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Centre for Open and Distance Learning

Mathematics Module 3

Calculus One

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Contents

Introduction

The importance of concepts of calculus cannot be overemphasized. The necessity of calculus in manifested in our dynamic, everyday surroundings. In chemistry, the rate of reaction is modelled by differentiation and in the biological sciences, growth and decay phenomena are modelled by differential equations which involve integration. *Introduction to Calculus* addresses calculus of a single value. It is designed for students beginning science programmes. This book will be a source of additional examples, activities and exercises. The textual material is presented in such a way that you are involved in the development of ideas through activities and examples. Knowledge of functions will be assumed.

UNIT 1

LIMITS AND CONTINUITY

1.1. Introduction

In this unit you will learn about limits and their properties. The notion of limits is core to the study of calculus. It is important, therefore, to acquire working concept of limits before venturing into other topics of calculus. You will also learn about continuity of a function at a point or in a given interval.

Consider the population of a species. The growth of this population is guaranteed if food is available. This growth will slow down with time and eventually, with other specie supporting elements dwindling, become "constant".
The unit contains the following topics:

- Definition of a Limit;
- Properties of Limits;
- Techniques of Evaluating Limits;
- Limits at Infinity;
- Continuity of a Function

1.1.1. Learning objectives. By the end of this unit you should be able to:

- define the limit of a function at a given point;
- explain properties of limits;
- evaluate limits of functions at prescribed points;
- explain if a function is continuous at a given point;

1.1.2. Prerequisite knowledge. Knowledge of concepts of functions is sufficient prerequisite to successfully do this unit.

1.1.3. Time. You should be able to complete this unit in $5\frac{1}{2}$ hours.

1.2. Motivation to Limits

Gradient can be defined as the ratio of change in height to change in horizontal distance. In mathematics, this is defined as ratio of change in y to change in x. If gradient is denoted by m and $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are any two points on a line, then

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}.$$

This can also be regarded as *rate of change of* y on the interval $[x_1,\ x_2]$.

Activity 1.1. *Find the gradient of a line that passes through points* $(2,\ -3)$ *and* $(-4,\ 3)$.

Most functions modelling phenomena met in natural sciences are nonlinear and the interest is to find the rate of change at a *particular value* of the independent variable. In a chemical reaction, for instance, one would wish to determine the rate of chemical reaction or rate of economic growth at time t.

Remark 1.1. Note that a rate of change can be *positive* indicating an increase or *negative*.

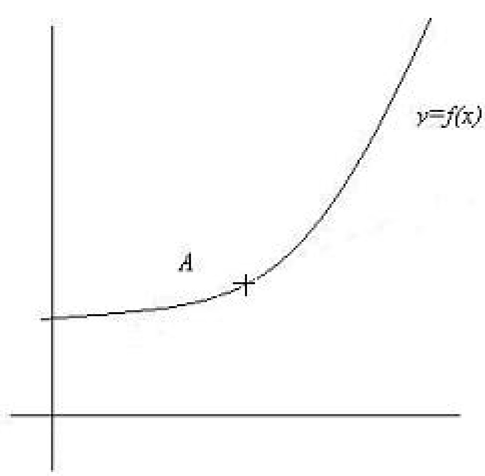


FIGURE 1.1. Gradient needed at point A

Consider the function in the Figure 1.1. To find the gradient of the curve of $f(x)$ at A, you need to think of a point B which is *very* close to A. The gradient of the join

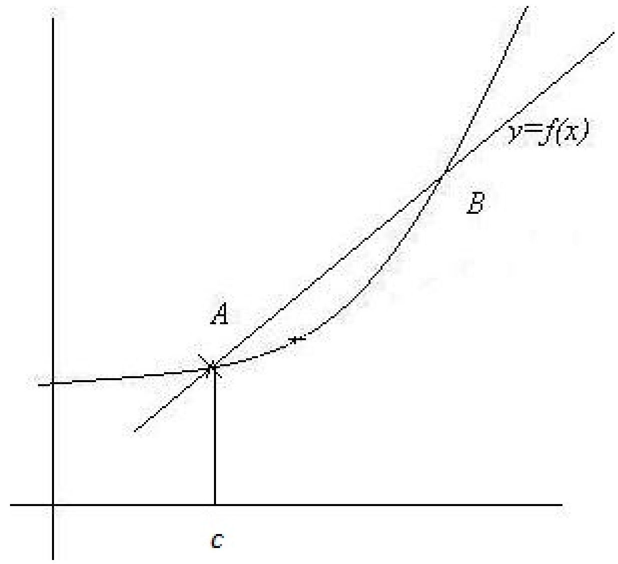


FIGURE 1.2. AB is a Secant line

of these points (the *secant*) will give an approximate value for the gradient at A. See Figure 1.2.

If, however, the point B is *moved closer* to A, then the approximation improves. Let c be the value of x at A and Δx be a small change in x on moving from A to B. Then the approximate gradient at A is given by

$$m = \frac{f(c+\Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Now the gradient of the curve at A is the same as the gradient of the tangent to the curve there. Thus as B is *moved closer* to A, the secant gets closer to the tangent line. See the illustration in Figure 1.3.

Activity 1.2. *When B is moved closer to A, what happens to Δx?*

Now let us turn to the notion of limits. Suppose you are asked to sketch the graph of the function given by

$$f(x) = \frac{x^2 - 1}{x + 1}, \; x \neq -1.$$

The function is not defined when $x = -1$. The line of $f(x)$ has a "hole" at $x = -1$. Look at the table below that shows various values of f corresponding values of x about $x = -1$. What can you say about $f(x)$ as x approaches -1?

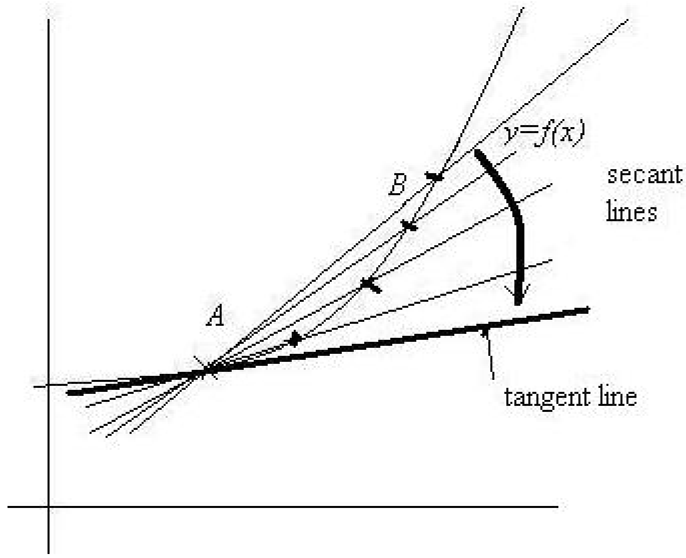


FIGURE 1.3. Secant line approaches tangent line

x	-0.75	-0.9	-0.99	-0.999	-1.001	-1.01	-1.1	-1.25
$f(x)$	-1.75	-1.9	-1.99	-1.999	-2.001	-2.01	-2.1	-2.25

Though x cannot be equal to -1, we can move arbitrarily close to -1 from left or right and consequently f moves closer to -2. We say the the *limit* of f as x approaches -1 is -2. This is written as

$$\lim_{x \to -1} f(x) = -2.$$

In general, if $f(x)$ becomes arbitrarily close to a single number K as x approaches a from either side, then we say the limit of $f(x)$ as x approaches a is K and is written as

$$\lim_{x \to a} f(x) = K.$$

The limit of a function can be estimated by evaluating the function at several points near a.

Activity 1.3. *Complete the tables and use the results to estimate the given limits.*

(1)

$$\lim_{x \to 2} \frac{x-2}{x^2 - x - 2}.$$

x	*1.9*	*1.99*	*1.999*	*2.001*	*2.01*	*2.1*
$f(x)$						

(2)

$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}.$$

x	-0.1	-0.01	-0.001	*0.001*	*0.01*	*0.1*
$f(x)$						

There are some functions that do not have limits at some points. Consider the function

$$g(x) = \frac{|x-1|}{x-1}.$$

Suppose we want to evaluate the limit

$$\lim_{x\to 1} g(x).$$

You note that for $x < 1$, $g(x) = -1$ and $g(x) = 1$ for $x > 1$.

That is, if $x \to 1$ from the left, $g(x) \to -1$ and when $x \to 1$ from the right, $g(x) \to 1$. So

$$\lim_{x\to 1} g(x)$$

does not exist.

Activity 1.4. *Discuss the existence of the limit*

$$\lim_{x\to 0}\frac{1}{x^2}.$$

You may need a graph of $\frac{1}{x^2}$ *to help you.*

We now give a formal $\epsilon - \delta$ definition of a limit.

Definition 1.1. Let $f(x)$ be a function defined on an open interval containing a (with the possibility of an exception at a) and let K be a real number and ϵ be an infinitesimal distance. Then

$$\lim_{x\to a} f(x) = K$$

means that $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$$|f(x) - K| < \epsilon \text{ whenever } 0 < |x-a| < \delta$$

Example 1.1. Find the indicated limit K and then find $\epsilon > 0$ such that $|f(x)-K| < 0.01$.

(1) $\lim_{x\to 2}(3x+2)$,

(2) $\lim_{x\to 4}(4-\frac{x}{2})$

Solution

(1) Now, $\lim_{x\to 2}(3x+2) = 8$. Thus $|3x+2-8| < 0.01$. But $|3x-6| = 3|x-2|$. It follows, therefore, that $|x-2| < \frac{1}{300}$ and that $\epsilon = \frac{1}{300}$.

(2) In this case, $\lim_{x\to 4}(4-\frac{x}{2}) = 2$. Therefore,

$$4 - \frac{x}{2} - 2 = \frac{1}{2}|x-4| = 0.01.$$

Thus, $|x-4| < 0.02$ and so $\delta = 0.02$.. Note that any delta greater that 0.02 for this problem.

1.3. Properties of Limits

It has been pointed out that $\lim_{x\to a} f(x)$ does not depend on the value of $f(x)$ at $x = a$. If it happens, however, that the limit is $f(a)$, then the limit is said to be evaluated by *direct substitution*. In that case,

$$\lim_{x\to a} f(x) = f(a).$$

Such a function is said to be *continuous at a*. See Section 1.7. Now we state and prove the following.

Theorem 1.1. *Let a be a real number and $g(x) = h(x)$ for all $x \neq a$ in an open interval containing a. If $\lim_{x\to a} h(x)$ exists, then $\lim_{x\to a} g(x)$ also exists and*

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

Proof

Suppose $\lim_{x\to a} h(x) = K$. Then for every infinitesimal distance $\epsilon > 0$, there exists another infinitesimal distance $\delta > 0$ such that $g(x) = h(x)$ in the intervals $(a-\delta,\ a)$ and $(a,\ a+\delta)$. Therefore,

$$|h(x) - K| < \epsilon \text{ whenever } 0 < |x-a| < \delta.$$

Since $g(x) = h(x)$ for $x \neq a$ in the open interval, we have

$$|g(x) - K| < \epsilon \text{ whenever } 0 < |x - a| < \delta.$$

We conclude that limit of $g(x)$ as $x \to a$ is also K.

Activity 1.5. *Show that the functions $f(x) = \frac{x^2+x-6}{x+3}$ and $g(x) = x - 2$ have the same values for all x other that $x = -3$.*

Remark 1.2. (1) Recognise limits that can be evaluated by direct substitution.

(2) If $\lim_{x\to a} f(x)$ cannot be evaluated by direct substitution, find a function $h(x)$ that agrees with $f(x)$ for all x except $x = a$. Then use Theorem 1.1 to get $\lim_{x\to a} f(x) = h(a)$.

Theorem 1.2. *Let a and b be real numbers, and n a positive integer, and let $f(x)$ and $g(x)$ be functions whose limits exist as $x \to a$.*

(1)
$$\lim_{x\to a}[b(f(x))] = b[\lim_{x\to a} f(x)],$$

(2)
$$\lim_{x\to a}[f(x) \pm h(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} h(x),$$

(3)
$$\lim_{x\to a}[f(x)h(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} h(x)]$$

(4)
$$\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} h(x)}, \textit{ provided } \lim_{x\to a} h(x) \neq 0.$$

(5)
$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

We provide the proof of Properties 2 and 3.

Proof

Let

$$\lim_{x\to a} f(x) = K \text{ and } \lim_{x\to a} h(x) = M$$

Property 2

Choose $\epsilon > 0$. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $0 < |x - a| < \delta_1$ implies

$|f(x) - K| < \frac{\epsilon}{2}$ and $0 < |x - a| < \delta_2$ implies $|h(x) - M| < \frac{\epsilon}{2}$. If $\delta = \min(\delta_1,\ \delta_2)$, then $0 < |x - a| < \delta$ means that

$$|f(x) - K| < \frac{\epsilon}{2}$$

and

$$|h(x) - M| < \frac{\epsilon}{2}.$$

Using the Triangle Inequality we have

$$|[f(x) + h(x)] - (K + M)| \leq |f(x) - K| + |h(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus

$$\lim_{x \to a}[f(x) + h(x)] = K + M = \lim_{x \to a} f(x) + \lim_{x \to a} h(x).$$

Property 3

Note that

$$f(x)h(x) = [f(x) - K][h(x) - M] + [Mf(x) + Kh(x)] - KM.$$

Now,

$$\lim_{x \to a}[f(x) - K] = 0 \text{ and } \lim_{x \to a}[h(x) - M] = 0.$$

Let $0 < \epsilon_1$. Then $\exists\ \delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x) - K - 0| < \epsilon \text{ and } |h(x) - M - 0| < \epsilon,$$

which implies

$$|f(x) - K||h(x) - M| < \epsilon\epsilon.$$

Therefore,

$$\lim_{x \to a}[f(x) - K][h(x) - M] = 0.$$

By Property 1, of the theorem,

$$\lim_{x \to a} Mf(x) = MK \text{ and } \lim_{x \to a} Kh(x) = KM.$$

Finally, by Property 2,

$$\lim_{x\to a} f(x)h(x) = \lim_{x\to a}[f(x)-K][h(x)-M] + \lim_{x\to a} Mf(x) + \lim_{x\to a} Kh(x) - \lim_{x\to a} KM.$$

Activity 1.6. *If* $\lim_{x\to a} f(x) = \frac{3}{2}$ *and* $\lim_{x\to a} h(x) = \frac{1}{2}$*, find the following.*

(1) $\lim_{x\to a}[4f(x)]$,

(2) $\lim_{x\to a} f(x)h(x)$,

(3) $\lim_{x\to a}[f(x) - h(x)]$

Remark 1.3. (1) If $p(x)$ is a polynomial function and $a \in \mathbb{R}$, then

$$\lim_{x\to a} p(x) = p(a).$$

(2) If $r(x)$ is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and a is a constant such that $q(a) \neq 0$, then

$$\lim_{x\to a} r(x) = r(a) = \frac{p(a)}{q(a)}.$$

(3) If $a > 0$ and n is counting number, or if $a \leq 0$ and n is an odd positive integer, then

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

(4) If $f(x)$ and $h(x)$ are functions such that

$$\lim_{x\to a} h(x) = M \text{ and } \lim_{x\to M}(x) = f(M)$$

then

$$\lim_{x\to a} f(h(x)) = fM.$$

(5) Limits involving basic trigonometric functions can be evaluated by direct substitution.

The items in Remark 1.3 help in the evaluation of limits of a large number of functions.

Exercise 1.1.

(1) Find the indicated limits

(a) $\lim_{x\to 0}(2x+1)$,

(b) $\lim_{x\to 2} \frac{1}{x}$,

(c) $\lim_{x \to \frac{5\pi}{3}} \cos x$

(2) If $\lim_{x \to a} f(x) = 4$, find the following.

(a) $\lim_{x \to 4} [f(x)]^2$,

(b) $\lim_{x \to 4} \sqrt{f(x)}$,

(c) $\lim_{x \to 4} [f(x)]^{\frac{2}{3}}$.

(3) Find two functions $f(x)$ and $h(x)$ such that

$$\lim_{x \to 0} f(x) \text{ and } \lim_{x \to 0} h(x)$$

do not exist, but

$$\lim_{x \to 0} [f(x) + h(x)]$$

does exist.

1.4. Techniques for Evaluating limits

There are two techniques used in the evaluation of limits: the *cancellation* method (see Theorem 1.1) and the *rationalisation* method. We illustrate how the techniques are used in the following examples.

Example 1.2. Find the limit (if it exists)

(1)

$$\lim_{x \to -2} \frac{x^3 + 8}{x + 2},$$

(2)

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}.$$

Solution

(1) Let $f(x) = \frac{x^3+8}{x+2}$. Note that the denominator is zero at $x = -2$ and that numerator can be factorised to give $(x^2 - 2x + 4)(x + 2)$. Thus

$$f(x) = \frac{x^3 + 8}{x + 2} = x^2 - 2x + 4.$$

Therefore,

$$\lim_{x \to -2} \frac{x^3 + 8}{x + 2} = \lim_{x \to -2} (x^2 - 2x + 4) = 12.$$

This should remind you of two functions that are equal, on an open interval that contains a, except when $x = a$.

(2) We simplify the numerator. This will give us

$$2x\Delta x + (\Delta x)^2 - 2\Delta x = \Delta(2x - 2 + \Delta x).$$

Thus

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 2(x+\Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} (2x - 2 + \Delta x) = 2x - 2.$$

Example 1.3.

(1) Find the limit (if it exists)

(a) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$,

(b) $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$,

(2) Let

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ x - 2, & x < 0 \end{cases}$$

Find

(a) $\lim_{x\to 0^+} f(x)$,

(b) $\lim_{x\to 0^-} f(x)$,

(c) $\lim_{x\to 0} f(x)$.

Solution

(1) (a) By direct substitution, we obtain the indeterminate form $0/0$. We rewrite the function $f(x) = \frac{\sqrt{x+1}-2}{x-3}$ by rationalising the numerator.

$$\begin{aligned} \frac{\sqrt{x+1}-2}{x-3} &= \left(\frac{\sqrt{x+1}-2}{x-3}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right) \\ &= \frac{1}{\sqrt{x+1}+2} \end{aligned}$$

Thus

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}.$$

(b) The same wisdom used in the first problem gives

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = 2\sqrt{2}.$$

(2) To evaluate limits in this case, you look at a graph of the piecewise function, $f(x)$.

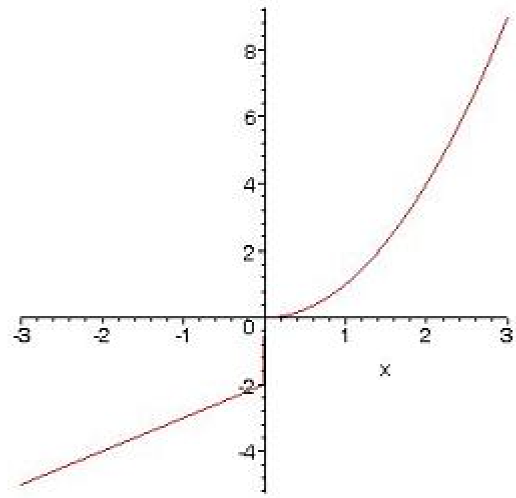


FIGURE 1.4. The piece-wise function

(a) Now,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x - 2) = -2$$

and

(b)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2) = 0.$$

(c) From the two results above, you can safely conclude that

$$\lim_{x \to 0} f(x)$$

does not exist.

When the indeterminate $0/0$ is encountered in a limit involving trigonometric functions, some technique is needed to evaluate the limit. The following *squeezing* theorem is useful.

Theorem 1.3. *Let $f(x) \leq g(x) \leq h(x)$ $\forall x$ in an open interval containing a except possibly at $x = a$. If*

$$\lim_{x \to a} f(x) = K = \lim_{x \to a} h(x).$$

Then

$$\lim_{x\to a} g(x) = K.$$

Proof

For $\epsilon > 0$, $\exists\ \delta_1$ and and δ_2 such that

$$|f(x) - K| < \epsilon \text{ whenever } 0 < |x - a| < \delta_1$$

and

$$|h(x) - K| < \epsilon \text{ whenever } 0 < |x - a| < \delta_2.$$

Let $\delta = \min(\delta_1,\ \delta_2)$. Then, if $0|x - a| < \delta$, it follows that $|f(x) - K| < \epsilon$ and $|h(x) - K| < \epsilon$. Therefore,

$$-\epsilon < f(x) - K < \epsilon \text{ and } -\epsilon < h(x) - K < \epsilon$$

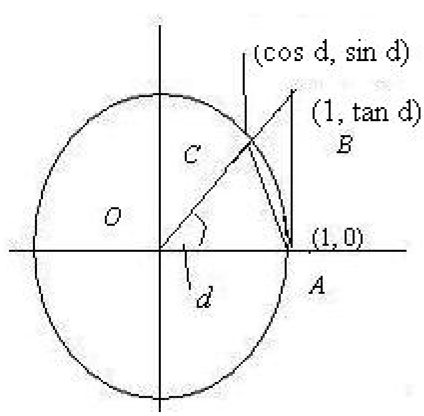
$$K - \epsilon < f(x) \text{ and } h(x) < K + \epsilon.$$

But $f(x) \leq g(x) \leq h(x)$, implies that $K - \epsilon < g(x) < K + \epsilon$ and $|g(x) - K| < \epsilon$. Thus

$$\lim_{x\to a} g(x) = K.$$

Activity 1.7. *Consider a unit circle in the figure below. We are interested in the areas of the sector OAC, triangles OAB and OAC. Use the formulas for areas when a is in radians to show that*

$$\frac{\tan d}{2} \geq \frac{d}{2} \geq \frac{\sin d}{2}.$$



The result in Activity 1.7 implies that

$$\frac{1}{\cos d} \geq \frac{d}{\sin d} \geq 1.$$

Now taking reciprocals yields

$$\cos d \leq \frac{\sin d}{d} \leq 1.$$

This inequality is valid for all $d \neq 0$ in the open interval $(\frac{-\pi}{2}, \frac{\pi}{2})$ since $\cos d = \cos(-d)$ and

$$\frac{\sin d}{d} = \frac{\sin(-d)}{(-d)}.$$

You also note that

$$\lim_{d\to 0} \cos d = \lim_{d\to 0} 1 = 1.$$

You can now use Theorem 1.3 to conclude that

$$\lim_{d\to 0} \frac{\sin d}{d} = 1.$$

Remark 1.4. Using similar arguments, it can be shown that

$$\lim_{d\to 0} \frac{\tan d}{d} = 1.$$

Activity 1.8. *Show, using rationalisation of the numerator, that*

$$\lim_{d\to 0} \frac{1 - \cos d}{d} = 0.$$

Example 1.4.

Determine the limit of the trigonometric function (if it exists).

(1)

$$\lim_{x\to 0} \frac{\sin x}{5x},$$

(2)

$$\lim_{x\to 0} \frac{\tan x^2}{x},$$

(3)

$$\lim_{x\to 0} \frac{(1 - \cos x)^2}{x}.$$

Solution

(1)

$$\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\frac{1}{5}\frac{\sin x}{x} = \frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x}.$$

But

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Thus

$$\lim_{x\to 0}\frac{\sin x}{5x} = \frac{1}{5}.$$

(2)

$$\lim_{x\to 0}\frac{\tan x^2}{x} = \lim_{x\to 0}\frac{x\tan x^2}{x^2}.$$

Now,

$$\lim_{x\to 0}\frac{x\tan x^2}{x^2} = \lim_{x\to 0} x \times \lim_{x\to 0}\frac{\tan x^2}{x^2}.$$

Note that $\lim_{x\to 0} x = 0$ and using the assertion in Remark 1.4 with $y = x^2$, you find that

$$\lim_{x\to 0}\frac{\tan x^2}{x^2} = 1.$$

Therefore,

$$\lim_{x\to 0}\frac{x\tan x^2}{x^2} = 0.$$

(3)

$$\lim_{x\to 0}\frac{(1-\cos x)^2}{x} = \lim_{x\to 0}\frac{(1-\cos x)}{x}(1-\cos x).$$

This is a limit of the product of two functions. Thus

$$\lim_{x\to 0}\frac{(1-\cos x)^2}{x} = \lim_{x\to 0}\frac{(1-\cos x)}{x}\lim_{x\to 0}(1-\cos x) = 0$$

because

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

and also

$$\lim_{x\to 0}(1-\cos x) = 0.$$

1.5. Infinite Limits

Let

$$f(x) = \frac{2}{3-x}.$$

x	2.5	2.9	2.99	2.999	3.001	3.01	3.1	3.5
$f(x)$	4	20	200	2000	-2000	-200	-20	-4

Consider the table above and note that as $x \to 3$ for the left, $f(x)$ increases without bound and as $x \to 3$ for the right $f(x)$ decreases without bound. These statements can be written as

$$\lim_{x\to 3^-} \frac{2}{3-x} = \infty$$

and

$$\lim_{x\to 3^+} \frac{2}{3-x} = -\infty.$$

respectively. Figure 1.5 can also help you understand this setting.

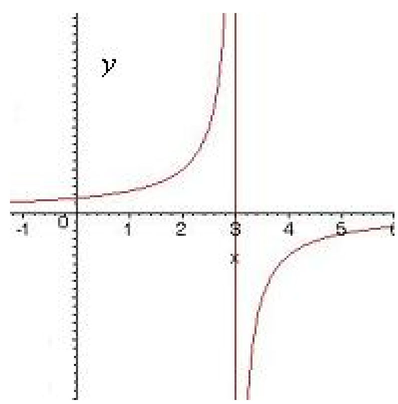


Figure 1.5. The function increases and decreases without bound as x approaches 3.

In a sense, $\lim_{x\to 3} f(x)$ does not exist.

Remark 1.5. The symbols $+\infty$ and $-\infty$ as used here are not *real numbers*. They only describe particular ways in which limits fail to exist. You should *not* manipulate these symbols using algebraic rules. For instance, it is wrong to suggest that $(+\infty) - (+\infty) = 0$.

Now consider a fairly different function $g(x)$ given by

$$g(x) = 1 - \frac{4}{x^2}.$$

This function has a graph shown in Figure 1.6.

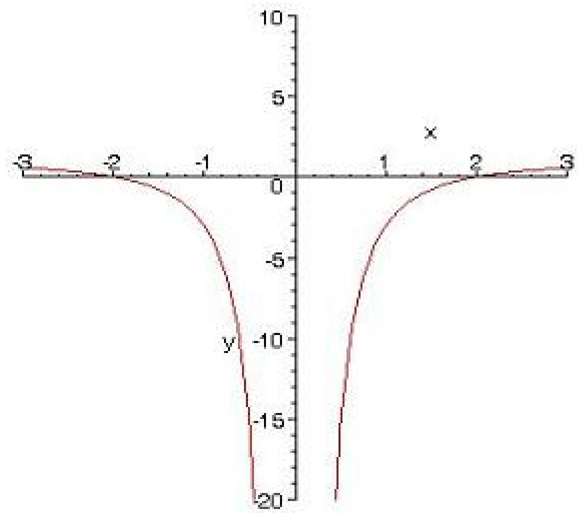


FIGURE 1.6. The function decreases without bound as x approaches 0.

