B \subset A$.

It is convenient to have a set with no elements in it. This set is called the empty set and is denoted by $\emptyset$. Note that the empty set is a subset of every set.

To construct new sets out of old sets, we can perform certain operations: the union $A \cup B$ of two sets $A$ and $B$ is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\};$$

the intersection of $A$ and $B$ is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

If $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 9\}$, then

$$A \cup B = \{1, 2, 3, 5, 9\} \quad \text{and} \quad A \cap B = \{1, 3\}.$$

We can consider the union and the intersection of more than two sets. In this case we write

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap \ldots \cap A_n$$

for the union and intersection, respectively, of the sets $A_1, \ldots, A_n$.

When two sets have no elements in common, they are said to be disjoint; for example, if $E$ is the set of even integers and $O$ is the set of odd integers, then $E$ and $O$ are disjoint. Two sets $A$ and $B$ are disjoint exactly when $A \cap B = \emptyset$.

Sometimes we will work within one fixed set $U$, called the universal set. For any set $A \subset U$, we define the complement of $A$, denoted by $A'$, to be the set

$$A' = \{x : x \in U \text{ and } x \notin A\}.$$

We define the difference of two sets $A$ and $B$ to be

$$A \setminus B = A \cap B' = \{x : x \in A \text{ and } x \notin B\}.$$

**Example 2.1.** *operations Let $\mathbb{R}$ be the universal set and suppose that*

$$A = \{x \in \mathbb{R} : 0 < x \leq 3\} \quad \textit{and} \quad B = \{x \in \mathbb{R} : 2 \leq x < 4\}.$$

*Then*

$$\begin{aligned} A \cap B &= \{x \in \mathbb{R} : 2 \leq x \leq 3\} \\ A \cup B &= \{x \in \mathbb{R} : 0 < x < 4\} \\ A \setminus B &= \{x \in \mathbb{R} : 0 < x < 2\} \\ A' &= \{x \in \mathbb{R} : x \leq 0 \textit{ or } x > 3\}. \end{aligned}$$

**Proposition 2.1.** *Let $A$, $B$, and $C$ be sets. Then*

(1) $A \cup A = A$, $A \cap A = A$, *and* $A \setminus A = \emptyset$;
(2) $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$;
(3) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$;
(4) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$;
(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
(6) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We will prove (1) and (3) and leave the remaining results to be proven in the exercises.
(1) Observe that

$$\begin{aligned} A \cup A &= \{x : x \in A \text{ or } x \in A\} \\ &= \{x : x \in A\} \\ &= A \end{aligned}$$

and

$$\begin{aligned} A \cap A &= \{x : x \in A \text{ and } x \in A\} \\ &= \{x : x \in A\} \\ &= A. \end{aligned}$$

Also, $A \setminus A = A \cap A' = \emptyset$.

(3) For sets $A$, $B$, and $C$,

$$\begin{aligned} A \cup (B \cup C) &= A \cup \{x : x \in B \text{ or } x \in C\} \\ &= \{x : x \in A \text{ or } x \in B, \text{ or } x \in C\} \\ &= \{x : x \in A \text{ or } x \in B\} \cup C \\ &= (A \cup B) \cup C. \end{aligned}$$

A similar argument proves that $A \cap (B \cap C) = (A \cap B) \cap C$. □

**Theorem 2.1** (De Morgan's Laws)**.** *Let $A$ and $B$ be sets. Then*

(1) $(A \cup B)' = A' \cap B'$*;*

(2) $(A \cap B)' = A' \cup B'$.

Proof. (1) We must show that $(A \cup B)' \subset A' \cap B'$ and $(A \cup B)' \supset A' \cap B'$. Let $x \in (A \cup B)'$. Then $x \notin A \cup B$. So $x$ is neither in $A$ nor in $B$, by the definition of the union of sets. By the definition of the complement, $x \in A'$ and $x \in B'$. Therefore, $x \in A' \cap B'$ and we have $(A \cup B)' \subset A' \cap B'$.

To show the reverse inclusion, suppose that $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$, and so $x \notin A$ and $x \notin B$. Thus $x \notin A \cup B$ and so $x \in (A \cup B)'$. Hence, $(A \cup B)' \supset A' \cap B'$ and so $(A \cup B)' = A' \cap B'$.

The proof of (2) is left as an exercise. □

**Example 2.2.** *other relations Other relations between sets often hold true. For example,*

$$(A \setminus B) \cap (B \setminus A) = \emptyset.$$

*To see that this is true, observe that*

$$\begin{aligned}(A \setminus B) \cap (B \setminus A) &= (A \cap B') \cap (B \cap A') \\ &= A \cap A' \cap B \cap B' \\ &= \emptyset.\end{aligned}$$

## 2.4. Cartesian Products and Mappings

Given sets $A$ and $B$, we can define a new set $A \times B$, called the Cartesian product of $A$ and $B$, as a set of ordered pairs. That is,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

**Example 2.3.** *cartesian products If $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \emptyset$, then $A \times B$ is the set*

$$\{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$

*and*

$$A \times C = \emptyset.$$

We define the Cartesian product of $n$ sets to be

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i \text{ for } i = 1, \ldots, n\}.$$

If $A = A_1 = A_2 = \cdots = A_n$, we often write $A^n$ for $A \times \cdots \times A$ (where $A$ would be written $n$ times). For example, the set $\mathbb{R}^3$ consists of all of 3-tuples of real numbers.

Subsets of $A \times B$ are called relations. We will define a mapping or function $f \subset A \times B$ from a set $A$ to a set $B$ to be the special type of relation in which for each element $a \in A$ there is a unique element $b \in B$ such that $(a, b) \in f$; another way of saying this is that for every element in $A$, $f$ assigns a unique element in $B$. We usually write $f : A \to B$ or $A \xrightarrow{f} B$. Instead of writing down ordered pairs $(a, b) \in A \times B$, we write $f(a) = b$ or $f : a \mapsto b$. The set $A$ is called the domain of $f$ and

$$f(A) = \{f(a) : a \in A\} \subset B$$

is called the range or image of $f$. We can think of the elements in the function's domain as input values and the elements in the function's range as output values.

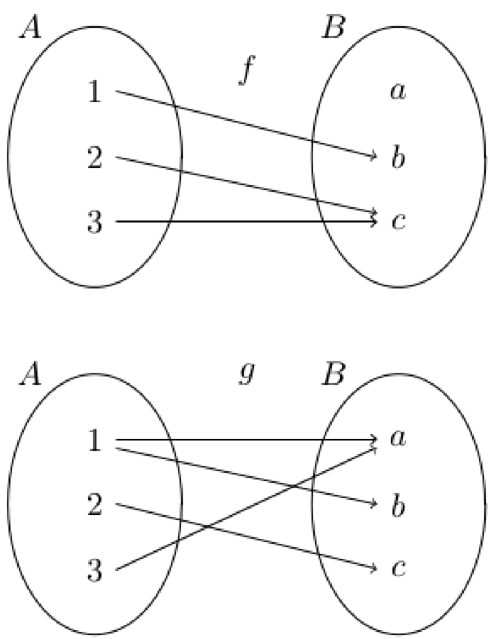



FIGURE 2.1. Mappings

**Example 2.4.** *Suppose $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. In Figure 2.1 we define relations $f$ and $g$ from $A$ to $B$. The relation $f$ is a mapping,* *but $g$ is not because $1 \in A$ is not assigned to a unique element in $B$; that is, $g(1) = a$ and $g(1) = b$.*

Given a function $f : A \to B$, it is often possible to write a list describing what the function does to each specific element in the domain. However, not all functions can be described in this manner. For example, the function $f : \mathbb{R} \to \mathbb{R}$ that sends each real number to its cube is a mapping that must be described by writing $f(x) = x^3$ or $f : x \mapsto x^3$. Consider the relation $f : \mathbb{Q} \to \mathbb{Z}$ given by $f(p/q) = p$. We know that $1/2 = 2/4$, but is $f(1/2) = 1$ or 2? This relation cannot be a mapping because it is not well-defined. A relation is well-defined if each element in the domain is assigned to a unique element in the range.
If $f : A \to B$ is a map and the image of $f$ is $B$, i.e., $f(A) = B$, then $f$ is said to be onto or surjective . In other words, if there exists an $a \in A$ for each $b \in B$ such that $f(a) = b$, then $f$ is onto. A map is one-to-one or injective if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. Equivalently, a function is one-to-one if $f(a_1) = f(a_2)$ implies $a_1 = a_2$. A map that is both one-to-one and onto is called bijective .

**Example 2.5.** *one to one onto Let $f : \mathbb{Z} \to \mathbb{Q}$ be defined by $f(n) = n/1$. Then $f$ is one-to-one but not onto. Define $g : \mathbb{Q} \to \mathbb{Z}$ by $g(p/q) = p$ where $p/q$ is a rational number expressed in its lowest terms with a positive denominator. The function $g$ is onto but not one-to-one.*

Given two functions, we can construct a new function by using the range of the first function as the domain of the second function. Let $f : A \to B$ and $g : B \to C$ be mappings. Define a new map, the composition of $f$ and $g$ from $A$ to $C$, by $(g \circ f)(x) = g(f(x))$.

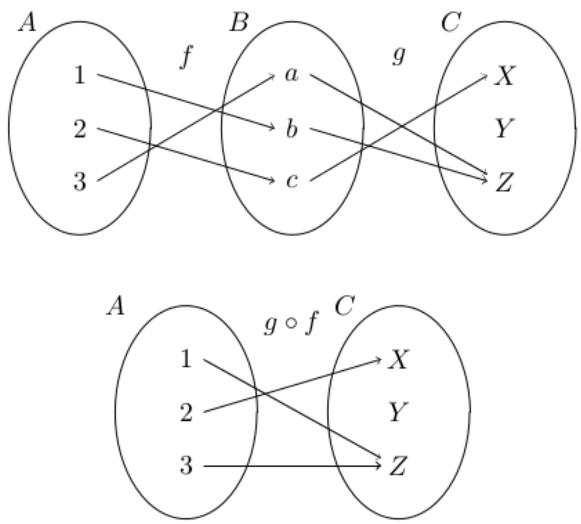



Figure 2.2. Composition of maps

**Example 2.6.** *Consider the functions $f : A \to B$ and $g : B \to C$ that are defined in Figure 2.2(a). The composition of these functions, $g \circ f : A \to C$, is defined in Figure 2.2(b).*

**Example 2.7.** *Let $f(x) = x^2$ and $g(x) = 2x + 5$. Then*

$$(f \circ g)(x) = f(g(x)) = (2x + 5)^2 = 4x^2 + 20x + 25$$

*and*

$$(g \circ f)(x) = g(f(x)) = 2x^2 + 5.$$

*In general, order makes a difference; that is, in most cases $f \circ g \neq g \circ f$.*

**Example 2.8.** *Sometimes it is the case that $f \circ g = g \circ f$. Let $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. Then*

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}\,) = (\sqrt[3]{x}\,)^3 = x$$

*and*

$$(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x.$$

**Example 2.9.** *Given a* $2 \times 2$ *matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*we can define a map* $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T_A(x, y) = (ax + by, cx + dy)$$

*for* $(x, y)$ *in* $\mathbb{R}^2$*. This is actually matrix multiplication; that is,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

*Maps from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *given by matrices are called linear maps or linear transformations .*

**Example 2.10.** *Suppose that* $S = \{1, 2, 3\}$*. Define a map* $\pi : S \to S$ *by*

$$\pi(1) = 2, \qquad \pi(2) = 1, \qquad \pi(3) = 3.$$

*This is a bijective map. An alternative way to write* $\pi$ *is*

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

*For any set* $S$*, a one-to-one and onto mapping* $\pi : S \to S$ *is called a permutation of* $S$*.*

**Theorem 2.2.** *Let* $f : A \to B$*,* $g : B \to C$*, and* $h : C \to D$*. Then*

(1) *The composition of mappings is associative; that is,* $(h \circ g) \circ f = h \circ (g \circ f)$*;*

(2) *If* $f$ *and* $g$ *are both one-to-one, then the mapping* $g \circ f$ *is one-to-one;*

(3) *If* $f$ *and* $g$ *are both onto, then the mapping* $g \circ f$ *is onto;*

(4) *If* $f$ *and* $g$ *are bijective, then so is* $g \circ f$*.*

Proof. We will prove (1) and (3). Part (2) is left as an exercise. Part (4) follows directly from (2) and (3).
(1) We must show that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

For $a \in A$ we have

$$\begin{aligned}(h \circ (g \circ f))(a) &= h((g \circ f)(a)) \\ &= h(g(f(a))) \\ &= (h \circ g)(f(a)) \\ &= ((h \circ g) \circ f)(a).\end{aligned}$$

(3) Assume that $f$ and $g$ are both onto functions. Given $c \in C$, we must show that there exists an $a \in A$ such that $(g \circ f)(a) = g(f(a)) = c$. However, since $g$ is onto, there is a $b \in B$ such that $g(b) = c$. Similarly, there is an $a \in A$ such that $f(a) = b$. Accordingly,

$$(g \circ f)(a) = g(f(a)) = g(b) = c.$$

□

If $S$ is any set, we will use $id_S$ or $id$ to denote the identity mapping from $S$ to itself. Define this map by $id(s) = s$ for all $s \in S$. A map $g : B \to A$ is an inverse mapping of $f : A \to B$ if $g \circ f = id_A$ and $f \circ g = id_B$; in other words, the inverse function of a function simply "undoes" the function. A map is said to be invertible if it has an inverse. We usually write $f^{-1}$ for the inverse of $f$.

**Example 2.11.** *The function $f(x) = x^3$ has inverse $f^{-1}(x) = \sqrt[3]{x}$.*

**Example 2.12.** *The natural logarithm and the exponential functions, $f(x) = \ln x$ and $f^{-1}(x) = e^x$, are inverses of each other provided that we are careful about choosing domains. Observe that*

$$f(f^{-1}(x)) = f(e^x) = \ln e^x = x$$

*and*

$$f^{-1}(f(x)) = f^{-1}(\ln x) = e^{\ln x} = x$$

*whenever composition makes sense.*

**Example 2.13.** *inverse matrix Suppose that*

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}.$$

*Then $A$ defines a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ by*

$$T_A(x, y) = (3x + y, 5x + 2y).$$

*We can find an inverse map of $T_A$ by simply inverting the matrix $A$; that is, $T_A^{-1} = T_{A^{-1}}$. In this example,*

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix};$$

*hence, the inverse map is given by*

$$T_A^{-1}(x, y) = (2x - y, -5x + 3y).$$

*It is easy to check that*

$$T_A^{-1} \circ T_A(x, y) = T_A \circ T_A^{-1}(x, y) = (x, y).$$

*Not every map has an inverse. If we consider the map*

$$T_B(x, y) = (3x, 0)$$

*given by the matrix*

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix},$$

*then an inverse map would have to be of the form*

$$T_B^{-1}(x, y) = (ax + by, cx + dy)$$

*and*

$$(x, y) = T \circ T_B^{-1}(x, y) = (3ax + 3by, 0)$$

*for all $x$ and $y$. Clearly this is impossible because $y$ might not be 0.*

**Example 2.14.** *Given the permutation*

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

*on $S = \{1, 2, 3\}$, it is easy to see that the permutation defined by*

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

*is the inverse of $\pi$. In fact, any bijective mapping possesses an inverse, as we will see in the next theorem.*

**Theorem 2.3.** *A mapping is invertible if and only if it is both one-to-one and onto.*

Proof. Suppose first that $f : A \to B$ is invertible with inverse $g : B \to A$. Then $g \circ f = id_A$ is the identity map; that is, $g(f(a)) = a$. If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Consequently, $f$ is one-to-one. Now suppose that $b \in B$. To show that $f$ is onto, it is necessary to find an $a \in A$ such that $f(a) = b$, but $f(g(b)) = b$ with $g(b) \in A$. Let $a = g(b)$.

Now assume the converse; that is, let $f$ be bijective. Let $b \in B$. Since $f$ is onto, there exists an $a \in A$ such that $f(a) = b$. Because $f$ is one-to-one, $a$ must be unique. Define $g$ by letting $g(b) = a$. We have now constructed the inverse of $f$. $\square$

### 2.5. Equivalence Relations and Partitions

A fundamental notion in mathematics is that of equality. We can generalize equality with the introduction of equivalence relations and equivalence classes. An equivalence relation on a set $X$ is a relation $R \subset X \times X$ such that

- $(x, x) \in R$ for all $x \in X$ (reflexive property);
- $(x, y) \in R$ implies $(y, x) \in R$ ( symmetric property);
- $(x, y)$ and $(y, z) \in R$ imply $(x, z) \in R$ ( transitive property).

Given an equivalence relation $R$ on a set $X$, we usually write $x \sim y$ instead of $(x, y) \in R$. If the equivalence relation already has an associated notation such as $=$, $\equiv$, or $\cong$, we will use that notation.

**Example 2.15.** *Let $p$, $q$, $r$, and $s$ be integers, where $q$ and $s$ are nonzero. Define $p/q \sim r/s$ if $ps = qr$. Clearly $\sim$ is reflexive and symmetric. To show that it is also transitive, suppose that $p/q \sim r/s$ and $r/s \sim t/u$, with $q$, $s$, and $u$ all nonzero. Then $ps = qr$ and $ru = st$. Therefore,*

$$psu = qru = qst.$$

*Since $s \neq 0$, $pu = qt$. Consequently, $p/q \sim t/u$.*

## Exercises

(1) Suppose that

$$A = \{x : x \in \mathbb{N} \text{ and } x \text{ is even}\},$$

$$B = \{x : x \in \mathbb{N} \text{ and } x \text{ is prime}\},$$

$$C = \{x : x \in \mathbb{N} \text{ and } x \text{ is a multiple of } 5\}.$$

Describe each of the following sets.

(a) $A \cap B$
(b) $B \cap C$
(c) $A \cup B$
(d) $A \cap (B \cup C)$

(2) If $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{x\}$, and $D = \emptyset$, list all of the elements in each of the following sets.

(a) $A \times B$
(b) $B \times A$
(c) $A \times B \times C$
(d) $A \times D$

(3) Find an example of two nonempty sets $A$ and $B$ for which $A \times B = B \times A$ is true.

(4) Prove $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(5) Prove $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(6) Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(7) Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(8) Prove $A \subset B$ if and only if $A \cap B = A$.

(9) Prove $(A \cap B)' = A' \cup B'$.

(10) Prove $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$.

(11) Prove $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

(12) Prove $(A \cap B) \setminus B = \emptyset$.

(13) Prove $(A \cup B) \setminus B = A \setminus B$.

(14) Prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(15) Prove $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

(16) Prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(17) Which of the following relations $f : \mathbb{Q} \to \mathbb{Q}$ define a mapping? In each case, supply a reason why $f$ is or is not a mapping.

(a) $f(p/q) = \dfrac{p+1}{p-2}$

(b) $f(p/q) = \dfrac{3p}{3q}$

(c) $f(p/q) = \dfrac{p+q}{q^2}$

(d) $f(p/q) = \dfrac{3p^2}{7q^2} - \dfrac{p}{q}$

(18) Determine which of the following functions are one-to-one and which are onto. If the function is not onto, determine its range.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + 3$

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$

(d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$

(19) Let $f : A \to B$ and $g : B \to C$ be invertible mappings; that is, mappings such that $f^{-1}$ and $g^{-1}$ exist. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(20) (a) Define a function $f : \mathbb{N} \to \mathbb{N}$ that is one-to-one but not onto.

(b) Define a function $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.

(a) If $f$ and $g$ are both one-to-one functions, show that $g \circ f$ is one-to-one.

(b) If $g \circ f$ is onto, show that $g$ is onto.

(c) If $g \circ f$ is one-to-one, show that $f$ is one-to-one.

(d) If $g \circ f$ is one-to-one and $f$ is onto, show that $g$ is one-to-one.

(e) If $g \circ f$ is onto and $g$ is one-to-one, show that $f$ is onto.

(21) Define a function on the real numbers by

$$f(x) = \frac{x+1}{x-1}.$$

What are the domain and range of $f$? What is the inverse of $f$? Compute $f \circ f^{-1}$ and $f^{-1} \circ f$.

(22) Let $f : X \to Y$ be a map with $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$.

(a) Prove $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(b) Prove $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Give an example in which equality fails.

(c) Prove $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$, where

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

(d) Prove $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(e) Prove $f^{-1}(Y \setminus B_1) = X \setminus f^{-1}(B_1)$.

UNIT 3

# POLYNOMIALS AND RATIONAL FUNCTIONS

**Definition 3.1.** *A polynomial function is a function defined by a polynomial expression. A polynomial function of degree $n$ is a function of the form*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Polynomial functions of degree 0 and 1 are functions of the form $P(x) = a_0$ and $P(x) = a_1 x + a_0$, respectively. These functions are linear.
As the degree of a polynomial function increases, the shape of its graph changes.

## 3.1. Quadratic Functions and Models

**Definition 3.2.** *A quadratic function is a polynomial function of degree 2 and is of the form*

$$P(x) = ax^2 + bx + c.$$

*A quadratic function can be expressed in standard form*

$$f(x) = a(x - h)^2 + k$$

*by completing the square.*

If we take $a = 1$ and $b = c = 0$, then $f(x) = x^2$, whose graph is a parabola. So any graph of a quadratic function is a parabola and can be obtained from the graph of $f(x) = x^2$ by transformations.
Let $f(x) = ax^2 + bx + c$ whose standard form is $f(x) = a(x - h)^2 + k$. The graph of $f$ is a parabola with vertex $(h, k)$. The parabola opens upward or is cupped up if $a > 0$ and opens down or cupped down if $a < 0$.