You note that for this function,

$$\lim_{x \to 0^-} [1 - \frac{4}{x^2}] = -\infty.$$

and

$$\lim_{x \to 0^+} [1 - \frac{4}{x^2}] = -\infty.$$

Thus, $\lim_{x \to 0} [1 - \frac{4}{x^2}] = -\infty$.

Remark 1.6. If a and K are real numbers and f and h are functions such that

$$\lim_{x \to a} f(x) = \infty \text{ and } \lim_{x \to a} h(x) = K$$

then the following properties are true:

(1) Sum or difference:

$$\lim_{x \to a} (f(x) \pm h(x)) = \infty,$$

(2) Product:

$$\lim_{x\to a}[f(x)h(x)] = \infty, \; K > 0$$

$$\lim_{x\to a}[f(x)h(x)] = \infty, \; K < 0.$$

(3) Quotient:

$$\lim_{x\to a} \frac{h(x)}{f(x)} = 0.$$

Similar properties hold for one-sided limits and for

$$\lim_{x\to a} f(x) = -\infty.$$

Activity 1.9. *Using a diagram where appropriate, determine whether $f(x)$ approaches ∞ or $-\infty$ as $x \to -3^+$.*

(1) $f(x) = \frac{1}{x^2-9}$,
(2) $f(x) = \frac{x^2}{x^2-9}$,
(3) $f(x) = (x^2 + \frac{4}{x+3})$

1.6. Limits at Infinity

If the value of x increases without bound, then we write $x \to +\infty$ and if x decreases without bound, then we write $x \to -\infty$. The behaviour of $f(x)$ as $x \to \pm\infty$ is known as the *end behaviour* of $f(x)$. Consider the function, $f(x)$, given by

$$f(x) = \frac{1}{x^2}.$$

You note from Figure 1.7 that as $x \to \pm\infty$, $f(x) \to 0$. This leads to a formal

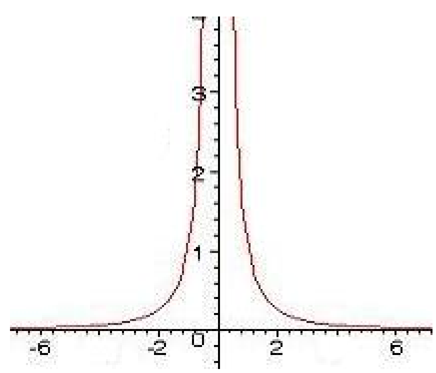


FIGURE 1.7. The graph of $f(x) = 1/x^2$

definition of a limit at infinity.

Definition 1.2. Let $f(x)$ be a function such and K is a constant. Then

$$\lim_{x\to\infty} f(x) = K$$

means that $\forall\epsilon > 0$, $\exists$ an $M > 0$ such that $|f(x) - K| < \epsilon$ whenever $x > M$. and the statement

$$\lim_{x\to-\infty} f(x) = K$$

means that $\forall\epsilon > 0$, $\exists$ an $N < 0$ such that $|f(x) - K| < \epsilon$ whenever $x < N$.

The line $y = K$ is called a *horizontal asymptote* of the graph of $f(x)$.

Activity 1.10. *Now, consider the function*

$$g(x) = \frac{x}{1-x}.$$

Determine $\lim_{x\to+\infty}$ *and* $\lim_{x\to-\infty}$ *using Figure 1.8.*

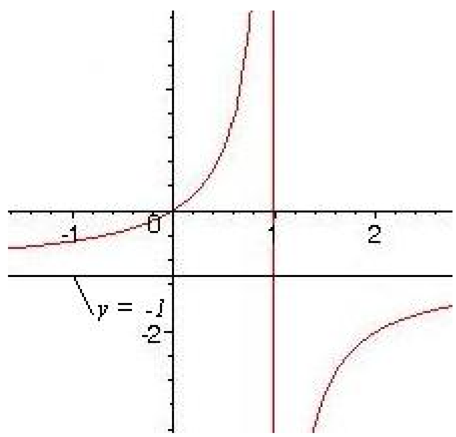
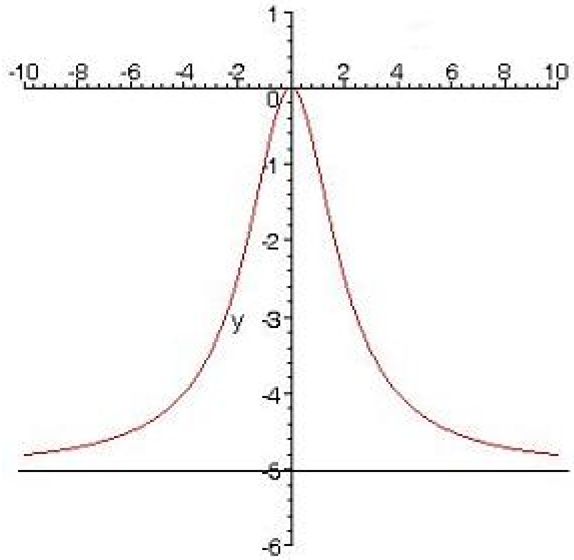


FIGURE 1.8. The graph of $f(x) = \frac{x}{1-x}$

From Activity 1.10 you note that there is a *horizontal asymptote* c for which $f(x) \to c$ as x increases (or decreases) without bound. Evaluation of limits at infinity requires some algebraic manipulation. For instance, suppose $f(x) = \frac{-5x^2}{4+x^2}$. The graph shows that

$$\lim_{x\to\infty} \frac{-5x^2}{4+x^2} = -5.$$



This limit is found as follows. The numerator approaches $-\infty$ as x approaches ∞ and denominator approaches ∞ as x approaches ∞. To circumvent this problem, divide both numerator and denominator by x^2. Then evaluate the limit as follows

$$\lim_{x\to\infty}\frac{-5x^2}{4+x^2}=\lim_{x\to\infty}\frac{-5}{\frac{4}{x^2}+1}=\frac{\lim_{x\to\infty}(-5)}{\lim_{x\to\infty}\frac{4}{x^2}+\lim_{x\to\infty}1}.$$

This gives the solution. Let us have a look at a few more problems.

Example 1.5.

Find the indicated limits:

(1)

$$\lim_{x\to\infty}\frac{5x^3+1}{10x^3-3x^2+7},$$

(2)

$$\lim_{x\to-\infty}\frac{-2x^2}{x+3},$$

(3)

$$\lim_{x\to\infty}\frac{\sin 5x}{x},$$

(4)

$$\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2-x}},$$

Solution

(1) Divide the numerator and denominator by x^3, a term in x with the highest power. Then evaluate the limit from there. That is

$$\begin{aligned}\lim_{x\to\infty} \frac{5x^3+1}{10x^3-3x^2+7} &= \lim_{x\to\infty} \frac{5+\frac{1}{x^3}}{10-\frac{3}{x}+\frac{7}{x^3}}, \\ &= \frac{\lim_{x\to\infty} 5+\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 10-\lim_{x\to\infty}\frac{3}{x}+\lim_{x\to\infty}\frac{7}{x^3}}, \\ &= \frac{5+0}{10-0+0}, \\ &= \frac{1}{2}\end{aligned}$$

(2)

$$\begin{aligned}\lim_{x\to-\infty} \frac{-2x^2}{x+3} &= \frac{\lim_{x\to-\infty} -2x}{\lim_{x\to-\infty} 1+\frac{3}{x}}, \\ &= \frac{\lim_{x\to-\infty} -2x}{\lim_{x\to-\infty} 1+\frac{3}{x}}, \\ &= \frac{\lim_{x\to-\infty} -2x}{1+0},\end{aligned}$$

The numerator has no limit. Thus $\lim_{x\to-\infty} \frac{-2x^2}{x+3}$ does not exist.

(3) To evaluate $\lim_{x\to\infty} \frac{\sin 5x}{x}$, we need to consider the behaviour of $\sin 5x$. This function lies in the interval $[-1,\ 1]$. Thus

$$\frac{-1}{x} \le \frac{\sin 5x}{x} \le \frac{1}{x}.$$

But

$$\lim_{x\to\infty} \frac{-1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

So by the Squeezing Theorem met earlier, we conclude that

$$\lim_{x\to\infty} \frac{\sin 5x}{x} = 0.$$

(4) Divide the numerator and the radical by x to get

$$\lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1-\frac{1}{x}}} = 2.$$

Let us conclude this section with some problems. Sketches may help you to you find limits.

Activity 1.11. (1) *Evaluate the following limits:*

(a)

$$\lim_{x\to\infty} (x+3)^{-2},$$

(b)

$$\lim_{x\to\infty} \frac{x-\cos x}{x},$$

(c)

$$\lim_{x\to\infty} \frac{x^2-x}{\sqrt{x^4+x}},$$

(2) *The efficiency of an internal combustion engine is defined to be*

$$\textit{efficiency}(\%) = 100[1 - \frac{1}{(v_1/v_2)^c}]$$

where v_1/v_2 is the ratio of uncompressed gas to the compressed gas and c is a constant dependent upon engine design. What is the limit of efficiency as the compression ratio approaches infinity?

1.7. Continuity

When we that $f(x)$ is continuous at $x = a$, it means that there is no interruption in the graph of the function at $x = a$. The graph has no jumps, gaps and holes at a. Consider the function which has a graph shown in Figure 1.9.

$$f(x) = \begin{cases} x^2, & x < 0 \\ 1, & x = 0 \\ -x+2, & x > 0 \end{cases}$$

The function is said to be discontinuous at $x = 0$.

Activity 1.12. *For the function shown in Figure 1.9 find*

(1)

$$\lim_{x\to 0^-} f(x),$$

(2)

$$\lim_{x\to 0^+} f(x),$$

(3)

$$f(0).$$

This activity leads to a formal definition of continuity.

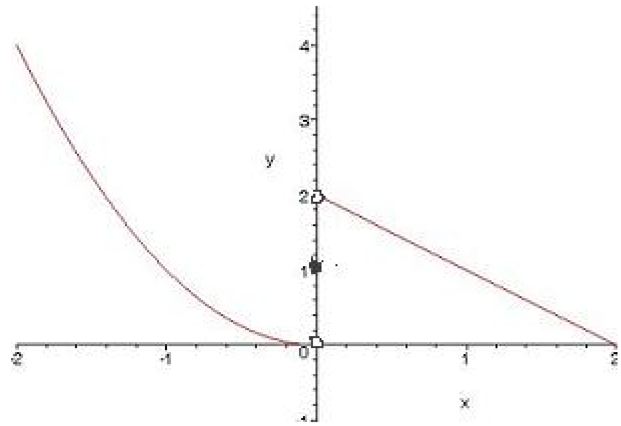


FIGURE 1.9. Jump at 0

Definition 1.3. A function $f(x)$ is called *continuous* at a if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a).$$

A function is called *continuous* on an open interval if it is continuous at each point in the interval. A function that is continuous for all real values of x is called *everywhere continuous*.

A function which is not continuous at a is said to be *discontinuous* there. A discontinuity is said to be *removable* if

$$\lim_{x \to a} f(x)$$

exists, but $f(a)$ is not defined. For instance, the function

$$f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$$

is *not* continuous at 1. However, you will note that

$$\begin{aligned} \lim_{x \to 1^-} \frac{x^2 - 1}{x + 1} &= \lim_{x \to 1^+} \frac{x^2 - 1}{x + 1} \\ &= \lim_{x \to 1} \frac{x^2 - 1}{x + 1} \\ &= 2 \end{aligned}$$

Now consider the function $f(x)$ given by

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x < 1, \\ 4 - x, & \text{if } x \geq 1 \end{cases}$$

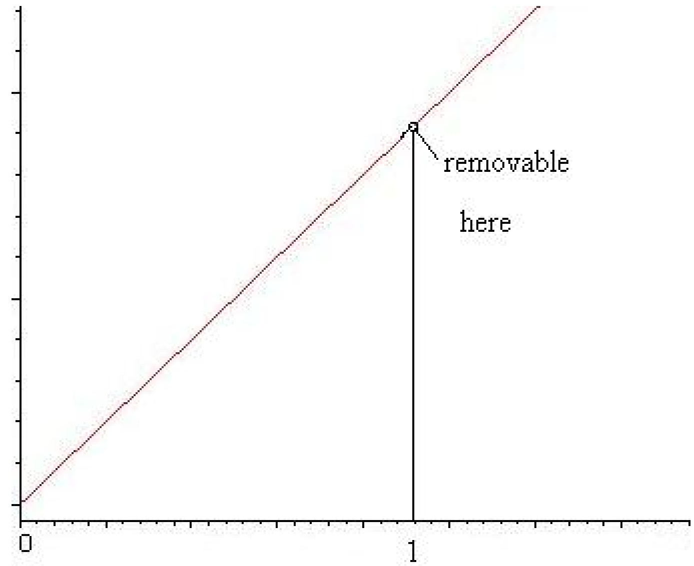


FIGURE 1.10. Removable at 1

The graph is show in Figure 1.11. This function is said to have a jump discontinuity at $x = 1$ because

$$\lim_{x\to 1^-} f(x) = 0 \neq \lim_{x\to 1^+} f(x) = 3$$

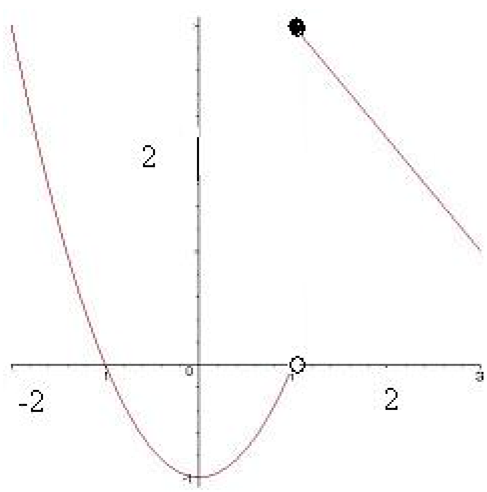


FIGURE 1.11. Jump discontinuity at 1

Activity 1.13. *Let*

$$f(x) = \frac{1}{x^2 - 1}.$$

Sketch its graph and state whether it is continuous or discontinuous at $x = -1$.

A discontinuity where $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = -\infty$ or $\lim_{x\to a} f(x) = \pm\infty$ is said to be an *infinite discontinuity.*

Remark 1.7. If λ is a constant and f and g are continuous at a, then the following functions are also continuous at a:

(1) Scalar multiple: λf,

(2) Sum and difference: $f(x) \pm g(x)$,

(3) Product: $f(x)g(x)$,

(4) Quotient:

$$\frac{f(x)}{g(x)}, \text{ if } g(x) \neq 0$$

(5) Composite function:

$$(f \circ g)(x) = f(g(x)).$$

(6) If $f(x)$ is any trigonometric function and a is any number in the natural domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

Let us look at an example.

Example 1.6. State whether the following functions are continuous or not

(1)

$$f(x) = \begin{cases} 2x+3, & x \leq 4 \\ 7 + \frac{16}{x}, & x > 4 \end{cases}$$

(2) $f(x) = |x-1|$, for all x,

(3)

$$f(x) = \begin{cases} -x^2, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ x-2, & \text{if } x > 1 \end{cases}$$

(4) $f(x) = 3x^2 + 5 - \frac{1}{\sqrt{-x}}$, at $x = -2$.

Solution

(1) For the function

$$f(x) = \begin{cases} 2x+3, & x \leq 4 \\ 7 + \frac{16}{x}, & x > 4 \end{cases}$$

you note that $f(4) = \lim_{x\to 4} = 11$. So this function is continuous everywhere.

(2) Take the function $f(x) = |x-1|$. The graph of this function is shown in Figure 1.12. You see that

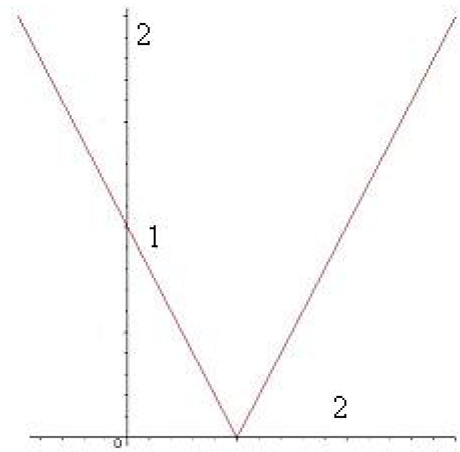


FIGURE 1.12. The Modulus function

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 0.$$

Thus $f(x) = |x-1|$ is continuous everywhere.

(3) Sketch a graph of the function and show that it is not continuous at $x = 1$ because there is a "hole"' there.

(4) $f(x) = 3x^2 + 5 - \frac{1}{\sqrt{-x}}$ is continuous at $x = -2$ since

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^=} f(x) = 19 + \frac{\sqrt{2}}{2}.$$

1.7.1. Intermediate Value Theorem. Let us conclude this section with a theorem that concerns the behaviour of continuous functions on $[b,\ c]$. We state it without proof.

Theorem 1.4. *If $f(x)$ is continuous on $[b,\ c]$ and $k \in [f(b),\ f(c)]$ the $\exists\ a \in [b,\ c]$ such that $f(a) = k$.*

This theorem can be viewed in this way. Suppose the weight of a baby at $2\frac{1}{2}$ years is $7\frac{3}{4}$ kg and at 4 the weight is 13 kg. Then for a weight, w kg, in $(7\frac{3}{4},\ 13)$ there must have been a corresponding age a years for which the weight was w kg.

Remark 1.8. (1) There may be more than one $a \in [b,\ c]$ that give $f(a) = k$. Consider the graph of $f(x) = x^3 - 3x - 2$ shown in Figure 1.13. In the interval $[-2,\ 2]$ there are three values of a that correspond to k.

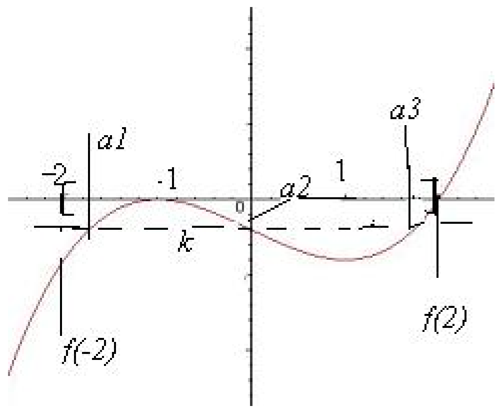


FIGURE 1.13. The Graph of $f(x) = x^3 - 3x - 2$

(2) There may be no value of a for a discontinuous function.
(3) The Intermediate Value Theorem is used to locate the zeros of a function that is continuous on $[b,\ c]$; especially if $f(b)$ and $f(c)$ have different signs.

Example 1.7.

(1) Prove that $f(x) = x^2 - 4x + 3$ has a zero in the interval $[2,\ 3]$.
(2) Use the Intermediate Value Theorem to approximate the zero of

$$f(x) = x^3 + x - 1 \text{ on } [0,\ 1].$$

Begin by locating the zero in a subinterval of length 0.1. Refine approximation by locating the zero in a subinterval of length 0.01.

(3) Verify the applicability of th IVT in $[\frac{5}{2},\ 4]$ and find the value of a so that $f(a) = 6$ where

$$f(x) = \frac{x^2 + x}{x - 1}.$$

Solution

(1) The IVT guarantees that $f(x) = x^2 - 4x + 3$ has a zero in $[2,\ 3]$ if $f(2)$ and $f(3)$ have different signs. Now $f(2) = -1$ and $f(3) = 0$. So $f(x)$ has a zero at 1 and 3.

(2) Now $f(0) = 0^3 + 0 - 1 = -1$ and $f(1) = 1^3 + 1 - 1 = 1$. Therefore, by the IVT, $f(x)$ has a zero on $[0,\ 1]$. Let us consider the interval $[0,\ 0.5]$. So $f(0.5) = -0.375$ and we cannot have a zero in this interval. Take $[0.6,\ 0.8]$. The value of $f(0.6)$ is -0.184 and that of $f(0.7)$ is 0.043. Proceeding in this manner gives an approximate in $[0.682327,\ 0.682328]$ and it is $f(0.6823278) = -.92^{-8}$.

(3) You note that for $f(x) = \frac{x^2+x}{x-1}$, $f(\frac{5}{2}) = \frac{29}{6}$ and $f(4) = \frac{17}{3}$. By the IVT, and considering the fact that $f(x)$ is continuous on this interval, there is no a that satisfies $f(a)$ because it does not fall in $[\frac{29}{6},\ \frac{17}{3}]$.

1.8. Unit Summary

This unit has given you an opportunity to explore the concept of a limit of a function. It has been discussed that there some functions that do not have limits at some points. You have also learnt how to find limits at infinity for various functions. This unit has given a definition of continuity of a function. There are some functions which are not defined at some point(s) on $\mathbb{R}$. such functions are said to discontinuous at such points. Several discussion examples and activities have been given so that you interact with the concepts.

1.9. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

1.10. Exercises

(1) Find the given limit (if it exists);

(a)

$$\lim_{x\to 2}(5x-3),$$

(b)

$$\lim_{t\to -2}\frac{t+2}{t^2-4},$$

(c)

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x},$$

(d)

$$\lim_{x\to -1^+}\frac{\sec x}{x},$$

(e)

$$\lim_{t\to -2}\frac{\sin[(\pi/6+\Delta x)+]-(1/2)}{\Delta x},$$

(2) Estimate the limit

$$\lim_{x\to 1^+}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}$$

by completing the following table.

x	1.1	1.01	1.001	1.0001
$f(x)$				

(3) Determine whether the given limit statement is true or false:

(a)

$$\lim_{x\to -2} f(x)=1$$

where

$$f(x)=\begin{cases} x-2, & x\le 3\\ -x^2+8x-14, & x>3\end{cases}$$

(b)

$$\lim_{x\to 0}\frac{|x|}{x=1},$$

(c)

$$\lim_{x\to 2} f(x)=3$$

where

$$f(x)=\begin{cases} 3, & x\le 2\\ 0, & x>2\end{cases}$$

(4) Determine the intervals on which the given function is continuous

(a)

$$f(x) = \frac{3x^2 - x - 2}{x - 1},$$

(b)

$$f(x) = \sqrt{\frac{x+2}{x}},$$

(c)

$$f(x) = \begin{cases} 5 - x, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

UNIT 2

DIFFERENTIATION

2.1. Introduction

Instantaneous rate of a chemical reaction in more important that the average rate of change because it gives you the behaviour of the reaction at a particular time t. Similarly, the rate of population growth of a species is not uniform: it changes with time and the availability of food, among other factors.
These rates can be computed successfully if we know the functions modelling these phenomena. This can be done using *differentiation* of the function and then evaluating the result at that material (or *instantaneous*) time.
The unit contains the following topics:

- Rates of Change;
- The Derivative;
- Properties of Differentiation;
- Chain Rule of Differentiation;
- Linear Approximations and Differentials

2.1.1. Learning objectives. By the end of this unit you should be able to:

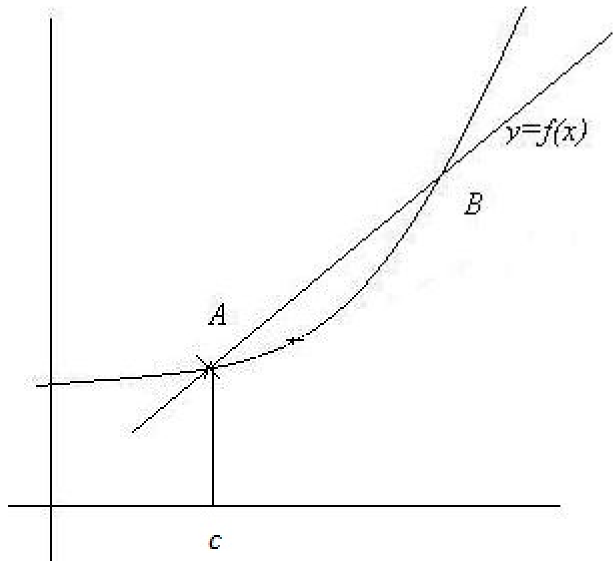
- find the instantaneous rate of change of a function with respect to an independent variable at a given point;
- find the derivative of a function from first principles;
- use product and quotient rules of differentiating products and quotients of two functions;
- differentiate compositions of functions using chain rule;
- find linear approximations to find values of functions at given points using differentials.

2.1.2. Prerequisite knowledge. you will need motivating concepts introduce in Unit One to appreciate the beauty of differentiation and its applications.

2.1.3. Time. You should be able to complete this unit in 7 hours.

2.2. Rates of Change

To motivate the discussion on differentiation, we present again Figure 1.2 of the previous unit. Let PQ be a tangent to the curve at A in Figure 2.1.



Suppose B is moved closer to A. Then the slope of AB is approximately equal to the slope of PQ, tangent to the curve. This means that if B is moved closer to A, then the slope of AB becomes a better approximation of the slope of the curve at A. Let $x_2 = x_1 + h$, m_1 the gradient of the line segment AB and m the slope of the tangent line PQ.

Then the value of (x) at A is $f(x_1)$ and $f(x_1 + h)$ at B and thus

$$m_1 = \frac{f(x_1 + h) - f(x_1)}{h}$$

so that

$$\lim_{h \to 0} m_1 = \lim_{h \to 0} \frac{f(x_1 + h) - f(x)}{h} = m. \tag{2.1}$$

Example 2.1. Find the slopes of tangent lines to the following curves at a point where $x = 2$

(1) $f(x) = x^2 + x$,

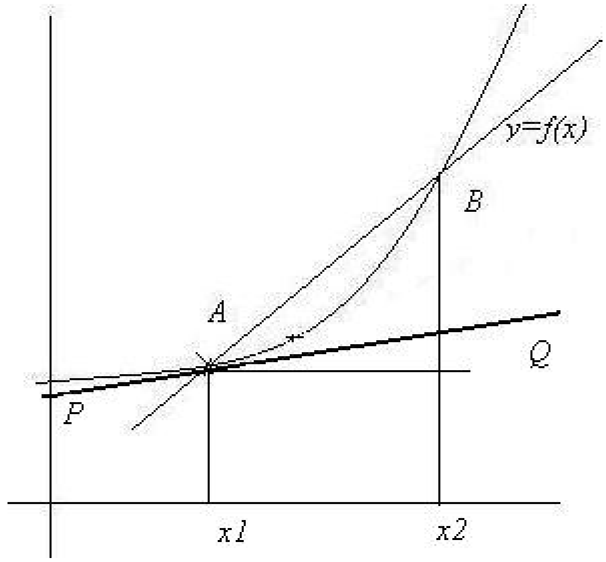


FIGURE 2.1. PQ is tangent at A

(2) $f(x) = \sqrt{x}$.

Solution

In both cases we will use Equation 2.1. Let the change in x be h. Then

(1)

$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{(2+h)^2+(2+h)-(2^2+2)}{h}\\ &= \lim_{h\to 0}\frac{[4+4h+h^2+2+h-4-2]}{h}\\ &= \lim_{h\to 0}\frac{[5+h]h}{h}\\ &= 5\end{aligned}$$

So the slope of $f(x) = x^2 + x$ at $(2,\ 6)$ is 5.

(2) Note that the numerator has to rationalised before evaluating the limit.

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h} &= \lim_{h\to 0}\frac{2+h-2}{h}\times\frac{1}{\sqrt{2+h}+\sqrt{2}}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}}\\ &= \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{2}}{4}\end{aligned}$$

Therefore, at $(2, \sqrt{2})$ the curve has a slope of $\frac{\sqrt{2}}{4}$.

Activity 2.1. *Consider the function $f(x) = \frac{1}{x^2}$. Find the equation of the tangent to $f(x)$ when $x = -2$.*

Let $s(t) = 2t^2$ be the distance travelled by a car in time t seconds. Recall that

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time taken}} = v_a.$$

Now, when $t = 1$, $s = 2$, and when $t = 2$, $s = 8$. So average velocity is $\frac{8-2}{2-1} = 6$ m/s. If we are interested in the velocity at time $t = 1$ second, then we would get a time close to $t = 1$ and the average velocity will be a good approximation to the velocity, v at time $t = 1$. Here is a summary for various values of t.

Time	Distance	Average Velocity (v_a)
1.1	2.42	4.2
1.01	2.0402	4.02
1.001	2.004002	4.002
1.0001	2.00040002	4.0002

You will note that as $t \to 1$ $v_a \to 4$ m/sec. In fact,

$$v = \lim_{\Delta t \to 0} v_a = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t \to 0}. \tag{2.2}$$

The velocity, v, as given in Equation 2.2 is called *instantaneous velocity* at $t = 1$ second.

In general, if $f(x)$ is a function, then

$$c_i = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2.3}$$

is called *instantaneous rate of change* of $f(x)$ with respect to x at a.

Example 2.2. A projectile is fired directly upward from the ground with an initial velocity of $u = 112$ m/sec. Its distance, s, above the ground is given by $s(t) = 112t - 16t^2$ metres.

(1) Find the velocity of the projectile at $t = 2$, $t = 3$.

(2) When does the projectile strike the ground?

(3) Find the velocity at the instant it strikes the ground.

Solution

(1)

$$\begin{aligned} v &= \lim_{h\to 0} \frac{112(2+h) - 16(2+h)^2 - (112\times 2 - 16\times 2^2)}{h} \\ &= \lim_{h\to 0} \frac{112h - 64h - 16h^2}{h} \\ &= 48 \text{ m/sec} \end{aligned}$$

(2) Projectile reaches maximum height when $v = 0$. Now, v is given by

$$\begin{aligned} v &= \lim_{h\to 0} \frac{112(t+h) - 16(t+h)^2 - (112t - 16t^2)}{h} \\ &= \lim_{h\to 0} \frac{112h - 32th - 16h^2}{h} \\ &= 112 - 32t \end{aligned}$$

When $v = 0$, $32t = 112$, Thus $t = 3.5$. The projectile reaches maximum height after 3.5 seconds.

The third part can be solved by finding t when $s = 0$ and then find the instantaneous velocity at that value of t.

2.3. The Derivative

We begin by giving the definition of a derivative.

Definition 2.1. Let $f(x)$ be a function. The function is said to be *differentiable* at x if

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is denoted by $\frac{df(x)}{dx}$ or $f'(x)$.

$\frac{df(x)}{dx}$ is called the *derivative* of $f(x)$ with respect to x. It is a function in its own right. Note that it is **WRONG** to write

(1) $f(x) \neq f(x) + f(h)$,

(2) $f(x+h) \neq f(x) + h$.

Example 2.3. Let $f(x) = x^4$. Find $f'(x)$.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{(x+h)^4-x^4}{h}\\
&= \lim_{h\to 0}\frac{[x^4+4x^3h+6x^2h^2+4xh^3+h^4]-x^4}{h}\\
&= \lim_{h\to 0} 4x^3+4x^2h+4xh^2+h^3\\
&= 4x^3.
\end{aligned}$$

The procedure used in Example 2.3 is called *differentiating* $f(x)$ *with respect to* x *from first principles*. Using this example you will also note that if $f(x) = x^n$ for $n \neq 0$, then

$$f'(x) = nx^{n-1}.$$

Consider a functions $f(x) = |x-5|$ and $g(x) = x^{1/2}$. The first function is continuous at $x = 5$, but

$$\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^+}\frac{|x-5|}{x-5} = 1 \text{ and}$$

$$\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5} = \lim_{x\to 5^-}\frac{|x-5|}{x-5} = 1.$$

have different values. These are derivatives of f from the *right* and *left* respectively. Because these derivatives are not equal,then the function, $f(x)$, is not differentiable at $x = 5$. You will also note that the graph of $f(x)$ has a sharpe corner at $x = 5$. Now, $g(x) = x^{1/2}$ is continuous at $x = 0$. However,

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{1}{x^{\frac{1}{2}}} = \infty.$$

Thus $g(x)$ is not differentiable at $x = 0$. The graph of $g(x)$ has a vertical tangent at $x = 0$. These examples lead us to the following.

Theorem 2.1. *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous there.*

The proof is left as an activity.

Activity 2.2. *Prove Theorem 2.1 using properties of limits and the fact that*

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left[(x-a)\left(\frac{f(x)-f(a)}{x-a}\right)\right].$$

Theorem 2.2. (1) *The derivative of a constant function is zero.*

(2) *Let $f(x)$ and $g(x)$ be differentiable functions. Then*

(a)

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x),$$

(b)

$$[\lambda f(x)] = \lambda f'(x),$$

(c)

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g',$$

(d)

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof

We leave the first two for your exercises.

(1)

(2) (a)

(b)

$$\begin{aligned}
[f(x)g(x)]' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \left\{ g(x)\frac{f(x+h) - f(x)}{h} \right\} + \lim_{h\to 0} \left\{ f(x+h)\frac{g(x+h) - g(x)}{h} \right\} \\
&= f'(x)g(x) + g'(x)f(x)
\end{aligned}$$

(c)

$$\begin{aligned}
\left[\frac{f(x)}{g(x)}\right]' &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \left\{ \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \right\} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

Theorems 2.2 2c and 2.2 2d are respectively called the *Product* and *Quotient* rules.

Example 2.4. Find the derivatives of the following

(1) $g(x) = 3x^2 + \sqrt[3]{x^4}$,

(2) $g(t) = \frac{\sqrt[5]{t^2}}{3t-5}$.

Solution

(1) $g'(x) = 3x^{2-1} + \frac{4}{3}x^{\frac{4}{3}-1} = 6x + \frac{4}{3}x^{\frac{1}{3}}$.

(2) For this one, we use the Quotient rule.

$$\begin{aligned} g'(t) &= \frac{\frac{2}{3}t^{-\frac{1}{3}}(3t-5) - 3t^{\frac{2}{3}}}{(3t-5)^2} \\ &= \frac{2t^{\frac{2}{3}} - 3t^{\frac{2}{3}} - \frac{10}{3}t^{-\frac{1}{3}}}{(3t-5)^2} \\ &= -\frac{3t^{\frac{2}{3}} + 10t^{-\frac{1}{3}}}{3(3t-5)^2} \end{aligned} \tag{2.4}$$

You will now note that

(1) if $f(x)$ is any differentiable function, then gradient of the curve of $f(x)$ at $x = a$ is $f'(a)$.

(2) the instantaneous rate of change of $g(x)$ with respect to x at a is $g'(a)$.

Activity 2.3. *Find the equation of the tangent line and normal to the curve* $f(x) = x^2 + 3x + 1$ *at* $x = 0$.

2.4. Chain Rule of Differentiation

Most functions in the natural sciences and in mathematics are compositions of two or more functions. For instance,

$$f(x) = \sqrt{x^3}$$

is a composition of $h(x) = x^3$ and $k(x) = \sqrt{(x)}$. To differentiate such functions, we use thr following.

Theorem 2.3. *If* $y = f(u)$ *and* $u = g(x)$ *and the derivatives of* y *and* u *exist, then*

$$y' = [f(g(x))]' = \frac{df}{du} \times \frac{du}{dx}.$$

Proof

$$\begin{aligned}
y' &= \lim_{h\to} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{df}{dg} \times g'(x), \quad \text{since } g(x+h) \to g(x) \text{ as } h \to 0
\end{aligned}$$

This result is Known as *Chain Rule*.

Example 2.5. Find the derivatives of the following

(1)

$$f(x) = \frac{x}{(x^2-)^4},$$

(2)

$$s(t) = \left(\frac{3t+4}{6t-7}\right)^3.$$

Solution

(1) $f(x) = \frac{x}{(x^2-1)^4} = x(x^2-1)^{-4}$. Thus

$$\begin{aligned}
\frac{df(x)}{dx} &= \frac{dx}{dx} \times (x^2-1)^{-4} + x\frac{d(x^2-1)^{-4}}{dx} \\
&= (x^2-1)^{-4} - 4x(x^2-1)^{-5}2x \\
&= (x^2-1)^{-4}\left[1 - 8x^2(x^2-1)^{-1}\right] \\
&= -\frac{7x^2+1}{(x^2-1)^5}
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{ds(t)}{dt} &= \left[(3t+4)^3 \times (6t-7)^{-3}\right]' \\
&= 3(3t+4)^2 \times 3(6t-7)^{-3} - 3(6t-7)^{-4} \times 6(3t+4)^3 \\
&= (6t-7)^{-3}\left[9(3t+4)^2 - 18(6t-7)^{-1}(3t+4)^3\right] \\
&= 9(6t-7)^{-3}(3t+4)^2\left[1 - 2\frac{(3t+4)}{6t-7}\right] \\
&= -135\frac{(3t+4)^2}{(6t-7)^4}
\end{aligned}$$

Sometimes it is desired to find a **higher-order derivative** of a function. In that case, a lower derivative is treated as a function on its own. For instance, acceleration is the rate of change of velocity. However, velocity itself is the rate of change of displacement. Thus if we let $s(t)$ denote displacement at time t, then

$$\begin{aligned} v(t) &= \frac{d}{dt}[s(t)] = s'(t) \\ a(t) &= \frac{d}{dt}[v(t)] = \frac{d}{dt}[s'(t)] = s''(t) \end{aligned}$$

$a(t)$ is called the *second derivative of* $s(t)$. Derivatives of higher-order are represented as follows:

First derivative	y',	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x(x)$
second derivative	y'',	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx2}[f(x)]$	$D_x^2(x)$
Third derivative	y''',	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3(x)$
Fourth derivative	$y^{(4)}$,	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4(x)$
nth derivative	$y^{(n)}$,	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n(x)$

Example 2.6. Find the indicated derivative for each of the following.

Given	Find
$f(x) = 2x^2 - 2$	$f''(x)$
$f'' = x^3$	$f'''(x)$
$f'' = 2 - \frac{2}{x}$	$f'''(x)$
$f''' = 2\sqrt{x-1}$	$f^5(x)$.

Solution

We only do the first and the last. For the other two, differentiate each of them once. To find $f''(x)$ differentiate $f(x)$ twice. Now, $f\prime(x) = 4x$. Thus, $f''(x) = 4$.

The fourth derivative is

$$\begin{aligned} f^{(4)}(x) &= \frac{d}{dx}[f'''(x)] \\ &= \frac{d}{dx}[2\sqrt{x-1}] \\ &= 2 \times \frac{1}{2}\frac{1}{\sqrt{x-1}} \\ &= \frac{1}{\sqrt{x-1}}. \end{aligned}$$

Thus

$$f^{(5)}(x) = -\frac{1}{2}\frac{1}{\sqrt{(x-1)^3}}.$$

2.5. Derivatives of Trigonometric Functions

There are six trigonometric functions, with sine and cosine as the basic ones. You will recall that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \text{ and } \lim_{h\to 0}\frac{1-\cos(h)}{h} = 0.$$

These quantities are required in finding the derivatives of trigonometric functions.

Theorem 2.4.

$$\frac{d}{dx}[\sin x] = \cos x \quad \text{and} \quad \frac{d}{dx}[\cos x] = -\sin x.$$

Proof

We prove the second part. You do the first one in the activity below.

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x\sin h}{h} \\
&= \lim_{h\to 0}\frac{-\cos x(1-\cos h)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h} \\
&= -\cos x\lim_{h\to 0}\frac{(\cos h - 1)}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} \\
&= -\sin x, \text{ as required.}
\end{aligned}$$

Activity 2.4. *Prove the first part of Theorem 2.4.*

Using the rules of differentiation derivatives of the other four trigonometric functions can be verified as follows:

$$
\begin{aligned}
\frac{d}{dx}[\tan x] &= \sec^2 x \\
\frac{d}{dx}[\cot x] &= -\csc^2 x \\
\frac{d}{dx}[\sec x] &= \sec x \tan x \\
\frac{d}{dx}[\csc x] &= -\csc x \cot x
\end{aligned}
$$

Example 2.7.

(1) Find $f'(x)$ if

  (a) $f(x) = \frac{5}{x^2} + 2\sin^2(x)$,

  (b) $f(x) = \sec^2(x) - \tan^2(x)$.

(2) Find the equation of the line tangent to the graph of $\sin x$ at $x = \frac{\pi}{3}$.

Solution

(1) (a)

$$
\begin{aligned}
\frac{d}{dx}[\frac{5}{x^2} + 2\sin^2(x)] &= -\frac{10}{x^3} + 4\sin x \cos x \\
&= -\frac{10}{x^3} + 2\sin 2x
\end{aligned}
$$

  (b)

$$
\begin{aligned}
\frac{d}{dx}[\sec^2(x) - \tan^2 x] &= 2\sec x \sec x \tan x - 2\tan x \sec^2 x \\
&= 2\sec^2 x \tan x - 2\tan x \sec^2 x \\
&= 0
\end{aligned}
$$

  This result is not unexpected because $\sec^2 x - \tan^2 x = 1$

(2) Let $f(x) = \sin x$. Then $f'(x) = \cos x$. When $x = \frac{\pi}{3}$, $f(x) = \frac{\sqrt{3}}{2}$ and $f'(x) = \frac{1}{2}$. Thus, the equation of the tangent to the curve is

$$
\frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{3}} = \frac{1}{2}
$$

which simplifies to $6y - 3x = 3\sqrt{3} - 2\pi$.

2.6. Implicit Differentiation

Consider the function

$$y = x^3 + 3.$$

You note that it is *explicitly* clear that y is a function of x. That is, if y can be made subject of the formula, then we say the function is explicit. On the other hand, from the relation

$$x^2y + y^2x = -2$$

you cannot solve for y. This is an *implicit* function. To find $\frac{dy}{dx}$ you need to use *implicit differentiation.*

Example 2.8. Find dy/dx given that $x^3 - 2x^2y + 3xy^2 = 38$.

Solution

First, differentiate both sides of the equation with respect to x.

$$\begin{aligned}
\frac{d}{dx}[x^3 - 2x^2y + 3xy^2] &= \frac{d}{dx}[38] \\
\frac{d}{dx}[x^3] - \frac{d}{dx}[2x^2y] + \frac{d}{dx}[3xy^2] &= \frac{d}{dx}[38] \\
3x^2 - [4xy + 2x^2\frac{dy}{dx}] + [3y^2 + 6xy\frac{dy}{dx}] &= 0
\end{aligned}$$

The collect the $\frac{dy}{dx}$ terms on one side, usually left hand side of the equation and and factor out this term.

$$\begin{aligned}
6xy\frac{dy}{dx} - 2x^2\frac{dy}{dx} &= 4xy - 3x^2 - 3y^2 \\
\frac{dy}{dx}[6xy - 2x^2] &= 4xy - 3x^2 - 3y^2
\end{aligned}$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{4xy - 3x^2 - 3y^2}{6xy - 2x^2}.$$

Let us consider another example on how you can find the gradient of a tangent to curve using implicit differentiation.

Example 2.9. Find the equation of a tangent to the curve $(4 - x)y^2 = x^3$ at $(2,\ 2)$.

Solution

Differentiate both sides with respect to x to find

$$\begin{aligned} \frac{d}{dx}[(4-x)y^2] &= \frac{d}{dx}[x^3] \\ -y^2 + 2(4-x)y\frac{dy}{dx} &= 3x^2 \end{aligned}$$

This gives

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2(4-x)y}.$$

At $(2,\ 2)$, $\frac{dy}{dx} = 2$. Thus the equation of the tangent to this curve is $y = 2(x-1)$.

Just like in explicit differentiation, we can find a second derivative implicitly. See problems in the following activity.

Activity 2.5. *Find* $\frac{d^2y}{dx^2}$ *in terms of* x *and* y *in each of the following:*

(1) $x^2 + xy = 5$,

(2) $x^2y^2 - 2x = 3$,

(3) $3(x^2 + y^2)^2 = 100xy$.

2.7. Unit Summary

In this unit you have learnt how to find derivatives from first principles. Rules of differentiation have also been introduced in this unit do not have limits at some points. We have also discussed how to carry out differentiation of trigonometric functions and implicit functions.

2.8. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

2.9. Exercises

(1) Use the definition of a derivative to find $f'(.)$:

  (a) $f(x) = 5x - 3$,

  (b) $f(t) = \frac{t+2}{t^2-4}$,

(c) $f(x) = \sqrt{4+x} - 2$

(2) An automobile's velocity starting from rest is given by

$$v = \frac{100t}{2t+15}$$

where v is measured in metres per second. Find the acceleration at tha following times (i) 6 seconds, (ii) 20 seconds.

(3) Find an equation of a tangent line of the given function at the given point.

(a) $y = x^4 - 3x^2 + 2$,

(b) $y = x^3 + x$

(4) Sketch the graphs of the two equations $y = x^2$ and $y = -x^2 + 6x - 5$ and sketch the two lines that are tangent to both graphs. Find the equations of these lines.

(5) Find the derivatives of the following trigonometric functions

(a) $y = 5x \csc x - \sin x$

(b) $f(\theta) = \frac{\sin\theta}{1-\cos\theta}$

(6) Find the second derivative of

(a)

$$f(x) = \frac{x^2}{\sqrt{x^2+9}},$$

(b)

$$f(t) = \left(\frac{1}{t-3}\right)^3$$

(7) Find dy/dx by implicit differentiation and evaluate the derivative at the given point.

(a) $2\sin x \cos y = 1, \quad (-\frac{\pi}{4},\ \frac{\pi}{4})$,

(b) $(x+y)^3 = x^3 + y^3, \quad (-1,\ 1)$,

(c) $xy = 4, \quad (-4,\ -1)$.

UNIT 3

APPLICATIONS OF DIFFERENTIATION

3.1. Introduction

This unit presents to you the concept of extreme values, curve sketching, and linear approximations and differential. You will meet activities and real life examples that illustrate the use of differentiation. Along with these problems, think of phenomena and scenarios that can fit in differentiation models.

Consider, for instance, a manufacturer of radios. He will look at the cost production and then worry about to maximise his profit. He might want to have a thin work force, or he would use cheap raw materials and such things that will ensure he realises more that the production cost.

The unit contains the following topics:

- Extrema on a given interval,
- The Mean Value Theorem,
- Newton's Method of approximating Zeros of Function,
- Linear Approximations and Differentials.

3.1.1. Learning objectives. By the end of this unit you should be able to:

- find the minimum or maximum value of function on a given interval,
- use the second derivative test to determine if a point in minimum or maximum turning point,
- sketch curves of functions,
- approximate zeros of s function using iterations of the Newton's Method,
- use linear approximations to evaluate numerical problems.

3.1.2. Prerequisite knowledge. The material in the unit is based on differentiation of functions.

3.1.3. Time. You should be able to complete this unit in 3 hours.

3.2. Extreme Values of a Function on an interval

Functions behave differently and most of the times you will be required to check if a function has (and indeed find) a minimum or maximum value on a given interval. A value of x that gives this value may also be required.

Definition 3.1. *Let $f(x)$ be defined on some interval I containing a. Then*

(1) *$f(a)$ is the maximum of f on the interval if $f(a) \geq f(x)$ $\forall x \in I$,*

(2) *$f(a)$ is the minimum of f on the interval if $f(a) \leq f(x)$ $\forall x \in I$*

The maximum or minimum values of the function on the interval are also called *extreme values* or *extrema* of the function on I.

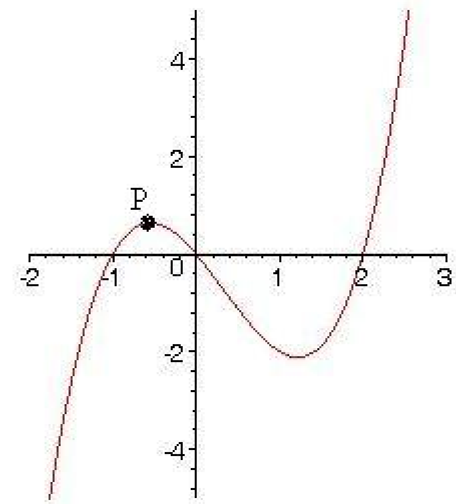


FIGURE 3.1. Extreme Values of a function

Activity 3.1. *Consider the graph in Figure 3.1. If you take the interval $[-2, 2]$, what is*

(1) *the minimum value of $f(x)$?*

(2) *the maximum value of the function?*

In this activity, you have seen that the end points of the interval can also give an extreme value of a function. If, however, we use the open interval $(-2, 2)$, the curve in Figure 3.1 has a maximum value at P, but it does *not* have a minimum value. You also note that if we use all x, the extrema would not be the same as those found in Activity 3.1. Since we are talking about extrema on an interval, we use the terms *relative maximum* or *relative minimum.* It is not always the case that an extremum of a function exists on a given *open* interval. Look at the following.

Remark 3.1. (1) If a function $f(x)$ is continuous on a closed interval $[b,\ c]$, then it has both a minimum and a maximum value on the interval.

(2) For a given interval, a is called a *critical number* if $f'(a) = 0$. The value of f corresponding to a is called the *critical value of* f. For a given interval these critical values occur at turning points or "hills" and "valleys" of the curve.

(3) To find the extrema of a continuous function on $[b,\ c]$, you try the following steps

   (a) Find the critical number(s) by solving $f'(a) = 0$ and find the corresponding values of $f(a)$,

   (b) Evaluate $f(x)$ at the end points of $[b,\ c]$,

Now go through the following examples.

Example 3.1. Locate the extrema of the function on the given interval.

(1) $f(x) = x^2 + 2x - 4,\quad [-1,\ 1]$,

(2) $f(x) = 3x^{\frac{2}{3}} - 2x\quad [-2,\ 2]$

Solution

(1) $f'(x) = 2x + 2$. Thus, at critical point, $x = -1$ and the critical value of f is -5. At $x = 1$, $f(x) = -1$ Thus the maximum value of $f(x) = x^2 + 2x - 4$ on $[-1,\ 1]$ is -1 and the minimum value is -5.

(2) The first derivative of $f(x) = 3x^{\frac{2}{3}} - 2x$ is $f'(x) = 2x^{-\frac{1}{3}} - 2x$. The critical number is 1 and the corresponding critical value of $f(x)$ is 1. Now, $f(-2) = 3\sqrt[3]{4} + 4$ and $f(2) = 3\sqrt[3]{4} - 4$. Note also that $f(0) = 0$. So the minimum value is 0 and the maximum value is $3\sqrt[3]{4} + 4$.

3.3. The Mean Value Theorem and Rolle's Theorem

We start with an activity.

Activity 3.2. *Consider the function $f(x) = -x^2 + 3x$ on $[0,\ 3]$. The graph is shown in Figure 3.2. Find the values of x for which $f(x) = 0$. Does the interval have an x*

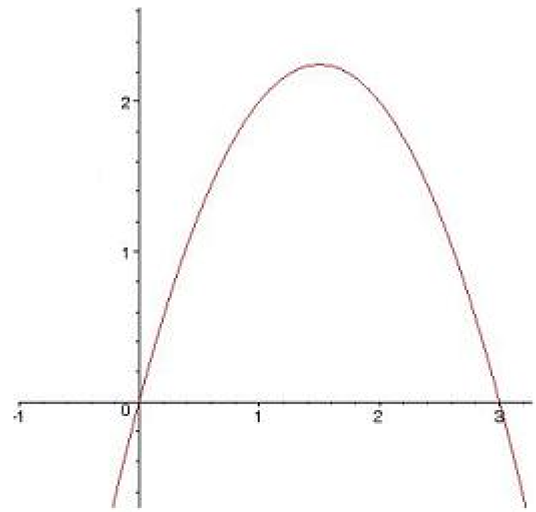


FIGURE 3.2. Rolle's Theorem

so that $f'(x) = 0$?

The findings of this activity are summarised in the following.

Theorem 3.1. *Rolle's Theorem Let $f(x)$ be a continuous function on $[b,\ c]$ and differentiable on $(b,\ c)$. If $f(b) = f(c)$ then $\exists a \in (b,\ c)$ so that $f'(a) = 0$.*

Now to illustrate Rolle's Theorem, look at the following example.

Example 3.2.

Determine whether Rolle's Theorem can be applied to f on the given interval. If it can be applied, find all values of a on $(b,\ c)$ such that $f'(a) = 0$.

(1)
$$f(x) = x^2 - 2x \quad [0,\ 2],$$

(2)
$$f(x) = \frac{x^2 - 2x - 3}{x + 2} \quad [-1,\ 3],$$

(3)
$$f(x) = \frac{6x}{\pi} - 4\sin^2 x \quad [0,\ \frac{\pi}{6}].$$

Solution

We do the last two only. Using arguments similar to the ones used here, yo should be able to handle the first problem.

(1)

(2) The numerator is factored to give $(x+1)(x-3)$. It follows that $f(-1) = f(3) = 0$. Thus Rolle's Theorem can be used. The first derivative of $f(x)$ is

$$f'(x) = \frac{x^2 + 4x - 1}{(x+2)^2}.$$

Thus $f'(x) = 0$ implies $x^2 + 4x - 1 = 0$ and this occurs when $x = \sqrt{5} - 2$. (Not that we could not take $-\sqrt{5} - 2$ because it falls outside the given interval).

(3) $f(0) = f(\frac{\pi}{6}) = 0$. So by Rolle's Theorem $(0,\ \frac{\pi}{6})$ contains an a so that $f'(a) = 0$. Now,

$$f'(x) = \frac{6}{\pi} - 4\sin x \cos x = \frac{6}{\pi} - 4\sin(2x).$$

Equating this derivative to zero and solving it gives

$$x = \frac{1}{2}\sin^{-1}(\frac{3}{2\pi}).$$

We now state (without proof) one of the most important theorems in calculus.