**Example 3.1.** *Let $f(x) = 2x^2 - 12x + 23$.*

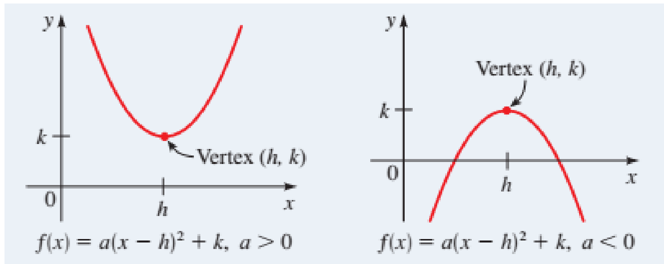



FIGURE 3.1.

(a) *Express $f$ in standard form.*

***Solution:***

*We need to complete the square:*

$$
\begin{aligned}
f(x) &= 2x^2 - 12x + 23 \\
&= 2(x^2 - 6x) + 23 \\
&= 2(x^2 - 6x + 9) + 23 - 2(9) \\
&= 2(x-3)^2 + 5
\end{aligned}
$$

(a) *Sketch the graph of $f$.*

***Solution:***

*We have to follow the steps below to sketch the graph of $f$.*

- *Take the parabola $y = x^2$.*
- *Shift it to the right 3 units.*
- *Stretch it by a factor of 2.*
- *Move it upward 5 units.*

*What we get is the following:*

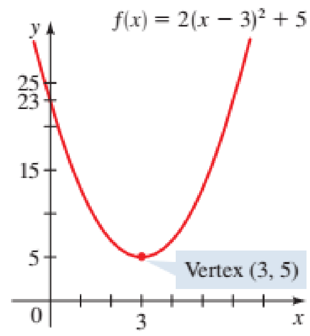



FIGURE 3.2.

Let's look at another example.

**Example 3.2.** *Sketch the graph of $f(x) = -x^2 + 6x - 8$ and identify the vertex and $x$- intercepts.*

***Solution:***

*To identify the vertex, we need to complete the square.*

$$\begin{aligned} f(x) &= -x^2 + 6x - 8 \\ &= -1(x^2 - 6x) - 8 \\ &= -1(x^2 - 6x + 9) - 8 - (-1)(9) \\ &= -1(x-3)^2 + 1 \end{aligned}$$

*The vertex is $(3, 1)$. Since $a < 0$, the parabola opens downward. The $x$-intercepts are found by equating $f(x)$ to zero.*

$$\begin{aligned} -x^2 + 6x - 8 &= 0 \\ -(x-2)(x-4) &= 0 \\ x - 2 = 0 &\implies x = 2 \\ x - 4 = 0 &\implies x = 4 \end{aligned}$$

*So the $x$-intercepts are $(2, 0)$ and $(4, 0)$. The graph is shown in the figure below:*

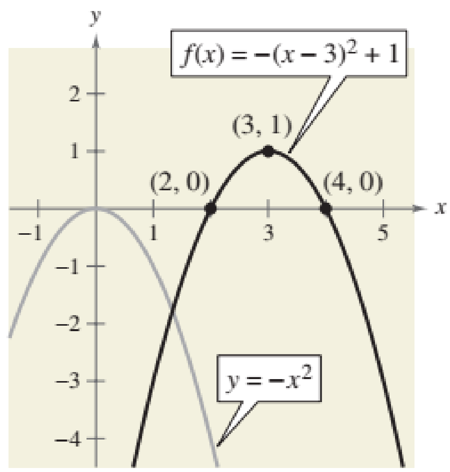



FIGURE 3.3.

**Activity 3.1.** *A quadratic function is given. (a) Express the quadratic function in standard form. (b) Find its vertex and its* $x$*- and* $y$*-intercept(s). (c) Sketch its graph.*

(1) $f(x) = x^2 - 6x$

(2) $f(x) = -x^2 + 6x + 4$

(3) $f(x) = -3x^2 + 6x - 2$

(4) $f(x) = 6x^2 + 12x - 5$

### 3.1.1. Maximum and Minimum Values of Quadratic Functions.

**Definition 3.3.** *Let $f$ be a quadratic function with standard form $f(x) = a(x-h)^2 + k$. The maximum or minimum value of $f$ occurs at $x = h$. If $a > 0$, $f(h) = k$ is the minimum value of $f$. If $a < 0$, $f(h) = k$ is the maximum value of $f$.*

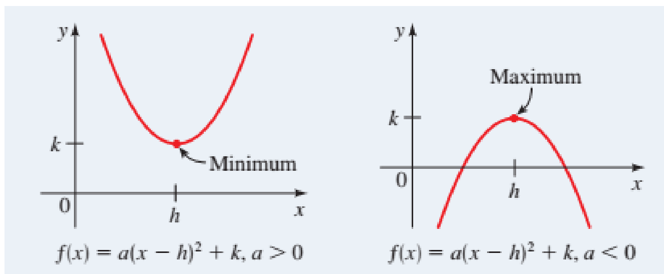



FIGURE 3.4.

The following example is about the minimum value of a function.

**Example 3.3.** *Let $f(x) = 5x^2 - 30x + 49$.*

(a) *Express $f$ in standard form.*

***Solution:***

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 \\ &= 5(x^2 - 6x + 9) + 49 - 5(9) \\ &= 5(x-3)^2 + 4 \end{aligned}$$

(b) *Sketch the graph of $f$.*

***Solution:***

– *Take the parabola $y = x^2$.*
– *Shift it to the right 3 units.*
– *Stretch it by a factor of 5.*

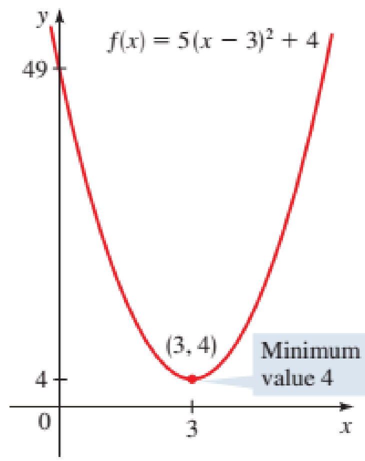



FIGURE 3.5.

– *Move it upward 4 units.*

(c) *Find the minimum value of $f$.*

***Solution:***

*Since the coefficient of $x^2$ is positive, $f$ has a minimum value. The minimum value is $f(3) = 4$.*

We give another example for a maximum value of a function.

**Example 3.4.** *Let $f(x) = -x^2 + x + 2$.*

(a) *Express $f$ in standard form.*

***Solution:***

$$\begin{aligned} f(x) &= -x^2 + x + 2 \\ &= -1(x^2 - x) + 2 \\ &= -1(x^2 - x + \frac{1}{4}) + 2 - (-1)\left(\frac{1}{4}\right) \\ &= -1(x - \frac{1}{2})^2 + \frac{9}{4} \end{aligned}$$

(b) *Sketch the graph of $f$.*

***Solution:***

*Since $a < 0$, the graph opens downward. To sketch the graph, we follow the steps below:*

- *Take the graph $y = x^2$.*
- *Shift to the right $\frac{1}{2}$ units.*
- *Stretch by a factor of 1.*
- *Move up $\frac{9}{4}$ units.*
- *We find the intercepts. It is clear that the $y$-intercept is $f(0) = 2$. The $x$-intercepts are are found by setting $f(x) = 0$. Thus,*

$$-x^2 + x + 2 = 0$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x - 1) = 0$$

*So the $x$-intercepts are $x = 2$ and $x = 1$.*

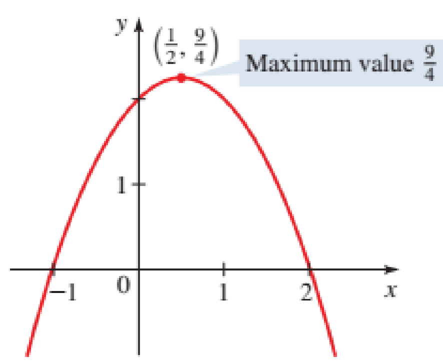



FIGURE 3.6.

(c) *Find the maximum value of $f$.*

***Solution:***

*The maximum value of $f$ is $f\left(\frac{1}{2}\right) = \frac{9}{4}$.*

If we are interested in just finding the minimum or maximum value, there is a formula for that and it is obtained by completing the square the function

$f(x) = ax^2 + bx + c$. Thus,

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b^2}{4a}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

So, $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$.

Hence, the maximum or the minimum value of a quadratic function occurs at

$$x = -\frac{b}{2a}.$$

**Example 3.5.** *Find the maximum or minimum value of each function.*

(a) $f(x) = x^2 + 4x$.

***Solution:***

*All we need are values of $a$ and $b$ as in $f(x) = ax^2 + bx + c$. So, $a = 1$ and $b = 4$. Since $a > 0$, the function has a minimum value. Then the minimum value of $f$ occurs at*

$$x = -\frac{b}{2a} = -\frac{4}{2(1)} = -2.$$

*Hence, the minimum value of $f$ is $f(-2) = (-2)^2 + 4(-2) = -4$.*

(b) $f(x) = -2x^2 + 4x - 5$.

***Solution:***

*We have $a = -2$ and $b = 4$. Since $a < 0$, $f$ has a maximum value at*

$$x = -\frac{b}{2a} = -\frac{4}{2(-2)} = 1.$$

*Hence, the maximum value is $f(1) = -2(1)^2 + 4(1) - 5 = -3$.*

**Activity 3.2.** *A quadratic function is given. (a) Express the quadratic function in standard form. (b) Sketch its graph. (c) Find its maximum or minimum value.*

(1) $f(x) = x^2 - 8x + 8$
(2) $f(x) = -x^2 - 3x + 3$
(3) $f(x) = 1 - 6x - x^2$
(4) $f(x) = 3 - x - \frac{1}{2}x^2$

**3.1.2. Modeling with Quadratic Functions.** We study some examples of real world phenomena that are modeled by quadratic functions.

**Example 3.6.** *Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage $M$ for a certain new car is modeled by the function*

$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \quad 15 \leq s \leq 70$$

*where $s$ is the speed in mi/h and $M$ is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?*

***Solution:***

*What we need is the maximum value of $M$. We have $a = -\frac{1}{28}$ and $b = 3$. The maximum value of the function $M$ occurs at*

$$s = -\frac{b}{2a} = -\frac{3}{2(-1/28)} = 42.$$

*The maximum value of the function is*

$$M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32.$$

*Therefore, the car's best mileage is* 32 *mi/gal when it is travelling at* 42 *mi/h.*

**Activity 3.3.** (1) *If a ball is thrown directly upward with a velocity of 40 ft/s, its height (in feet) after $t$ seconds is given by $y = 40t - 16t^2$. What is the maximum height attained by the ball?*

(2) *A manufacturer finds that the revenue generated by selling $x$ units of a certain commodity is given by the function $R(x) = 80x - 0.4x^2$, where the revenue $R(x)$ is measured in dollars. What is the maximum revenue, and how many units should be manufactured to obtain this maximum?*

(3) *The effectiveness of a television commercial depends on how many times a viewer watches it. After some experiments an advertising agency found that if the effectiveness $E$ is measured on a scale of 0 to 10, then*

$$E(n) = \frac{2}{3}n - \frac{1}{90}n^2$$

*where $n$ is the number of times a viewer watches a given commercial. For a commercial to have maximum effectiveness, how many times should a viewer watch it?*

(4) *A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost \$6, and the society sells an average of 20 per week at a price of \$10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it loses 2 sales per week.*

   (a) *Find a function that models weekly profit in terms of price per feeder.*

   (b) *What price should the society charge for each feeder to maximize profits? What is the maximum weekly profit?*

### 3.2. Polynomial Functions and their Graphs

We know that a polynomial function of degree $n$ is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a non-negative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called **coefficients** of the polynomial. In the polynomial, $a_0$ is known as the **constant coefficient** or **constant term**, $a_n$ is known as the **leading coefficient** and $a_n x^n$ is known as the **leading term**.

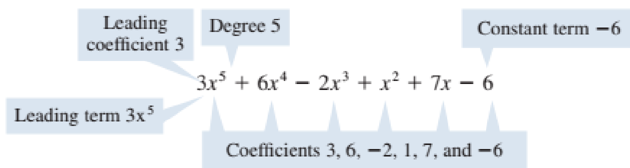



FIGURE 3.7.

**Example 3.7.**

If a polynomial consists of a single term, it is known as a **monomial**. Examples are $f(x) = x^3$, $f(x) = x^5$. In general, any polynomial of the form $f(x) = ax^n$ where $a$ and $n$ are integers is a monomial.

**3.2.1. Graphing Basic Polynomial Functions.** We know that graphs of polynomials of degree 0 and 1 are lines and those of degree 2 are parabolas. The greater the degree of a polynomial, the more complicated its graph can be. However, the graph of a polynomial function is continuous. The graph has no breaks or holes. It is a smooth curve. Check the figure below:

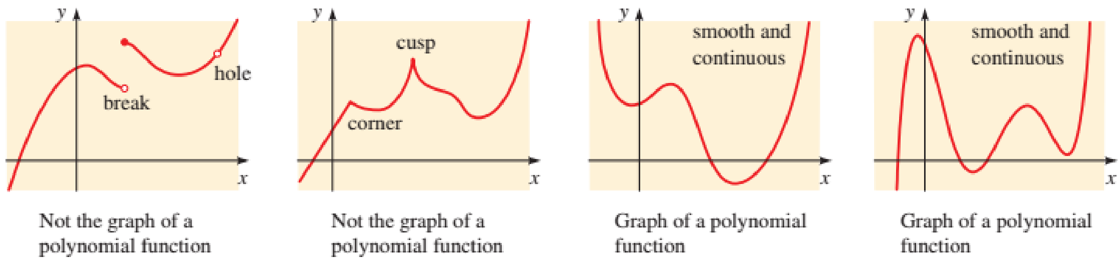



FIGURE 3.8.

The simplest polynomial functions are the monomials $P(x) = x^n$. Some of their graphs are shown in the figure below:

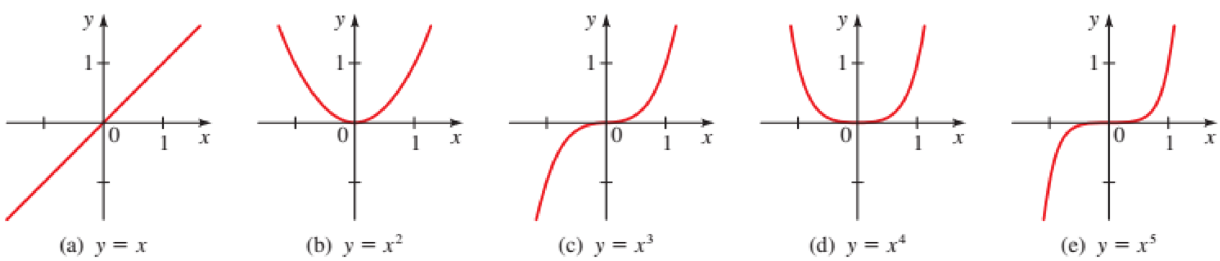



FIGURE 3.9.

As the figure shows, when $n$ is even, the shape of the graph is similar to the shape of the graph of $f(x) = x^2$, and when $n$ is odd, the shape of the graph is similar to the shape of $f(x) = x^3$.

The following are examples of some graphs of polynomials with degree greater than 2.

**Example 3.8.** *Sketch the graphs of the following functions:*

(a) $P(x) = -x^3$,

***Solution:***

*The graph of* $P(x) = -x^3$ *is just the reflection of the graph of* $y = x^3$ *in the* $x$*-axis.*

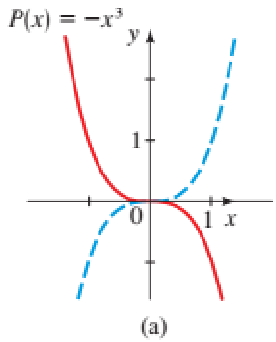



FIGURE 3.10.

(b) $Q(x) = (x-2)^4$,

***Solution:***

*This is just the graph of $y = x^4$ shifted to the right 2 units.*

(c) $R(x) = -2x^5 + 4$.

***Solution:***

*We start with the graph of $y = x^5$. The negative in the first term of $R(x) = -2x^5 + 4$ means we reflect the graph of $y = x^5$ on the $x$-axis, the 2 in that first term means we stretch the graph vertically 2 units, and lastly, the $+4$ means we shift the graph 4 units upwards. See the figure below.*

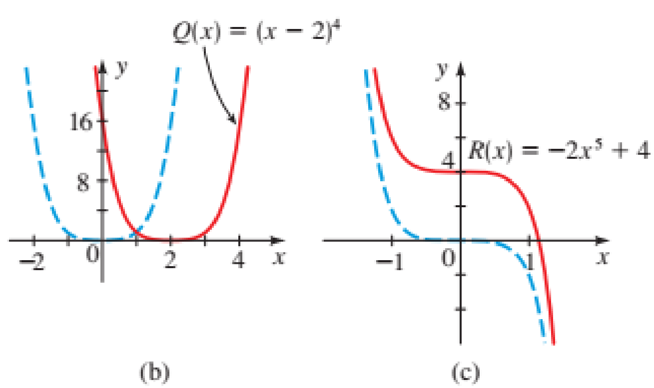



FIGURE 3.11.

**Activity 3.4.** *Sketch the graphs of the following functions.*

(1) $P(x) = x^2 - 4$

(2) $Q(x) = (x-4)^2$

(3) $P(x) = x^4 - 16$

(4) $Q(x) = -2(X+2)^4$

**3.2.2. End-Behaviour of the Leading Term.** The **end-behaviour** is just the description of what happens as $x$ becomes large in the positive or negative direction, i.e.,

- $x \to \infty$ means "$x$ becomes large in the positive direction,
- $x \to \infty$ means "$x$ becomes large in the negative direction.

**Example 3.9.** *Look at the function $y = x^2$. No matter the direction that the value of $x$ gets large, the value of $y$ is always positive. Thus as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to \infty$.*

The end-behaviour is determined by the term that contains the highest power of $x$ because when $x$ is large, the other terms are relatively insignificant in size. Below are the four possible types of end-behaviours:

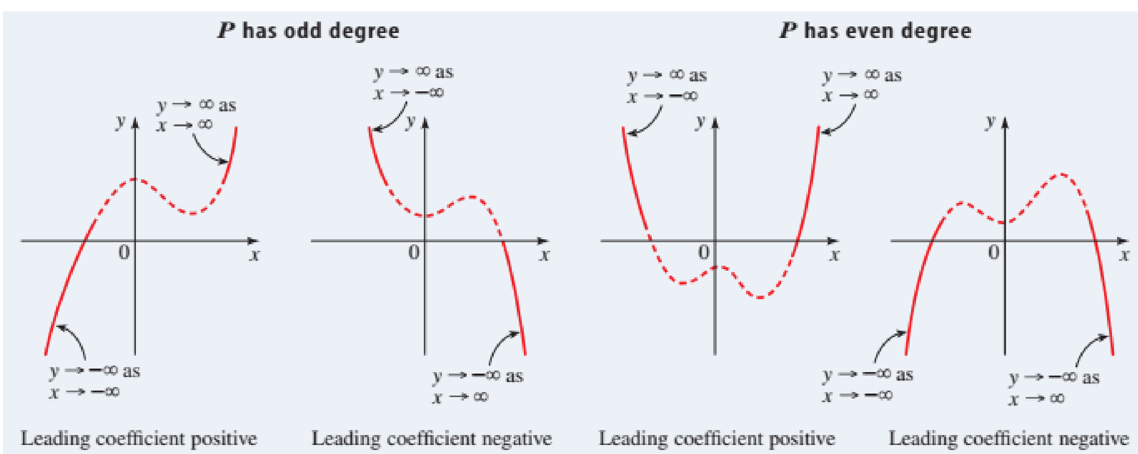



FIGURE 3.12. End-behaviour possible types

**Example 3.10.** *Determine the end-behaviour of the following polynomials:*

(a) $P(x) = -2x^4 + 5x^3 + 4x - 7$,

***Solution:***

*$P$ has degree 4 and leading coefficient $-2$. Thus, $P$ has even degree and negative leading coefficient. So, $y \to -\infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.*

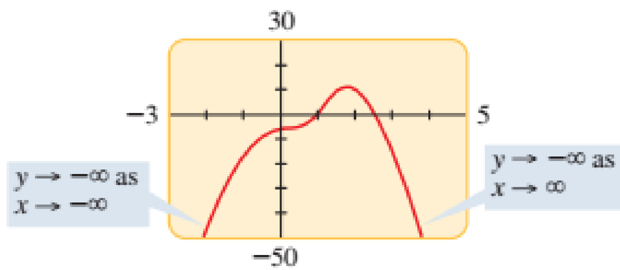



FIGURE 3.13.