Theorem 3.2. *Mean Value Theorem Let $f(x)$ be continuous on $[b,\ c]$ and differentiable on $(b,\ c)$. Then $\exists a \in (b,\ c)$ such that*

$$f'(a) = \frac{f(c) - f(b)}{c - b}.$$

As an illustration of the Mean Value Theorem, consider the graph of $f(x) = x^2 - x - 2$ in Figure 3.3. Take the interval $[-1,\ 3]$. The secant line has a gradient which is equal to the gradient of the tangent to the curve at Q. The point A has coordinates $(-1,\ 0)$ and the point B has coordinates $(3,\ 4)$. Thus slope of the secant line that passes through A and B is, m_s, given by

$$m_s = \frac{0 - 4}{-1 - 4} = 1.$$

At P, $x = 1$ so that

$$f'(x) = 2x - 1 = 1.$$

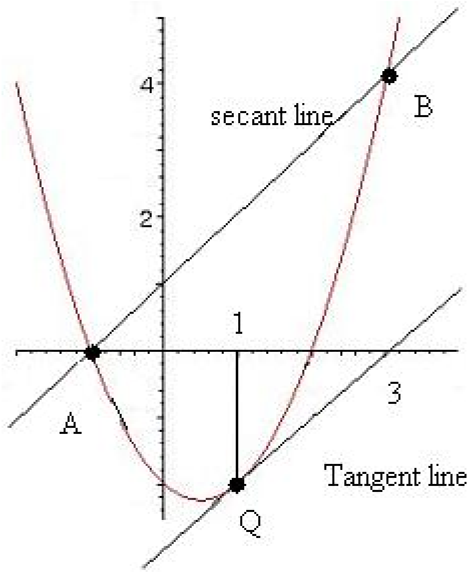


FIGURE 3.3. Mean Value Theorem

Activity 3.3. *Show that the function $f(x) = x^3 + x - 4$ satisfies the hypotheses of the Mean-Value Theorem on the interval $[-1,\ 2]$ and find all the values of a in $(-1,\ 2)$ at which the tangent line to the graph of $f(x)$ is parallel to the secant line joining $(-1,\ f(-1))$ and $(2,\ f(2))$.*

3.4. Curve Sketching

3.4.1. Increasing and decreasing functions. The curve in Figure 3.4 shows what happens to a function as x increases.

Definition 3.2. *Let $f(x)$ be a function defined on an interval $(b,\ c)$ containing x_1 and x_2 such that $x1 < x2$. Then $f(x)$ is said to be*

(1) *increasing if $f(x_1) < f(x_2)$.*
(2) *decreasing if $f(x_1) > f(x_2)$.*

You note from Figure 3.4 that as x increases from -4 to $-2, f(x)$ increases and its graph moves up. From -2 to 1, you see that the function decreases and its graph moves down.

Activity 3.4. *Figure 3.5 is a graph of $f(x) = x^2 - 6x + 8$. Identify an open interval on which $f(x)$ is increasing or decreasing.*

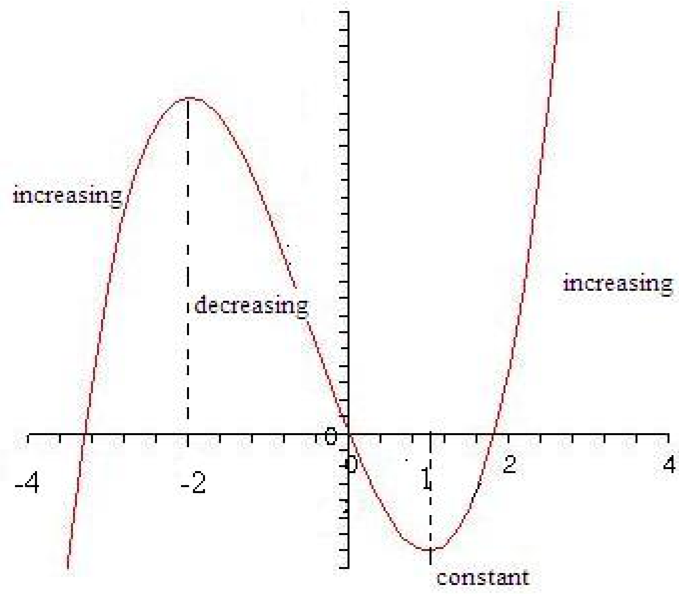


FIGURE 3.4. Increasing or Decreasing with x

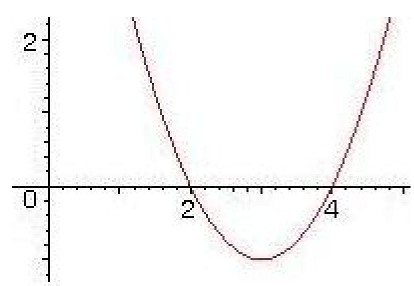


FIGURE 3.5. Graph of $f(x) = x^2 - 6x + 8$

What can you say about the sign of $f'(x)$ on the interval the function is decreasing/increasing?

Remark 3.2. From Activity 3.4 you note that, for a function $f(x)$ differentiable on $(b,\ c)$,

- if $f'(x) > 0\ \forall\ x\ \in (b,\ c)$ then $f(x)$ is increasing on $(b,\ c)$,
- if $f'(x) < 0\ \forall\ x\ \in (b,\ c)$ then $f(x)$ is decreasing on $(b,\ c)$,
- if $f'(x) = 0\ \forall\ x\ \in (b,\ c)$ then $f(x)$ is constant on $(b,\ c)$.

As proof to the first statement, assume that $f'(x) > 0\ \forall\ x\ \in (b,\ c)$ and take $x_1 < x_2$ in $(b,\ c)$. The Mean Value Theorem guarantees the existence of an a in $(x_1,\ x_2)$ such that

$$f'(a) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $f'(a) > 0$. Thus $f(x_2) > f(x_1)$ since $x_2 - x_1 > 0$ by assumption.

To find the open interval on which $f(x)$ is increasing or decreasing, you

(1) locate critical points and use these to get test intervals;

(2) determine the sign of $f'(x)$ at one value in each of the test intervals.

Example 3.3. Find critical numbers of $f(x)$ and the open intervals on which the function is increasing or decreasing and locate all relative extrema.

(1) $f(x) = 2x^3 - 3x^2 - 12x$,

(2) $f(x) = -2x^2 + 4x + 3$,

(3) $f(x) = \frac{x^2-3x-4}{x-2}$.

Solution

(1) $f(x) = 2x^3 - 3x^2 - 12x$. Thus $f'(x) = 6x^2 - 6x - 12$. At critical points,

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \text{ so that} \\ 6(x+1)(x-2) &= 0 \\ \text{giving } x = -1 & \quad x = 2. \end{aligned}$$

We set up a table for testing whether the function is increasing or decreasing in a given interval.

Interval	$-\infty < x < -1$	$-1 < x < 2$	$2 < x < \infty$
Test Value	-2	0	3
Sign of $f'(x)$	$f'(-2) = 24 > 0$	$f'(0) = -12 < 0$	$f'(x) = 24 > 0$
Conclusion	Increasing	Decreasing	Increasing

Thus the function is increasing on $(-\infty,\ -1) \cup (2,\ \infty)$. The relative maximum is $f(-1) = 7$ and relative minimum is $f(2) = -20$.

(2) You can try this one.

(3) $f(x) = \frac{x^2-3x-4}{x-2}$. So $f'(x)$ is given by

$$f'(x) = \frac{(2x-3) \times (x-2) - (x^2 - 3x - 4)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}.$$

Now setting the numerator to zero does not give us real roots. So there are no critical numbers for this function. However, you will note that the function increases on $(\infty, \ 2)$ and there is vertical asymptote at $x = 2$. On $(2, \ \infty)$ $f(x)$ is increasing. See Figure 3.6.

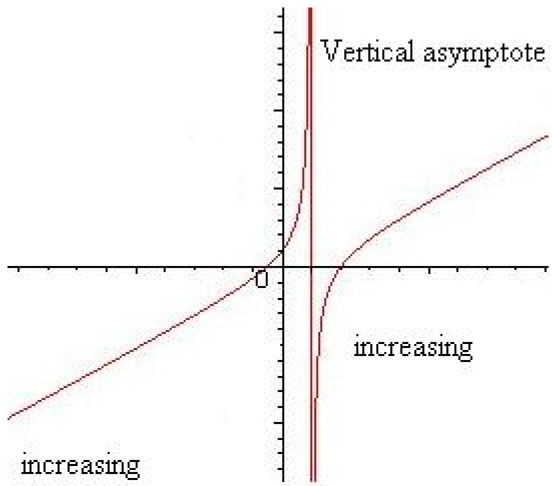


FIGURE 3.6. Graph of $f(x) = \frac{x^2-3x-4}{x-2}$

3.4.2. Concavity of a Graph.

Let us start the subsection with an activity.

Activity 3.5. *Consider graph of $f(x) = x^3 - 3x^2 + 3$ shown in Figure 3.7. Take the interval $(-1, \ \frac{1}{2})$ on which A is found and $(1, \ 3)$ where we find B. Find $f'(x)$.*

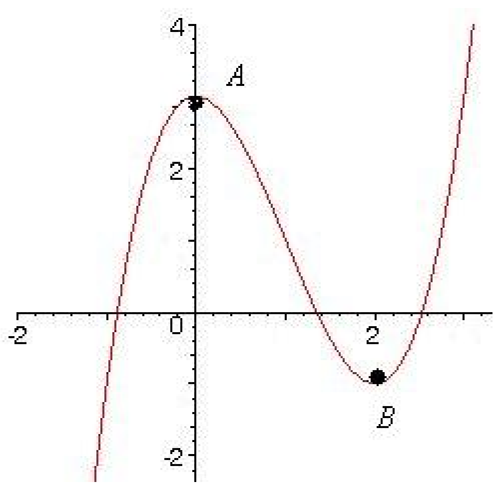


FIGURE 3.7. Graph of $f(x) = x^3 - 3x^2 + 3$

Determine whether $f'(x)$ is increasing or decreasing on $(-1, \ \frac{1}{2})$ and $(1, \ 3)$.

From Activity 3.5, you see that the point A is a "hill" or *is concave downward* and B is a "trough" or *is concave upward.* You also note that $f'(x)$ is decreasing at points around A and increasing at points around B. This is generalised in the following.

Theorem 3.3. *Test for Concavity Let $f(x)$ be a function with a second derivative on $(b,\ c)$.*

(1) *The the graph of $f(x)$ is concave upward on $(b,\ c)$ if $f''(x) > 0 \ \forall\ x\ \in\ (b,\ c)$.*

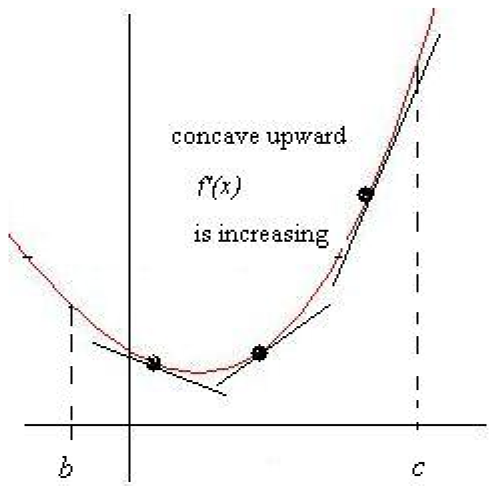


FIGURE 3.8. Concave Upward

(2) *The the graph of $f(x)$ is concave downward on $(b,\ c)$ if $f''(x)\ <\ 0$ $\forall\ x\ \in\ (b,\ c)$.*

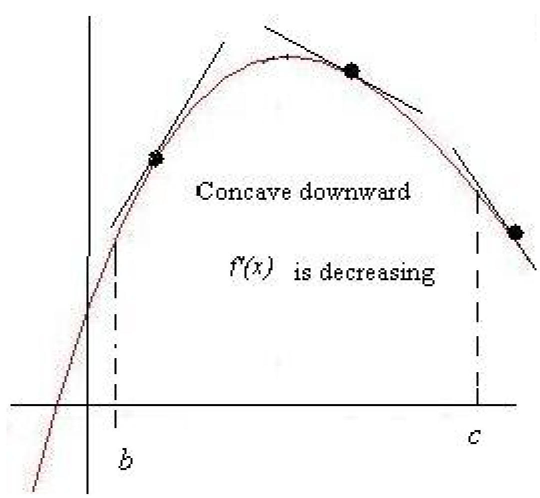


FIGURE 3.9. Concave Downward

(3) *The the graph of $f(x)$ changes concavity at $a \in (b,\ c)$ if $f''(a) = 0$ or $f''(x)$ is undefined. This is called the point of inflection. For Figure 3.10, the points of inflection are M and N.*

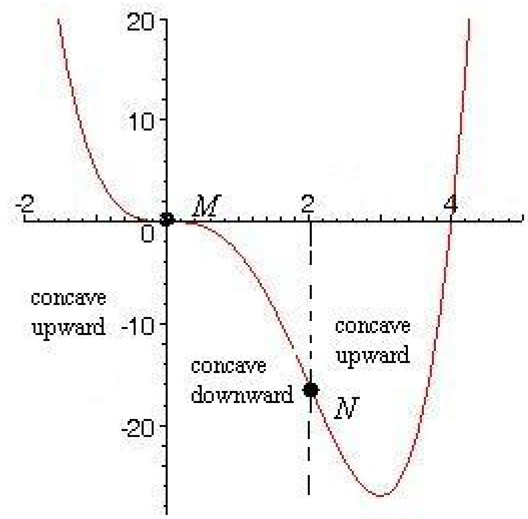


FIGURE 3.10. Points of inflection

(4) *Let $f(x)$ be a function such that $f'(a) = 0$ and $f''(x)$ on $(b,\ c)$ containing a. Then*

(a) *$f(a)$ is a relative minimum if $f''(a) > 0$,*

(b) *$f(a)$ is a relative maximum if $f''(a) < 0$. If $f''(a) = 0$, then $(a,\ f(a))$ is a point of inflection.*

These results will also help you sketch the curves of given functions. Follow the arguments in the following examples.

Example 3.4. For the following functions, identify all relative extrema. State whether it is minimum or maximum.

(1) $f(x) = x^3 - 9x^2 + 27x - 26$,

(2) $f(x) = \sqrt{x^2 + 1}$,

(3) $f(x) = 2\sin x + \cos 2x, \quad 0 \leq x \leq 2\pi$.

Solution

(1) $f(x) = x^3 - 9x^2 + 27x - 26$. This is a cubic function and it should have two extrema. Now $f'(x) = 3x^2 - 18x + 27$. At turning points, $x = 3$. There is a repeated root. The second derivative of $f(x)$ is $6x - 18$. You note that $f''(3) = 3$. Therefore, there is no extremum. The point $(3,\ 1)$ is a point of inflection.

(2) We start with critical numbers of the function.

$$\begin{aligned} f'(x) &= \frac{x}{\sqrt{x^2+1}} = 0 \\ x &= 0. \end{aligned}$$

Furthermore,

$$f''(x) = \frac{1}{\sqrt{(x^2+1)^2}}.$$

This quantity is always positive. Thus the minimum value of $f(x)$ is $f(0) = 1$.

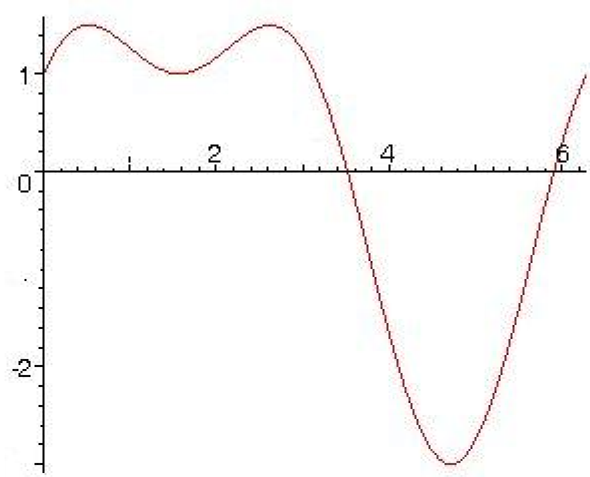


FIGURE 3.11. Graph of $f(x) = 2\sin x + \cos 2x$

.

(3) $f(x) = 2\sin x + \cos 2x$. Now $f'(x) = 2\cos x - 2\sin 2x$. Equating this to zero gives $x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{4}$. The turning points are $A(\frac{3\pi}{2},\ -3)$, $B(5\frac{\pi}{6},\ \frac{3}{2})$, $C(\frac{\pi}{2},\ 1)$ and $D(\frac{\pi}{6},\ \frac{3}{2})$. See the graph in Figure 3.11. Now, $f''(x) = -2\sin x - 4\cos 2x$. At A, $f''(x) = 6 > 0$. Thus -3 is relative minimum. $f''(x) = -3 < 0$ at B and this implies that $\frac{3}{2}$ is a relative maximum. The rest can be classified in the same way.

Example 3.5.

Sketch the graph of the given function. Ensure all relative extrema and points of inflection are identifiable.

(1) $f(x) = -\frac{1}{3}(x^3 - 3x + 2)$,

(2) $f(x) = (x+1)(x-2)(x-5)$,

(3) $f(x) = \frac{x}{\sqrt{x^2+7}}$,

(4) $y = \cos x - \frac{1}{2}\cos 2x$.

Solution

(1) $f(x) = -\frac{1}{3}(x^3 - 3x + 2)$. Now $f'(x) = -\frac{1}{3}(3x^2 - 3) = -(x^2 - 1)$. At turning points $f'(x) = 0$ implying $x = 1$ or $x = -1$. Furthermore, $f''(x) = -2x$. Here is a table of values and curve's behaviour,

	-3	-2	0	1
$f(x)$	$\frac{16}{3}$	0	$-\frac{3}{2}$	0
$f''(x)$	$-$	$f''(-2) > 0$		$f''(1) < 0$
Conclusion	decreasing	Concave upward	Increasing	Concave downward

The curve is shown in Figure 3.12.

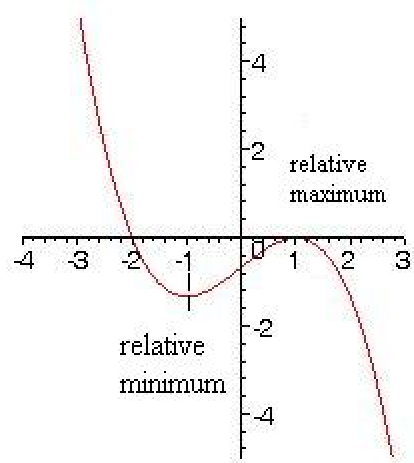


FIGURE 3.12. Graph of $f(x) = -\frac{1}{3}(x^3 - 3x + 2)$

(2) $f(x) = (x+1)(x-2)(x-5)$.The curve crosses x-axis when $x = -1,\ 2,\ 5$. Now $f'(x) = 3x^2 - 12x + 3$ and $f''(x) = 6x - 12$. At critical points, $f'(x) =$

$3x^2-12x+3=0$ so that $x=2+\sqrt{3}$ or $2-\sqrt{3}$. A table of this information is given below.

	$2-\sqrt{3}$	$2+\sqrt{3}$
$f(x)$	$6\sqrt{3}$	$-6\sqrt{3}$
$f''(x)$	$f''(2-\sqrt{3})<0$	$f''(2+\sqrt{3})>0$
Conclusion	Concave downward	Concave upward

We have a relative maximum at $(2-\sqrt{3},\ 6\sqrt{3})$ and a relative minimum at $(2+\sqrt{3},\ -6\sqrt{3})$. The function is increasing on $(-\infty,\ 2-\sqrt{3})$, decreasing on $((2-\sqrt{3},\ 2+\sqrt{3})$ and so on. See the curve in Figure 3.13.

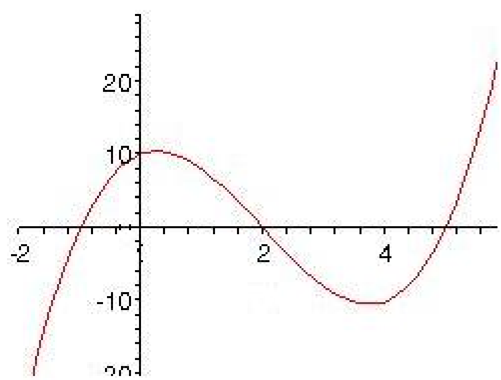


FIGURE 3.13. Graph of $f(x)=-\frac{1}{3}(x^3-3x+2)$

(3) If $f(x)=\frac{x}{\sqrt{x^2+7}}$, then

$$f'(x)=\frac{x^2-x+7}{\sqrt{(x^2+7)^3}}.$$

This quantity is never 0 since completion of squares in the numerator gives us $(x-\frac{1}{2})^2+\frac{27}{4}>0$. This means that the curve does not have relative maximum and minimum turning points. By inspection and using the fact that

$$f''(x)=\frac{-21x}{\sqrt{(x^2+7)^5}}=0$$

at $(0,\ 0)$. So this is a point of inflection. Look at Figure 3.14.

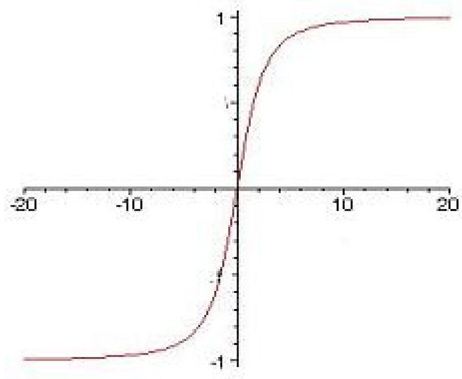


FIGURE 3.14. Graph of $f(x) = \frac{x}{\sqrt{x^2+7}}$

A closer look will reveal that

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\left[\frac{x}{\sqrt{x^2+7}}\right] \\ &= \lim_{x\to\infty}\left[\frac{1}{\sqrt{1+\frac{7}{x^2}}}\right] = 1 \end{aligned}$$

similarly you see that $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left[\frac{x}{\sqrt{x^2+7}}\right] = -1.$

(4) $y = \cos x - \frac{1}{2}\cos 2x$. Thus $f'(x) = -\sin x + \sin 2x$ and $f''^{(x)} = -\cos x + 2\cos 2x$ Now $f'(x) = -\sin x + \sin 2x = 0$ means $x = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$.

	0	$\frac{\pi}{3}$	π	$\frac{5\pi}{6}$
$f(x)$	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{3}{4}$
$f''(x)$	$1 > 0$	$-\frac{3}{2} < 0$	$3 > 0$	$-\frac{3}{2}$
Conclusion	Concave upward	Concave downward	Concave upward	Concave downward

The graph of this function is given in Figure 3.15. It has relative minima at $(0,\ 1/2)$, $(\pi, -\ 3/2)$, $(2\pi,\ 1/2)$ and relative maxima at $(\pi/3,\ 3/4)$ and $(5\pi/3,\ 3/4)$.

3.5. Solving Minimum and maximum Problems

In many areas you encounter a problem which demands a minimum or maximum possible quantity to fit a particular setting. A manufacturer would like to maximise profit or minimise losses. This is just one of the many problems that involves the

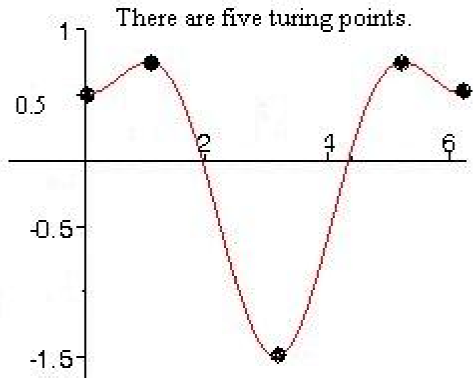


FIGURE 3.15. Graph of $f(x) = \cos(x) - \frac{1}{2}\cos(2x)$

determination of maximum and minimum values.

Here is an outline of possible steps in solving such problems.

(1) Write an equation in the unknown quantity that you wish to optimised.
(2) Determine the feasible interval of the equation.
(3) Find th first and second derivatives of the function to get critical points and their behaviours (relative minimum or relative maximum?). Compare the relative extreme values with those at endpoints of feasible interval.

Go through the following examples and see how these steps have been followed.

Example 3.6. A dairy farmer plans to fence in a rectangular pasture adjacent to a rive. The pasture must contain 180,000 square metres in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is required along the river?

Solution

Let the width of the fence be y and the length be x. Suppose the length of fence is C and the enclosed area A. Then

$$A = xy = 180000 \quad \text{and} \quad P = x + 2y.$$

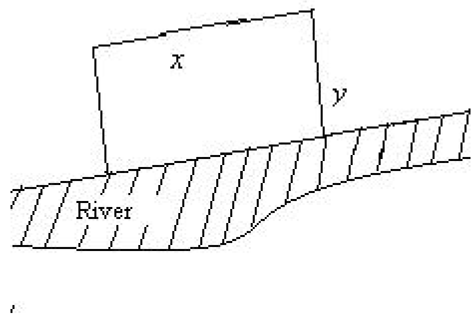


FIGURE 3.16. Fence of Pasture

Now express y in terms of x in area. We get $y = \frac{180000}{x}$. Therefore, P can be expressed as a function of x alone. Thus

$$P(x) = x + \frac{360000}{x} \quad \text{and} \quad P'(x) = 1 - \frac{360000}{x^2}.$$

For $P(x)$ to be extreme, its first derivative must be zero. At this point $x = 600$ and so $y = 300$. At this value,

$$P''(x) = 720000/x = 1200 > 0.$$

Thus the minimum value of rectangular fencing is $P(600) = 600 + \frac{360000}{600} = 1200$ metres. Dimensions are: width 300 metre and length is 600 metres.

Example 3.7. The profit for a certain company is given by

$$P(x) = 2300 + 20x - \frac{1}{2}s^2$$

where x is the amount (in thousands of Kwacha) spent on advertising. What amount of advertising gives a maximum profit?

Solution

$P'(x) = 20 - s$. For the profit to be maximum, $x = 20$ and and the maximum profit is $MK\ 430,\ 000$.

Activity 3.6. *A real estate office handles* 50 *apartment units. When the rent is* $K5,\ 400$ *per month, all units are occupied. However, on the average, for* $K300$ *increase in rent one unit becomes vacant. Each occupied unit requires an average of K360 per month for service and repairs. What rent should be charged to realised the most profit?*

3.6. Newton's Method

There are some equations whose zeros cannot be determined exactly. Such equations have their solutions approximated. The Newton's Method is one of them.

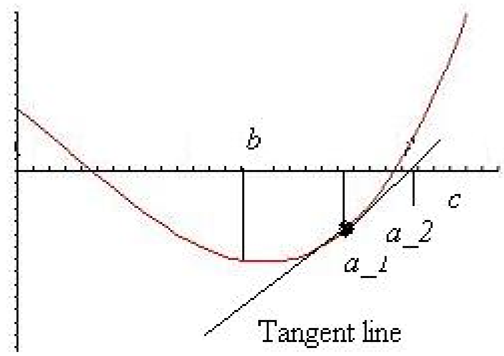


FIGURE 3.17. x-intercept of tangent gives a_2

How does the method work?

Consider a function $f(x)$ that is continuous on $[b,\ c]$ and differentiable on $(b,\ c)$. If $f(b)$ and $f(c)$ differ in sign, the Intermediate Value Theorem guarantees the existence of a zero in $(b,\ c)$. You make an estimate of a zero in the interval. Take $x = a_1$. See Figure 3.17. Draw a tangent to the curve of f where $x = a_1$. Find the x-intercept for this tangent line and use it to estimate a second zero of f. The equation of the line tangent to the curve at $(a_1,\ f(a_1))$ is given by

$$\frac{y - f(a_1)}{x - a_1} = f'(x) \tag{3.1}$$

Now, $y = 0$ when the line crosses x-axis. From Equation 3.1 we find

$$x = a_1 - \frac{f(a_1)}{f'(a_1)}$$

This gives us a new estimate for the zero of f, a_2

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}.$$

A third estimate can be found as an improvement on a_3,

$$a_3 = a_2 - \frac{f(a_2)}{f'(a_2)}.$$

In general, if $f(a) = 0$, with $f(x)$ differentiable, then to approximate a you apply the following procedure:

(1) Draw a sketch of the graph of $f(x)$ and make an initial estimate a_n

(2) Find a new estimate using a_n

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}.$$

(3) The *iteration* terminates when $|a_n - a_{n+1}| < r_n$, where r_n is the prescribed error. Note that

$$r_n = \left| \frac{f(a_n)}{f'(a_n)} \right|$$

Look at the following examples.

Example 3.8. Complete one iteration of the Newton's Method for the given function using the provided a_1.

Function	a_1
$f(x) = x^2 - 3$	1.7
$g(x) = \tan x$	0.1

Solution

Now $f'(x) = 2x$. Since $a_1 = 1.7$, $f(1.7) = 1.7^2 - 3$ and $f'(1.7) = 3.7$. Thus

$$a_2 = 1.7 - \frac{1.7^2 - 3}{2 \times 1.7} = 1.73235.$$

The derivative of $g(x)$ $g'(x) = \sec^2(x)$. Thus with $a_1 = 0.1$ we have

$$\begin{aligned} a_2 &= 0.1 - \frac{\tan(0.1)}{\sec^2(0.1)} \\ &= 0.1 - \frac{.1003346721}{1.010067046} \\ &= 0.00067 \end{aligned}$$

Example 3.9. Use the Newton's Method and continue the iteration until two successive approximations differ by 0.001.

(1) $f(x) = x^3 + x - 1$,

(2) $f(x) = x^5 + x - 1$

Solution

We do the second part only.

(1)

(2) A graph of $f(x) = x^5 + x - 1$ is show in Figure 3.18. $f'(x) = 5x^4 + 1$. From the figure we start with $a_1 = 1$. Then

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}.$$

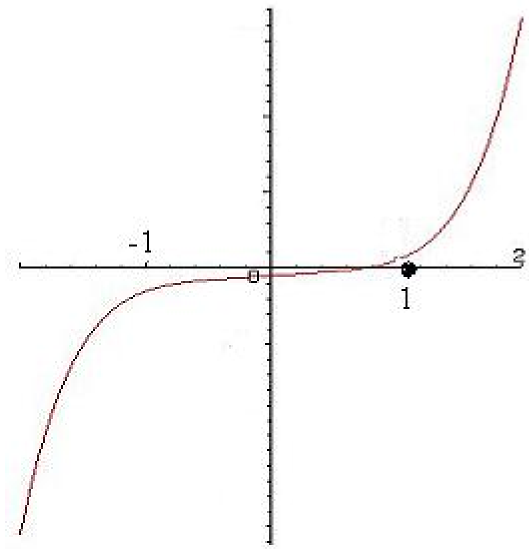


FIGURE 3.18. Curve crosses x-axis close to $x = 1$.

Thus

$$\begin{aligned} a_2 &= 1 - \frac{f(1)}{5+1} = 0.83333 \\ a_3 &= 0.76438 \\ a_4 &= 0.75502 \\ a_5 &= 0.75494 \end{aligned}$$

In this case, $r_n = 0.001$. Now $a_4 - a_5 = 8 \times 10^{-05}$. Thus the approximate zero is $a_5 = 0.75494$.

Remark 3.3. (1) The Newton's Method is based on the assumption that the curve of $f(x)$ and the tangent at $(a_1,\ f(a_1))$ cross $x - axis$ at close points.

(2) Newton's Method does not always work. For instance, if $f'(a_1) = 0$, then the value a_1 cannot be used, and so the method fails. This method can be checked for applicability for a given function by using

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

on an open interval containing the zero. As an example, consider the function $f(x) = 2x^3 - 6x^2 + 6x - 1$ on the interval $(0,\ 2)$.

$$\frac{f(x) \times f''(x)}{f'(x)} = \frac{(x^3 - 6x^2 + 6x - 1) \times (12x - 12)}{(6x^2 - 12x + 6)}$$

which is not defined when $x = 1$ because $f'(1) = 0$.

Activity 3.7. *Apply Newton's Method using the indicated a_1, and explain why it fails.*

(1) $f(x) = 4x^3 - 12x^2 + 12x - 3,\ a_1 = \frac{3}{2}$.
(2) $g(x) = 2\sin x + \cos 2x,\ a_1 = \frac{3\pi}{2}$.

3.7. Linear Approximations and Differentials

We begin this section with an activity.

Activity 3.8. *Let $y = f(x)$ and the variable x have an initial x_0. If x is changed from x_0 to x_1 by an increment of Δx, find the corresponding change in y.*

The function in the activity is presented in Figure 3.19.
From Activity 3.8 you note that

$$\Delta y = f(x + \Delta x) - f(x) \tag{3.2}$$

and that gradient, m_{AB}, of line segment AB is given by

$$m_{AB} = \frac{\Delta y}{\Delta x}.$$

Now Equation 3.2 provides that

$$\begin{aligned} f(x_1) &= f(x_0) + \Delta y \\ &= f(x_0) + m_{AB}\Delta x \\ &\approx f(x_0) + f'(x_0)\Delta x. \qquad (3.3) \end{aligned}$$

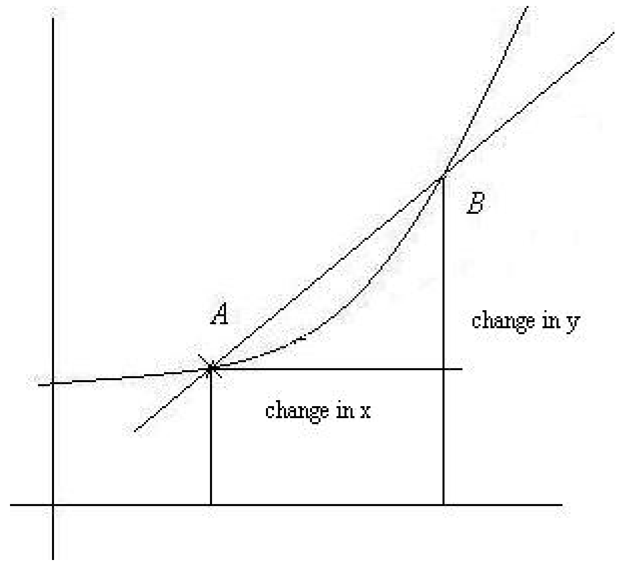


FIGURE 3.19. Changes in y and x

Thus in general, if $y = f(x)$, with f differentiable at $x = x_0$, then

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0), \text{ for } x_1 \text{close to} x_0. \tag{3.4}$$

Example 3.10. Use linear approximations to estimate $f(b)$ if the independent variable changes from from a to b.

(1) $f(x) = 4x^5 - 6x^4 + x^2 - 5$, $a = 1$, $b = 1.03$
(2) $f(x) = x^4$, $a = 2$, 1.98

Solution

(1) We wish to find an approximate value for $f(1.03)$. Now $f'(x) = 20x^4 - 24x^3 + 6x$. Thus

$$\begin{aligned} f(1.03) &= [4 - 6 + 3 - 5] + [20 - 24 + 6] \times (0.03) \\ &= -3.94 \end{aligned}$$

(2) $f'(x) = 4x^3$. Therefore, $f(1.98) = 2^4 + 4 \times 2^3 \times (-0.02) = 14.72$

Now see of you can work through the following activity.

Activity 3.9. *Use linear approximations to evaluate the following:*

(1) $\sqrt[3]{63.07}$,

(2) $\sqrt[5]{32.05}$.

Remark 3.4. (1) We know that

$$\frac{\Delta y}{\Delta x} \approx f'(x_0), \text{ if } \Delta x \to 0.$$

Therefore

$$\Delta y \approx f'(x_0)\Delta x \tag{3.5}$$

We can use this equation to compute the *propagated error* (Δy) in y using the *error in measurement*, Δx.

(2) Change or error can be *absolute* when we are interested in Δy, it can be *relative* if we consider

$$\frac{\Delta y}{y_0}$$

and it can also be *percentage change* where we have

$$\frac{\Delta y}{y_0} \times 100.\%$$

Example 3.11.

(1) Find the approximate relative change in T for the function $T = 4 + 3u - 2u^2$ when u is increased from 2 to 2.05.

(2) The height of a cone is 20 cm. The radius of its circular base was measured to be 10 cm. If the measurement is correct to within 0.01 cm, estimate the percentage error in the volume of the cone.

Solution

(1)

$$\begin{aligned} \Delta T &\approx T'(2) \times \Delta u \\ &\approx (3 - 8) \times 0.05 = -0.25 \end{aligned}$$

So the approximate relative change in T is $\frac{-0.25}{4+3\times 2-2\times 2^2} = -0.125$. It is negative because T decreases as u increases after $u = \frac{3}{4}$.

(2) Let volume, V, of a cone be given by

$$V(r) = \frac{1}{3}\pi r^2 h, \;\; h \text{ is fixed.}$$

Now $r = 10$ cm and $|\Delta r| \leq 0.01$. Then $V'(r) = \frac{2}{3}\pi r h$. Therefore,

$$\begin{aligned} \Delta V &\approx V'(10)\Delta r \\ &\approx \frac{2}{3} \times \pi \times 10 \times 20 \times (\pm 0.01) \\ &= \pm\frac{4}{3}\pi \end{aligned}$$

So the percentage error in volume corresponding to a measurement error of ± 0.01 in radius is $\frac{1}{5}\%$.

3.8. Differentials

From Equation 3.5 of Remark 3.4, you have

$$\Delta y \approx f'(x)\Delta x.$$

Usually in such approximations, Δx is denoted by dx and is called the *differential of* x, and the expression

$$f'(x)dx$$

is denoted by dy and is called the *differential of* y.

We can use the definition of a differential to express derivative rules in *differential form*. Suppose $f(x)$ and $g(x)$ are differentiable functions of x. Then the product rule of differentiation provides that

$$d[fg] = df(x)g(x)dx + dg(x)f(x)dx$$

in differential form. Now look at the following.

Example 3.12.

(1) Find the differential of the given function.

(a) $f(x) = 2x^{3/2}$

(b) $\frac{x+1}{2x-1}$,

(2) Let $x = 2$ and use the function $f(x) = x^3$ and the value of $\Delta x = dx$ to complete the table.

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$\frac{dy}{\Delta y}$
1.000				
0.500				
0.100				
0.010				
0.001				

Solution

(1) We do the second part only. You can use this solution to work out the first one as well.

(a)

(b)

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{\frac{(2x-1)d(x+1)}{dx} - (x+1)\frac{d(2x-1)}{dx}}{(2x-1)^2} \\ &= \frac{(2x-1) - 2x - 2}{(2x-1)^2} \\ &= -\frac{3}{(2x-1)^2}\end{aligned}$$

Thus in differential form, we have $df(x) = -\frac{3}{(2x-1)^2}dx$.

(2) Since $f(x) = x^3$, it follows that $\frac{df(x)}{dx} = 3x^2$. Thus $df(x) = 3x^2dx$. Now, $\Delta y = f(x + \Delta x) - f(x)$. In the first case, $x = 2$ and $\Delta x = dx = 1$. Therefore, $df(x) = 3 \times 2^2 \times 1 = 12$ and $\Delta y = (2 + 1)^3 - 2^3 = 19$. The rest are computed using similar arguments to give the following table.

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$\frac{dy}{\Delta y}$
1	12	19	7	0.6316
0.5	6	7.625	1.625	0.7869
0.1	1.2	1.261	0.061	0.9516
0.01	0.12	0.120601	0.000601	0.9950
0.001	0.012	0.012006001	6.00×10^{-6}	0.9995

Activity 3.10. *Express the following in differential form*

(1)

$$(\lambda f(x))', \ \lambda \neq 0,$$

(2)

$$(f(x) \pm g(x))',$$

(3)

$$\left(\frac{f(x)}{g(x)}\right)'$$

3.9. Unit Summary

In this unit you have learnt how to determine whether a function is increasing on an interval. You have also seen how concavity of a curve can be classified as upward or downward depending using the sign of the second derivative.
The Mean Value Theorem and the Rolles theorem have also been discussed. We often find approximate solutions to equations of the form $f(x)0$ especially when we cannot factorise the quantity or use some formula. The Newton's Method is one of the techniques used in this approximation.

3.10. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.
(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

3.11. Exercises

(1) The height of a ball t seconds after it is thrown is given by

$$f(t) = -16t^2 + 48t + 32.$$

   (a) Verify that $f(1) = f(2)$.
   (b) According to Rolle's Theorem, what must be the velocity in the interval $[1,\ 2]$?

(2) A retailer has determined that the cost $C(x)$ for ordering and storing x units of a certain product is

$$C(x) = 2x + \frac{300000}{x},\ 0 \le x \le 300.$$

Find the order size that will minimise cost if the delivery truck can bring a maximum of 300 units per order.

(3) A company introduces a new product for which the number of units sold S is given by

$$S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

where t is the time in months.

(a) Find the average rate of change of $S(t)$ during the first year.

(b) During what month does $S'(t)$ equal its average rate of change during the first year?

(4) Consider the graph shown in Figure 3.20 below. Identify open intervals on which the function is increasing or decreasing.

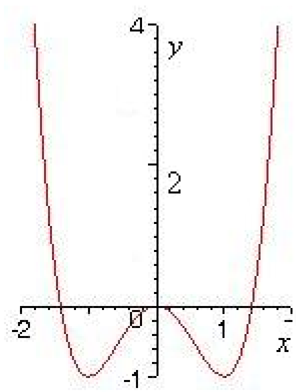


FIGURE 3.20. Graph of $f(x) = x^4 - 2x^2$

(5) A fast food restaurant sells x hamburgers to make a profit $P(x)$ given by

$$P(x) = 2.44x - \frac{x^2}{20000} - 500, \; 0 \le x \le 35000.$$

Find the open interval on which $P(x)$ is increasing or decreasing.

(6) Sketch the graph of

(a) $2x^4 - 8x^2 + 3$,

(b) $\frac{x-2}{x^2-4x+3}$.

(7) Sketch the graph of a function $f(x)$ which has the following characteristics

(a)

$$\begin{aligned}
f(2) &= f(4) = 0 \\
f'(x) &< 0 \text{ if } x < 3 \\
f'(3) & \quad \text{is undefined} \\
f'(x) &> 0 \text{ if } x > 3 \\
f''(x) &< 0 \ x \neq 3
\end{aligned}$$

(b)

$$\begin{aligned}
f(0) &= f(0) = 0 \\
f'(x) &> 0 \text{ if } x < 1 \\
f'(1) &= 0 \\
f'(x) &< 0 \text{ if } x > 1 \\
f''(x) &< 0.
\end{aligned}$$

(8) A manufacturer has determined that the total cost, $C(x)$, of operating a certain facility is given by

$$C(x)0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimum? (The average cost per unit is given by $\frac{C}{x}$).

(9) An open box is to be made from a rectangular piece of material by cutting squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume if the material has dimensions of 60 cm by 90 cm.

(10) Use the Newton's Method to approximate the zeros of the following functions. Continue the process until two successive approximations differ by less than 0.001.

(a) $f(x) = x^3 + x - 1$,

(b) $f(x) = x + \sin(x + 1)$, shown in Figure 3.21.

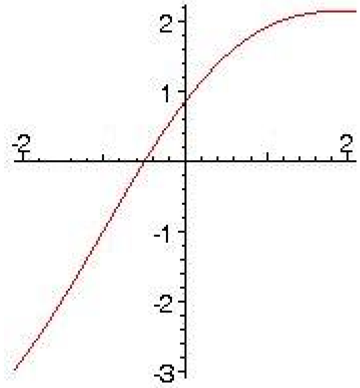


FIGURE 3.21. $f(x) = x + \sin(x+1)$

(11) Approximate π to three decimal points using Newton's Method and the function $f(x) = 1 + \cos x$.

(12) The profit $P(x)$ for a company is given by

$$P(x) = (500x - x^2) - (\frac{1}{2}x^2 - 77x + 3000).$$

Approximate the change in profit as production changes from $x = 115$ to $x = 120$ units. Approximate the percentage change in profit when x changes from $x = 115$ to $x = 120$ units.

UNIT 4

DERIVATIVES OF TRANSCENDENTAL FUNCTIONS

4.1. Introduction

A function can be classified as *algebraic* if it can be expressed as a finite sum, difference, multiple, quotient and radicals involving x^n (or power of any independent variable). For instance, $f(x) = \sqrt[3]{x2 + 2x - 5}$ is algebraic. A function that is *not* algebraic is called a *transcendental function.* For instance, trigonometric functions, logarithmic functions, exponentials and inverse trigonometric functions are transcendental functions. You will learn how to differentiate such functions. The unit contains the following topics:

- Derivatives of Logarithmic Functions,
- Derivatives of Exponential and Inverse Trigonometric Functions,
- L'Hopital's Rule for Indeterminate Forms.

4.1.1. Learning objectives. By the end of this unit you should be able to:

- use implicit differentiation to find derivatives of transcendental functions,
- find derivatives of functions that are of indeterminate form using L'Hopital's Rule.

4.1.2. Prerequisite knowledge. The material in the unit is based on differentiation of algebraic functions.

4.1.3. Time. You should be able to complete this unit in 3 hours.

4.2. Derivatives of Logarithmic Functions

It is desired to find the derivative $f(x) = \ln x$ by applying the derivative definition to $f(x)$. Recall that this function uses base e which is given by the following limit:

$$\lim x \to \infty \left(1 + \frac{1}{x}\right)^x = e. \tag{4.1}$$

This is an important definition.

Activity 4.1. *Express the following limits in terms of e.*

(1)

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x},$$

(2)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n},$$

(3)

$$\lim_{v\to 0}(1+v)^{1/v}.$$

4.2.1. $\frac{dx}{dx}(\ln x)$. We differentiate $f(x)=\ln x$ from first principles.

$$\begin{aligned}
\frac{d}{dx}(\ln x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\\
&= \lim_{u\to 0}\frac{1}{ux}\ln(1+v),\ \text{let } u=h/x\\
&= \frac{1}{x}\lim_{u\to 0}\frac{1}{u}\ln(1+u)\\
&= \frac{1}{x}\lim_{u\to 0}\ln(1+u)^{\frac{1}{u}}\\
&= \frac{1}{x}\ln\left(\lim_{n\to\infty}(1+\frac{1}{n})^{n}\right),\ \text{where } n=\frac{1}{u}\\
&= \frac{1}{x}
\end{aligned}$$

Example 4.1. Find $\frac{dy}{dx}$ where

(1) $y=\ln(2+\sqrt{x})$,

(2) $y=\ln|x^2..1|$,

(3) $y=\ln(\tan x)$,

(4) $y=\frac{x^2}{1+\log x}$.

Solution

(1)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ln(2+\sqrt{x}) \\
&= \frac{1}{2+\sqrt{x}} \times \frac{1}{2} \times \frac{1}{\sqrt{x}} \quad \text{using chain rule} \\
&= \frac{1}{2(x+2\sqrt{x})}
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ln|x^2-1| \\
&= \frac{d}{dx}\ln(x^2-1) \quad \text{property of derivative of logs} \\
&= \frac{1}{x^2-1} \times 2x \quad \text{chain rule} \\
&= \frac{2x}{x^2-1}
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ln(\tan x) \\
&= \frac{1}{\tan x}\sec^2 x \quad \text{chain rule} \\
&= \csc(x)\sec(x)
\end{aligned}$$

(4)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{x^2}{1+\log x}\right] \\
&= \frac{2x(1+\log x) - x^2\frac{1}{x}}{(1+\log x)^2} \quad \text{quotient rule} \\
&= \frac{2x\log x + x}{(1+\log x)^2}
\end{aligned}$$

Remark 4.1.
- In some cases, it could be helpful to expand the logarithmic function.
- *Logarithmic differentiation* is a technique used for differentiating functions that are composed of powers, products and quotients.

Example 4.2. Find $\frac{dy}{dx}$ for the following functions

(1)

$$y = \ln\left[\frac{\cos x}{\sqrt{4-3x^2}}\right]$$

(2)

$$y = \frac{(x^2-8)^{\frac{1}{3}}\sqrt{x^3+1}}{x^6-7x+5}$$

Solution

(1) We first expand the function.

$$\begin{aligned} y &= \ln\left[\frac{\cos x}{\sqrt{4-3x^2}}\right] \\ &= \ln(\cos x) - \frac{1}{2}\ln(4-3x^2) \end{aligned}$$

Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\ln\left[\frac{\cos x}{\sqrt{4-3x^2}}\right] \\ &= \frac{d}{dx}\ln(\cos x) - \frac{1}{2}\frac{d}{dx}\ln(4-3x^2) \\ &= -\tan x - \frac{1}{2} \times \frac{1}{4-3x^2} \times (-6x) \\ &= -\tan x + \frac{3x}{4-3x^2} \end{aligned}$$

(2) For the function $y = \frac{(x^2)^{\frac{1}{3}}\sqrt{x^3+1}}{x^6-7x+5}$, we take natural logarithms of both sides. Thus

$$\ln y = \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \ln(x6-7x+5).$$

Differentiating both sides with respect to x we have

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{3}\frac{2x}{x^2-8} + \frac{1}{2}\frac{3x^2}{x^3+1} - \frac{6x^5-7}{x^6-7x+5} \\ &= \frac{2x}{3(x^2-8)} + \frac{3x2}{2(x3+1)} - \frac{6x^5-7}{x^6-7x+5} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= y\left[\frac{2x}{3(x^2-8)} + \frac{3x2}{2(x3+1)} - \frac{6x^5-7}{x^6-7x+5}\right] \\ &= \left[\frac{2x}{3(x^2-8)} + \frac{3x2}{2(x3+1)} - \frac{6x^5-7}{x^6-7x+5}\right]\left(\frac{(x^2-8)^{\frac{1}{3}}\sqrt{x^3+1}}{x^6-7x+5}\right) \end{aligned}$$

4.3. Derivatives of Exponential and Inverse Trigonometric Functions

Let $y = f(x)$ be a one-one differentiable function on $(b,\ c)$. Then $f^{-1}(x)$ is differentiable at any point x at which $f'(f^{-1}(x)) \neq 0$, and its derivative is

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{dx/dy} \tag{4.2}$$

Example 4.3. Determine $\frac{d}{dx}[f^{-1}(x)]$ if $f(x) = x^3 + 1$.

Solution

Let $y = x^3 + 1$. Then $x = f^{-1}(y) = \sqrt[3]{y-1}$. Therefore

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{3}(y-1)^{-\frac{2}{3}}$$

If we put $x = \sqrt[3]{y-1}$ in dy/dx we get

$$\frac{dy}{dx} = 3(\sqrt[3]{y-1}) = 3(y-1)^{2/3} = \frac{1}{\frac{1}{3}(y-1)^{-2/3}} = \frac{1}{dx/dy}$$

4.3.1. Derivatives of Exponential Functions. Consider the general exponential function $y = r^x$, where $r \in (0,\ 1)$. From this function we have

$$\ln y = x \ln r. \tag{4.3}$$

Implicit differentiation of Equation 4.3 yields

$$\frac{1}{y}\frac{dy}{dx} = \ln r.$$

Thus

$$\frac{dy}{dx} = r^x \ln r.$$

In the case where $y = e^x$, the derivative becomes

$$\frac{dy}{dx} = e^x.$$

Example 4.4. Differentiate the following functions with respect to x.

(1) $y = x^3 e^x$,

(2) $y = \ln(\cos(e^{2x-4}))$.

Solution

(1)

$$\frac{dy}{dx} = 3x^2e^x + x^3e^x = x^2e^x(3+x) \text{ using the product rule.}$$

(2)

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cos(e^{2x-4})} \times (-\sin(e^{2x-4})) \times (2) \\ &= 2\tan(e^{2x-4})e^{2x-4}\end{aligned}$$

4.3.2. Derivative of Inverse Trigonometric Functions. We need to restrict intervals for the domain of trigonometric functions. For instance, the function $f(x) = \sin x$ is not one-to-one on $[-2\pi,\ 2\pi]$. Look at a sketch of this function in Figure 4.1.

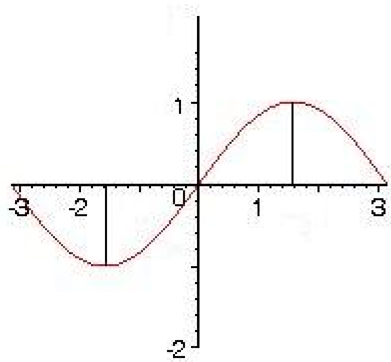


FIGURE 4.1.

Now, the inverse function of $x = \sin y$ is

$$y = \arcsin x,\ -1 \leq x \leq 1$$

when $-2\pi \leq y \leq 2\pi$.

Remark 4.2. • Under certain restrictions all the trigonometric functions possess inverses.

Function	*Domain*	*Range*
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \tan^{-1} x$	$-\infty \leq x \leq \infty$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cot^{-1} x$	$-\infty \leq x \leq \infty$	$0 \leq y \leq \pi$
$y = \sec^{-1} x$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \csc^{-1} x$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

- Central to the discussion on derivatives of inverse functions is the basic identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{4.4}$$

The derivatives of the six trigonometric functions are presented in the following table. Let u be a differentiable function of x.

$$\begin{array}{ll}
\frac{d}{dx}\sin^{-1}u = \frac{u'}{\sqrt{1-u^2}} & \frac{d}{dx}\cos^{-1}u = -\frac{u'}{\sqrt{1-u^2}} \\
\frac{d}{dx}\tan^{-1}u = \frac{u'}{1+u^2} & \frac{d}{dx}\cot^{-1}u = -\frac{u'}{1+u^2} \\
\frac{d}{dx}\sec^{-1}u = \frac{u'}{|u|\sqrt{u^2-1}} & \frac{d}{dx}\csc^{-1}u = -\frac{u'}{|u|\sqrt{u^2-1}}
\end{array}$$

We only prove the third formula in the first column.

$$\frac{d}{dx}[\sec^{-1}u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

Proof

Let $\sec^{-1}u = y$. Then $u = \sec y$. Therefore,

$$\begin{aligned}
u' &= \frac{d}{dx}\sec y \\
&= \sec y \tan y \frac{dy}{dx}, \text{ implicit differentiation} \\
&= |u| \tan y \frac{dy}{dx}, \text{ domain has to be positive}
\end{aligned}$$

Now, $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{u^2-1}$. Therefore, $u' = |u|\sqrt{u^2-1}$. Thus

$$\frac{dy}{dx} = \frac{u'}{|u|\sqrt{u^2-1}}.$$

Example 4.5. Find the derivative of the given function

(1) $f(x) = 2\sin^{-1}(x-1)$;

(2)

$$fx = \frac{1}{2}\left(\ln\frac{x+1}{x-1} + \tan^{-1}2x\right)$$

Solution

(1) $f(x) = 2\sin^{-1}(x-1)$.