(b) $P(x) = 3x^5 - 5x^3 + 2x$,

***Solution:***

*The leading coefficient is positive and the leading term has an odd degree 5. Then the end behaviour of the function is: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.*

**Activity 3.5.** *Determine the end behavior of the following functions.*

(1) $P(x) = 3x^3 - x^2 + 5x + 1$

(2) $Q(x) = x^4 - 7x^2 + 5x + 5$

(3) $P(x) = -x^5 + 2x^2 + x$

(4) $Q(x) = x^{11} - x^9$

**3.2.3. Using Zeros to Graph Polynomials.** If $P$ is a polynomial, then $c$ is called a **zero** if $P(c) = 0$. In other words, zeros of a polynomial are solutions to the equation $P(x) = 0$. These zeros are the $x$-intercepts of the polynomial.

**Example 3.11** (Real Zeros of a Polynomial)**.** *If $P$ is a polynomial and $c$ is a real number, then the following are equivalent:*

1. *$c$ is a zero of $P$;*

- *$x = c$ is a solution of the equation $P(x) = 0$;*
- *$x - c$ is a factor of $P(x)$;*
- *$c$ is an $x$-intercept of the graph of $P$.*

**Theorem 3.1** (Intermediate Value Theorem for Polynomials)**.** *If $P$ is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $P(c) = 0$.*

The following are the guidelines for graphing polynomial functions:

**GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS**

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the $x$-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the $x$-axis on the intervals determined by the zeros. Include the $y$-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

FIGURE 3.14.

**Example 3.12.** *Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.*

***Solution:***

- *The zeros are $-2$, $1$ and $3$.*
- *These zeros provide the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$.*
- *It is clear that $P(x)$ has odd degree. Thus, the end behavior is that $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.*

- *We get test points in these intervals to get information in the sign diagram below.*

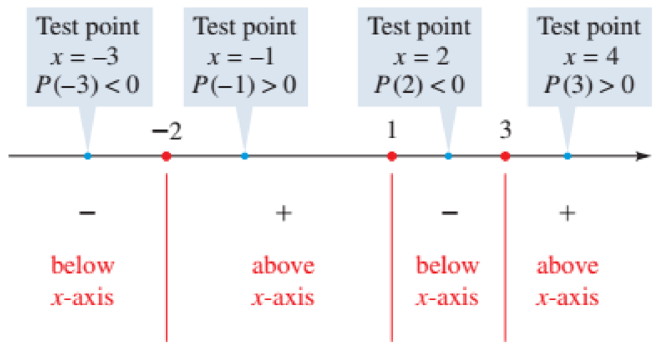



FIGURE 3.15.

*Plotting a few additional points and connecting them with a smooth curve helps us to complete the graph as below.*

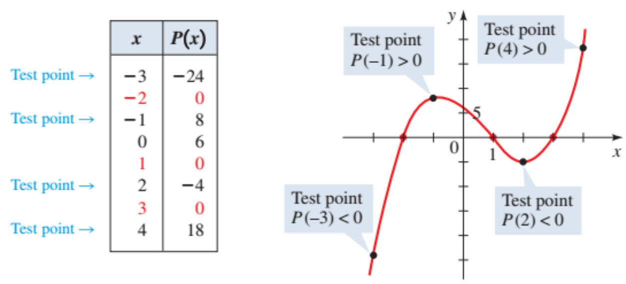



FIGURE 3.16.

**Example 3.13.** *Let* $P(x) = x^3 - 2x^2 - 3x$.

(a) *Find zeros of* $P$.

***Solution:***

*We need to factor* $P(x)$ *completely.*

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) \\ &= x(x-3)(x+1) \end{aligned}$$

*So the zeros are $x = 0$, $x = 3$ and $x = -1$.*

(b) *Sketch the graph of $P$.*

***Solution:***

*From the zeros, we come up with the following intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 3)$ and $(3, \infty)$. Since $P$ has odd degree, the end behavior is as follows: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$. With plotting a few more points and connecting them by a smooth curve, we get the following graph:*

| | $x$ | $P(x)$ |
|---|---|---|
| Test point → | $-2$ | $-10$ |
| | $-1$ | $0$ |
| Test point → | $-\frac{1}{2}$ | $\frac{7}{8}$ |
| | $0$ | $0$ |
| Test point → | $1$ | $-4$ |
| | $2$ | $-6$ |
| | $3$ | $0$ |
| Test point → | $4$ | $20$ |

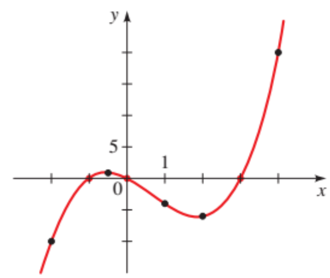



FIGURE 3.17.

**Activity 3.6.** *Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.*

(1) $P(x) = x(x-3)(x+2)$
(2) $P(x) = (2x-1)(x+1)(x+3)$
(3) $P(x) = \frac{1}{12}(x+2)^2(x-3)^2$
(4) $P(x) = x^3(x+2)(x-3)^2$

**3.2.4. Shape of the Graph Near a Zero.** If $c$ is a zero of $P$ and the corresponding factor $x - c$ occurs exactly $m$ times in the factorisation of $P$, then $c$ has **multiplicity** $m$. For example, for $P(x) = x^4(x-2)^3(x+1)^2$, $x = 2$ has multiplicity 3 and $x = -1$ has multiplicity 2.

Near $x = c$, the graph has the same general shape as the graph of $y = A(x-c)^m$. If $c$ is a zero of $P$ of multiplicity $m$, then the shape of the graph of $P$ near $C$ is as follows:

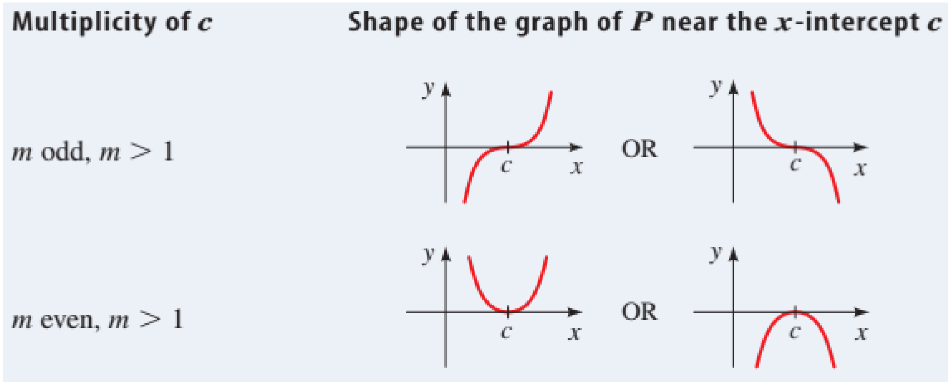



FIGURE 3.18.

**Example 3.14.** *Graph the polynomial* $P(x) = x^4(x-2)^3(x+1)^2$.

***Solution:***

- *The zeros are* $-1$, $0$ *and* $2$ *with multiplicity 2, 4 and 3.*
- *2 has odd multiplicity, so the graph crosses the* $x$*-axis at 2. But* $0$ *and* $-1$ *have even multiplicity so the graph does not cross the* $x$*-axis at 0 and* $-1$.
- $P$ *has odd degree 9 and has positive leading coefficient. So the end behavior is:* $y \to \infty$ *as* $x \to \infty$ *and* $y \to -\infty$ *as* $x \to -\infty$
- *The graph is as follows*

**Activity 3.7.** *Graph the following polynomials functions.*

(1) $P(x) = x^3 + 2x^2 - 8x$
(2) $P(x) = x^4 - 3x^3 + 2x^2$
(3) $P(x) = x^5 - 9x^3$
(4) $P(x) = 2x^3 - x^2 - 18x + 9$

**3.2.5. Local Maxima and Minima of Polynomials.** Let $f$ be a polynomial function and $(a, f(a))$ be the highest point on the graph of $f$ within some viewing rectangle, then $f(a)$ is a **local maximum value** of $f$ and $(a, f(a))$ is a **local maximum point**.

| $x$ | $P(x)$ |
|---|---|
| $-1.3$ | $-9.2$ |
| $-1$ | $0$ |
| $-0.5$ | $-3.9$ |
| $0$ | $0$ |
| $1$ | $-4$ |
| $2$ | $0$ |
| $2.3$ | $8.2$ |

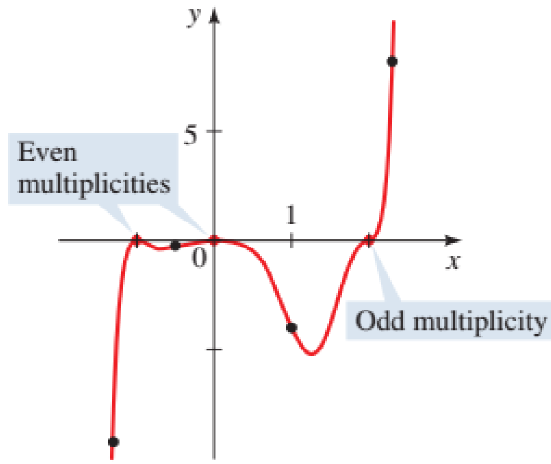



FIGURE 3.19.

If $(a, f(a))$ is the lowest point on the graph of $f$, then $f(a)$ is the **local minimum value** of $f$ and $(a, f(a))$ is a **local minimum point** of $f$.

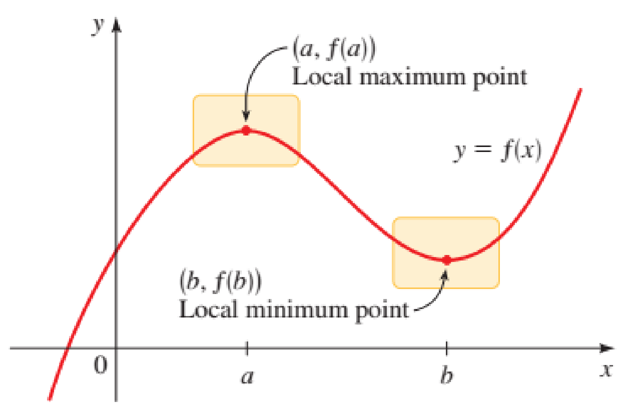



FIGURE 3.20.

**Local Extrema of Polynomials**

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree $n$, then the graph of $P$ has at most $n-1$ local extrema.

**Example 3.15.** *Determine how many local extrema each polynomial has.*

(a) $P(x) = x^4 + x^3 - 16x^2 - 4x + 48$

***Solution:***

*$P$ has two local minimum points and one local maximum point making it a total of 3 local extrema.*

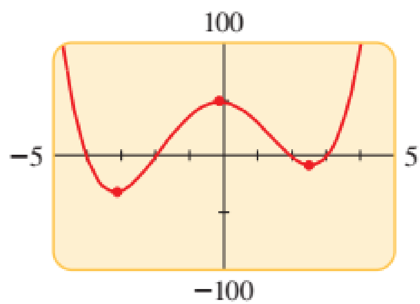



FIGURE 3.21.

(b) $Q(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

***Solution:***

*$Q$ has two local minimum points and two local maximum points making it a total of 4 local extrema.*

(c) $R(x) = 7x^4 + 3x^2 - 10x$

***Solution:***

*$R$ has one local extrema, a local minimum point.*

**Activity 3.8.** *Graph the following polynomial functions and find all local extrema.*

(1) $y = x^3 - x^2 - x$

(2) $y = x^4 - 5x^2 + 4$

(3) $P(x) = (x-1)(x-3)(x-4)$

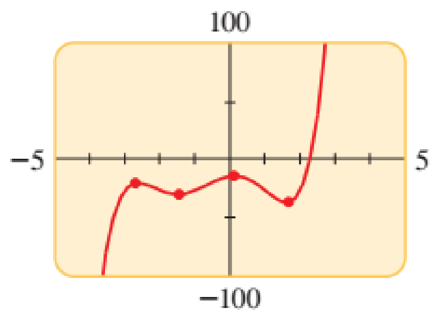



FIGURE 3.22.

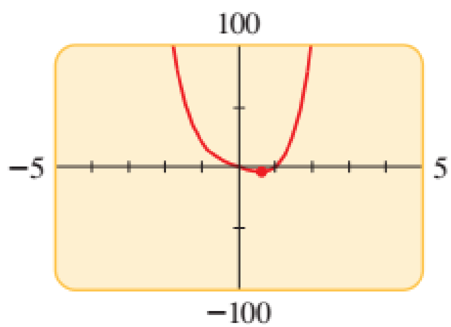



FIGURE 3.23.

(4) $P(x) = (x-1)(x-3)(x-4)+5$

## 3.3. Dividing Polynomials

**Division Algorithm**

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exists unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$P(x) = D(x) \cdot Q(x) + R(x).$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively. The polynomials $Q(x)$ and $R(x)$ are called the **quotient** and the **remainder**, respectively.

When dividing polynomials, there are two methods that are used, long division and synthetic division. We will mostly use synthetic division.

**3.3.1. Synthetic Division.** Synthetic division is a quick method of dividing polynomials. This method can be used when the divisor is of the form $x-c$. Only the essential parts are written. Look at the example below:

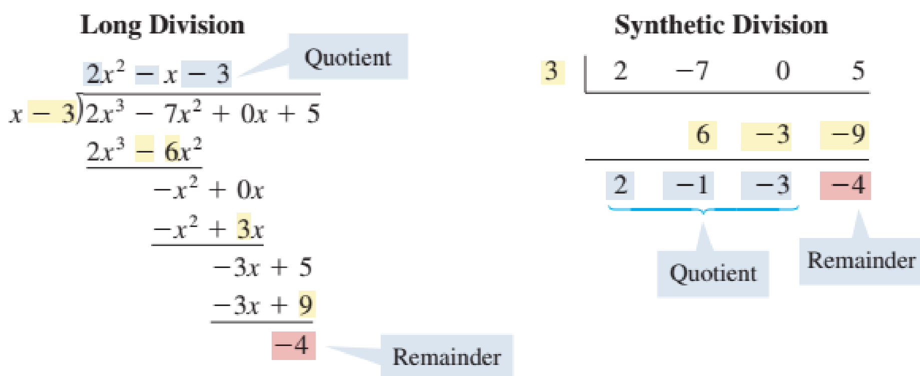



FIGURE 3.24.

The process starts with writing down the coefficients as follows:

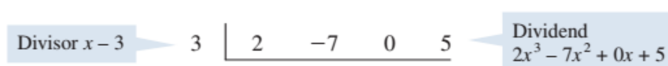



FIGURE 3.25.

We then bring down the 2 and multiply it with 3, i.e., $2 \cdot 3 = 6$. We then write 6 below $-7$ and add getting $-1$.

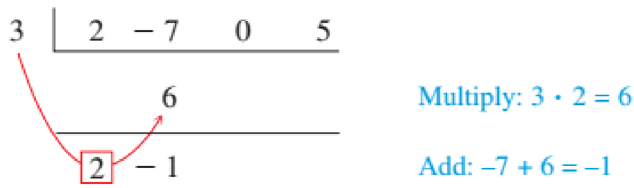



FIGURE 3.26.

We then multiply 3 with $-1$, i.e., $3 \cdot -1 = -3$ and write it below 0. We then add getting $-3$. We do the same for the last column.

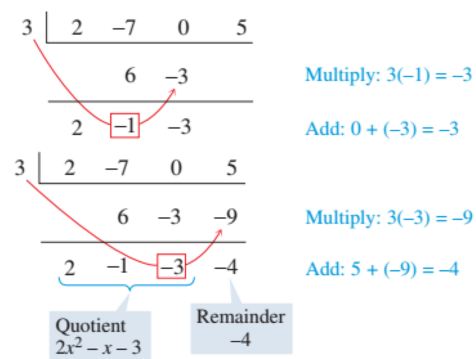



FIGURE 3.27.

**Activity 3.9.** *Two polynomials $P(x)$ and $D(x)$ are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$.*

(1) $P(x) = 3x^2 + 5x - 4,\ D(x) = x + 3$
(2) $P(x) = 2x^3 - 3x^2 - 2x,\ D(x) = 2x - 3$
(3) $P(x) = x^4 - x^3 + 4x + 2,\ D(x) = x^2 + 3$
(4) $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3,\ D(x) = x^2 - 2$

### 3.3.2. Remainder and Factor Theorem.

**Theorem 3.2** (Remainder Theorem). *If a polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.*

**Example 3.16.** *Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.*

- *Find the quotient and remainder when $P$ is divided by $x + 2$.*

  ***Solution:***

  *We use synthetic division as follows*

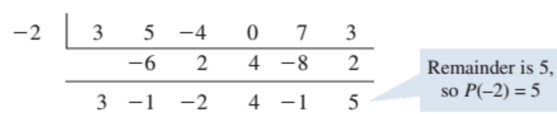
  
  

  FIGURE 3.28.

  *So the quotient is $Q(x) = 3x^4 - x^3 - 2x^2 + 4x - 1$ and the remainder is 5.*

- *Use the Remainder Theorem to find $P(-2)$.*

  ***Solution:***

  *By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. Thus, the remainder is $P(-2) = 5$.*

**Theorem 3.3** (Factor Theorem). *$c$ is a zero of $P$ if and only if $x - c$ is a factor of $P(x)$.*

**Example 3.17.** *Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$ and use this fact to factor $P(x)$ completely.*

**Solution:**

We have that

$$P(1) = (1)^3 - 7(1) + 6 = 0.$$

So $x - 1$ is a factor of $P(x)$. We then use synthetic division as follows

$$
\begin{array}{r|rrrr}
1 & 1 & 0 & -7 & 6 \\
  &   & 1 & 1 & -6 \\
\hline
  & 1 & 1 & -6 & 0
\end{array}
$$

FIGURE 3.29.

So the quotient is $Q(x) = x^2 + x - 6$. Then

$$
\begin{aligned}
P(x) &= (x-1)(x^2 + x - 6) \\
&= (x-1)(x-2)(x+3)
\end{aligned}
$$

**Example 3.18.** *Find a polynomial of degree 4 that has zeros $-3, 0, 1$ and 5.*

***Solution:***

*By Factor Theorem, $x+3$, $x-0$, $x-1$ and $x-5$ are factors. Then*

$$
\begin{aligned}
P(x) &= (x-0)(x+3)(x-1)(x-5) \\
&= x^4 + 3x^3 - 13x^2 + 15x
\end{aligned}
$$

**Activity 3.10.** (1) *Use synthetic division and the Remainder Theorem to evaluate $P(c)$.*
   (a) $P(x) = 4x^2 + 12x + 5$, $c = -1$
   (b) $P(x) = x^7 - 3x^2 - 1$, $c = 3$

(2) *Find a polynomial of degree 3 that has zeros 1, $-2$, and 3 and in which the coefficient of $x^2$ is 3.*

(3) *Find a polynomial of degree 4 that has integer coefficients and zeros $1, -1, 2,$ and $\frac{1}{2}$.*

### 3.3.3. Real zeros of Polynomials.

**Theorem 3.4** (Real Zeros Theorem)**.** *If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ has integer coefficients, then every rational zero of $P$ is of the form $p/q$, where $p$ is a factor of the constant coefficient $a_0$ and $q$ is a factor of the leading coefficient of $a_n$.*

**Finding the Rational Zeros of a Polynomial**

1. **List all possible zeros.** List all possible rational zeros, using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for the rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

**Activity 3.11.** *Find all rational zeros of the polynomial, and write the polynomial in factored form.*

(1) $P(x) = x^3 + 3x^2 - 4$
(2) $P(x) = x^3 - 6x^2 + 12x - 8$
(3) $P(x) = x^3 - 4x^2 - 7x + 10$
(4) $P(x) = x^4 - 5x^2 + 4$
(5) $P(x) = 4x^3 - 7x + 3$
(6) $P(x) = 4x^4 - 25x^2 + 36$
(7) $P(x) = 6x^3 + 11x^2 - 3x - 2$
(8) $P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36$

## 3.4. Rational Functions

**Definition 3.4.** *A **Rational Function** is a function of the form*

$$r(x) = \frac{p(x)}{Q(x)}$$

*where $P$ and $Q$ are polynomials. We assume $P(x)$ and $Q(x)$ have no factor in common.*

When graphing rational functions, we pay attention to the behavior of the graph near the $x$-values.

**Example 3.19.** *Graph the rational function* $f(x) = \frac{1}{X}$, *and state the domain and range.*

***Solution:***

*The function* $f(x) = \frac{1}{x}$ *is not defined for* $x = 0$. *We come up with table of values for values of* $x$ *closer to* $x = 0$ *from the negative side and from the positive side.*

| $x$ | $f(x)$ |
|---|---|
| $-0.1$ | $-10$ |
| $-0.01$ | $-100$ |
| $-0.00001$ | $-100{,}000$ |

Approaching $0^-$ — Approaching $-\infty$

| $x$ | $f(x)$ |
|---|---|
| $0.1$ | $10$ |
| $0.01$ | $100$ |
| $0.00001$ | $100{,}000$ |

Approaching $0^+$ — Approaching $\infty$

FIGURE 3.30.

*We also show how* $f(x)$ *changes as* $|x|$ *becomes large.*

| $x$ | $f(x)$ |
|---|---|
| $-10$ | $-0.1$ |
| $-100$ | $-0.01$ |
| $-100{,}000$ | $-0.00001$ |

Approaching $-\infty$ — Approaching $0$

| $x$ | $f(x)$ |
|---|---|
| $10$ | $0.1$ |
| $100$ | $0.01$ |
| $100{,}000$ | $0.00001$ |

Approaching $\infty$ — Approaching $0$

FIGURE 3.31.