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\left[2\sin^{-1}(x-1)\right] \\ &= 2\frac{(x-1)'}{\sqrt{1-(x-1)^2}}, \text{ using derivative rule} \\ &= \frac{2}{\sqrt{2x-x^2}}\end{aligned}$$

(2) $f(x) = \frac{1}{2}\left(\ln\frac{x+1}{x-1} + \tan^{-1} 2x\right)$. The right hand member of this function can be expanded so that

$$\begin{aligned}\frac{d}{dx}[f(x)] &= \frac{d}{dx}\frac{1}{2}\left(\ln\frac{x+1}{x-1} + \tan^{-1} 2x\right) \\ &= \frac{1}{2}\left(\frac{d}{dx}[\ln(x+1) - \ln(x-1)] + \frac{d}{dx}\tan^{-1} 2x\right) \\ &= \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{1+4x^2}\right) \\ &= \frac{1}{1+4x^2} + \frac{1}{1-x^2}\end{aligned}$$

4.4. L'Hopital's for Indeterminate Forms

You saw in Unit 1 that algebraic techniques can be used to evaluate limits of indeterminate forms $0/0$ or ∞/∞. For instance

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

has been computed by dividing numerator and denominator by $x-3$. It is not always that we can manipulate such expressions by algebraic techniques. For instance, consider the limit

$$\lim_{x\to 0}\frac{e^{-2x}-1}{x}.$$

There is no term that can divide both numerator and denominator. However, using estimation, the limit seems to be -2. See values in the table below.

Activity 4.2.

x	-0.01	-0.001	-0.0001	*0*	0.0001	0.001	0.01
$\frac{e^{-2x}-1}{x}$	-2.02013	–	-2.0002	–	–	$1:9980$	-1.9801

Use a calculator to complete the table. What is the value of the function when $x=0$?

The limit in the above activity is evaluated by using the *L'Hopital's Rule* presented in the following.

Theorem 4.1. *Let $f(x)$ and $g(x)$ be differentiable functions on $(b,\ c)$ except not necessarily at $a \in (b,\ c)$. If*

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

produces an indeterminate of the form $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to} \frac{f'}{g(x)}$$

provided

$$\lim_{x\to a} \frac{f(f'(x))}{g'(x)}$$

exists.

This theorem applies to all forms of the indeterminate, namely$0/0$, ∞/∞, $-\infty/\infty$, $\infty/-\infty$, and $-\infty/-\infty$. If you encounter indeterminate forms 1^∞, ∞^0 and 0^0 use logarithmic differentiation discussed in Section 4.2.

Example 4.6. Evaluate the following the limits using L'Hopital's Rule where necessary.

(1)

$$\lim_{x\to\infty} \frac{x^2}{e^x},$$

(2)

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}},$$

(3)

$$\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right),$$

(4)

$$\lim_{x\to\infty} (1+x)^{1/x},$$

(5)

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx}$$

Solution

(1) We use L'Hopital's Rule since the indeterminate is of the form ∞/∞. This gives us

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2}{e^x} &= \lim_{x\to\infty}\frac{2x}{e^x} \\
&= \lim_{x\to\infty}\frac{2}{e^x},\ \text{using L'Hopital's Rule again} \\
&= 0
\end{aligned}$$

(2) For this limit it is not necessary to use L'Hopital's Rule. We divide both numerator and denominator by x so that

$$\begin{aligned}
\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{x/x}{\sqrt{1+1/x^2}} \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1
\end{aligned}$$

(3) Write the expression as a single fraction.

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right) &= \lim_{x\to 1^+}\frac{3(x-1)-2\ln x}{1-1/x+\ln x} \\
&= \lim_{x\to 1^+}\frac{3-2/x}{1-1/x+\ln x},\ \text{using L'Hopital's Rule.}
\end{aligned}$$

We cannot use L'Hopital's Rule again. So the limit does not exist.

(4) For this limit we use logarithmic differentiation because the indeterminate is of the form ∞^0 . Let $y=\lim_{x\to\infty}(1+x)^{1/x}$. Then

$$\begin{aligned}
\ln y &= \lim_{x\to\infty}\frac{\ln(1+x)}{x} \\
&= \lim_{x\to\infty}\frac{1/(1+x)}{1},\ \text{using the rule} \\
&= 0
\end{aligned}$$

Thus $y=1$ so that

$$\lim_{x\to\infty}(1+x)^{1/x}=1.$$

(5) If you apply L'Hopital's Rule you easily get

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\frac{a}{b}.$$

4.5. Unit Summary

In this unit you have learnt how to differentiate logarithmic functions. It has been shown that for a given differentiable function $f(x)$ on $(b,\ c)$, the derivative of the inverse function $f^{-1}(x)$ is given by

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{dx/dy}$$

Using this knowledge, you have gained concepts of how to differentiate exponential and inverse trigonometric functions. In the last section of this unit, you have seen examples on how to use L'Hopital's Rule in the evaluation of limits of indeterminate forms $0/0$, ∞/∞, $-\infty/\infty$, $\infty/-\infty$ and many other such forms.

4.6. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998), Calculus of a Single Variable. Houghton Miffin Com- pany,Boston.

4.7. Exercises

(1) Find the derivative of the function

(a)

$$y = \ln\left(\frac{\sqrt{4+x^2}}{x^3}\right),$$

(b)

$$y = \sin 2x \ln x^2.$$

(2) Find the derivative of the function

(a)

$$y = x^2 e^{-x^3}$$

(b)

$$y = \ln(1 + e^{2x - x^3})$$

(3) Differentiate the following functions:

(a) $h(t) = \sin(\sec^{-1}(x))$,

(b) $g(x) = \cot^{-1}(\sqrt{x^2 - 2})$.

(4) In the following limits, L'Hopital's Rule is not used correctly. Describe the error:

(a)

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2e^x = 2.$$

(b)

$$\begin{aligned} \lim_{x\to\infty} x\cos\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\cos(\frac{1}{x})}{\frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{[-\sin(1/x)]}{\frac{1}{x^2}} \end{aligned}$$

(5) Complete the table to show that e^x eventually "overtakes" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

UNIT 5

INTEGRATION

5.1. Introduction

This unit discusses integration of a function over a given interval. Consider the following problem. Let $F(x)$ be a function such that

$$\frac{d}{dx}F(x) = f(x) = 5x^4$$

From you knowledge of derivatives, what is the function $F(x)$? This will be one of the many problems whose solutions will be sought. Technique of how to acquire these solutions will be presented.
The unit has the following topics:

- Antiderivatives,
- Basic integration Rules
- Area: Upper and Lower Sums,
- Rieman Sums and Definite Integrals
- fundamental Theorem of Calculus.

5.1.1. Learning objectives. By the end of this unit you should be able to:

- find the antiderivative of a given function,
- use the basic rules of integration for the various functions in getting their antiderivatives,
- sketch curves of functions,
- find the upper and lower sums of a region,
- use properties of definite integrals in evaluating given integrals.

5.1.2. Prerequisite knowledge. The material in the unit is base on differentiation of functions.

5.1.3. Time. You should be able to complete this unit in $3\frac{1}{2}$ hours.

5.2. Antiderivatives

From differentiation, you know that $f(x) = 5x^4$ is the derivative iof the function x^5. That is

$$f(x) = 5x^4 = \frac{d}{dx}(x^5)$$

You would correctly argue, in the introductory statement, that $F(x) = x^5$. $F(x)$ is said to be an *antiderivative* of $f(x)$. In general, a function $F(x)$ is called an *antiderivative* of $f(x)$ if for every x in the domain of $f(x)$

$$F'(x) = f(x).$$

Activity 5.1. *Consider the functions* $F(x) = x^5 + 4$, $G(x) = x^5 - 7$, $H(x) = x^5 + K$,*where* K *is a constant. Find* $G'(x)$, $F'(x)$, $H'(x)$.

The results in Activity 5.1 can be generalised as follows.

Remark 5.1. If $F(x)$ is an antiderivative of $f(x)$ on $(b,\ c)$, then a function $H(x)$ is an antiderivative of $f(x)$ on the interval iff $H(x)$ is of the form

$$H(x) = F(x) + K,\ \forall\ x\ \in\ (b,\ c)$$

where K is a constant.

$H(x)$ is called the *general antiderivative* of $f(x)$. The equation

$$H'(x) = 3x^2$$

can be regarded as a *differential equation* with

$$H(x) = x^3 + K$$

as the general solution. If we let $F(x) = y$, then

$$dy = f(x)dx.$$

From this equation we find y by *integration*, which is denoted by $\int$ and its general is denoted by

$$\int f(x)dx = F(x) + K.$$

In this case, $f(x)$ is called *integrand* and K is the *constant of integration*. It is also important to note that integration is the *inverse of differentiation* and *vice–versa*.

5.3. Some Basic Properties of Integration

The section gives the relationship between functions you have seen in differentiation and their antiderivatives. To ensure that these properties are understood, a number of examples will follow.

$$
\begin{array}{rl}
\text{Differentiation} & \text{Integration} \\
\dfrac{d}{dx}[K] = 0 & \displaystyle\int 0dx = K \\
\dfrac{d}{dx}[\lambda x] = \lambda & \displaystyle\int \lambda dx = \lambda x + K \\
\dfrac{d}{dx}[\lambda f(x)] = \lambda f'(x) & \displaystyle\int \lambda f(x)dx = \lambda \int f(x)dx \\
\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) & \displaystyle\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx \\
\dfrac{d}{dx}[x^n] = nx^{n-1} & \displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + K,\ n \neq -1 \ \ \text{Power Rule} \\
\dfrac{d}{dx}[\sin \lambda x] = \lambda \cos \lambda x & \displaystyle\int \cos \lambda x dx = \frac{1}{\lambda} \sin \lambda x + K \\
\dfrac{d}{dx}[\cos \lambda x] = -\lambda \sin \lambda x & \displaystyle\int \sin \lambda x dx = -\frac{1}{\lambda} \cos \lambda x + K \\
\dfrac{d}{dx}[\tan \lambda x] = \lambda \sec^2 \lambda x & \displaystyle\int \sec^2 \lambda x dx = \frac{1}{\lambda} \tan \lambda x + K \\
\dfrac{d}{dx}[\sec \lambda x] = \lambda \sec \lambda x \tan \lambda x & \displaystyle\int \sec \lambda x \tan \lambda x dx = \frac{1}{\lambda} \sec \lambda x + K \\
\dfrac{d}{dx}[\cot \lambda x] = -\lambda \csc^2 \lambda x & \displaystyle\int \csc^2 \lambda x dx = -\frac{1}{\lambda} \cot \lambda x + K \\
\dfrac{d}{dx}[\csc \lambda x] = -\lambda \csc \lambda x \cot \lambda x & \displaystyle\int \csc \lambda x \cot \lambda x dx = -\frac{1}{\lambda} \csc \lambda x + K
\end{array}
$$

You need to realise that in the power rule, n can be any real number except -1. Thus, when evaluating integrals that have surds, you need to rewrite the integrand in th form x^n and then proceed with the evaluation. You also need to expand integrands that come as products of two functions. Now go through the following examples.

Example 5.1. Complete the following table:

Given	Rewrite	Integrate	Simplify
$\int \sqrt[5]{x2}dx$			
$\int x^3(x^4 + \frac{1}{x^2})dx$			

Solution

We expand and/or express the integrand in the form x^n.

Given	Rewrite	Integrate	Simplify
$\int \sqrt[5]{x^2}dx$	$\int x^{\frac{2}{5}}dx$	$\frac{x^{7/5}}{7/5} + K$	$\frac{5x^{7/5}}{7}dx$
$\int x^3(x^4 + \frac{1}{x})dx$	$\int(x^7 + x^2)dx$	$\frac{x^8}{8} + \frac{x^3}{3} + K$	$x^3(\frac{x^5}{8} + \frac{1}{3})$

Example 5.2. Evaluate the indefinite integral and check your answer by differentiation.

(1) $\int(x^{3/2} + 2x + 1)dx$,
(2) $\int \frac{x^2+x+1}{\sqrt{x}}dx$,
(3) $\int(x + 2)(3x - 1)dx$,
(4) $\int(2\sin 3x - 5\cos 2x)dx$,
(5) $\int(\theta^2 + \sec^2\theta)d\theta$,
(6) $\int \sec y(\tan y - \sec y)dy$,
(7) $\int \frac{\sin(\frac{1}{2}x)}{1-\sin^2(\frac{1}{2}x)}dx$.

Solution

(1) In $\int(x^{3/2} + 2x + 1)dx$, take the integrand to be the sum of three functions: $x^{3/2}$, $2x$ and 1. Thus

$$\begin{aligned}\int(x^{3/2} + 2x + 1)dx &= \int x^{3/2}dx + \int 2xdx + \int dx \\ &= \frac{x^{5/2}}{5/2} + \frac{2x^2}{2} + x + K \\ &= \frac{2x^{\frac{5}{2}}}{5} + x^2 + x + K\end{aligned}$$

(2) The integrand is express as individual fractions and treated as the sum of three functions so that

$$\begin{aligned}\int \frac{x^2+x+1}{\sqrt{x}}dx &= \int (\frac{x^2}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}})dx \\ &= \int x^{3/2}dx + \int x^{1/2}dx + \int x^{-1/2}dx \\ &= \frac{x^{\frac{5}{2}}}{5/2} + \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + K \\ &= \frac{2x^{\frac{5}{2}}}{5} + \frac{2x^{\frac{3}{2}}}{3} + 2x^{\frac{1}{2}} + K\end{aligned}$$

(3) We expand the integrand and integrate.

$$\begin{aligned}\int (x+2)(3x-1)dx &= \int (3x^2+5x-2)dx \\ &= \int (3x^2)dx + \int 5xdx + \int (-2)dx \\ &= x^3 + \frac{5x}{2} - 2x + K\end{aligned}$$

(4) From the table of basic properties of integration, we have $\int \sin kxdx = -\frac{1}{\lambda}\cos\lambda x$ and $\int \cos\lambda xdx = \frac{1}{\lambda}\sin\lambda x$. This means that

$$\begin{aligned}\int (2\sin 3x - 5\cos 2x)dx &= \int (2\sin 3x)dx - \int (5\cos 2x)dx \\ &= 2\int \sin 3xdx - 5\int \cos 2xdx \\ &= -\frac{2\cos 3x}{3} - \frac{5\sin 2x}{2} + K\end{aligned}$$

(5)

$$\begin{aligned}\int (\theta^2 + \sec^2\theta)dx &= \int \theta^2 d\theta + \int \sec^2\theta \\ &= \frac{\theta^3}{3} + \tan\theta + K\end{aligned}$$

(6) Expand so that we have two separate functions.

$$\begin{aligned}\int \sec y(\tan y - \sec y)dy &= \int \sec y \tan ydx - \int \sec^2 ydy \\ &= \sec y - \tan y + K\end{aligned}$$

(7) The denominator is $\cos^2 x$. Thus

$$\begin{aligned}\int \frac{\sin(\frac{1}{2}x)}{1-\sin^2(\frac{1}{2}x)}dx &= \int \frac{\sin\frac{1}{2}x}{\cos^2 x} \\ &= \int \sec\frac{1}{2}\tan\frac{1}{s}x \\ &= 2\sec\frac{1}{2}x\end{aligned}$$

5.4. The Area Problem

5.4.1. Summation Properties and some formulas. We begin this section with some summation formulas as they will be needed in approximating the area under the curve of $f(x)$. The sum $x_1 + x_2 + x_3 + \ldots x_n$ indicates that we have n terms added together. In mathematical notation this is written as

$$x_1 + x_2 + x_3 + \ldots x_n = \sum_{i=1}^{n} x_i$$

where i is index of summation, 1 and n are bounds of summation.
Sums of powers occur frequently and their formulas are given below.

$$\sum_1^n k = kn \qquad \sum_1^n i = \tfrac{n(n+1)}{2}$$
$$\sum_1^n i^2 = \tfrac{n(n+1)(2n+1)}{6} \qquad \sum_1^n i^3 = \tfrac{n^2(n+1)^2}{4}$$

The operator $\sum$ has the property that $\sum(f(x) \pm g(x)) = \sum f(x) + g(x)$.

Example 5.3. Find the given sum

(1)

$$\sum_{i=1}^{7}(2i+1),$$

(2)

$$\sum_{n=1}^{10}\frac{3}{n+1}.$$

Solution

(1)

$$\begin{aligned}
\sum_{i=1}^{7}(2i+1) &= \sum_{i=1}^{7}(2i)+\sum_{i=1}^{7}(1)\\
&= 2\sum_{i=1}^{7} i+7\\
&= 2\times\frac{7(7+1)}{2}+7, \text{ using second formula}\\
&= 63
\end{aligned}$$

(2)

$$\begin{aligned}
\sum_{n=1}^{10}\frac{3}{n+1} &= 3\sum_{n=1}^{10}\frac{1}{n+1}\\
&= 3\left[\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{11}\right]\\
&= 3\times\frac{55991}{27720}=\frac{55991}{9240}
\end{aligned}$$

5.4.2. Area. Let $f(x)$ be a nonnegative continuous function on the interval $[b,\ c]$. We wish to estimate the area under the curve of $f(x)$ on $[b,\ c]$ and bounded below by x-axis. See Figure 5.1. The interval $[b,\ c]$ is divided into n equal subintervals each

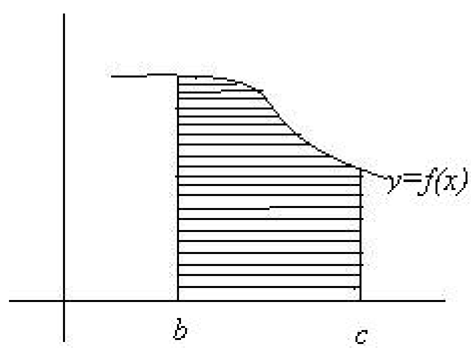


FIGURE 5.1. Area under the curve of $f(x)$

of length $h=\frac{c-b}{n}$. The partitions are as follows:

$$\underbrace{b}_{x_0}<\underbrace{b+h}_{x_1}<\underbrace{b+2h}_{x_2}<\underbrace{b+3h}_{x_3}<\ldots<\underbrace{b+nh}_{x_n=c}.$$

Above each subinterval, construct a rectangle up to any point above the curve. You may use the maximum value as the height or the minimum value as the height. The area under the curve is approximated by the sum of the areas of the rectangles. This is called *rectangle method* of approximating the area under the curve of $f(x)$. Your conclusion that the approximation improves with an increase in the number of subintervals is correct. In fact, if A is the exact area and A_n is the approximate area, then

$$A = \lim_{n\to\infty} A_n$$

As an illustration, let us find an approximate area under the curve $f(x) = x^3 + 1$ on $[0,\ 1]$. The enclosure is shown in Figure 5.2.

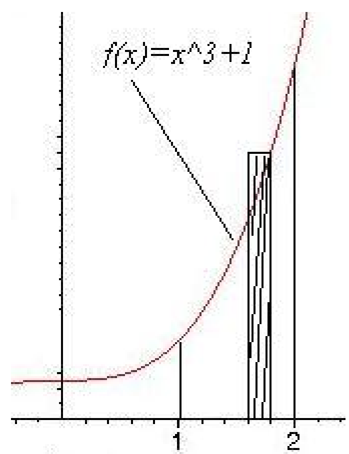


FIGURE 5.2. Approximate Area

The endpoints of subintervals are

$$0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \ldots,\ \frac{n-2}{n},\ \frac{n-1}{n},\ 1.$$

If we take the circumscribed rectangles we get

$$\begin{aligned} A_n &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + 1\right] + \frac{1}{n}\left[\left(\frac{2}{n}\right)^3 + 1\right] + \ldots + \frac{1}{n}\left[\left(\frac{n-1}{n}\right)^3 + 1\right] + \frac{2}{n} \\ &= \frac{1}{n}\left[n + 1 + \frac{1}{n^3}\sum_{i=1}^{n-1} i^3\right] \\ &= \frac{1}{n}\left[n + 1 + \frac{1}{n^3}\frac{n^2(n-1)^2}{4}\right] \end{aligned}$$

Thus

$$
\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty}\left[1+\frac{1}{n}+\frac{n^4-2n^3+n^2}{4n^4}\right]\\
&= 1+\frac{1}{4}=\frac{5}{4}
\end{aligned}
$$

Example 5.4. Use the upper and lower sums to approximate the area of the given region using th indicated number of intervals

(1) $y=\sqrt{x},\ x\in[0\ 1],\ 5$ stripes,

(2) $y=\frac{1}{x-2},\ x\in[4\ 1],\ 4$ stripes.

(1) The required area is shown in Figure 5.3. The interval limits are

$$
0,\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ 1.
$$

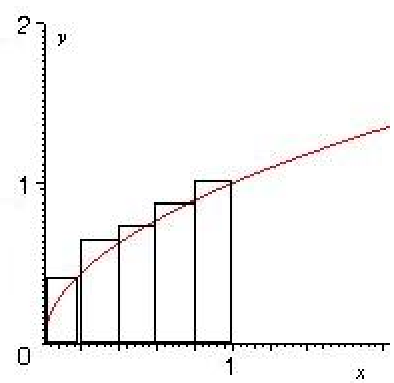


FIGURE 5.3. Circumscribed rectangle for $y=\sqrt{x}$

Let S_u and S_l denote upper and lower sums that approximate the area. Then

$$
S_l=\frac{1}{5}\times\left[\sqrt{\frac{1}{5}}+\sqrt{\frac{2}{5}}+\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{5}}\right]=0.5497
$$

and the upper sums are given by

$$
S_u=\frac{1}{5}\times\left[\sqrt{\frac{1}{5}}+\sqrt{\frac{2}{5}}+\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{5}}+1\right]=0.7497
$$

(2) For $y = \frac{1}{x-2}$ on $[4,\ 6]$ the subintervals are of width $\frac{6-4}{4} = \frac{1}{2}$. Therefore,

$$S_l = \frac{1}{2}\left[\frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right] = \frac{533}{840},$$

and

$$S_u = \frac{1}{2}\left[\frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}\right] = \frac{319}{420}.$$

Activity 5.2. *Consider the triangle of area 2 bounded by the graphs of $y = x$, $y = 0$ and $x = 2$.*

(1) *Sketch the graph of the region.*

(2) *Divide the interval $[0,\ 2]$ into n equal subintervals and show that the end points are*

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \ldots < (n-2)\left(\frac{2}{n}\right) < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right)$$

Remark 5.2.

- Let $f(x)$ be a continuous nonnegative function on $[b,\ c]$. The limits as $n \to \infty$ of both S_l and S_u exist and are equal to each other.

$$\begin{aligned}\lim_{n\to\infty} S_l &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x \\ &= \lim_{n\to\infty} S_u \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x\end{aligned}$$

  where $\Delta x = \frac{c-b}{n}$, $f(m_i)$ and $f(M_i)$ are extreme values of $f(x)$ on the interval.

- If $f(x)$ is continuous and nonnegative on $[c,\ b]$, then the area of the region bounded by the graph of $f(x)$, the $x-$axis and the vertical lines $x = b$ and $x = c$ is given by

$$\text{Area} = \lim_{n\to\infty} f(a_i)\Delta x,\ x_{i-1} \leq a_i \leq x_i. \tag{5.1}$$

Example 5.5. Use the limit process to find the area of the region between the graph $f(x) = 2x^2$ and the lines $x = 1$ and $x = 3$.

Solution

For the interval $[1,\ 3]$, $\Delta x = \frac{2}{n}$ so that the endpoints of the subintervals are

$$1,\ 1 + \frac{2}{n},\ 1 + \frac{4}{n},\ \ldots\ , 1 + \frac{2(n-2}{n},\ 1 + \frac{2(n-1)}{n},\ 1 + \frac{2n}{n} = 3.$$

If we take the upper endpoints for a_i and use Equation 5.1 we get the following

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\left(\frac{n+2i}{n}\right)^2 \frac{2}{n} \\
&= \lim_{n\to\infty}\left[\frac{4}{n^3}\sum_{i=1}^{n}(n^2+4ni+4i^2)\right] \\
&= 4\lim_{n\to\infty}\frac{1}{n^3}\left[n^3+2n^2(n+1)+\frac{n(n+1)(2n+1)}{3}\right], \text{ using summation formulas} \\
&= \frac{52}{3}
\end{aligned}$$

5.5. Rieman Sums and Definite Integrals

We begin with a definition of the Rieman Sum.

Definition 5.1. *Consider a function $f(x)$ which is defined on $[b,\ c]$ with Δ as any partition of the interval such that*

$$b = x_0 < x_1 < x_2 < \ldots x_{i-1} < x_i \ldots < x_{n-2} < x_{n-1} < x_n = c.$$

Let Δx_i be the width of the ith subinterval and a_i any point in that subinterval. Then

$$\sum_{i=1}^{n} f(a_i)\Delta x_i,\ \ x_{i-1} \le a_i \le x_i \tag{5.2}$$

is called a Rieman sum of $f(x)$ for this partition.

From this definition, you will note that the intervals do not have to be of equal width. The sums discussed in the previous section are special cases of the Rieman sum when the partition is *regular*. If partitions are not equal, we denote the largest interval by $||\Delta||$. The *definite integral* is the limit of the sum in Equation 5.2.

Definition 5.2. *Let $f(x)$ be defined on $[b,\ c]$ and the limit as $\Delta x \to 0$ of Equation 5.2 exist. Then $f(x)$ is said to be integrable on $[b,\ c]$ and the limit is denoted by*

$$\lim_{||\Delta||\to 0}\sum_{i=1}^{n} f(a_i)\Delta x_i = \int_b^c f(x)dx$$

where b and c are the limits of integration.

Remark 5.3. Note that as $||\Delta|| \to 0$, the number of partitions n tends to infinity.

Now look at the following example.

Example 5.6. Evaluate the definite integral

$$\int_1^2 (x^2+1)dx$$

by the limit definition.

Solution

The function is continuous on $[1,\ 2]$ and so by definition it is integrable there. If we assume that the subintervals are regular then

$$\Delta x_i = \frac{2-1}{n} = \frac{1}{n}, \text{ and } a_i = 1 + \frac{i}{n}.$$

Therefore,

$$\begin{aligned}
\int_1^2 (x^2+1)dx &= \lim_{||\Delta||\to 0} \sum_{i=1}^n f(a_i)\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^n f(a_i)\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left[\left(1+\frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left[\frac{n^2+2ni+i^2}{n^2} + 1\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{n^2}\left(n^3 + n^2(n+1) + \frac{n(n+1)(2n+1)}{6}\right) + n\right] \\
&= 2\left(2+\frac{1}{3}\right) + 1 = \frac{17}{3}
\end{aligned}$$

5.5.1. Properties of Definite Integrals. Let $f(x)$ and $g(x)$ be integrable functions on $[b,\ c]$. Suppose further that $\lambda$is a constant. Then

- $$\int_b^b f(x)dx = 0,$$
- $$\int_b^c f(x)dx = -\int_c^b f(x)dx,$$
- $$\int_b^c f(x)dx = \int_b^t f(x)dx + \int_t^c f(x)dx,$$

- $$\int_b^c \lambda f(x)dx = \lambda \int_b^c f(x)dx,$$
- $$\int_b^c [f(x) \pm g(x)]dx = \int_b^c f(x)dx \pm \int_b^c g(x)dx,$$
- If $f(x) \leq g(x) \; \forall \; x \; \in \; [b, \; c]$ then
$$\int_b^c f(x)dx \leq \int_b^c g(x)dx.$$

The Property $\int_b^c [f(x) - g(x)]dx = \int_b^c f(x)dx - \int_b^c g(x)dx$ is illustrated in Figure 5.4. This property is used in determining the area between two curves.