*So* $f(x) \to 0$ *as* $x \to -\infty$ *and* $f(x) \to 0$ *as* $x \to -\infty$. *The following is a table of values for* $f(x) = \frac{1}{x}$.

| $x$ | $f(x) = \frac{1}{x}$ |
|---|---|
| $-2$ | $-\frac{1}{2}$ |
| $-1$ | $-1$ |
| $-\frac{1}{2}$ | $-2$ |
| $\frac{1}{2}$ | $2$ |
| $1$ | $1$ |
| $2$ | $\frac{1}{2}$ |

FIGURE 3.32.

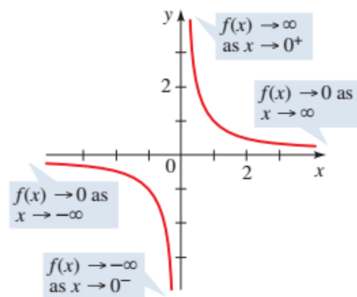



FIGURE 3.33.

The behavior explained in the example above has the following meaning:

| Symbol | Meaning |
|---|---|
| $x \to a^-$ | $x$ approaches $a$ from the left |
| $x \to a^+$ | $x$ approaches $a$ from the right |
| $x \to -\infty$ | $x$ goes to negative infinity; that is, $x$ decreases without bound |
| $x \to \infty$ | $x$ goes to infinity; that is, $x$ increases without bound |

FIGURE 3.34.

The line $x = 0$ in the above example is called a **vertical asymptote** and the line $y = 0$ is a **horizontal asymptote**.

**3.4.1. Transformations of $f(x) = \frac{1}{x}$.** A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

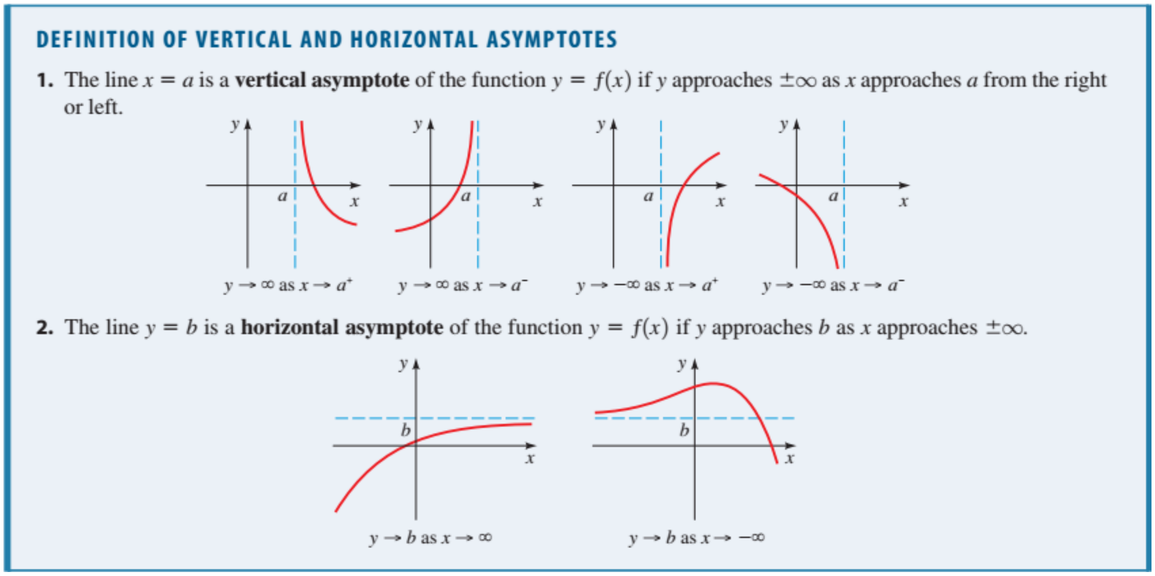



FIGURE 3.35.

can be graphed by shifting, stretching and/or reflecting the graph of $f(x) = \frac{1}{x}$. These functions are known as fractional transformations.

**Example 3.20.** *Graph the following functions:*

(a) $r(x) = \frac{2}{x-3}$

***Solution:***

*We can express* $r(x)$ *in terms of* $f(x) = \frac{1}{x}$

$$r(x) = \frac{2}{x-3} = 2\left(\frac{1}{x-3}\right) = 2(f(x-3))$$

*So we shift the graph of* $f(x) = \frac{1}{x}$ *3 units to the right and stretch vertically by a factor of 2.*

*The domain of* $r(x)$ *is* $\{x|x \neq 3\}$ *since the function* $r(x)$ *is defined for all* $x$ *other than 3. The range of* $r(x)$ *is* $\{y|y \neq 0\}$

(b) $s(x) = \frac{3x+5}{x+2}$

***Solution:***

*Let* $f(x) = \frac{1}{x}$. *Using long division, we get* $s(x) = 3 - \frac{1}{x+2}$. *So* $s(x) = 3 - f(x+2)$.

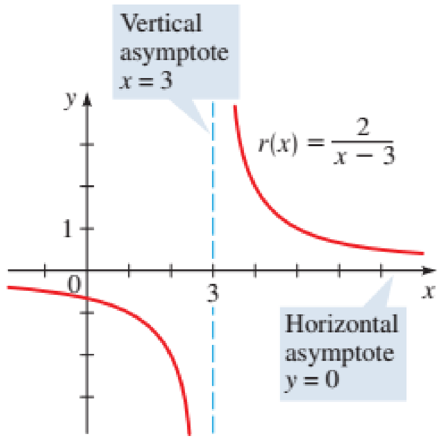



FIGURE 3.36.

*We therefore reflect the graph of $f(x)$ on the $x$-axis. We also shift it to the left 2 units and shift it upwards 3 units.*

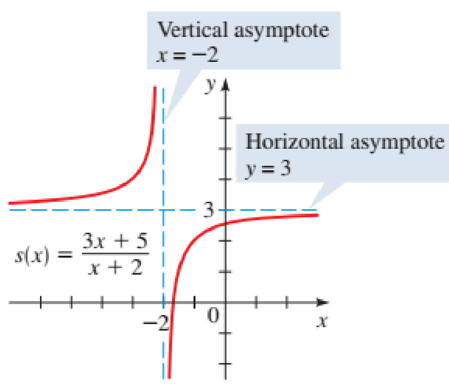



FIGURE 3.37.

**Activity 3.12.** *Use transformations of the graph of* $y = \frac{1}{x}$ *to graph the rational function.*

(1) $r(x) = \frac{1}{x+4}$
(2) $s(x) = \frac{2x-3}{x-2}$
(3) $t(x) = \frac{3x-3}{x+2}$
(4) $r(x) = \frac{2x-9}{x-4}$

**FINDING ASYMPTOTES OF RATIONAL FUNCTIONS**

Let $r$ be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

**1.** The vertical asymptotes of $r$ are the lines $x = a$, where $a$ is a zero of the denominator.

**2. (a)** If $n < m$, then $r$ has horizontal asymptote $y = 0$.

**(b)** If $n = m$, then $r$ has horizontal asymptote $y = \frac{a_n}{b_m}$.

**(c)** If $n > m$, then $r$ has no horizontal asymptote.

FIGURE 3.38.

### 3.4.2. Asymptotes of Rational Functions.

**Example 3.21.** *Find the vertical and horizontal asymptotes of*

$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}.$$

***Solution:***

*Factor the denominator:*

$$2x^2 + 3x - 2 = (2x - 1)(x + 2).$$

*Thus,* $x = \frac{1}{2}$ *or* $x = -2$*. So the vertical asymptotes are the lines* $x = \frac{1}{2}$ *and* $x = -2$*.*

*To find the horizontal asymptotes, we look at the degrees of the numerator and denominator and compare them according to the conditions mentioned earlier. Thus we find that the degrees of the numerator and the denominator* *are the same. Then the horizontal asymptote is the line $y = \frac{3}{2}$, i.e.,*

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

*If we graph the function using a graphing device, we get*

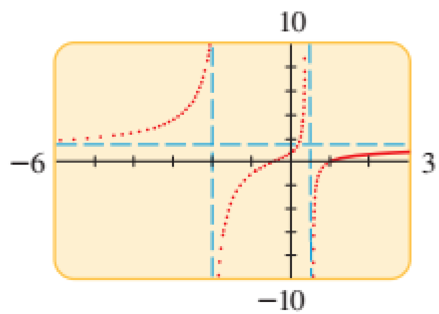



FIGURE 3.39.

**Activity 3.13.** *Find all horizontal and vertical asymptotes (if any).*

(1) $r(x) = \frac{5}{x-2}$
(2) $r(x) = \frac{2x-4}{x^2+x+1}$
(3) $r(x) = \frac{6x^3-2}{2x^3+5x^2+6x}$
(4) $r(x) = \frac{x^3+3x^2}{x^2-4}$

**3.4.3. Graphing Rational Functions.** The guidelines for sketching the graphs of rational functions are outlined below:

**SKETCHING GRAPHS OF RATIONAL FUNCTIONS**

1. **Factor.** Factor the numerator and denominator.
2. **Intercepts.** Find the $x$-intercepts by determining the zeros of the numerator and the $y$-intercept from the value of the function at $x = 0$.
3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote by using test values.
4. **Horizontal Asymptote.** Find the horizontal asymptote (if any), using the procedure described in the box on page 282.
5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

FIGURE 3.40.

**Example 3.22.** *Graph* $r(x) = \frac{2x^2+7x-4}{x^2+x-2}$*, and state the domain and range.*

***Solution:***

***Factor:*** *We factor the numerator and denominator.*

$$y = \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \frac{(2x-1)(x+4)}{(x-1)(x+2)}.$$

$x$***-intercepts:*** *The* $x$ *intercepts are the zeros of the numerator,* $x = \frac{1}{2}$ *and* $x = -4$.

$y$***-intercepts:*** *We substitute* $x = 0$ *in* $r(x)$ *to find the* $y$*-intercept:*

$$y = \frac{2(0)^2 + 7(0) - 4}{0^2 + 0 - 2} = \frac{-4}{-2} = 2.$$

***Vertical asymptotes:*** *These occur where the denominator is 0. So the vertical asymptotes are the zeros of the denominator,* $x = 1$ *and* $x = -2$.

***Behavior near vertical asymptotes:*** *We need to determine the behavior of the function towards each asymptote. To do that, we use test values. The following table shows whether* $y \to \infty$ *or* $y \to -\infty$ *from each side of the vertical asymptotes.*

| As $x \to$ | $-2^-$ | $-2^+$ | $1^-$ | $1^+$ |
|---|---|---|---|---|
| the sign of $y = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)}$ is | $\dfrac{(-)(+)}{(-)(-)}$ | $\dfrac{(-)(+)}{(-)(+)}$ | $\dfrac{(+)(+)}{(-)(+)}$ | $\dfrac{(+)(+)}{(+)(+)}$ |
| so $y \to$ | $-\infty$ | $\infty$ | $-\infty$ | $\infty$ |

FIGURE 3.41.

***Horizontal asymptote:*** *We look at the degrees of the numerator and denominator. The degrees of the numerator and denominator are the same. So, the horizontal asymptote is*

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2$$

*Thus, the horizontal asymptote is the line $y = 2$.*

***Graph:*** *With some additional information as in the table below, And then*

| $x$ | $y$ |
|---|---|
| $-6$ | 0.93 |
| $-3$ | $-1.75$ |
| $-1$ | 4.50 |
| 1.5 | 6.29 |
| 2 | 4.50 |
| 3 | 3.50 |

FIGURE 3.42.

*we plot the graph:*

**Activity 3.14.** *Find the intercepts and asymptotes, and then sketch a graph of the rational function and state the domain and range.*

(1) $r(x) = \frac{4x-4}{x+2}$
(2) $r(x) = \frac{18}{(x-3)^3}$
(3) $r(x) = \frac{2x-4}{x^2+x-2}$
(4) $r(x) = \frac{(x-1)(x+2)}{(x+1)(x-3)}$

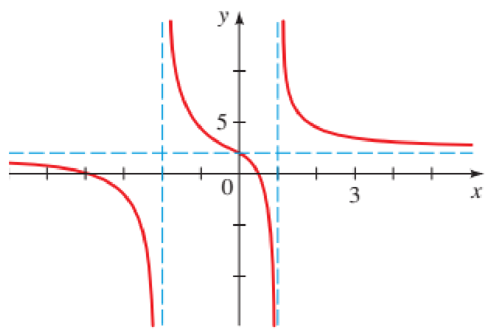



FIGURE 3.43.

**3.4.4. Slant Asymptotes and End Behavior.** For a rational function $r(x) = P(x)/Q(x)$, in which the degree of the numerator is one more than the degree of the denominator, we use the Division Algorithm to express $r(x)$ in the form

$$r(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of $R(x)$ is less than the degree of $Q(x)$. It means as $x \to \pm\infty$, $R(x)/Q(x) \to 0$. This means for large values of $|x|$ the graph of $r(x)$ approaches the graph of the line $y = ax + b$.

The line $y = ax+b$ is known as a **slant asymptote** or **oblique asymptote**.

**Example 3.23.** *Graph the rational function* $r(x) = \frac{x^2-4x-5}{x-3}$.

***Solution:***

***Factor:***

$$r(x) = \frac{(x+1)(x-5)}{x-3}.$$

$x$***-intercepts:***

$$-1 \text{and} 5.$$

$y$***-intercepts:***

$$r(0) = \frac{5}{3}.$$

***Horizontal asymptote:*** *There is no horizontal asymptote since the degree of the numerator is greater than the degree of the denominator.*

***Vertical asymptote:***

$$x = 3$$

***Behavior near vertical asymptote:*** *$y \to \infty$ as $x \to 3^-$ and $y \to -\infty$ as $x \to 3^+$.*

***Slant asymptote:*** *Using long division, Then*

$$\begin{array}{r} x-1 \\ x-3\overline{)x^2-4x-5} \\ \underline{x^2-3x} \\ -x-5 \\ \underline{-x+3} \\ -8 \end{array}$$

FIGURE 3.44.

$$r(x) = x - 1 - \frac{8}{x-3}.$$

*Thus, the slant asymptote is $y = x - 1$.*

***Graph:*** *With a few additional data points as below*

| $x$ | $y$ |
|---|---|
| −2 | −1.4 |
| 1 | 4 |
| 2 | 9 |
| 4 | −5 |
| 6 | 2.33 |

FIGURE 3.45.

*we then plot the graph:*

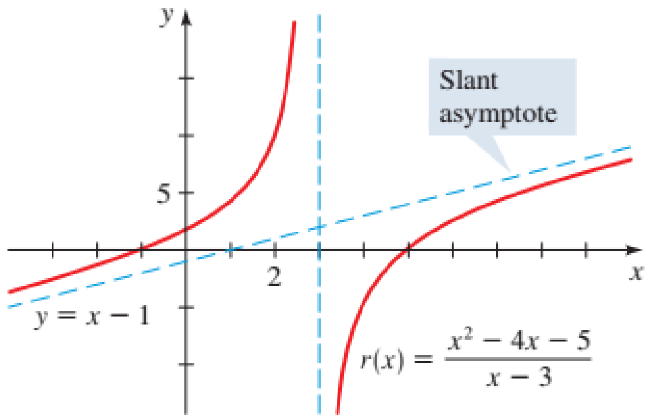



FIGURE 3.46.

**Activity 3.15.** *Find the slant asymptote, the vertical asymptotes, and sketch a graph of the function.*

(1) $r(x) = \frac{x^2}{x-2}$
(2) $r(x) = \frac{x^2-2x-8}{x}$
(3) $r(x) = \frac{x^3+4}{2x^2+x-1}$
(4) $r(x) = \frac{2x^3+2x}{x^2-1}$

UNIT 4

# EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**Exponential Functions** are functions where the independent variable is in the exponent, e.g., $f(x) = 2^x$. These are used for modeling many real world phenomena such as growth of a population. They can be used to predict population size at any given time. To find out when a population will reach a certain level, we use the inverse functions of exponential functions called logarithmic functions.

## 4.1. Exponential Functions

**Definition 4.1.** *The exponential function with base $a$ is defined for all real numbers $x$ by*

$$f(x) = a^x$$

*where $a > 0$ and $a \neq 1$.*

We assume $a \neq 1$ because $f(x) = 1^x = 1$ is just a constant function.

**Example 4.1.** *Let $f(x) = 3^x$, and evaluate the following:*

(a) $f(2)$,

***Solution:***

$f(2) = 3^2 = 9$.

(b) $f(-2/3)$.

***Solution:***

$f(-2/3) = 3^{-2/3} \approx 0.4807$.

**Activity 4.1.** *Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.*

(a) $f(x) = 4^x$; $f(0.5), f(\sqrt{2}), f(-\pi), f(\frac{1}{[}3])$.

(b) $g(x) = \left(\frac{3}{4}\right)^{2x}$; $g(0.7), g(\sqrt{7}/2), g(1/\pi), g(2/3)$.

**4.1.1. Graphs of Exponential Functions.** The exponential function

$$f(x) = a^x \quad (a > 0, a \neq 1)$$

has domain $\mathring{R}$ and range $(0, \infty)$. The line $y = 0$ (the $x$-axis) is a horizontal asymptote of $f$. The graph of $f$ has one of the following shapes:

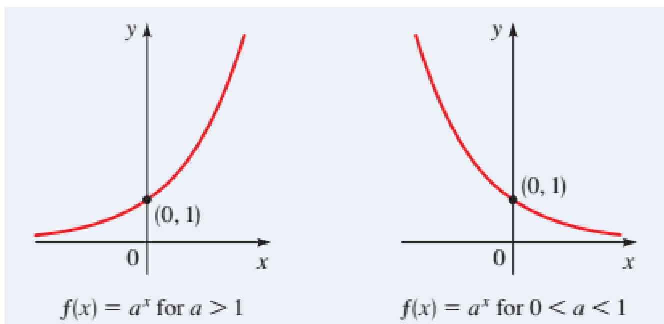



FIGURE 4.1.

**Example 4.2.** *Find the exponential function $f(x) = a^x$ whose graph is given.*

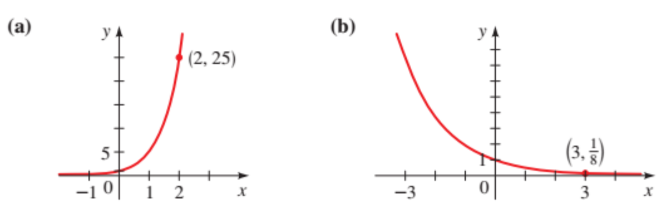



FIGURE 4.2.

**Solution:**

(a) From the point $(2, 25)$, we get that $f(2) = a^2 = 25$. Then base $a = 5$. So $f(x) = 5^x$.

(b) From the point $(3, 1/8)$, we also get that $f(3) = a^3 = \frac{1}{8}$. Then base $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

**Activity 4.2.** *Sketch the graph of the function by making a table of values. Use a calculator if necessary.*

(a) $g(x) = 8^x$
(b) $f(x) = \left(\frac{1}{3}\right)^x$
(c) $h(x) = 2\left(\frac{1}{4}\right)^x$
(d) $f(x) = (1.1)^x$

**4.1.2. Transformations of Exponential Functions.**

4.1.2.1. *Functions of the form $f(x) = -a^x$ and $g(x) = a^{-x}$.* To graph the function $f(x) = -a^x$, start with the graph of $y = a^x$ and reflect it on the $x$-axis.

**Example 4.3.** *Graph the function $h(x) = -2^x$.*

***Solution:***

*Start with the graph of $f(x) = 2^x$ and reflect it on the $x$-axis to get the graph of $h(x) = -2^x$.*

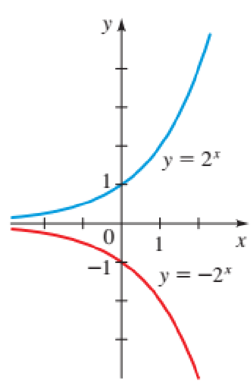



FIGURE 4.3.

To graph the function $g(x) = a^{-x}$, start with the graph of $y = a^x$ and reflect it on the $y$-axis.

**Example 4.4.** *Graph the function $g(x) = 2^{-x}$.*

***Solution:***

*Start with the graph of $f(x) = 2^x$ and reflect it on the $y$-axis to get the graph of $g(x) = 2^{-x}$.*

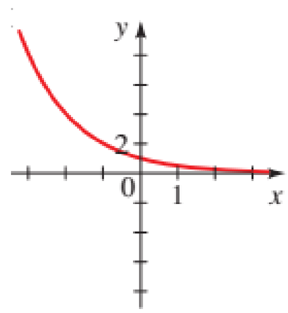



FIGURE 4.4.

4.1.2.2. *Graphs of functions of the form $f(x) = b + a^x$.* To graph these functions, we need to start with the graph of $y = a^x$ and shift it $b$ units upwards if $b > 0$ or shift it $b$ units downwards if $b < 0$.