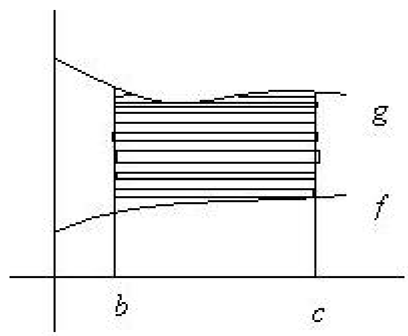


FIGURE 5.4. $\int_b^c g(x)dx - \int_b^c f(d)dx$

Activity 5.3. (1) *Given* $\int_0^5 f(x)dx = 10$ *and* $\int_5^7 f(x)dx = 3$, *find*

(a) $\int_0^7 f(x)dx$,

(b) $\int_5^0 f(x)dx$

(2) *Given* $\int_2^6 f(x)dx = 10 \int_2^6 g(x)dx = -2$, *find*

(a) $\int_2^6 [f(x) - 2g(x)]dx$,

(b) $\int_2^6 [\frac{1}{2} f(x) + \frac{3}{2}]dx$.

5.6. The Fundamental Theorem of Calculus

It would be tedious to evaluate definite integrals using the techniques discussed so far. The Fundamental Theorem of Calculus provides us with an elegant way of evaluating definite integrals. We state the theorem without proof. The proof is not difficult though.

Theorem 5.1. *Fundamental Theorem of Calculus, First Part If $f(x)$ is continuous on $[b,\ c]$ and $F(x)$ is any derivative of $f(x)$ on $[b,\ c]$, then*

$$\int_b^c f(x)dx = F(c) - F(b) \tag{5.3}$$

It is convenient to use the notation

$$\begin{aligned}\int_b^c f(x)dx &= F(x)]_b^c \\ &= F(c) - F(b)\end{aligned}$$

You will note the constant of integration is not necessary in this case. Now look at the following example.

Example 5.7. Evaluate the definite integrals

(1)
$$\int_{-2}^1 (x^2 - 6x + 12)dx,$$

(2)
$$\int_0^{\frac{\pi}{4}} \sec^2\theta d\theta,$$

(3)
$$\int_0^3 |5 - 2x|dx.$$

Solution

(1) Using the properties of integration met in the previous sections, we have

$$\begin{aligned}\int_{-2}^1 (x^2 - 6x + 12)dx &= \int_{-2}^1 x^2dx - 6\int_{-2}^1 xdx + \int_{-2}^1 12dx \\ &= \left[\frac{x^3}{3} - \frac{6x^2}{2} + 12x\right]_{-2}^1 \\ &= [\frac{1}{3} - \frac{(-2)^3}{3}] - [3 - 3(-2)^2] + [12 - (-36)] = 48\end{aligned}$$

(2)
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sec^2\theta d\theta &= \tan\theta]_0^{\pi/4} \\ &= 1.\end{aligned}$$

(3) For this problem, we need to express the integrand so that modulus signs do not appear. Recall that

$$|5-2x| = \begin{cases} (5-2x), & x < \frac{5}{2} \\ -(5-2x), & x \geq \frac{5}{2} \end{cases}$$

Thus the integral becomes

$$\begin{aligned} \int_0^3 |5-2x|dx &= \int_0^{5/2} (5-2x)dx - \int_{5/2}^3 (5-2x)dx \\ &= \frac{25}{4} + \frac{1}{4} = \frac{13}{2} \end{aligned}$$

5.6.1. The Mean Value Theorem for Integrals. Consider a continuous non-negative function $f(x)$ on $[b,\ c]$ and m and M minimum and maximum values of $f(x)$ on this interval. This means that $\forall\ x\ \in\ [b,\ c]$,

$$m \leq f(x) \leq M.$$

Therefore,

$$\begin{array}{ccccc} \int_b^c m dx & \leq & \int_b^c f(x)dx & \leq & \int_b^c M dx \\ (c-b)m & \leq & \int_b^c f(x)dx & \leq & (c-b)M \\ m & \leq & \frac{1}{c-b}\int_b^c f(x)dx & \leq & M \end{array}$$

By the Intermediate-Value Theorem, $\exists\ a_1\ \in\ [b,\ c]$ such that

$$\frac{1}{c-b}\int_b^c f(x)dx = f(a_1). \tag{5.4}$$

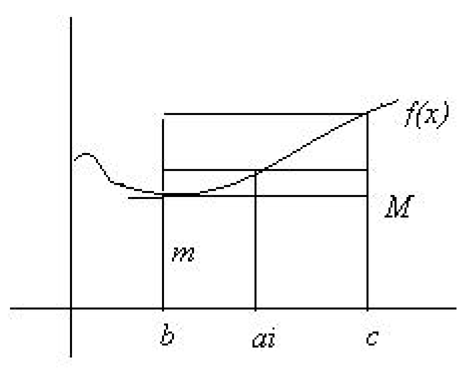


FIGURE 5.5. Illustrating the Mean Value Theorem for Integrals

This reasoning has proved the following.

Theorem 5.2. *Mean Value Theorem for Integrals Let $f(x)$ be continuous on $[b,\ c]$. The $\exists\ a_1\ \in\ (b,\ c)$ such that*

$$\int_b^c f(x)dx = f(a_1)(c-b).$$

The quantity if Equation 5.4 is called *average of* $f(x)$ on $[b,\ c]$.

Example 5.8. Find the average value of $f(x) = 3x^2 - 2x$ on $[1,\ 4]$.

Solution

Let A_a be the average of the function. Then

$$\int_1^4 (3x^2 - 2x)dx = \left[x^3 - x^2\right]_1^4 = 48.$$

Thus

$$A_a = \frac{48}{4-1} = 16.$$

You have seen that a definite integral involves two constants c and b which we have called *upper* and *lower* limits of integration respectively. Consider the integral in the following activity.

Activity 5.4. *Evaluate the function*

$$F(x) = \int_b^x \sin t dt.$$

It is often of interest to differentiate a function of the form shown in the activity. Let

$$F(x) = \int_b^x (t^2 - 2t)dt.$$

Then

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_b^x (t^2 - 2t)dt \\ &= \frac{d}{dx}\left[\frac{1}{3}t^3 - t^2\right]_b^x \\ &= \frac{d}{dx}(\frac{1}{3}x^3 - x^2 - \frac{1}{3}b^3 + b^2) \end{aligned} \tag{5.5}$$

Terms involving b are regarded as constants. Therefore,

$$F'(x) = x^2 - 2x.$$

You note that $x^2 - 2x$ is just the integrand with t replaced by *upper variable* limit of integration, x. This is a particular case of the *Second Fundamental Theorem of Calculus* which is generalised in the following.

Theorem 5.3. *Fundamental Theorem of Calculus, Second Part Let $f(x)$ be continuous on $(b,\ c)$ containing a_1. Then* $\forall\ x\ \in\ (b,\ c)$

$$\frac{d}{dx}\int_{a_1}^{x} f(t)dt = f(x)$$

Example 5.9. Find $F'(x)$ in each of the following:

(1)

$$F(x) = \int_{-1}^{x} \sqrt{t^4+1},$$

(2)

$$F(x) = \int_{0}^{x} t\tan t^{-4},$$

(3)

$$F(x) = \int_{1}^{x} \frac{t^2}{t^2+1}.$$

Solution:

(1)

$$\begin{aligned} \frac{d}{dx}F(x) &= \frac{d}{dx}\int_{-1}^{x}\sqrt{t^4+1}dt \\ F'(x) &= x^4+1, \text{ by the 2nd Fund. Thm} \end{aligned}$$

(2) Applying the preceding theorem, you find

$$F'(x) = \tan x^{-4}.$$

The other problem can be solved in similar manner.

5.7. Integration by Substitution

There are integrals where the integrand is a product of two functions. You would expand the integrand where possible. If, however, this operation is not feasible, then check whether the integrand is of the form $f(g(x))g'(x)$.

Activity 5.5. *Consider the integral*

$$\int 3x^2(x^3+7)^3dx.$$

The integrand is a product of two functions.

(1) *Would you identify them?*

(2) *Which one is composite?*

(3) *Can you differentiate the composite function with respect to x?*

(4) *What do you note?*

You have seen from the activity that it is not difficult to identify the "composite function" in the integrand. To complete the process of finding antiderivatives using substitution technique you follow the following steps:

- Denote the inner part of a *composite* function, $g(x)$, by u so that $u = g(x)$.
- Find the differential $du = g'(x)dx$.
- Rewrite the integral in terms of u.
- Find an antiderivative in terms of u.
- Replace u by $g(x)$ to have the antiderivative in terms of x.
- For definite integrals, the values of integration limits need to change. For instance, to evaluate the integral

$$\int_0^2 2x(x^2+1)dx$$

we make the substitution $u = x^2 + 1$. Thus the new limits will be:

$$\begin{array}{ccc} x & \to & g(x) \\ 0 & \text{to} & 1 \\ 1 & \text{to} & 5. \end{array}$$

Example 5.10. Evaluate the indefinite integrals

(1)
$$\int 5x\sqrt[3]{1-x^2}dx,$$

(2)
$$\int \sin 2x \cos 2x dx,$$

(3)

$$\int t^2 \left(t - \frac{2}{t}\right) dt.$$

Solution

(1)

$$\int 5x\sqrt[3]{1-x^2}dx = 5\int x\sqrt[3]{1-x^2}dx.$$

Let $u = 1 - x^2$. Then $u' = -2x$ and $-\frac{1}{2}du = xdx$. Therefore,

$$\begin{aligned}\int 5x\sqrt[3]{1-x^2}dx &= -\frac{5}{2}\int \sqrt[3]{u}du \\ &= -\frac{5}{2}\int u^{3/2} \\ &= -u^{\frac{5}{2}}\end{aligned}$$

Thus

$$\int 5x\sqrt[3]{1-x^2}dx = -\sqrt[5]{1-x^2}.$$

(2) In $\int \sin 2x \cos 2x dx$, let $u = \sin 2x$. This gives the differential $du = 2\cos 2x dx$. Thus

$$\begin{aligned}\int \sin 2x \cos 2x dx &= \int \left(\frac{1}{2}u\right) du \\ &= \frac{1}{2} \times \frac{1}{2}u^2 \text{ using power rule} \\ &= \frac{1}{4}\sin^2 2x \quad \text{replace } u \text{ by } \sin 2x.\end{aligned}$$

(3) The integrand can be 'modified'.$t^2\left(t - \frac{2}{t}\right) = t\left(t^2 - 2\right)$. Now let $u = t^2 - 2$. The differential of u is $du = 2tdt$.Therefore,

$$\begin{aligned}\int t^2\left(t - \frac{2}{t}\right) dt &= \int t(t^2-2)dt \\ &= \int \frac{1}{2}udu \\ &= \frac{1}{4}u^2 \quad \text{using power rule}\end{aligned}$$

Let us close the section with a definite integral involving substitution.

Example 5.11. Evaluate the definite integral

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx.$$

Solution

Let $u = 1+\sqrt{x}$. Then the limits will be $u = 1+\sqrt{1} = 2$ and $u = 1+\sqrt{9} = 4$. Furthermore, $du = \frac{1}{2\sqrt{x}}dx$. Thus

$$\begin{aligned}\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx &= \int_2^4 \frac{2}{u^2}du \\ &= \int_2^4 2u^{-2}du \\ &= \left[-2u^{-1}\right]_2^4 \\ &= \frac{1}{2}\end{aligned}$$

5.8. Integration of Natural logarithmic and Exponential Functions

5.8.1. Integration of Logarithmic Functions. From differentiation, you know that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

This gives us the following integration rules.

Remark 5.4. Let u be a differentiable function of x. Then

(1)
$$\int \frac{1}{x}dx = \ln|x| + K \quad \text{and}$$

(2)
$$\int \frac{1}{u}du = \int \frac{u'}{u}dx = \ln|u| + K.$$

This is usually called the *integration rule for logarithms.*

Let us now discuss some examples.

Example 5.12. Evaluate the following integrals:

(1)
$$\int \frac{x+5}{x}dx$$

(2)
$$\int \frac{x^2}{3-x^3}dx$$

(3)
$$\int \frac{x+3}{x^2+6x+7}dx$$
(4)
$$\int \frac{\sec x \tan x}{\sec x - 1}dx.$$

Solution

It is important to note that there are some problems that may be evaluated without using the integration rule for logarithms.

(1) We first simplify the integrand so that
$$\frac{x+5}{x} = 1 + \frac{5}{x}.$$
This gives us
$$\begin{aligned} \int \frac{x+5}{x}dx &= \int \left(1 + \frac{5}{x}\right) dx \\ &= \int dx + 5\int \frac{1}{x}dx \\ &= x + 5\ln x + K \end{aligned}$$

(2) Denote the denominator by v so that $\frac{dv}{dx} = -3x^2$. Thus $-\frac{dv}{3} = x^2 dx$. Then
$$\begin{aligned} \int \frac{x^2}{3-x^3}dx &= -\frac{1}{3}\int \frac{1}{v}dv \\ &= -\frac{1}{3}\ln|v| + K \quad \text{by integration rule of logs} \\ &= -\frac{1}{3}\ln|3-x^2| + K \quad \text{back substitution} \end{aligned}$$

(3) This has the same form as the preceding example. You let $u = x^2 + 6x + 7$ to obtain the differential $du = 2(x+3)dx$. Thus
$$\int \frac{x+3}{x^2+6x+7}dx = \frac{1}{2}\ln|x^2+6x+7| + K.$$

(4) Recall that $\frac{d}{dx}\sec x = \sec x \tan x$. Therefore, if we let $w = \sec x - 1$, we obtain $dw = \sec x \tan x$. This gives
$$\begin{aligned} \int \frac{\sec x \tan x}{\sec x - 1}dx &= \int \frac{1}{w}dw \\ &= \ln|w| + K \\ &= \ln|\sec x - 1| + K \quad \text{back substitution.} \end{aligned}$$

Lets have a look at an application example.

Example 5.13. The population of bacteria is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. Assuming that the initial population is 1000, write an equation that gives the population size at any time t, and then find the population when $t = 3$ days.

Solution

This is an example of *differential equations*. Now express the equation as a differential of P. Thus

$$\begin{aligned} dP &= \frac{3000}{1 + 0.25t} dt \quad \text{so that} \\ P &= \int \frac{3000}{1 + 0.25t} dt \end{aligned}$$

Now, if we let $u = 1 + 0.25t$, we find the differential $du = 0.25dt$. This gives

$$P = \int \frac{3000}{1 + 0.25t} dt = 12000 \ln |1 + 0.25t| + K.$$

When $t = 0$, $P = K = 1000$ because $\ln 1 = 0$. Therefore,

$$P(t) = 12000 \ln |1 + 0.25| + 1000.$$

When $t = 3$ days, $P(3) = 12000 \ln(1.75) + 1000 = 7715$.

Remark 5.5. We present the integration formulas for the six trigonometric functions

$$\begin{aligned} \int \sin u du &= -\cos u + K & \int \cos u du &= \sin u + K \\ \int \tan u du &= -\ln|\cos u| + K & \int \cot u du &= \ln|\sin u| + K \\ \int \sec u du &= \ln|\sec u + \tan u| + K & \int \csc u du &= -\ln|\csc u + \cot u| K \end{aligned}$$

Only one of these is shown. Others can also be easily proved.

Show that

$$\int \sec u du = \int |\sec u + \tan u| + K.$$

Proof:

$$\begin{aligned}\int \sec u du &= \int \sec u \left(\frac{\sec u + \tan u}{\sec u + \tan u}\right) du \\ &= \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} du.\end{aligned}$$

From differentiation, we know that $\frac{d}{dx}\sec x = \sec x \tan x$ and $\frac{d}{dx}\tan x = \sec^2 x$. Thus if we let $v = \sec u + \tan u$ we find the differential of v to be $dv = (\sec u \tan u + \sec^2 u)du$. Therefore,

$$\begin{aligned}\int \sec u du &= \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} du \\ &= \int \frac{1}{v} dv \\ &= \ln |v| + K \\ &= \ln |\sec u + \tan u| + K.\end{aligned}$$

5.8.2. Integration of Natural Exponential Functions. Let u be a differentiable function of the variable x. Then

$$\int e^x dx = e^x + K \qquad \int e^u du = e^u + K.$$

Example 5.14. Evaluate the following integrals

(1)

$$\int_0^1 e^{-2x} dx,$$

(2)

$$\int x^2 e^{x^3} dx,$$

(3)

$$\int \frac{e^{-x}}{1+e^{-x}} dx.$$

Solution

(1) Let$u = -2x$. The limits of integration are $u = 0$ and $u = -2$ and the differential of u is $du = -2dx$. Therefore,

$$\int_0^1 e^{-2x} dx = \int_0^{-2} (-\frac{1}{2}) e^u du = \frac{1}{2}\left(1 - e^{-2}\right).$$

(2) Consider the substitution $v = x^3$. The differential of v is $dv = 3x^2dx$. Thus

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^d v = \frac{1}{3}e^{x^3}.$$

(3) The substitution $w = 1 + x^{-x}$ give $dw = -e^{-x}dx$. The integration, therefor, simplifies to

$$\begin{aligned}\int \frac{e^{-x}}{1+e^{-x}}dx &= -\int \frac{1}{w}dw \\ &= -\ln|w| + K \\ &= -\ln(1+e^{-x})\end{aligned}$$

5.9. Unit Summary

In this unit you have learnt the basic rules of integration for finding an indefinite integral. You have also seen how to partition an interval and the use of these subintervals in determining the sums of a region. The use of Rieman Sums has also been covered in this unit

The Fundamental Theorems of Calculus have also been discussed and examples have been given on how to use these theorems.You have also met th Mean Value Theorem of integration and the average of a function on an interval. Some integrands are of the form

$$f(g(x))g'(x).$$

You have seen that in such problems, substitutions of the form

$$u = g(x)$$

will give a differential $du = g'(x)dx$ so that

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Integration of logarithmic and exponential has also been discussed.

5.10. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

5.11. Exercises

(1) Evaluate the following integrals:
   (a) $\int_{-1}^{1}(t^3-9)dt$,
   (b) $\int_0^4 |x^2-4x+3|dx$,
   (c) $\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}d\theta$,

(2) Find the value of a_1 guaranteed by the MVT for integrals for the given function.
   (a) $f(x)=x^3,$ on $[0,\ 2]$
   (b) $f(x)=2\sec^2 x,$ on $[-\pi/4,\ \pi/4]$.

(3) Find the area bounded by $y=1+\sqrt{x},\ x=0,\ x=4,\ y=0.$

(4) Use the limit process to find the area of the region between the graph of $y=x^2-x^3$ and x-axis on $[-1,\ 1]$.

(5) Find $y=f(x)$ that satisfies the conditions: $f''(x)=x^2$, $f'(0)=6$, $f(0)=6$.

(6) Evaluate the indefinite integrals and check you answer by integration
   (a) $\int \frac{y^2}{\sqrt{y}}dy$
   (b) $\int(\sec^2 x-\sin x)dx$
   (c) $\int(t^3-9)^2dt$,

(7) Use the properties of sigma notation to find a formula for the given sum of n terms. The use the formula to find the limit as $n\to\infty$.
   (a)
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right),$$
   (b)
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$

UNIT 6

APPLICATIONS INTEGRATION

6.1. Introduction

Apart from computation of areas under curves, this unit discusses other applications in which definite integration is used. These applications include, but not limited to, distance, volume of solid of revolution and work done in moving an object. Let a_c be the acceleration of a vehicle after time t. Then the velocity reached at this time is given by

$$v(t) = a_c t + u$$

where is constant of integration. Continuing in this manner you can verify that displacement at time t is $s(t) = ut + \frac{1}{2}a_c t^2$, with u as initial velocity.
The unit has the following topics:

- Area of a region between two curves
- Volume of solid of revolution,
- Arc Length and Area of Surface of Revolution,
- Work done in moving an object.

6.1.1. Learning objectives. By the end of this unit you should be able to:

- find the area of a region between two curves,
- calculate the volume of a solid of revolution,
- determine the length of a segment of a curve between two points;
- find the amount of work done by applying varying force on an object.

6.1.2. Prerequisite knowledge. The material in the unit is base on integration of functions.

6.1.3. Time. You should be able to complete this unit in 3 hours.

6.2. Area of a Region Between Two Curves

In Section **??** you have learnt that if $f(x)$ is a nonnegative function on $(b,\ c)$, then

$$\int_b^c f(x)dx$$

is the area of the region between the curve of $f(x)$ and x-axis, $x = b$ and $x = c$.

Theorem 6.1. *If $f(x)$ and $g(x)$ are continuous functions on $[b,\ c]$ and $g(x) \leq f(x)$ $\forall\ x\ \in\ [b,\ c]$, then the area of the region bounded by the graph of $f(x)$ and $g(x)$ and the lines $x = b$ and $x = c$ is*

$$A = \int_b^c [f(x) - g(x)]dx.$$

Activity 6.1. *Sketch the region between $f(x) = 3x^3 - x^2 - 10x$ and $g(-x^2 + 2x)$. Identify the points of intersection. Verify that the area between $f(x)$ and $g(x)$ is 24 sq. units.*

There is a need to note that

- the assumption is that $g(x) \leq f(x)$,
- depending on the functions involved, it might be easier to evaluate the integrals by integrating with respect to y.

Now look at some examples.

Example 6.1.

Find the area of the given region.

(1) $f(x) = x^2 - 6x$, $g(x) = 0$.

(2) $f(x) = x^2$, $g(x) = x^3$.

(3) $f(x) = (x - 1)^3$, $g(x) = x - 1$.

Solution

(1) For this pair of functions, the region in question is between $x = 0$ and $x = 6$ and on this interval, $g(x) > f(x)$. The region is illustrated in Figure 6.1.

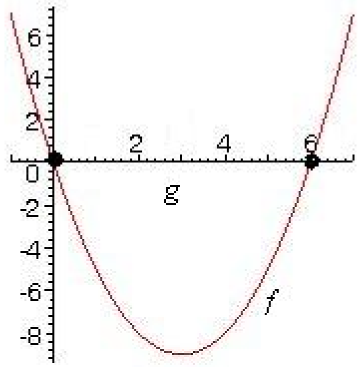


FIGURE 6.1.

Let area be A. Then

$$\begin{aligned}\int_0^6 [g(x) - f(x)]dx &= \int_0^6 (6x - x^2)dx \\ &= \left[3x^2 - \frac{1}{3}x^3\right]_0^6 \\ &= 36.\end{aligned}$$

(2) We compute the area of the region between $f(x) = x^2$ and $g(x) = x^3$ on $[0,\ 1]$. On this interval, $f(x) > g(x)$ (Figure 6.2). Thus the area is

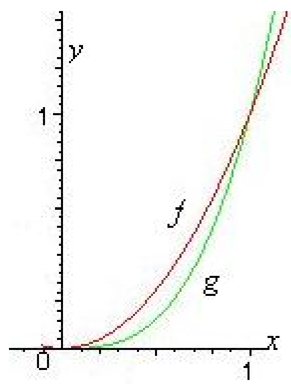


FIGURE 6.2.

$$\begin{aligned}\int_0^1 [f(x) - g(x)]dx &= \int_0^1 [x2 - x^3]dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 \\ &= \frac{1}{12}.\end{aligned}$$

(3) You note that there are two portions of the area between the two curve. On $(0,\ 1)$, $f(x) > g(x)$ and on $(1,\ 2)$ $g(x) > f(x)$ (See Figure 6.3). In this

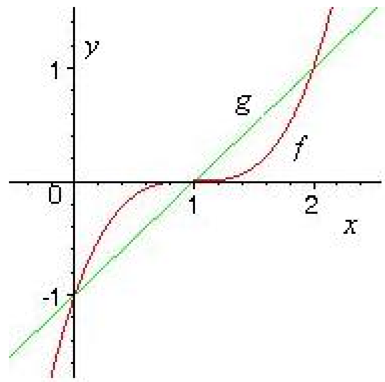


FIGURE 6.3.

case area is found as a sum of two areas: $\int[f(x) - g(x)]dx$ on $(0,\ 1)$ and $\int[g(x) - f(x)]dx$ on $(1,\ 2)$. To find these intervals you solve $f(x) - g(x)$ and then apply the technique used in Activity 6.1. The area is given by

$$\begin{aligned} A &= \int_0^1 [(x-1)^3 - (x-1)]dx + \int_1^2 [(x-1) - (x-1)^3]dx \\ &= \int_0^1 (x^3 - 3x^2 + 2x)dx + \int_1^2 (-x^3 + 3x^2 - 2x)dx \\ &= \frac{1}{2}. \end{aligned}$$

Note that in some cases you need to find the points of intersection of curves of the functions in question. We present two more examples for your consideration.

Example 6.2. Sketch the region bounded by the graphs of $f(y) = y^2$ and $g(y) = y+2$ and find the area of the region.

Solution

The curves intersect at $(1,\ -1)$ and $(4,\ 2)$. The graph is shown in Figure 6.4.

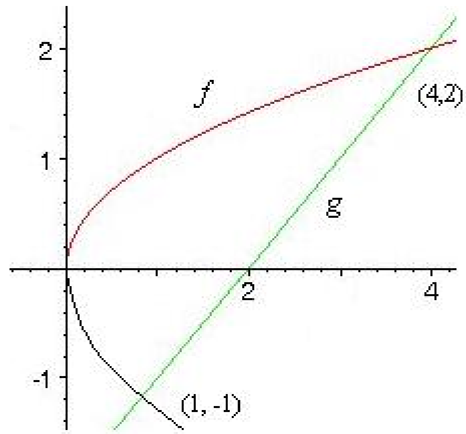


FIGURE 6.4.

Since $f(y) < g(y)$ on this interval, area of the region is

$$
\begin{aligned}
A &= \int_{-1}^{2} [(y+2) - y^2]dy \\
&= \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\right]_{-1}^{2} \\
&= \frac{9}{2}.
\end{aligned}
$$

Example 6.3. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves can be found by a single integral. Explain why this is so and write an integral for this area.