**Example 4.5.** *Sketch the graph of $f(x) = 1 + 2^x$.*

***Solution:***

*Start with the graph of $y = 2^x$ and shift it 1 unit upward to get the graph of $f(x) = 1 + 2^x$.*

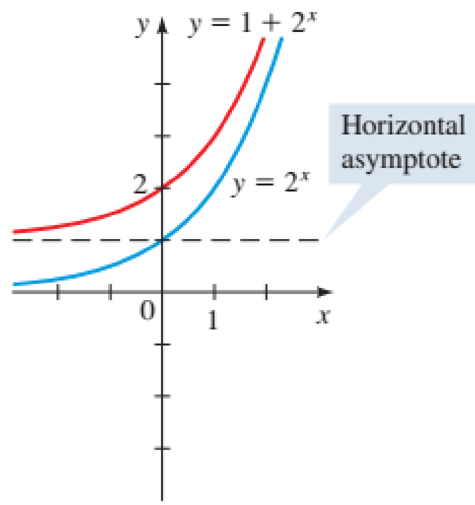



FIGURE 4.5.

4.1.2.3. *Graphs of functions of the form $f(x) = a^{x+b}$.* To graph functions of this form, we start with the graph of $y = a^x$ and shift it $b$ units left if $b > 0$ or shift it $b$ units right if $b < 0$.

**Example 4.6.** *Sketch the graph of $k(x) = 2^{x-1}$.*

***Solution:***

*Start with the graph of $y = 2^x$, shift it to the right 1 unit to get the graph of $k(x) = 2^{x-1}$.*

**Activity 4.3.** *Graph the following functions. State the domain, range and asymptote.*

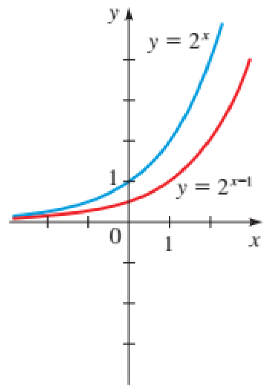



FIGURE 4.6.

(a) $f(x) = 2^x - 3$

(b) $h(x) = 4 + \left(\frac{1}{2}\right)^x$

(c) $g(x) = 10^{x+3}$

(d) $y = 2^{x-4} + 1$

### 4.1.3. Compound Interest.

**Definition 4.2.** *Compound Interest is calculated by the formula*

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

*where $A(t)$ is the amount after $t$ years, $P$ is the principal, $r$ is the interest rate per year, $n$ is the number of times interest is compounded per year and $t$, the number years.*

**Example 4.7.** *A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.*

**Solution:**

We use the compound interest formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

with $P = \$1000$, $r = 0.12$, and $t = 3$.

| Compounding | $n$ | Amount after 3 years |
|---|---|---|
| Annual | 1 | $1000\left(1+\frac{0.12}{1}\right)^{1(3)} = \$1404.93$ |
| Semiannual | 2 | $1000\left(1+\frac{0.12}{2}\right)^{2(3)} = \$1418.52$ |
| Quarterly | 4 | $1000\left(1+\frac{0.12}{4}\right)^{4(3)} = \$1425.76$ |
| Monthly | 12 | $1000\left(1+\frac{0.12}{12}\right)^{12(3)} = \$1430.77$ |
| Daily | 365 | $1000\left(1+\frac{0.12}{365}\right)^{365(3)} = \$1433.24$ |

FIGURE 4.7.

**Activity 4.4.**

1. *Sketch the graphs of the following functions and explain the relationship between them.*
   (a) $f(x) = 2^x$ *and* $g(x) = 3(2^x)$*;*
   (b) $f(x) = 9^{x/2}$ *and* $g(x) = 3^x$.
2. *If \$10,000 is invested at an interest rate of 3% per year, compounded semiannually, find the value of the investment after the given number of years. (a) 5 years (b) 10 years (c) 15 years.*

### 4.2. The Natural Exponential Function

The natural exponential function is similar to the other exponential functions except for the base. It uses a number $e$ as the base.

**Definition 4.3.** *The number $e$ is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as $n$ becomes large.*

The table below shows the value that $\left(1+\frac{1}{n}\right)^n$ approaches for increasingly large values of $n$.
In particular, $e \approx 2.71828182845904523536$.

**Definition 4.4.** *The **Natural Exponential Function** is the function $f(x) = e^x$ with base $e$. It is often referred to as just the exponential function.*

| $n$ | $\left(1+\frac{1}{n}\right)^n$ |
|---:|---|
| 1 | 2.00000 |
| 5 | 2.48832 |
| 10 | 2.59374 |
| 100 | 2.70481 |
| 1000 | 2.71692 |
| 10,000 | 2.71815 |
| 100,000 | 2.71827 |
| 1,000,000 | 2.71828 |

FIGURE 4.8.

Since $2 < e < 3$, the graph of $f(x) = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

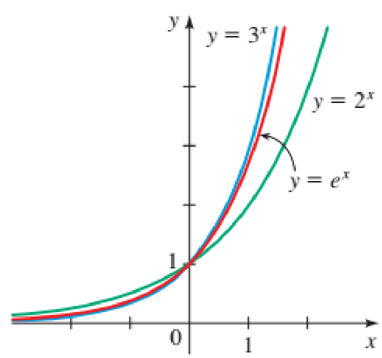



FIGURE 4.9.

The natural exponential function can be evaluated using a calculator.

**Example 4.8.** *Evaluate each expression rounded to 5 decimal places:*

(a) $e^3 \approx 20.08554$.

(b) $2e^{-0.53} \approx 1.17721$.

(c) $e^{4.8} \approx 121.51042$.

**Activity 4.5.** *Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.*

(a) $f(x) = e^x$; $f(3), f(0.23), f(1), f(-2)$.

(b) $h(x) = e^{-2x}$; $h(1), h(\sqrt{2}), h(-3), h(\frac{1}{2})$.

**4.2.1. Transformation of Exponential Functions.** The transformations of these functions are the same as those done on other exponential functions as explained earlier in the chapter.

**Example 4.9.** *Sketch the graph of each function:*

(a) $f(x) = e^{-x}$,

***Solution:***

*We start with the graph of $y = e^x$ and reflect it on the $y$-axis.*

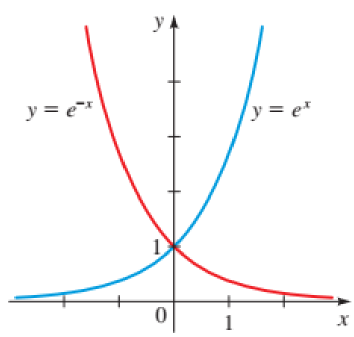



FIGURE 4.10.

(b) $f(x) = 3e^{0.5x}$,

***Solution:***

*We come up with a table of values as below. Then we plot the resulting points and connect them with a smooth curve.*

**Activity 4.6.** 1. *Graph the function, not by plotting points, but by starting from the graph of $y = e^x$. State the domain, range and asymptote.*

(a) $y = 1 - e^x$

(b) $f(x) = e^{-x} - 1$

(c) $y = e^{x-3} + 4$

| $x$ | $f(x) = 3e^{0.5x}$ |
|---|---|
| −3 | 0.67 |
| −2 | 1.10 |
| −1 | 1.82 |
| 0 | 3.00 |
| 1 | 4.95 |
| 2 | 8.15 |
| 3 | 13.45 |

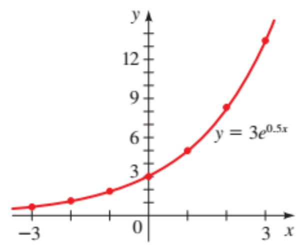



FIGURE 4.11.

(d) $f(x) = -e^{x-1} - 2$

2. *Find the local maximum and minimum values of the function and the value of $x$ at which each occurs. State each answer correct to two decimal places.*
   (a) $g(x) = x^x$
   (b) $g(x) = e^x + e^{-3x}$
3. *A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time $t$ is given by*

$$v(t) = 80(1 - e^{-0.2t})$$

*where $t$ is measured in seconds and $v(t)$ is measured in $ft/sec$.*
   (a) *Find the initial velocity of the sky diver.*
   (b) *Find the velocity after 5 s and after 10 s.*
   (c) *Draw a graph of the velocity function $v(t)$.*
   (d) *The maximum velocity of a falling object with wind resistance is called its terminal velocity. From the graph in part (c) find the terminal velocity of this sky diver.*

### 4.3. Logarithmic Functions

Every exponential funtion $f(x) = a^x$ with $a > 0$ and $a \neq 1$ is a one-to-one function by the horizontal line test and therefore has an inverse function.

The inverse function $f^{-1}$ of the exponential function is called the **logarithm function** with base $a$ and is denoted by $\log_a$.

**Definition 4.5.** *Let $a$ be a positive number with $a \neq 1$. The* ***logarithm function*** *with base $a$ denoted by $\log_a$ is defined by*

$$\log_a x = y \iff a^y = x.$$

*So $\log_a x$ is the exponent to which the base $a$ must be raised to give $x$.*

**Example 4.10.** *The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.*

| Logarithmic form | Exponential form |
|---|---|
| $\log_{10} 100{,}000 = 5$ | $10^5 = 100{,}000$ |
| $\log_2 8 = 3$ | $2^3 = 8$ |
| $\log_2\left(\frac{1}{8}\right) = -3$ | $2^{-3} = \frac{1}{8}$ |
| $\log_5 s = r$ | $5^r = s$ |

FIGURE 4.12.

**Example 4.11.** *Evaluate the following logarithms:*

(a) $\log_1 01000 = 3$ *because* $10^3 = 1000$.

(b) $\log_2 32 = 5$ *because* $2^5 = 32$.

With the following properties of logarithms, we will find it easier to understand the above examples.

**Properties of Logarithms:**

1. $\log_a 1 = 0$,
2. $\log_a a = 1$,
3. $\log_a a^x = x$,
4. $a^{\log_a x} = x$.

**Activity 4.7.** (1) *Evaluate the expression.*

(a) $\log_2 32$

(b) $\log_8 8^{27}$

(c) $\log_{49} 7$

(d) $3^{\log_3 8}$

(2) *Use the definition of the logarithmic function to find $x$.*

(a) $\log_2 x = 5$

(b) $\log_4 2 = x$

(c) $\log_2 16 = x$

(d) $\log_{10} 0.1 = x$

**4.3.1. Graphing Logarithmic Functions.** Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathring{R}$ and range $(0, \infty)$, its inverse function $f^{-1}(x) = \log_a x$ has domain $(0, \infty)$ and range $\mathring{R}$.
The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$.

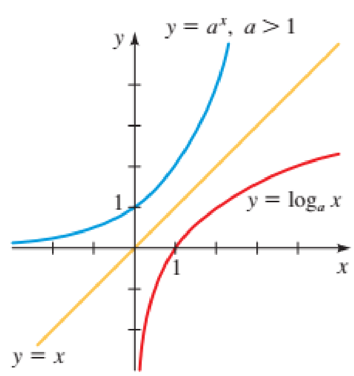



FIGURE 4.13.

**Example 4.12.** *Sketch the graph of $f(x) = \log_2 x$.*

**Solution:**

We make a table of values and choose the $x$ values to be powers of 2 so we can find their logarithms easily.

| $x$ | $\log_2 x$ |
|---|---|
| $2^3$ | 3 |
| $2^2$ | 2 |
| 2 | 1 |
| 1 | 0 |
| $2^{-1}$ | $-1$ |
| $2^{-2}$ | $-2$ |
| $2^{-3}$ | $-3$ |
| $2^{-4}$ | $-4$ |

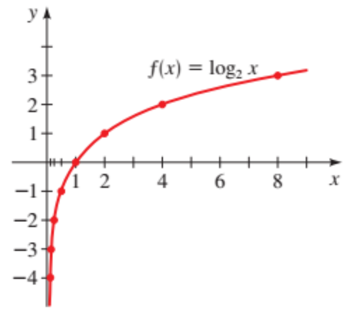



FIGURE 4.14.

The following graphs are graphs of logarithmic functions with bases 2,3, 5, and 10.

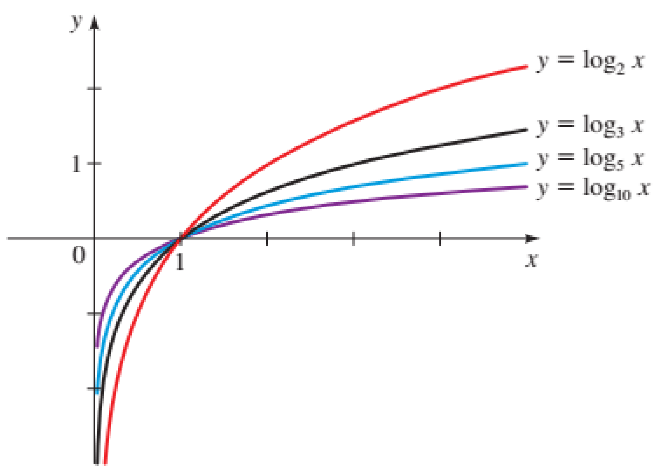



FIGURE 4.15.

It is clear to see that these graphs are obtained by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ in the line $y = x$.

**4.3.2. Transformations of Graphs of Logarithmic Functions.** Transformations of graphs of logarithmic functions follow the same suit as exponential functions.

As we will see in the following example, reflection in the $x$-axis happens when $f(x) = -\log_a x$ while reflection in the $y$-axis happens when $f(x) = \log_a(-x)$.

**Example 4.13.** *Sketch the graphs of the following functions:*

(a) $g(x) = -\log_2 x$.

(b) $h(x) = \log_2(-x)$.

**Solution:**

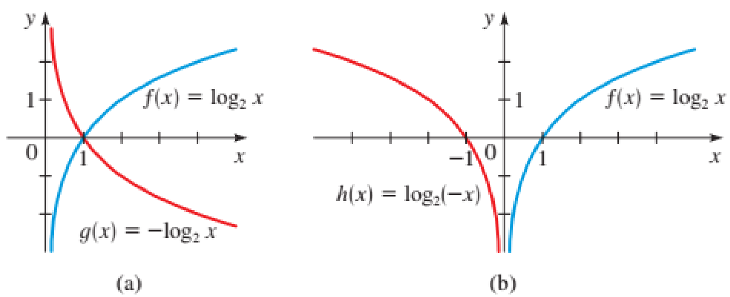



FIGURE 4.16.

The graph of any logarithmic function of the form $f(x) = b + \log_a x$ where $b$ is an integer, is just the graph of $y = \log_a x$ shifted upward $b$ units.
If $f(x) = \log_a(x - b)$, then the graph is just the graph of $y = \log_a x$ shifted $b$ units to the right. See the examples below.

**Example 4.14.** *Find the domain of each function and sketch the graph.*

(a) $g(x) = 2 + \log_5 x$.

***Solution:***

*The graph of $g$ is obtained from the graph of $f(x) = \log_5 x$ by shifting it upwards 2 units. The domain of $f$ is $(0, \infty)$.*

(b) $h(x) = \log_{10}(x - 3)$.

***Solution:***

*The graph of $h$ is obtained from the graph of $f(x) = \log_{10} x$ by shifting to the right 3 units. The line $x = 3$ is the vertical asymptote. The domain is*

$$\{x|x - 3 > 0\} = \{x|x > 3\} = (3, \infty)$$

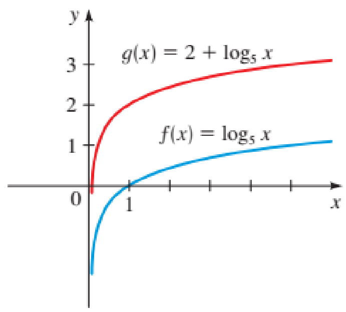



FIGURE 4.17.

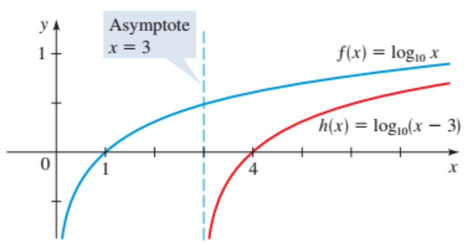



FIGURE 4.18.

**Definition 4.6.** *The logarithm with base 10 is called the* ***common logarithm*** *and is denoted by omitting the base*

$$\log x = \log_{10} x$$

Common logarithms can be evaluated using calculators.

**Activity 4.8.** *Graph the function. State the domain, range, and asymptote.*

(a) $f(x) = \log_2(x - 4)$

(b) $y = 1 - \log x$

(c) $y = 2 + \log_3 x$

(d) $y = \log_3(x - 1) - 2$

## 4.4. The Natural Logarithm Function

**Definition 4.7.** *The logarithm with base e is called the* ***natural logarithm*** *and is denoted by* ln.

$$\ln x = \log_e x.$$

So $\ln x = y \iff e^y = x$.

**Properties of Natural Logarithmic Functions**

The properties are just similar to the properties of the logarithmic function.

1. $\ln 1 = 0$.
2. $\ln e = 1$.
3. $\ln e^x = x$.
4. $e^{\ln x} = x$.

**Example 4.15.** *Evaluate*

(a) $\ln e^8 = 8$;

(b) $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$;

(c) $\ln 5 \approx 1.609$.

**Example 4.16.** *Find the domain of the function* $f(x) = \ln(4 - x^2)$.

***Solution:***

$$\begin{aligned}\{x|4 - x^2 > 0\} &= \{x|x^2 < 4\}\\ &= \{x||x| < 2\}\\ &= \{x| -2 < x < 2\}\\ &= (-2, 2)\end{aligned}$$

**Activity 4.9.** *Draw the graph of the function in a suitable viewing rectangle, and use it to find the domain, the asymptotes, and the local maximum and minimum values.*

(a) $y = x + \ln x$

(b) $y = \frac{\ln x}{x}$

(c) $y = x(\ln x)^2$

(d) $y = \ln(x^2 - x)$

## 4.5. Laws of Logarithms

Let $a$ be a positive number, with $a \neq 1$. Let $A$, $B$, and $C$ be any real numbers with $A > 0$ and $B > 0$,

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C\log_a A$.

**Example 4.17.** *Evaluate the following:*

(a) $\log_4 2 + \log_4 32$

***Solution:***

$$\begin{aligned} \log_4 2 + \log_4 32 &= \log_4(2 \times 32) \\ &= \log_4 64 \\ &= \log_4 4^3 \\ &= 3\log_4 4 \\ &= 3 \end{aligned}$$

(b) $\log_2 80 - \log_2 5$

***Solution:***

$$\begin{aligned} \log_2 80 - \log_2 5 &= \log_2\left(\frac{80}{5}\right) \\ &= \log_2 16 \\ &= \log_2 2^4 \\ &= 4\log_2 2 \\ &= 4 \end{aligned}$$

(c) $-\frac{1}{3}\log 8$

***Solution:***

$$\begin{aligned}
-\frac{1}{3}\log 8 &= \log 8^{1/3} \\
&= \log\left(\frac{1}{8^{1/3}}\right) \\
&= \log\frac{1}{2} \\
&\approx -0.301
\end{aligned}$$

**4.5.1. Expanding and Combining Logarithmic Functions.** We can use the Laws of Logarithms to expand or combine logarithmic expressions.

**Example 4.18** (Expanding expressions)**.** *Use the Laws of Logarithms to expand each expression:*

(a) $\log_2(6x) = \log_2 6 + \log_2 x$*;*
(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$*;*
(c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln(\sqrt[3]{c}) = \ln a + \ln b - \frac{1}{3}\ln c$*.*

**Example 4.19** (Combining expressions)**.** *Combine the following into a single logarithmic expression.*

(a) $3\log x + \frac{1}{2}\log(x+1)$*;*

***Solution:***

$$\begin{aligned}
3\log x + \frac{1}{2}\log(x+1) &= \log x^3 + \log(x+1)^{1/2} \\
&= \log(x^3\sqrt{x+1})
\end{aligned}$$

(b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$*;*

***Solution:***

$$\begin{aligned} 3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1) &= \ln s^3 + \ln\sqrt{t} - \ln(t^2+1)^4 \\ &= \ln(s^3\sqrt{t}) - \ln(t^2+1)^4 \\ &= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right) \end{aligned}$$

**Activity 4.10.** (1) *Use the Laws of Logarithms to expand the expression.*

(a) $\log_2(2x)$
(b) $\log_2(ab^2)$
(c) $\log_a\left(\frac{x^2}{yz^3}\right)$
(d) $\log\left(\frac{10^x}{x(x^2+1)(x^4+2)}\right)$

(2) *Use the Laws of Logarithms to combine the expression.*

(a) $\log_2 A + \log_2 B - 2\log_2 C$
(b) $\log 12 + \frac{1}{2}\log 7 - \log 2$
(c) $\ln(a+b) + \ln(a-b) - 2\ln c$
(d) $\frac{1}{3}\log(x+2)^3 + \frac{1}{2}\left[\log x^4 - \log(x^2-x-6)\right]$

**4.5.2. Change of Base Formula.** Sometimes we would want to change from logarithms in one base to logarithms in another base. We can use the following to accomplish this.

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

**Example 4.20.** *Use the change of base formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.*

(a) $\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$;
(b) $\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$.

**Activity 4.11.** *Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places. Use either natural or common logarithms.*

(a) $\log_2 5$

(b) $\log_5 2$

(c) $\log_4 125$

(d) $\log_{12} 2.5$

## 4.6. Exponential and Logarithmic Equations

We first look at exponential equations and then we will look at logarithmic equations. The following are the guidelines for solving exponential equations.

**GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS**

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

FIGURE 4.19.

**Example 4.21.** *Solve* $3^{x+2} = 7$.

***Solution:***

$$\begin{aligned} 3^{x+2} &= 7 \\ \log(3^{x+2}) &= \log 7 \\ (x+2)\log 3 &= \log 7 \\ x+2 &= \frac{\log 7}{\log 3} \\ x &= \frac{\log 7}{\log 3} - 2 \end{aligned}$$

**Example 4.22.** *Solve the equation* $8e^{2x} = 20$.

***Solution:***

$$8e^{2x} = 20$$
$$e^{2x} = \frac{20}{8}$$
$$\ln e^{2x} = \ln 2.5$$
$$2x = \ln 2.5$$
$$x = \frac{\ln 2.5}{2}$$
$$x \approx 0.458$$

**Example 4.23.** *Solve* $e^{2x} - e^x - 6 = 0$.

***Solution:***

*Let* $y = e^x$, *then*

$$e^{2x} - e^x - 6 = 0$$
$$y^2 - y - 6 = 0$$
$$(y-3)(y+2) = 0$$

*So either* $y = 3$ *or* $y = -2$. *Thus,* $e^x = 3 \implies x = \ln 3$. *We do not consider the second case because* $e^x > 0$ *for all* $x$, *so the second case gives no solution.*

We now look at logarithmic equations. The following are the guidelines for solving logarithmic equations.

**GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS**

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

FIGURE 4.20.

**Example 4.24.** *Solve each equation for* $x$.

(a) $\ln x = 8$.

***Solution:***

$x = e^8 = 2981$.

(b) $\log_2(25 - x) = 3$.

***Solution:***

$$\begin{aligned}
\log_2(25 - x) &= 3 \\
25 - x &= 2^3 \\
25 - x &= 8 \\
x &= 17
\end{aligned}$$

(c) $4 + 3\log(2x) = 16$.

***Solution:***

$$\begin{aligned}
4 + 3\log(2x) &= 16 \\
3\log(2x) &= 12 \\
\log(2x) &= \frac{12}{3} \\
\log(2x) &= 4 \\
2x &= 10^4 \\
x &= \frac{10000}{2} \\
x &= 5000
\end{aligned}$$

(d) $\log(x + 2) + \log(x - 1) = 1$.

***Solution:***

$$\begin{aligned}
\log(x+2)+\log(x-1) &= 1\\
\log\left((x+2)(x-1)\right) &= 1\\
(x+2)(x-1) &= 10\\
x^2+x-2 &= 10\\
x^2+x-12 &= 0\\
(x+4)(x-3) &= 0
\end{aligned}$$

*So $x=-4$ or $x=3$.*

**Activity 4.12.** (1) *Solve the equation.*

(a) $e^{2x}+3e^x+2=0$

(b) $x^2 2^x - 2^x = 0$

(c) $e^{1-4x}=2$

(d) $2^{3x+1}=3^{x-2}$

(2) *Solve the logarithmic equation.*

(a) $\log(3x+5)=2$

(b) $\log_2(x^2-x-2)=2$

(c) $\log x+\log(x-1)=\log(4x)$

(d) $\log_9(x-5)+\log_9(x+3)=1$

## 4.7. Modeling with Exponential and Logarithmic Functions

**4.7.1. Exponential Growth.** Suppose we start with a single bacterium which divides every hour. After one hour we have 2 bacteria, after 2 hours $2^2$ bacteria, after 3 hours $2^3$. We can model the bacteria population after $t$ hours by $f(t)=2^t$. If we start with $n_0$ bacteria, then $f(t)=n_0 2^t$. If the initial size of a population is $n_0$ and the doubling time is $a$, then the size of the population at time $t$ is

$$n(t)=n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

**Example 4.25.** *Under ideal conditions a certain bacteria population doubles every three hours. Initially there are 1000 bacteria in a colony.*

(a) *Find a model for the bacteria population after t hours.*

***Solution:***

*The population at time $t$ is modeled by*

$$n(t) = 1000 \cdot 2^{t/3}.$$

(b) *How many bacteria are in the colony after 15 hours?*

***Solution:***

$$n(15) = 1000 \cdot 2^{15/3} = 1000 \cdot 2^5 = 32000.$$

(c) *When will the bacteria count reach 100,000?*

***Solution:***

*We need to find the time $t$ given $n(t) = 100,000$. So*

$$\begin{aligned} 100,000 &= 1000 \cdot 2^{t/3} \\ 100 &= 2^{t/3} \\ \log 100 &= \log 2^{t/3} \\ 2 &= \frac{t}{3} \log 2 \\ t &= \frac{6}{\log 2} \\ t &\approx 19.93 \end{aligned}$$

**4.7.2. Exponential Growth (Relative Growth Rate).** We can also model the population growth with an exponential function in any base.

**Definition 4.8.** *A population that experiences* ***exponential growth*** *increases according to the model*

$$n(t) = n_0 e^{rt}$$

*where $n(t)$ is the population at time $t$, $n_0$ is the initial size of the population, $r$ is the relative rate of growth expressed as a proportion of the population, and $t$ the time.*

**Example 4.26.** *The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.*

(a) *Find a function that models the number of bacteria after $t$ hours.*

***Solution:***

*We have $n_0 = 500$ and $r = 0.4$. Then*

$$n(t) = 500 \cdot e^{0.4t}$$

*where $t$ is in hours.*

(b) *What is the estimated count after 10 hours?*

***Solution:***

$$n(t) = 500 \cdot e^{0.4t} = 500 \cdot e^{0.4(10)} = 500 \cdot e^4 \approx 27,300$$

(c) *When will the bacteria count reach 80,000?*

***Solution:***

*We set $n(t) = 80,000$ and solve for $t$.*

$$\begin{aligned} 80,000 &= 500 \cdot e^{0.4t} \\ 160 &= e^{0.4t} \\ \ln 160 &= 0.4t \\ t &= \frac{\ln 160}{0.4} \\ t &\approx 12.68 \end{aligned}$$

(d) *Sketch the graph of $n(t)$.*

***Solution:***

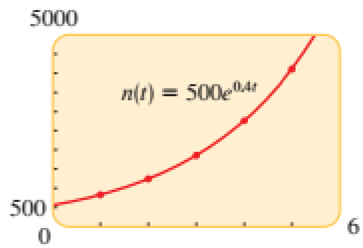



FIGURE 4.21.

**4.7.3. Radioactive Decay.** Radioactive substances decay by spontaneously omitting radiation. The rate of decay is proportional to the mass of the substance. Physicists express the rate of decay in terms of **half-life** (amount of time taken for the decay of half of the substance).

In general, for a radioactive substance with mass $m_0$ and half-life $h$, the amount remaining at time $t$ is modeled by

$$m(t) = m_0 2^{-t/h}.$$

If $m_0$ is the initial mass of a radioactive substance with half-life $h$, then the mass remaining at time $t$ is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \frac{\ln 2}{h}$.

**Activity 4.13.** (1) *A certain species of bird was introduced in a certain county 25 years ago. Biologists observe that the population doubles every 10 years, and now the population is 13,000.*
   (a) *What was the initial size of the bird population?*
   (b) *Estimate the bird population 5 years from now.*
   (c) *Sketch a graph of the bird population.*

(2) *The population of a certain city was 112,000 in 2006, and the observed doubling time for the population is 18 years.*

(a) *Find an exponential model* $n(t) = n_0 2^{t/a}$ *for the population* $t$ *years after 2006.*

(b) *Find an exponential model* $n(t) = n_0 2^{rt}$ *for the population* $t$ *years after 2006.*

(c) *Sketch the graph of the population at time* $t$.

(d) *Estimate when the population will reach 500,000.*

(3) *The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample.*

(a) *Find a function* $m(t) = m_0 2^{-t/h}$ *that models the mass remaining after* $t$ *years.*

(b) *Find a function* $m(t) = m_0 2^{-rt}$ *that models the mass remaining after* $t$ *years.*

(c) *How much of the sample will remain after 80 years?*

(d) *After how long will only 2 g of the sample remain?*

UNIT 5

# Trigonometry

## 5.1. Angles

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the initial side of the angle, and the position after rotation is the **terminal side**. The **endpoint** of the ray is the vertex of the angle.

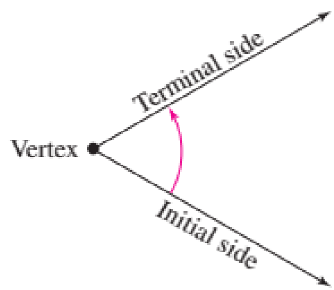



FIGURE 5.1.

Positive angles are generated by counterclockwise rotation, and negative angles by clockwise rotation.

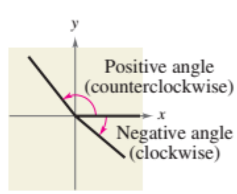
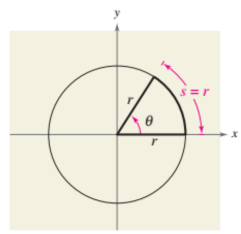



FIGURE 5.2.

**5.1.1. Radian Measure.** So far, it is commonly known that angles are measured in degrees. We introduce another way of measuring angles in this section known as the radian measure.

Suppose we have a circle with radius $r$ and $s$ an arc length on the circle. Let $\theta$ be an angle whose initial point is the $x$-axis and its terminal point is at a distance $s$ from the $x$-axis. A **radian** is the measure of a central angle that intercepts an arc equal in length to the radius of the circle. See the circle below:




*Arc length = radius when θ = 1 radian*

FIGURE 5.3.

Algebraically,

$$\theta = \frac{s}{r}$$

where $\theta$ is measured in radians.

Since the circumference of a circle is given by $2\pi r$, a central angle of one full revolution corresponds to an arc length of

$$s = 2\pi r.$$

Therefore, the radian measure of an angle of one full revolution is $2\pi$. We can then obtain the following:

$$\frac{1}{2}\text{ revolution} = \frac{2\pi}{2} = \pi \text{ radians,}$$
$$\frac{1}{4}\text{ revolution} = \frac{2\pi}{4} = \frac{1}{2}\text{ radians,}$$
$$\frac{1}{6}\text{ revolution} = \frac{2\pi}{6} = \frac{1}{3}\text{ radians.}$$

In the $xy$-coordinate system, there are four quadrants numbered $I, II, III$ and $IV$ as shown below.

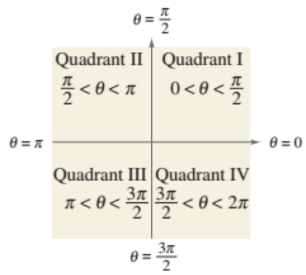



FIGURE 5.4.

It should be noted that the angles $0, \pi/2, \pi$ and $3\pi/2$ do not lie in any quadrant.

Two angles are **coterminal** if they have the same initial and terminal sides. For instance, the angles 0 and $2\pi$ are coterminal. You can find an angle that is coterminal to $\theta$ by adding $2\pi$ if $\theta$ is positive and subtracting $2\pi$ if $\theta$ is negative.

**Example:** The coterminal angle to $\theta = \frac{\pi}{6}$ is $\frac{\pi}{6} - 2\pi = \frac{13\pi}{6}$. The coterminal angle to $\theta = -\frac{2\pi}{3}$ is $-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}$.

Two positive angles $\alpha$ and $\beta$ are **complementary** if their sum is $\pi/2$, i.e., $\alpha + \beta = \pi/2$. Two angles are **supplementary** if their sum is $\pi$.

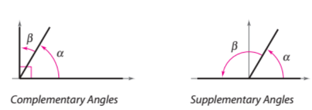



FIGURE 5.5.

**Activity 5.1.** *Find (if possible) the complement and supplement of each angle.*

(1) $\frac{\pi}{3}$

(2) $\frac{3\pi}{2}$

(3) $\frac{\pi}{12}$

(4) $\frac{11\pi}{12}$

### 5.1.2. Conversion Between Degrees and Radians.

(1) To convert degrees to radians, multiply degrees by $\frac{\pi \text{ rad}}{180}$.

(2) To convert radians to degrees, multiply radians by $\frac{180^0}{\pi \text{ rad}}$.

**Example:**

(a.) Express $60^0$ in radians (b.) Express $\frac{\pi}{6}$ rad in degrees.

**Solution 5.1.** *(a.)* $60^0 = 60\left(\frac{\pi}{180}\right)\text{rad} = \frac{\pi}{3}$ rad, *(b.)* $\frac{\pi}{6}$ rad $= \left(\frac{\pi}{6}\right)\left(\frac{180}{\pi}\right) = 30^0$.

**Activity 5.2.** (1) *Rewrite each angle in radian measure as a multiple of (Do not use a calculator).*

(a) $30^0$

(b) $150^0$

(c) $315^0$

(d) $-240^0$

(2) *Rewrite each angle in degree measure. (Do not use a calculator.)*

(a) $\frac{3\pi}{2}$

(b) $\frac{7\pi}{3}$

(c) $-\frac{7\pi}{12}$

(d) $-\frac{11\pi}{30}$

## 5.2. The Unit Circle

The **unit circle** is the circle of radius 1 centered at the origin in the xy-plane. Its equation is

$$x^2 + y^2 = 1.$$

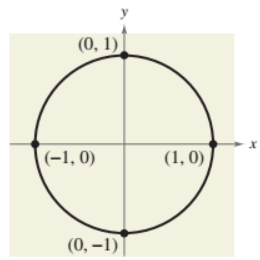



FIGURE 5.6.

Suppose $t$ is a real number. Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping as shown below:

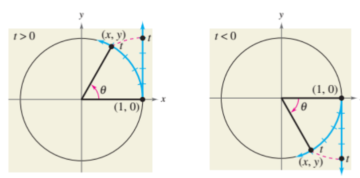
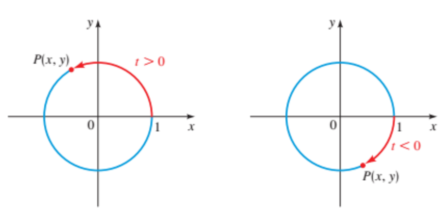



FIGURE 5.7.

As the real number line is wrapped around the unit circle, each real number $t$ corresponds to a point on the circle. The point obtained in this way is called the **terminal point** determined by the real number $t$.




(a) Terminal point $P(x, y)$ determined by $t > 0$ (b) Terminal point $P(x, y)$ determined by $t < 0$

FIGURE 5.8.

The circumference of the unit circle is $C = 2\pi$. So if a point starts at $(1, 0)$ and moves counterclockwise all the way around the unit circle and returns to $(1, 0)$, it travels a distance of $2\pi$. Check the figure below for instance:

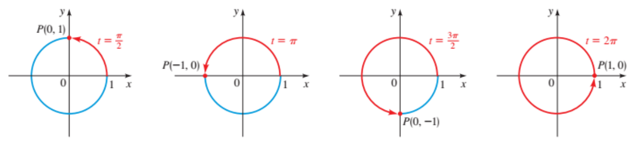



FIGURE 5.9.

Let $t$ be a real number. The **reference number** $\bar{t}$ associated with $t$ is the shortest distance along the unit circle between the terminal point determined by $\bar{t}$ and the $x$-axis. To find the reference number $\bar{t}$, its helpful to know the quadrant in which the terminal point determined by $t$ lies. If the terminal point lies in quadrants I or IV, where $x$ is positive, we find by moving along the circle to the positive $x$-axis. If it lies in quadrants II or III, where $x$ is negative, we find by moving along the circle to the $t$ negative x-axis.

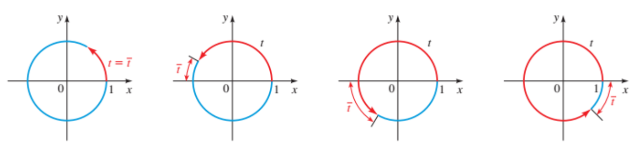


FIGURE 5.10.

**Example:**

Find the reference number for each value of $t$. (a.) $t = \frac{5\pi}{6}$, (b.) $t = \frac{7\pi}{4}$, (c.) $t = -\frac{2\pi}{3}$.

**Solution 5.2.** *(a.)* $\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$,
*(b.)* $\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$,
*(c.)* $\bar{t} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

In general, each real number also corresponds to a central angle whose radian measure is $t$. The following table shows some common $t$ values and their corresponding terminal points.

| $t$ | Terminal point determined by $t$ |
|---|---|
| 0 | $(1,0)$ |
| $\frac{\pi}{6}$ | $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ |
| $\frac{\pi}{4}$ | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ |
| $\frac{\pi}{3}$ | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ |
| $\frac{\pi}{2}$ | $(0,1)$ |

FIGURE 5.11.

**Activity 5.3.** (1) *Find the point $(x, y)$ on the unit circle that corresponds to the real number $t$.*

(a) $t = \frac{\pi}{4}$

(b) $t = \frac{5\pi}{3}$

(c) $t = \pi$

(d) $t = \frac{3\pi}{2}$

(2) *Find the reference number for each $t$ in question 1 above.*

## 5.3. Trigonometric Functions

Remember that for a given real number $t$, to find the terminal point $P(x, y)$, we move along the unit circle a distance of $t$ starting at the point $(1, 0)$. If $t$ is positive, we move counterclockwise and if $t$ is negative, we move clockwise. The $x$ and $y$ coordinates of the terminal point $P(x, y)$ can be used to define trigonometric functions.

$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}(x \neq 0)$

$\csc t = \frac{1}{y}(y \neq 0) \quad \sec t = \frac{1}{x}(x \neq 0) \quad \cot t = \frac{x}{y}(y \neq 0)$

**Examples:** Find the six trigonometric functions of each real number $t$. (a)$t = \frac{\pi}{3}$ (b) $t = \frac{\pi}{2}$

**Solution 5.3.** (a) *The terminal point determined by $t$ is $P(\frac{1}{2}, \frac{\sqrt{3}}{2})$. We have*

$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{\pi}{3} = \frac{1}{2}, \quad \tan \frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3}, \quad \sec \frac{\pi}{3} = 2, \quad \cot \frac{\pi}{3} = \frac{1/2}{\sqrt{3}/2} = \frac{\sqrt{3}}{3}.$

(b) *The terminal point determined by $\pi/2$ is $P(0, 1)$. So we have*

$\sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0, \quad \csc \frac{\pi}{2} = 1, \quad \cot \frac{\pi}{2} = \frac{0}{1} = 0.$

*Note that $\tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are undefined because $x = 0$ appears in the denominator.*

The following table shows some special values of the trigonometric functions.

| $t$ | $\sin t$ | $\cos t$ | $\tan t$ | $\csc t$ | $\sec t$ | $\cot t$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | — | 1 | — |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | 2 | $\frac{2\sqrt{3}}{3}$ | $\sqrt{3}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ | 1 |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{2\sqrt{3}}{3}$ | 2 | $\frac{\sqrt{3}}{3}$ |
| $\frac{\pi}{2}$ | 1 | 0 | — | 1 | — | 0 |

FIGURE 5.12.

**Activity 5.4.** (1) *Evaluate (if possible) the six trigonometric functions of the real number.*

(a) $t = \frac{3\pi}{4}$

(b) $t = 5\pi$

(c) $t = -\frac{15\pi}{2}$

(d) $t = -\frac{9\pi}{4}$

(2) *Use* $\sin t = \frac{1}{3}$ *to evaluate the indicated functions.*

(a) $\sin(-t)$

(b) $\csc(-t)$

(c) $\sin(\pi - t)$

(d) $\sin(t + \pi)$

### 5.3.1. Domains of Trigonometric Functions.

| **Function** | **Domain** |
|---|---|
| sin,cos | All real numbers |
| tan, sec | All real numbers other than $\frac{\pi}{2} + n\pi$ for any integer $n$ |
| cot,csc | All real numbers other than $n\pi$ for any integer $n$. |

### 5.3.2. Values of Trigonometric Functions.

When computing the values of trigonometric functions, we first determine their signs. The sign of a trigonometric function is determined by the quadrant in which the terminal point of $t$ lies. The following device becomes useful to remember the sign of any trigonometric function in any quadrant.

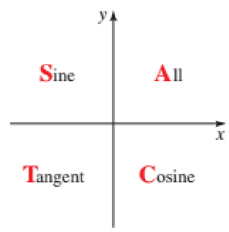



FIGURE 5.13.

In the first quadrant, all trigonometric functions are positive. In the second, only sine is positive. In the third, only tangent is positive and in the fourth, only cosine is positive. It should be noted that where sin is positive, then csc is also positive since they are just reciprocals of each other. The same applies to tan and cot, and to cos and sec.

**Example:** Find each value:

(a) $\cos\frac{2\pi}{3}$ (b) $\tan -\frac{\pi}{3}$ (c) $\sin\frac{19\pi}{4}$

**Solution 5.4.** (a) *Since the terminal point for* $\frac{2\pi}{3}$ *is in quadrant II,* $\cos\frac{2\pi}{3}$ *is negative. We find the reference number to be* $\frac{\pi}{3}$.

*Then*

$$\cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}.$$

(b) *The reference number for* $-\frac{\pi}{3}$ *is* $\frac{\pi}{3}$. *Since the terminal point of* $-\frac{\pi}{3}$ *is in quadrant IV,* $\tan -\frac{\pi}{3}$ *is negative. Thus,*

$$\tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

(c) *Since* $(19\pi/4) - 4\pi = 3\pi/4$, *the terminal points determined by* $19\pi/4$ *and* $3\pi/4$ *are the same. The reference number for* $3\pi/4$ *is*

*$\pi/4$. The terminal point is in quadrant II, therefore, $\sin(3\pi/4)$ is positive. Thus,*

$$\sin\left(\frac{19\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = +\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

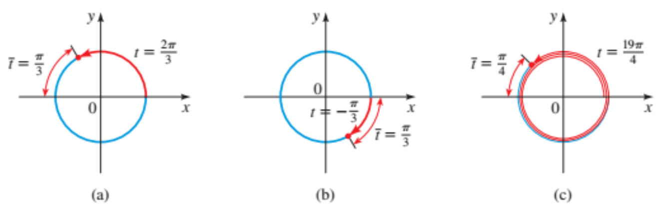



FIGURE 5.14.

**Activity 5.5.** *Find the values of the trigonometric functions of $t$ from the given information.*

(1) $\sin t = \frac{3}{5}$, *terminal point of $t$ is in quadrant II.*
(2) $\cos t = -\frac{4}{5}$, *terminal point of $t$ is in quadrant III.*
(3) $\tan t = -\frac{3}{2}$, $\cos t > 0$.
(4) $\sec t = 2$, $\sin t < 0$.

**5.3.3. Even-Odd Trigonometric Functions.** Recall that if a function $f$ satisfies $f(-x) = f(x)$ for every $x$ in its domain, then $f$ is called an even function. If a function $f$ satisfies $f(-x) = -f(x)$ for every $x$ in its domain, then it is an odd function.

As is every function, trigonometric functions are either even or odd. Sine, cosecant, tangent, contangent are all odd functions while cosine and secant are even functions, i.e.,

$$\sin(-t) = -\sin t, \qquad \cos(-t) = \cos t, \qquad \tan(-t) = -\tan t,$$

$$\csc(-t) = -\csc t, \qquad \sec(-t) = \sec t, \qquad \cot(-t) = -\cot t.$$

**Activity 5.6.** *Determine whether the function is even, odd, or neither.*

(1) $f(x) = x^2 \sin x$

(2) $f(x) = \sin x \cos x$

(3) $f(x) = x \sin^3 x$

(4) $f(x) = \cos(\sin x)$

## 5.4. Right Triangle Trigonometry

We also look at trigonometric functions from a right triangle perspective. Consider the following right triangle:


FIGURE 5.15.

Based on this triangle, the six trigonometric functions are defined as follows:

$$\sin\theta = \frac{opp}{hyp}, \qquad \cos\theta = \frac{adj}{hyp}, \qquad \tan\theta = \frac{opp}{adj},$$

$$\csc\theta = \frac{hyp}{opp}, \qquad \sec\theta = \frac{hyp}{adj}, \qquad \cot\theta = \frac{adj}{opp}.$$

The abbreviations $opp, adj$ and $hyp$ represent the lengths of the three sides of a right triangle, i.e.,

- $opp$ = length of the side opposite $\theta$,
- $adj$ = length of the side adjacent to $\theta$,
- $hyp$ = length of the hypotenuse.

It must be clear that the opposite and adjacent sides are always in relation to $\theta$. In other words, they can be different depending on the position of $\theta$ while the hypotenuse is always the same.

It is clear to see that the functions in the second row above are just reciprocals of the functions in the first row.

**Example:** Consider the following triangle: Find the six trigonometric functions.

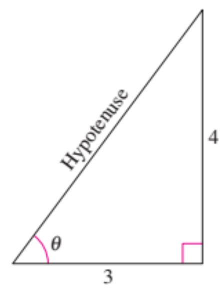



FIGURE 5.16.

**Solution 5.5.** *We first have to compute the value of the hypotenuse using the Pythagorean Theorem. Thus*

$$\begin{aligned} hyp &= \sqrt{4^2 + 3^2} \\ &= \sqrt{25} \qquad (5.1) \\ &= 5. \end{aligned}$$

*Thus, the six trigonometric functions of $\theta$ are*

$$\sin\theta = \frac{4}{5}, \qquad \cos\theta = \frac{3}{5}, \qquad \tan\theta = \frac{4}{3},$$

$$\csc\theta = \frac{5}{4}, \qquad \sec\theta = \frac{5}{3}, \qquad \cot\theta = \frac{3}{4}.$$

**Example:** Find the values of $\sin 45^0$, $\cos 45^0$ and $\tan 45^0$.

Construct a right triangle with $45^0$ as one of its acute angles, the adjacent and opposite sides to the $45^0$ angle should have length 1 each. Then the hypotenuse is computed to be $\sqrt{2}$. The triangle is shown below:

**Solution 5.6.** *Thus,*

$$\begin{aligned} \sin 45^0 &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \cos 45^0 &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \tan 45^0 &= \frac{1}{1} = 1 \end{aligned}$$

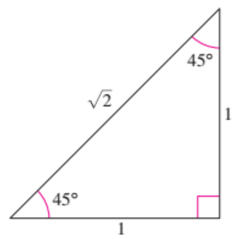



FIGURE 5.17.

**Example:** Use the equilateral triangle as shown below to find the values of $\sin 60^0$, $\cos 60^0$, $\sin 30^0$, and $\cos 30^0$.

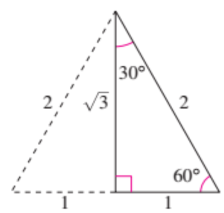



FIGURE 5.18.

**Solution 5.7.** *The lengths of the sides can be verified using the Pythagorean Theorem.*

$$\sin 60^0 = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60^0 = \frac{1}{2},$$

$$\sin 30^0 = \frac{1}{2} \quad \text{and} \quad \cos 60^0 = \frac{\sqrt{3}}{2}.$$

The following table shows the sine, cosine and tangent values of the common angles, $30^0$, $45^0$ and $60^0$.

The angles used in the two previous examples are most common. From the right triangle definitions, it can be shown that cofunctions of complementary angles are equal. That is, if $\theta$ is an acute angle, then the following relationships are true.

**Sines, Cosines, and Tangents of Special Angles**

| | | |
|---|---|---|
| $\sin 30° = \sin \frac{\pi}{6} = \frac{1}{2}$ | $\cos 30° = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ | $\tan 30° = \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$ |
| $\sin 45° = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ | $\cos 45° = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ | $\tan 45° = \tan \frac{\pi}{4} = 1$ |
| $\sin 60° = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ | $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$ | $\tan 60° = \tan \frac{\pi}{3} = \sqrt{3}$ |

FIGURE 5.19.

$$\sin(90^0 - \theta) = \cos\theta \qquad \cos(90^0 - \theta) = \sin\theta,$$

$$\tan(90^0 - \theta) = \cot\theta \qquad \cot(90^0 - \theta) = \tan\theta,$$

$$\sec(90^0 - \theta) = \csc\theta \qquad \csc(90^0 - \theta) = \sec\theta.$$

**Activity 5.7.** (1) *Find the exact values of the six trigonometric functions of the angle shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)*

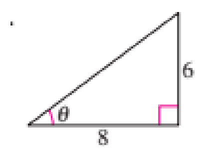



FIGURE 5.20.

(2) *Use the given function value(s), and trigonometric identities (including the cofunction identities), to find the indicated trigonometric functions.*

$$\sin 60^0 = \frac{\sqrt{3}}{2}, \quad \cos 60^0 = \frac{1}{2}$$

(a) $\tan 60^0$
(b) $\sin 30^0$
(c) $\cos 30^0$
(d) $\cot 60^0$

(3) *Find the values of $\theta$ in degrees $(0^0 < \theta < 90^0)$ and radians $(0 < \theta < \pi/2)$ without the aid of a calculator.*

   (a) $\sin\theta = \frac{1}{2}$
   (b) $\cos\theta = \frac{\sqrt{2}}{2}$
   (c) $\tan\theta = \sqrt{3}$
   (d) $\csc\theta = \frac{2\sqrt{3}}{3}$

**5.4.1. The Law of Sines.** Trigonometric ratios are used to solve right triangles. Trigonometric functions can be used to solve oblique triangles, thus, triangles without right angles. We follow the convention of labeling angles of a triangle as A, B, C and the lengths of the corresponding opposite sides as $a, b$, and $c$ as shown below.

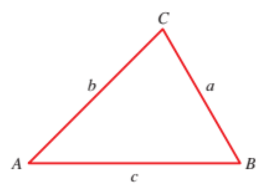



FIGURE 5.21.

The Law of Sines states that in a triangle $ABC$, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This law is used to solve two cases:

- Given one side and two angles, (ASA or SAA), e.g., given $a$ and $A, B$, find $b$.
- Given two sides and an angle opposite one of those sides, (SSA), e.g., given $a, b$ and $B$, find $A$.

**Example:** Solve for $a$, $b$ and $B$ in the following triangle.

**Solution 5.8.** *We find $\angle B$:*

$$\angle B = 180^0 - (20^0 + 25^0) = 135^0.$$

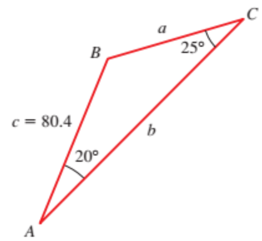



FIGURE 5.22.

*To find $a$, we need $A$, $C$ and $c$ to use the sine rule. Thus SAA, two angles and one side.*

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$a = \frac{c \sin A}{\sin C}$$
$$a = \frac{80.4 \sin 20^0}{\sin 25^0}$$
$$a \approx 65.1$$

*And to find $b$, we need $B$, $C$ and $c$. Thus*

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$
$$b = \frac{c \sin B}{\sin C}$$
$$b = \frac{80.4 \sin 135^0}{\sin 25^0}$$
$$b \approx 134.5$$

**Activity 5.8.** (1) *Use the Law of Sines to find the indicated side $x$.*

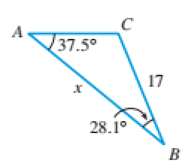



FIGURE 5.23.

(2) *Use the Law of Sines to find the indicated angle $\theta$.*

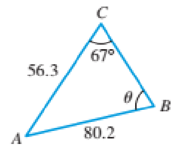



FIGURE 5.24.

(3) *Sketch each triangle, and then solve the triangle using the Law of Sines.*

(a) $\angle A = 50^0$, $\angle B = 68^0$, $c = 230$.

(b) $\angle A = 30^0$, $\angle C = 65^0$, $b = 10$.

**5.4.2. The Cosine Rule.** The Law of Cosines applies for the following two cases:

- Given two sides and the included angle, (SAS),
- Given all three sides, (SSS).

The law states that in any triangle $ABC$, we have

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

$$b^2 = a^2 + c^2 - 2ac\cos B,$$

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

**Example:** A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in the figure below. Use the surveyors data to approximate the length of the tunnel.

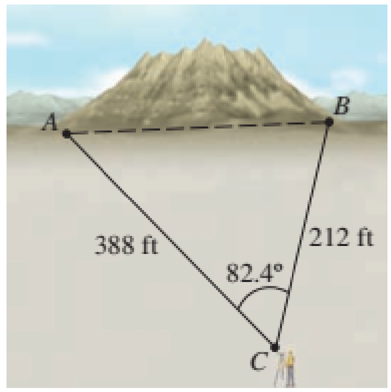



FIGURE 5.25.

**Solution 5.9.** *We have to approximate the length c. We have two sides and an angle in between, this is the case SAS. So, we use the Law of Cosines.*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 388^2 + 212^2 - 2(388)(212)\cos 82.4^0$$

$$c^2 \approx 173730.2367$$

$$c^2 \approx \sqrt{173730.2367}$$

$$c \approx 416.8$$

*Thus the tunnel will be approximately 417 ft long.*

**Activity 5.9.** (1) *Use the Law of Cosines to determine the indicated side $x$.*

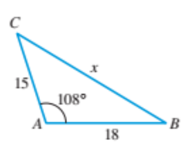



FIGURE 5.26.

(2) *Use the Law of Cosines to determine the indicated angle $\theta$.*

(3) *Sketch each triangle, and then solve the triangle using the Law of Cosines.*

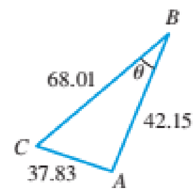



FIGURE 5.27.

(a) $a = 3.0,\ b = 4.0,\ \angle C = 53^0$

(b) $b = 125,\ c = 162,\ \angle B = 40^0$

**5.4.3. Navigation: Heading and Bearing.** In navigation, direction is often given as a bearing, that is, an acute angle measured from due north or due south. Check the examples below:

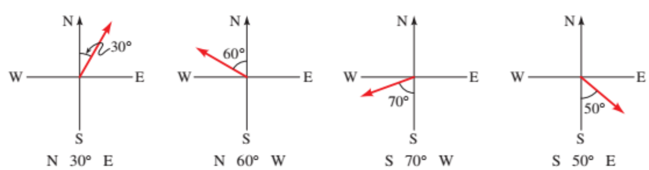



FIGURE 5.28.

**Example:** A pilot sets out from an airport and heads in the direction N $20^0$ E, flying at 200 mi/h. After one hour, he makes a course correction and heads in the direction N $40^0$ E. Half an hour after that, engine trouble forces him to make an emergency landing.

(a) Find the distance between the airport and his final landing point.

(b) Find the bearing from the airport to his final landing point.

We will need to sketch the course first.

**Solution 5.10.** (a) *Using the Law of Cosines, we find b,*

$$b^2 = 200^2 + 100^2 - 2(200)(100)\cos 160^0$$

$$b^2 \approx 87587.70$$

$$b \approx 295.95$$

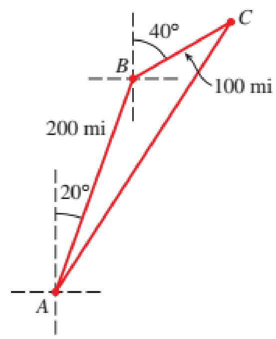



FIGURE 5.29.

*The pilot lands about 296 mi from his starting point.*

(b) *We need to find* $\angle A$ *using the Law of Sines.*

$$\frac{\sin A}{100} = \frac{\sin 160^0}{295.95}$$
$$\sin A = 100 \cdot \frac{\sin 160^0}{295.95}$$
$$\sin A \approx 0.11557$$
$$A \approx \sin^{-1}(0.11557)$$
$$A \approx 6.636^0$$

*The line from the airport to the final landing site points in the direction* $20^0 + 6.636^0 = 26.636^0$ *east of due north. The bearing is about* $N\ 26.6^0\ E$.

**Activity 5.10.** (1) *A short-wave radio antenna is supported by two guy wires, 165 ft and 180 ft long. Each wire is attached to the top of the antenna and anchored to the ground, at two anchor points on opposite sides of the antenna. The shorter wire makes an angle of 67! with the ground. How far apart are the anchor points?*

(2) *A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is* $58^0$. *A guy wire is to be attached to the top of the tower and to the ground,* $100m$

*downhill from the base of the tower. The angle a in the figure is determined to be* $12^0$. *Find the length of cable required for the guy wire.*

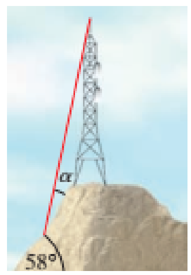



FIGURE 5.30.

(3) *Two straight roads diverge at an angle of* $65^0$. *Two cars leave the intersection at 2:00 P.M., one traveling at 50 mi/h and the other at 30 mi/h. How far apart are the cars at 2:30 P.M.?*

(4) *Two boats leave the same port at the same time. One travels at a speed of 30 mi/h in the direction* $N50^0E$ *and the other travels at a speed of 26 mi/h in a direction* $S70^0E$ *(see the figure). How far apart are the two boats after one hour?*

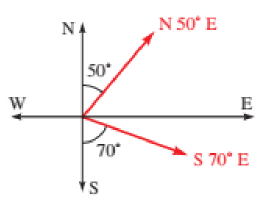



FIGURE 5.31.

## 5.5. Trigonometric Identities

Trigonometric functions are related to each other through equations called trigonometric identities.

**Reciprocal identities:**

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

**Quotient Identities:**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

**Pythagorean Identities:**

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$$

**Sum Identities:**

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

**Double Angle Identities:**

If in the sum identities above, $A = B$, then the identities become the following:

$$\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A,$$

$$\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A.$$

**Example:** If $\cos\theta = \frac{3}{5}$ and $\theta$ is in quadrant IV, Find the values of all the trigonometric functions at $\theta$.

**Solution 5.11.** *Using* $\cos\theta$*, we can find* $\sin\theta$ *using the Pythagorean identity* $\sin^2\theta + \cos^2\theta = 1$*. Thus*

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1 \\ \sin^2\theta &= 1 - \left(\frac{9}{25}\right) = \frac{16}{25} \\ \sin\theta &= \pm\frac{4}{5} \end{aligned} \tag{5.2}$$

*Since* $\theta$ *is in quadrant IV,* $\sin\theta$ *is negative, so* $\sin\theta = -\frac{4}{5}$*. Now we can use the reciprocal identities to find values of the other functions at* $\theta$*.*

$$\sin\theta = -\frac{4}{5}, \qquad \cos\theta = \frac{3}{5}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4/5}{3/5} = -\frac{4}{3},$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{4}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}, \quad \cot\theta = \frac{1}{\tan\theta} = -\frac{3}{4}.$$

**Example:**

Simplify $\sin t + \cot t \cos t$.

**Solution 5.12.**

$$\begin{aligned} \sin t + \cot t \cos t &= \sin t + \left(\frac{\cos t}{\sin t}\right)\cos t \\ &= \frac{\sin^2 t + \cos^2 t}{\sin t} \\ &= \frac{1}{\sin t} \\ &= \csc t \end{aligned}$$

**Example:**

Simplify $\frac{1-\sin^2 x}{\csc^2 x - 1}$.

**Solution 5.13.** *We know that* $1 - \sin^2 x = \cos^2 x$ *and* $\csc^2 x - 1 = \cot^2 x$. *Then*

$$\begin{aligned}\frac{1-\sin^2 x}{\csc^2 x - 1} &= \frac{\cos^2 x}{\cot^2 x}\\ &= \cos^2 x \cdot \left(\frac{\sin^2 x}{\cos^2 x}\right)\\ &= \sin^2 x\end{aligned}$$

**Activity 5.11.** (1) *Use the given values to evaluate (if possible) all six trigonometric functions.*

(a) $\sin x = \frac{\sqrt{3}}{2}$

(b) $\tan\theta = \frac{3}{4}$

(2) *Use the fundamental identities to simplify the expression. There is more than one correct form of each answer.*

(a) $\cot\theta \sec\theta$

(b) $\sec^2 x(1-\sin^2 x)$

(c) $\sec\alpha \cdot \frac{\sin\alpha}{\tan\alpha}$

(d) $\frac{\tan^2\theta}{\sec^2\theta}$

(e) $\sin^4 x - \cos^4 x$

(f) $\tan^2 x - \tan^2 x \sin^2 x$

## 5.6. Trigonometric Graphs

**5.6.1. Sine and Cosine Functions.** A function $f$ is periodic if there is a positive number $p$ such that $f(t+p) = f(t)$ for every t. The Sine and Cosine functions are periodic in nature. They repeat values every $2\pi$. Thus,

$$\sin(t + 2n\pi) = \sin(t) \quad \text{for any integer n}$$

$$\cos(t + 2n\pi) = \cos(t) \quad \text{for any integer n}$$

To sketch their graphs, we only sketch one period $2\pi$. To draw their graphs more accurately, we use a table of values below. Note that we could still find other values by using a calculator.

Now we use this information to graph these functions.

| $t$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin t$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $0$ |
| $\cos t$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |

FIGURE 5.32.

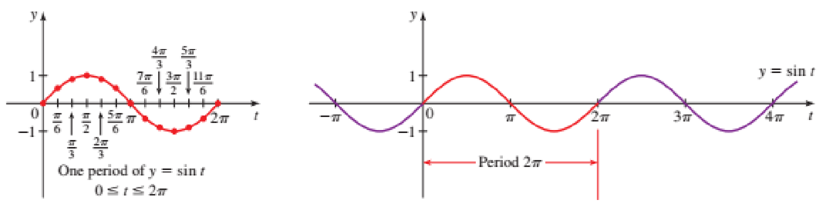



FIGURE 5.33.

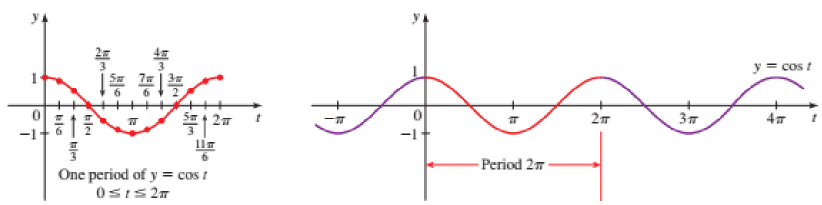



FIGURE 5.34.

It is easy to see that the graph repeats itself after every $2\pi$ period.

NB: It is customary to use the letter $x$ to denote the variable in the domain of a function, so we will use this notation onwards, i.e., $y = \sin x$.

**5.6.2. Graphs of $y = a\sin x$ and $y = a\cos x$.** In the graphs of $y = a\sin x$ and $y = a\cos x$, the number $|a|$ is called its amplitude. This number dictates the height of the curve. If $|a| < 1$, then the graphs are shrunk vertically. If $|a| > 1$, then the graphs are stretched vertically.

**Example:**

Look at the graph of $y = 2\sin x$ and $y = \frac{1}{2}\sin x$. We first start with the graph of $y = \sin x$ and then multiply the $y$-coordinates by 2 and $\frac{1}{2}$ respectively. When we multiply by 2, the graph is stretched vertically by a factor of 2 and when we multiply by $\frac{1}{2}$, the graph is shrunk vertically by a factor of $\frac{1}{2}$.

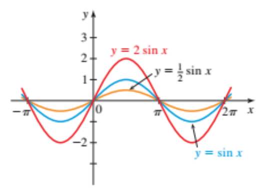



FIGURE 5.35.

This applies to cosine graphs as well.

**Activity 5.12.** *Sketch the graphs of the following functions.*

(1) $y = 3\cos x$

(2) $y = \frac{1}{2}\sin x$

(3) $y = -\frac{1}{3}\cos x$

(4) $y = -3\sin x$

**5.6.3. Graphs of $y = \sin kx$ and $y = \cos kx$.** In these functions, if $k \neq 0$, then both $y = \sin kx$ and $y = \cos kx$ have period given by $\frac{2\pi}{k}$. If $0 < |k| < 1$, the graphs are stretched horizontally and if $|k| > 1$, the graphs are shrunk horizontally.

**Example:**

Let's graph the sine curves $y = \sin 2x$ and $y = \sin \frac{1}{2}x$.

For $y = \sin 2x$, we find that the period is $\frac{2\pi}{2} = \pi$, so the graph completes one period in the interval $0 \leq x \leq \pi$.

For $y = \sin \frac{1}{2}x$, we find that the period is $2\pi \div \frac{1}{2} = 4\pi$, so the graph completes one period in the interval $0 \leq x \leq 4\pi$.

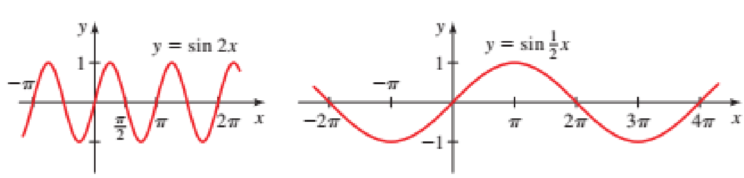



FIGURE 5.36.

The same principles apply to the cosine curves as well.

**Activity 5.13.** *Sketch the graphs of the following functions.*

(1) $y = 2\cos 2x$

(2) $y = -\cos 4x$

(3) $y = -\sin \frac{2\pi x}{3}$

(4) $y = \sin \frac{\pi x}{4}$

**5.6.4. Graphs of $y = a\sin k(x-b)$ and $y = a\cos k(x-b)$.** These graphs have the same shape as the graphs of $y = a\sin kx$ and $y = a\cos kx$ respectively but shifted c units to the right if $c > 0$ and $|c|$ units to the left if $c < 0$. The number $c$ is called the phase shift of the sine or cosine graph. An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

**Example:**

Shown below are graphs of $y = \sin\left(x - \frac{\pi}{3}\right)$ and $y = \sin\left(x + \frac{\pi}{6}\right)$.

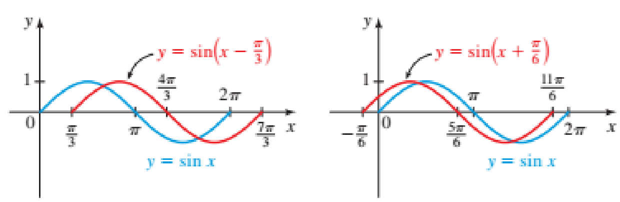



FIGURE 5.37.

**Example:**

Find the amplitude, period and phase shift of $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$ and graph one complete period.

The amplitude is $|a| = |3| = 3$. The period is $\frac{2\pi}{k} = \frac{2\pi}{2} = \pi$. The phase shift is $\frac{\pi}{4}$ to the right. One complete period occurs on the interval $\left[\frac{\pi}{4}, \frac{\pi}{4} + \pi\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

**Activity 5.14.** *Sketch the graphs of the following functions.*

(1) $y = 3\cos(x + \pi)$

(2) $y = -3 + 5\cos \frac{\pi t}{12}$

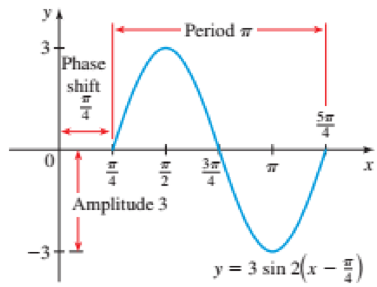



FIGURE 5.38.

(3) $y = -2\sin(4x + \pi)$

(4) $y = -4\sin\left(\frac{2}{3}x - \frac{\pi}{3}\right)$

**5.6.5. Graphs of Tangent, Cotangent, Secant and Cosecant.** The graphs of tangent and cotangent have period $\pi$:

$$\tan(x + \pi) = \tan x \qquad \text{and} \qquad \cot(x + \pi) = \cot x.$$

The graphs of cosecant and secant have period $2\pi$:

$$\csc(x + 2\pi) = \csc x \qquad \text{and} \qquad \sec(x + 2\pi) = \sec x.$$

5.6.5.1. *Tangent and Cotangent Graphs.* We will start with the graph of the tangent function. Since the tangent function has a period of $\pi$, we can only sketch the graph on any interval of length $\pi$. We sketch the graph on the interval $(-\pi/2, \pi/2)$. Note that as $x$ approaches $-\pi/2$ and $\pi/2$, $\cos x$ approaches 0 and $\sin x$ approaches 1. Thus, $\tan x = \frac{\sin x}{\cos x}$ gets large.
The graph of $\tan x$, thus, approaches the vertical lines $x = \pi/2$ and $x = -\pi/2$. These lines are vertical asymptotes. Graphing the cotangent function follows the same arguments. The figure below shows the graph of $y = \tan x$ on the interval $-\pi/2 < x < \pi/2$ and $y = \cot x$ on the interval $0 < x < \pi$.

5.6.5.2. *Cosecant and Secant graphs.* To graph these functions, we use the reciprocal identities. Just as their reciprocals, they have a period of $2\pi$. So the graphs of these two functions look as below:

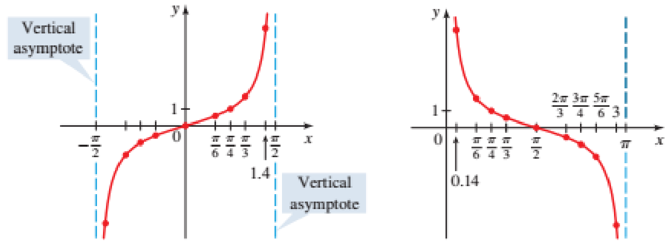



FIGURE 5.39.

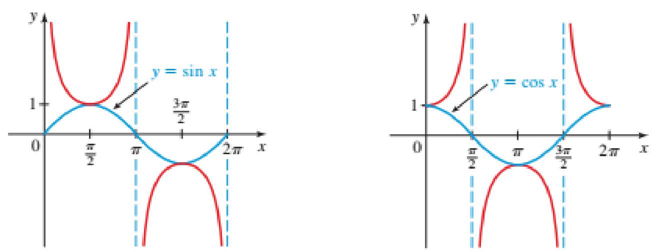



FIGURE 5.40.

All the graphs described above are shown below. Note the vertical asymptotes.

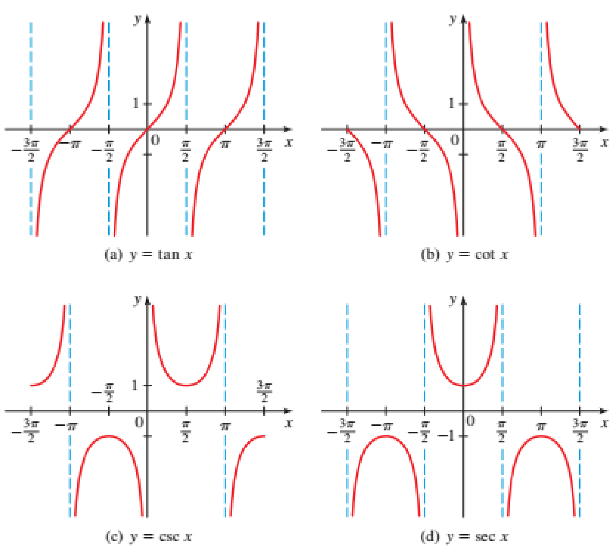



FIGURE 5.41.

## 5.7. Graphs of Transformations of Tangent and Cotangent

**5.7.1. Graphs of $y = a\tan kx$ and $y = a\cot kx$.** The functions $y = a\tan kx$ and $y = a\cot kx$, $k > 0$ have period $\pi/k$. Thus, one complete period of these functions occurs on any interval of length $\pi/2$.

- To graph one period of $y = a\tan kx$, an appropriate interval is $\left(-\frac{\pi}{2k}, \frac{\pi}{2k}\right)$.
- To graph one period of $y = a\cot kx$, an appropriate interval is $\left(0, \frac{\pi}{k}\right)$.

**Example:**

Graph the following tangent functions:

a. $y = \tan 2x$,

**Solution 5.14.** *The period is $\pi/2$ and the appropriate interval is $(-\pi/4, \pi/4)$. The graph has the same shape as that of the tangent function, but is shrunk horizontally by a factor of $\frac{1}{2}$. We then repeat that portion of the graph to the left and to the right.*

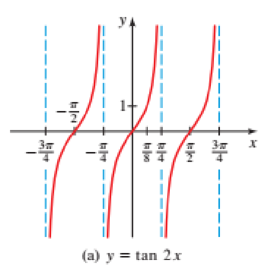



FIGURE 5.42.

b. $y = \tan 2\left(x - \frac{\pi}{4}\right)$.

**Solution 5.15.** *This function completes one period as* $2\left(x-\frac{\pi}{4}\right)$ *varies from* $-\frac{\pi}{2}$ *to* $\frac{\pi}{2}$*. So the start of period is*

$$\begin{aligned} 2\left(x-\frac{\pi}{4}\right) &= -\frac{\pi}{2} \\ x-\frac{\pi}{4} &= -\frac{\pi}{4} \\ x &= 0 \end{aligned}$$

*and the end of period is*

$$\begin{aligned} 2\left(x-\frac{\pi}{4}\right) &= \frac{\pi}{2} \\ x-\frac{\pi}{4} &= \frac{\pi}{4} \\ x &= \frac{\pi}{2} \end{aligned}$$

*The graph is the same as the graph in (a) but it is shifted to the right by* $\pi/4$*.*

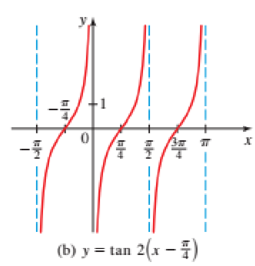



FIGURE 5.43.

**Activity 5.15.** *Graph the following functions.*

(1) $y=\frac{1}{2}\tan(\pi x-\pi)$
(2) $y=\tan\frac{1}{2}\left(x+\frac{\pi}{4}\right)$
(3) $y=\cot\left(2x-\frac{\pi}{2}\right)$
(4) $y=\cot\frac{\pi}{2}x$

## 5.8. Graphs of Transformations of Cosecant and Secant Functions

The functions $y = a \csc kx$ and $y = a \sec kx$ for any integer $k > 0$ have period $2\pi/k$.

The following are examples of such graphs. (a.) $y = \frac{1}{2} \csc 2x$.

We see here that the period is $2\pi/2 = \pi$. An appropriate interval is $[0, \pi]$ and the asymptotes occur in this interval whenever $\sin 2x = 0$. Thus, the asymptotes are $x = 0$, $x = \pi/2$ and $x = \pi$. In this interval, we sketch a graph with the same shape as that of one period of the cosecant function. The complete graph is obtained by repeating this shape to the left and to the right.

(b.) $y = \frac{1}{2} \csc \left(2x + \frac{\pi}{2}\right)$.

We first notice that a graph of $y = \csc x$ completes one period between $x = 0$ and $x = 2\pi$. So

$$\begin{aligned} 2x + \frac{\pi}{2} &= 0 \\ 2x &= -\frac{\pi}{2} \\ x &= -\frac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned} 2x + \frac{\pi}{2} &= 2\pi \\ 2x &= \frac{3\pi}{2} \\ x &= \frac{3\pi}{4} \end{aligned}$$

So we graph the function $y = \frac{1}{2} \csc \left(2x + \frac{\pi}{2}\right)$ on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$. So the graph is the same as that in (a.) but shifted to the left $\pi/4$.

**Activity 5.16.** *Graph the following functions.*

(1) $y = 5 \csc 3x$

(2) $y = \frac{1}{2} \sec \left(x - \frac{\pi}{6}\right)$

(3) $y = \csc 2 \left(x + \frac{\pi}{2}\right)$

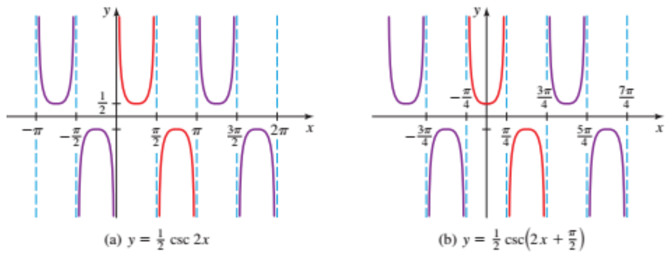



FIGURE 5.44.

(4) $y = 2\sec\left(\frac{1}{2}x - \frac{\pi}{3}\right)$

### 5.9. Inverse Trigonometric Functions and their Graphs

**5.9.1. The Inverse Sine Function.** The sine function is one-to-one on the interval [
$pi/2, \pi/2]$ and it attains all the values in its range on this interval. So, the sine function has an inverse in this interval.
The inverse sine function is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \iff \sin y = x.$$

The inverse function is also called arcsine, denoted by arcsin.
The graph of $\sin^{-1} x$ is obtained by reflecting the graph of $\sin x$, $-\pi/2 \leq x \leq \pi/2$, in the line $y = x$. This is shown below:

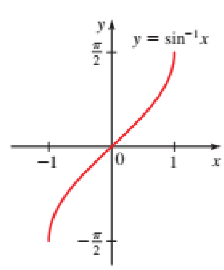



FIGURE 5.45.

We have the following cancellation properties:

- $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$,
- $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

**Example:**

Find each value:

(a) $\sin^{-1} \frac{1}{2}$,

**Solution 5.16.** *The number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* $\frac{1}{2}$ *is* $\pi/6$.

(b) $\sin^{-1}\left(-\frac{1}{2}\right)$,

**Solution 5.17.** *The number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* $-\frac{1}{2}$ *is* $-\pi/6$.

(c) $\sin^{-1}\left(\frac{3}{2}\right)$.

**Solution 5.18.** *Since* $\frac{3}{2} > 1$, *it is not in the domain of* $\sin^{-1} x$, *so* $\sin^{-1}\left(\frac{3}{2}\right)$ *is not defined.*

**Activity 5.17.** *Find the exact value of the expression, if it is defined.*

(1) $\sin\left(\sin^{-1} \frac{1}{2}\right)$

(2) $\sin^{-1}(-1)$

**5.9.2. The Inverse Cosine Function.** We restrict the domain of the cosine function to the interval $[0, \pi]$ because on it, the function attains each of its values exactly once. Thus, the cosine function is one-to-one in the interval $[0, \pi]$ and so has an inverse.

The inverse cosine function is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} y = x \iff \cos x = y.$$

The inverse cosine function is also called the arccosine, denoted by arccos.

The graph of $\sin^{-1} x$ is obtained by reflecting $y = \cos x$, $0 \le x \le \pi$, in the line $y = x$.

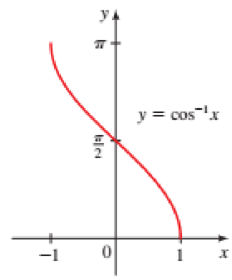



FIGURE 5.46.

We have the following cancellation properties that follow from the inverse properties:

- $\cos(cos^{-1}x) = x$ for $-1 \le x \le 1$,
- $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$.

**Example:**

Find each value:

- $\cos^{-1}\frac{\sqrt{3}}{2}$,

  **Solution 5.19.** *In the interval* $[0, \pi]$, $\cos^{-1}\frac{\sqrt{3}}{2} = \pi/6$.

- $\cos^{-1} 0$,

  **Solution 5.20.** *In the interval* $[0, \pi]$, $\cos^{-1} 0 = \pi/2$.

- $\cos^{-1}\frac{5}{7}$.

  **Solution 5.21.** *No rational multiple of* $\pi$ *has cosine* $\frac{5}{7}$*, so we use a calculator in radian measure to find the value approximately*

$$\cos^{-1}\frac{5}{7} \approx 0.77519.$$

**Activity 5.18.** *Find the exact value of each expression, if it is defined.*

(1) $\cos^{-1}(-1)$
(2) $\cos^{-1}\frac{1}{2}$
(3) $cos^{-1}\left(\cos\frac{5\pi}{6}\right)$
(4) $\cos(\sin^{-1} 0)$

**5.9.3. The Inverse Tangent Function.** Recall that the tangent function $\tan x$ has period $\pi$, and when graphing the function, we used the interval $(-\pi/2, \pi/2)$. We restrict the domain of the tangent function to this interval, i.e., $(-\pi/2, \pi/2)$ to obtain a one-to-one function.
The inverse tangent function is the function $\tan^{-1}$ with domain $\mathring{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \iff \tan y = x.$$

The inverse tangent function is also called arctangent, denoted by arctan.
The following cancellation properties follow:

- $\tan(\tan^{-1} x) = x$ for $x \in \mathring{R}$,
- $\tan^{-1}(\tan x) = x$ for $-\pi/2 < x < \pi/2$.

The graph is shown below:

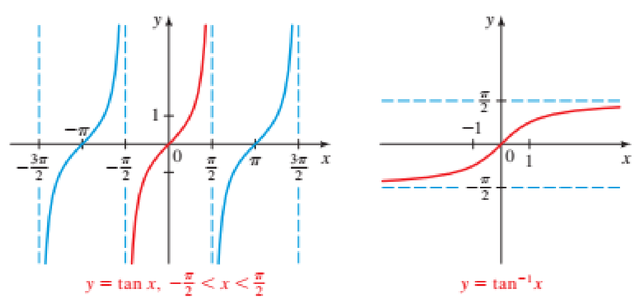



FIGURE 5.47.

**Example:**

Find each value:

- $\tan^{-1} 1$,

  **Solution 5.22.** *In the interval* $(-\pi/2, \pi/2)$, *we find* $\tan^{-1} 1 = \pi/4$.

- $\tan^{-1} \sqrt{3}$,

**Solution 5.23.** *In the interval* $(-\pi/2, \pi/2)$*, we find* $\tan^{-1}\sqrt{3} = \pi/4$.

- $\tan^{-1} 20$.

**Solution 5.24.** *Use a calculator in radian measure,* $\tan^{-1} 20 \approx -1.52084$.

**Activity 5.19.** *Find the exact value of each expression, if it is defined.*

(1) $\tan^{-1}\sqrt{3}$
(2) $\tan^{-1}\frac{\sqrt{3}}{3}$
(3) $\tan\left(\sin^{-1}\frac{1}{2}\right)$
(4) $\sin(\tan^{-1}(-1))$

**5.9.4. The Inverse Secant, Cosecant and Cotangent Functions.** It is better to restrict the domain of the inverse function to an interval in which the function is one-to-one and on which it can attain all its values. We display the graphs of inverse secant, cosecant and cotangent functions below.

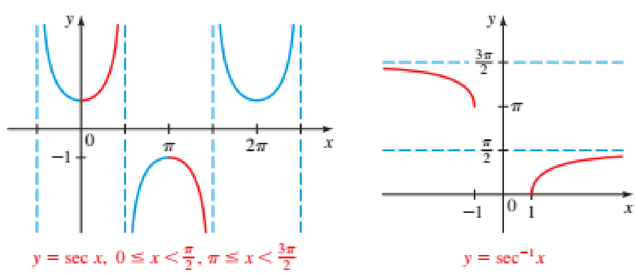



FIGURE 5.48.

**Activity 5.20.** *Find the exact value of each expression, if it is defined.*

(1) $\sec\left(\arctan\left(-\frac{3}{5}\right)\right)$
(2) $\csc\left(\arctan\left(-\frac{5}{12}\right)\right)$
(3) $\cot\left(\arctan\frac{5}{8}\right)$

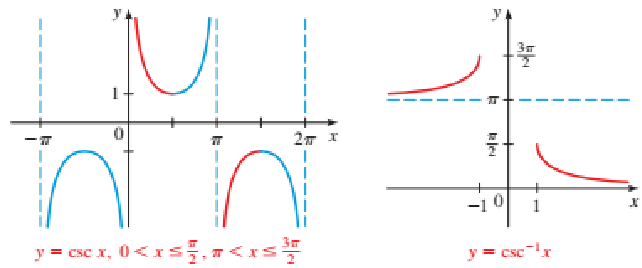



FIGURE 5.49.

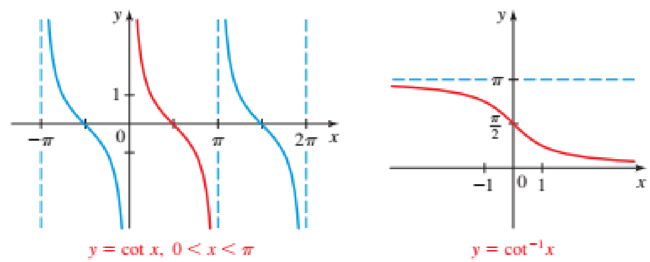



FIGURE 5.50.

(4) $\sec\left(\arcsin \frac{4}{5}\right)$