Solution

If you look at graphs of the two functions, you note that there are *only three* points of intersection; they are $(-1,\ 0)$, $(0,\ 1)$ and $(1,\ 0)$ (Figure 6.5 illustrates the functions). The area of the region between the two curves is

$$
\begin{aligned}
A &= \int_{-1}^{1} [1 - x^2 - (x^4 - 2x + 1)]dx \\
&= \int_{-1}^{1} [x^2 - x^4]_{-1}^{1} dx \\
&= \frac{4}{5}
\end{aligned}
$$

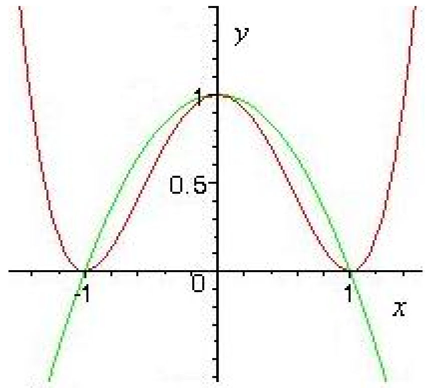


FIGURE 6.5.

6.3. Volume of a Solid of Revolution

If a region in the plane is revolved about a line, the resulting solid is called a *solid of revolution*. Now work on the following activity.

Activity 6.2. *Consider a disc of radius R. If it is h units thick, what is the volume of the disc?*

We introduce the volume of a solid of revolution by looking at a representative disc generated by a representative triangle (Figure 6.6).

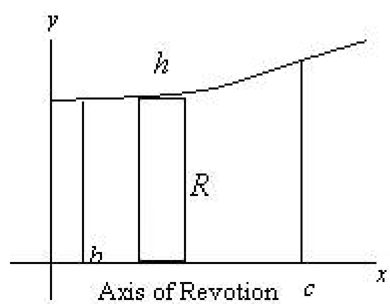


FIGURE 6.6.

When this rectangle is revolved about the axis of revolution, it generates a representative disc. The volume of this disc, ΔV, is given by

$$\Delta V = \pi R^2 \Delta x.$$

If we divide the solid into n such discs with thickness Δx and radius $f(x)$, then the approximate volume of the solid, V_a, is given by

$$V_a = \sum_{i=1}^{n} \pi [f(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x$$

Thus volume of the solid of revolution is

$$V = \lim_{n\to\infty} \pi \sum_{i=1}^{n} [f(x_i)]^2 \Delta x = \pi \int_b^c [f(x)]^2 dx. \tag{6.1}$$

Remark 6.1. (1) In finding the volume of the solid, $f(x)$ provides the radius because it is the height at a particular value of x.

(2) The volume formula given in Equation 6.1 is for a revolution about x-axis. If the solid is generated by a revolution about y-axis, the formula becomes

$$V = \pi \int_b^c [f(x)]^2 dx. \tag{6.2}$$

(3) If the region is bounded by bounded by an outer radius $f(x)$ and an inner radius $g(x)$. Then the volume of the solid formed by revolving this region about its axis is

$$V = \pi \int_b^c ([f(x)]^2 - [g(x)]^2) dx$$

Now look at how the volumes have been determined in the following examples.

Example 6.4. Find the volume of the solid formed by revolving the given region about x-axis in each of the following figures.

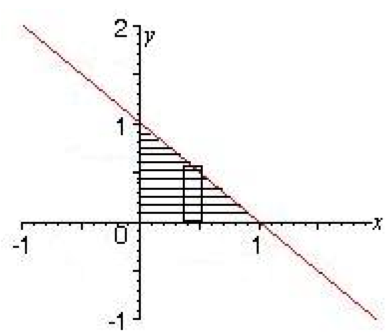


FIGURE 6.7. $y = -x + 1$

(1)

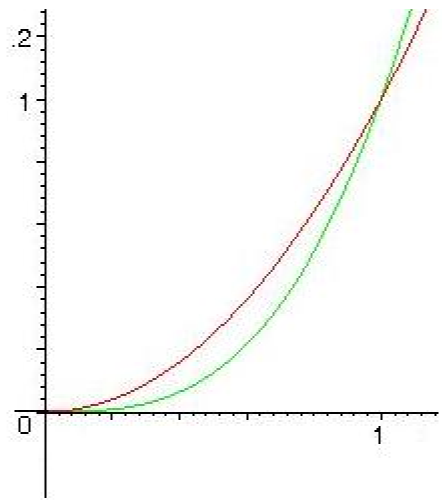


FIGURE 6.8. $y = x^2,\ y = x^3$

(2)

Solution

(1) Since the revolution is about x-axis, we let $f(x) = -x + 1$. Thus, the volume of revolution generated by revolving $f(x)$ about the axis is

$$\begin{aligned} V &= \pi \int_0^1 [f(x)]^2 dx \\ &= \pi \int_0^1 (1-x)^2 dx \\ &= \frac{\pi}{3}. \end{aligned}$$

(2) The region is bounded by two radii: $f(x) = x^2$ and $g(x) = x^3$. Thus

$$\begin{aligned} V &= \pi \int_0^1 \left([f(x)]^2 - [g(x)]^2\right) dx \\ &= \pi \int_0^1 \left((x^2)^2 - (x^3)^2\right) dx \\ &= \frac{2\pi}{35}. \end{aligned}$$

Example 6.5. Find the volume of the solid formed by revolving the given region about y-axis.

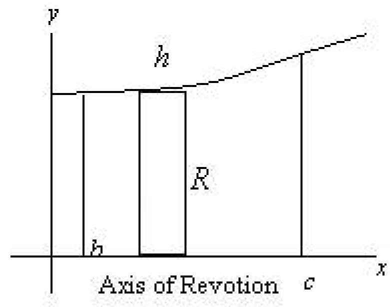


FIGURE 6.9. $y = \sqrt{16 - x^2}$

Solution

$$\begin{aligned} V &= \pi \int_0^4 [f(y)]^2 dy \\ &= \pi \int_0^4 \sqrt{16 - y^2} dy \\ &= \pi \int_0^4 (16 - y^2) dy \\ &= \frac{128\pi}{3}. \end{aligned}$$

We close the section with an activity.

Activity 6.3. *The region bounded by the parabola $y = 4x - x^2$ and x-axis is revolved about the x-axis. Find the volume of the resulting solid.*

6.4. Arc Length and Surface Areas of Revolution

6.4.1. Arc Length. Let $f(x)$ be a continuous differentiable function on $[b,\ c]$. Suppose s is the length of the curve of f on this interval. Divide the interval into n portions

$$b = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = c.$$

Then we can approximate the graph of $f(x)$ by the n line segments. The ith segment has length

$$s_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Thus letting $x_i - x_{i-1} = \Delta x_i$ and $y_i - y_{i-1} = \Delta y_i$ gives

$$s \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i).$$

Therefore, the arc length of $f(x)$ between b and c is given by

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i) \\ &= \int_b^c \sqrt{1 + [f'(x)]^2} dx \end{aligned} \tag{6.3}$$

For a smooth curve given by $x = g(y)$, the length formula becomes

$$s = \int_t^r \sqrt{1 + [g'(y)]^2} dy$$

Example 6.6. Find the arc length of the graph of the given functions over indicated intervals

(1) $y = \frac{x^4}{8} + \frac{1}{4x^2}$, $[1,\ 2]$,

(2) $\frac{1}{2}(e^x + e^{-x})$, $[0,\ 2]$

Solution

Let s be the arc length and $f(x)$ the function. Then

(1) $f'(x) = \frac{x^3}{2} - \frac{1}{2x^3}$. Therefore,

$$\begin{aligned} s &= \int_1^2 \sqrt{1 + [f'(x)]^2} dx \\ &= \int_1^2 \sqrt{1 + [\frac{x^3}{2} - \frac{1}{2x^3}]^2} dx \\ &= \frac{33}{16} \end{aligned}$$

(2) $f'(x) = \frac{1}{2}(e^x - e^{-x})$, so that

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + [f'(x)]^2} dx \\ &= \int_0^2 \sqrt{1 + [\frac{1}{2}(e^x - e^{-x})]^2} dx \\ &= \frac{1}{2} \left[e^x - e^{-x}\right] \\ &= 3.62 \end{aligned}$$

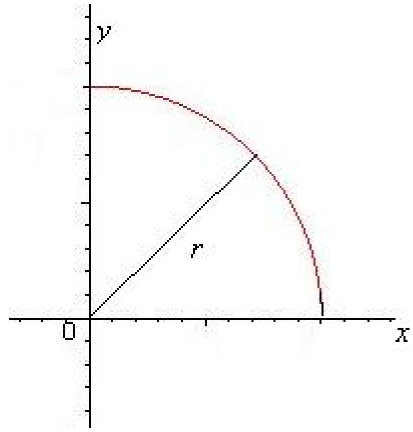


FIGURE 6.10. Graph of $y = \sqrt{r^2 - x^2}$ in first quadrant

Example 6.7.

Find the perimeter of a circle of radius r and center the origin.

Solution

The equation of a circle is given by

$$x^2 + y^2 = r^2.$$

We consider the portion of the circle that lies in the first quadrant shown in Figure 6.10. This gives one quarter of of the circle's perimeter. Now, $y = \sqrt{r^2 - x^2}$. Therefore,

$$y' = \frac{1}{2}\frac{1}{\sqrt{r^2 - x^2}}(-2x) = -\frac{x}{\sqrt{r^2 - x^2}}$$

and

$$\begin{aligned} \sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{r^2 - x^2}} \\ &= \frac{r}{\sqrt{r^2 - x^2}} \end{aligned}$$

Thus the arc length, s, is given by

$$\begin{aligned} s &= \int_0^r \frac{r}{\sqrt{r^2 - x^2}} dx) \\ &= \left[r \sin^{-1}(x/r)\right]_0^r \\ &= \frac{\pi r}{2}. \end{aligned}$$

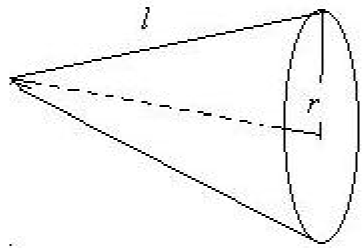
So the perimeter of the whole circle is $4 \times \frac{\pi r}{2} = 2\pi r$.

Now see if you can do the following

Activity 6.4. *Find the arc length of the graph of* $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

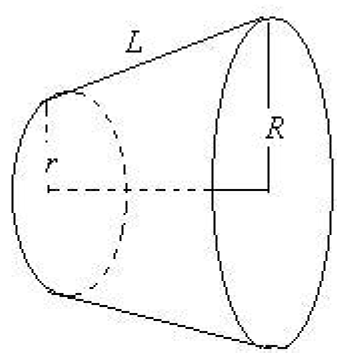
6.4.2. Area of Surface of Revolution. Now we turn our attention to the problem of finding the *surface of revolution* generated when the curve of $y = f(x)$ is rotated about the $x$axis.

Activity 6.5. *When the graph of* $f(x) = kx$ *is rotated about* x*-axis between* $x = 0$ *and* $x = a$*, the solid generated has the shape show below.*



What is the name of the solid? What is the formula of the surface area?

Now consider the surface area of a frustrum of a right circular cone. Let L be the length of slant surface, R and r be radii of wider and narrower end. Extend from the narrower end to complete a cone of radius r with slant length of m.



Then the curved surface area of the frustrum, A_s, is given by

$$\begin{aligned} A_s &= \pi R(L+m) - \pi r m \\ &= \pi R L + \pi(R-r)m \\ &= \pi R L + \pi \frac{rL}{R-r}(R-r), \text{ by similar triangles} \\ &= \pi(R+r)L \end{aligned}$$

Consider a general function $y = f(x) \geq 0$ and let the graph of $f(x)$ be rotated about x-axis between $x = b$ and $x = c$ as show in Figure 6.11.

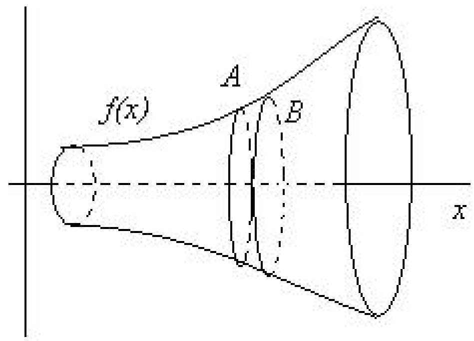


FIGURE 6.11.

Divide the solid into thin slices. Let the surface area of the slice be δs. Then

$$\delta s \approx \pi(y + y + \delta y) \times AB$$

where y is value of $f(x)$ at A and $y + \delta y$ is the value of the function at B. Now,

$$AB = \sqrt{(\delta x)^2 + (\delta y)^2}, \text{ by Pythagoras' Theorem}$$

Then

$$\delta s \approx \pi(2y + \delta y)\sqrt{(\delta x)^2 + (\delta y)^2}$$

which is also written as

$$\frac{\delta s}{\delta x} \approx \pi(2y + \delta y)\sqrt{1 + (\frac{\delta y}{\delta x})^2}.$$

So, with $\delta s \to 0$, we have

$$\frac{ds}{dx} = 2\pi y)\sqrt{1 + (y')^2}.$$

Therefore, the surface area of revolution, s, is given by

$$s = 2\pi \int_b^c y\sqrt{1 + (y')^2}dx \tag{6.4}$$

Example 6.8. Find the surface area of revolution generated by revolving the curve about the x-axis:

(1) $y = \frac{1}{3}x^3,\ [0,\ 3]$,

(2) $y = \sqrt{x}$, $(1,\ 4)$.

Solution

(1) $y' = x^2$. Therefore, the area of surface of revolution,s, is given by

$$\begin{aligned} s &= 2\pi \int_0^3 y\sqrt{1 = (y')^2}dx \\ &= 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1+x^4}dx \end{aligned}$$

Now, letting $1 + x^4 = u$ gives $du = 4x^3 dx$. Thus $du/4 = x^3 dx$ and the limits of integration will be $u = 1$ and $u = 82$. Therefore,

$$\begin{aligned} s &= 2\pi \int_0^3 y\sqrt{1+(y')^2}dx \\ &= 2\pi \int_1^{82} \frac{1}{12}\sqrt{u}du \\ &= \left[\frac{1}{6}\frac{u^{3/2}}{3/2}\right]_1^{82} \\ &= \frac{(82\sqrt{82}-1)\pi}{9} \end{aligned}$$

(2) $y' = \frac{1}{2\sqrt{x}}$. Thus

$$\begin{aligned} s &= 2\pi \int_1^4 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)}dx \\ &= 2\pi \int_1^4 \frac{1}{2}\sqrt{4x+1}dx \end{aligned}$$

Let $v = 4x + 1$ to get $\frac{1}{4}dv = dx$ so that

$$\begin{aligned} s &= 2\pi \int_1^4 \frac{1}{2}\sqrt{4x+1}dx \\ &= \frac{\pi}{4}\int_5^1 7\sqrt{u}du \\ &= \frac{(17\sqrt{17}-5\sqrt{5})\pi}{6} \end{aligned}$$

Look at the following example where the solid of revolution is about y-axis.

Example 6.9. Find the area of surface of revolution generated by revolving the curve about y-axis

(1) $y = \sqrt[3]{x} + 2$, $[1,\ 8]$

(2) $y = 4 - x^2$, $[0,\ 2]$

Solution

(1) Since $y = \sqrt[3]{x} + 2$, it follows that $x = (y-2)^3$ and $x' = 3(y-2)^2$ and so

$$\begin{aligned} s &= 2\pi \int_3^4 x\sqrt{1+(x')^2}dy \\ &= 2\pi \int_3^4 (y-2)^2\sqrt{1+9(y-2)^4}dy \end{aligned}$$

Let $w = 1 + 9(y-2)^4$ so that $\frac{dw}{36} = (y-2)^3 dy$. Thus

$$\begin{aligned} s &= 2\pi \int_3^4 x\sqrt{1+(x')^2}dy \\ &= 2\pi \int_{10}^{145} \frac{1}{36}\sqrt{w}dw \\ &= \left[\frac{\frac{2}{3}u^{3/2}\pi}{18}\right] \\ &= \frac{(145\sqrt{145} - 10\sqrt{10})\pi}{27}. \end{aligned}$$

(2) From $y = 4 - x^2$ we find $x = \sqrt{4-y}$ and $x' = -\frac{1}{2\sqrt{4-y}}$. This gives

$$\begin{aligned} s &= 2\pi \int_4^0 x\sqrt{1+(x')^2}dy \\ &= 2\pi \int_4^0 \sqrt{4-y}\sqrt{1+\frac{1}{4(4-y)}}dy \\ &= \pi \int_4^0 \sqrt{16-4y}dy \\ &= \frac{32\pi}{3} \end{aligned}$$

6.5. Work

If an object is moved a distance, d, in the direction of an applied *constant* force F, then work done by the force is given by $W = Fd$.
Suppose a variable force, $F(x)$, is applied to an object so that it is moved from $x = b$ to $x = c$. To find the total amount of work done we require concepts from calculus since force changes with position. For instance the force of attraction between two objects increases as they are brought closer to each other.

Activity 6.6. *Name other situations where force changes with position.*

We now give a definition of work done by a variable force.

Definition 6.1. *Let an object be moved along a straight line by a variable force $F(x)$. Then the amount of work done by the force as the object is moved from $x = b$ to $x = c$ is given by*

$$W = \int_b^c F(x)dx$$

There are three standard formulas for work. These include, but not limited to:

(1) *Hooke's Law*: The force required to compress or stretch a spring is proportional to distance, x, that the spring is compressed or stretched from its original length. In that case,

$$F(x) = kx, \quad k \text{ constant.}$$

(2) *Law of Gravitation*: The force of attraction, F, between two particles of mass m_1 and m_2 is proportional to product of masses and inversely proportional to the square of the distance, x, between them and is given by

$$F(x) = k\frac{m_1 m_2}{x^2}, \quad k \text{ constant.}$$

(3) *Cuolomb's Law*: The force of attraction between two charges q_1 and q_2 in a vacuum is proportional to the product of the charges and inversely proportional to the square of the distance, x, between the charges. It is given by

$$F(x) = k\frac{q_1 q_2}{x^2}$$

Example 6.10. A force of 750 newtons compresses a spring 7.5 cm from its natural length of 37.5 cm. Find the work done in compressing the spring an additional 7.5 cm.

Solution

Let $F(x)$ be the force required to compress the spring x units. By Hooke's Law, $F(x) = kx$. Using the information given, we have

$$F(3) = 750 = k(7.5).$$

This gives $k = 100$ and thus $F(x) = 100x$. Now the spring is compressed from $x = 7.5$ to $x = 15$. Therefore, the amount of work, W, done by the force is given by

$$\begin{aligned} W &= \int_{7.5}^{15} 100x dx \\ &= \left[50x^2\right]_{7.5}^{15} = 8437.5 \end{aligned}$$

6.6. Unit Summary

In this unit you have learnt how to find the area of a region between two curves using integration. You have also seen how to compute arc length and area of surface of revolution in a given interval. Determination of volume of solid of revolution has also been covered in this unit.

6.7. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

6.8. Exercises

(1) Sketch the region bounded by the graphs of the given functions and find the area of the region.
   (a) $f(x) = 3 - 2x-^2, \quad g(x) = 0,$
   (b) $f(x) = 3x^2 + 2x, \quad g(x) = 8,$
   (c) $f(y) = y^2, \quad g(y) = y + 2.$

(2) Find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about indicated lines.
   (a) $y = 2x^2, \quad y = 0, \quad x = 2$ about the x-axis,
   (b) $y = 6 - 2x - x^2, \quad y = x + 6$ about the y-axis.

(3) A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centred along the axis of revolution, is drilled through this solid so that one-quarter of the volume is removed. Find the diameter of the hole.

(4) Find the arc length of

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval $[\frac{1}{2},\ 2\frac{1}{2}]$ as indicated in Figure 4.

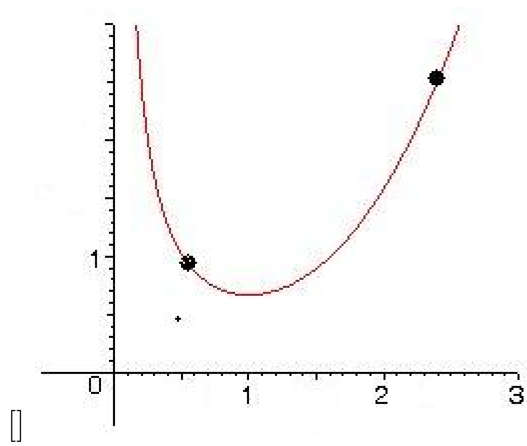


FIGURE 6.12. $y = \frac{x^3}{6} + \frac{1}{2x}$

(5) Find the area of the surface of revolution generated by revolving the given plane curve about the x-axis

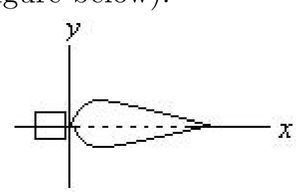
(a) $y = \frac{x^3}{6} + \frac{1}{2x}, \quad [1,\ 2],$

(b) $y = \frac{x}{2}, \quad [0,\ 6]$

(6) An ornamental light bulb is designed by revolving the graph of

$$y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$$

about the x-axis where x and y are measured in centimetres. Find the surface area of the bulb (see the figure below).



(7) A locomotive of freight train pulls its cars with a constant force 9 tons while travelling at constant rate of 88 km/hr on a level track. How much work does the locomotive do in a distance of 3 km?

UNIT 7

TECHNIQUES OF INTEGRATION

7.1. Introduction

So far you have seen that integration of simple functions is a straight forward exercise. Suppose, however, that you are required to evaluate the following integrals

$$\int_2^3 \frac{1}{\sqrt{1-x^2}}dx,$$

or

$$\int_3^5 \frac{1}{x^2-x-2}dx,$$

All these cannot be evaluated until we discuss some techniques that will help us in the evaluation.

The unit has the following topics:

- Partial fractions
- Integration by partial fractions
- Integration by substitution– a revisit
- Integration by parts.

7.1.1. Learning objectives. By the end of this unit you should be able to:

- decompose a fraction to separate fractions,
- evaluate integrals using partition fraction,
- carry out integration using trigonometric substitution,
- integrate by parts.

7.1.2. Prerequisite knowledge. The material in the unit is based on integration of functions.

7.1.3. Time. You should be able to complete this unit in 3 hours.

7.2. Integration by Partial Fractions

Suppose we require to evaluate the integral

$$\int_3^5 \frac{1}{x^2 - x - 2}dx.$$

You note that we cannot invoke any of the basic facts on integration you have met so far. To complete the evaluation, we need to note that the integrand can be written as the sum of two *partial fractions.* The following subsection presents to you a technique for decomposing a fraction whose denominator has factors.

7.2.1. Partial Fractions.

From basic algebra, you know that

$$\frac{3}{5} + \frac{2}{3}$$

can be expressed as a single fraction. You do this by finding a common denominator and then you find $\frac{19}{15}$. What we need, however, is a technique for *decomposing* a given fraction into *partial* fractions. The following will be needed.

Definition 7.1. *Suppose f and g are defined on D, a subset of the set of reals. Then we say f is identically equal to g on D if $f(x) = g(x)\ \forall\ x \in D$ and we write $f(x) \equiv g(x)$.*

The symbols "=" and "≡" have different meanings.

- $x^4 = 1$ is true *only* when $x = \pm 1$
- $x^4 - 1 \equiv (x-1)(x+1)(x^2+1)$ is true for all x.

The following theorem can easily be proved by differentiating both sides equal number of times and set $x = 0$.

Theorem 7.1. *Equating coefficients. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$ be polynomials such that $p \equiv q$ on $(b,\ c)$. Then*

$$a_n = b_n,\ a_{n-1} = b_{n-1},\ \ldots,\ a_0 = b_0$$

This theorem can be extended to rational functions with equal denominators.

Example 7.1. Find the unknown constants in the following identities

(1) $x^3 + 4x - 5 \equiv Ax^3 + Bx^2 + Cx + K$,

(2) $\frac{1}{(x-3)(x+5)} \equiv \frac{Ax+B}{(x-3)(x+5)}$.

Solution

(1) $x^3 + 4x - 5 \equiv Ax^3 + Bx^2 + Cx + K$. By Theorem 7.1, $A = 1$, $B = 0$, $C = 4$ and $K = -5$.

(2) In this identity, we equate numerators so that $1 \equiv Ax + B$. The left hand side does not have a term in x. Thus $A = 0$ and $B = 1$.

Example 7.2. Decompose the following into partial fractions

(1)
$$\frac{6x+6}{x^3+4x^2+x-6}$$

(2)
$$\frac{x^3+5x^2+7x+6}{x^2+3x+2}.$$

Solution

(1) In $\frac{6x+6}{x^3+4x^2+x-6}$, we first factorise the denominator. Now $x^3 + 4x^2 + x - 6 = (x-1)(x+2)(x+3)$. Therefore,

$$\begin{aligned}
\frac{6x+6}{x^3+4x^2+x-6} &= \frac{6x+6}{(x-1)(x+2)(x+3)} \\
&= \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \\
&= \frac{A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)}{x^3+4x^2+x-6} \qquad (7.1)
\end{aligned}$$

It follows that $6x + 6 \equiv A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$. Letting $x = -2$ gives $-6 = -3B$ which yields $B = 2$. If we put $x = -3$, we find $C = -3$ and $A = 1$ is found by setting $x = 1$. So

$$\frac{6x+6}{x^3+4x^2+x-6} = \frac{1}{x-1} + \frac{2}{x+2} - \frac{3}{x+3}.$$

(2) For $\frac{x^3+5x^2+7x+6}{x^2+3x+2}$, the numerator has degree more than the denominator. We, therefore, first of all write the fraction in its proper form by dividing. This

gives us

$$\begin{aligned}\frac{x^3+5x^2+7x+6}{x^2+3x+2} &= x+2+\frac{2-x}{(x+1)(x+2)}\\ &= x+2+\frac{A(x+2)+B(x+1)}{x^2+3x+2}\end{aligned}$$

We find $A = 3$ and $B = 4$. Hence

$$\frac{x^3+5x^2+7x+6}{x^2+3x+2} = x+2+\frac{3}{x+1}-\frac{4}{x+2}.$$

Now look at the following example in which the denomination has repeating factions or one of them has degree two.

Example 7.3. Decompose the following fractions into partial fractions

(1)

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)},$$

(2)

$$\frac{5x^2+20x+6}{x^3+2x^2+x}.$$

Solution

(1) The denominator of $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$ can be factored. That is, $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$. Therefore

$$\begin{aligned}\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} &= \frac{2x^3-4x-8}{(x^2-x)(x^2+4)},\\ &= \frac{2x^3-4x-8}{x(x-1)(x^2+4)}\\ &= \frac{A}{x}+\frac{B}{(x-1)}+\frac{Cx+D}{x^2+4}.\end{aligned}$$

Now multiplying by $x(x-1)(x^2+4)$ we get

$$2x^3-4x-8 \equiv A(x-1)(x^2+4)Bx(x^2+4)+(Cx+D)x(x-1). \tag{7.2}$$

To find A, set $x = 0$ so that $-8 = -4A$. Thus $A = 2$. The value of B is found by letting $x = 1$ and we find $-10 = 5B$. Thus $B = -2$. To find C and D use the fact that Equation 7.2 is an identity and so letting $x = -1$ and $x = 2$ yields

$$2 = -C+D \quad \text{and} \quad 8 = 2C+D$$

from which we have $C = 2$ and $D = 4$. Thus

$$\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} = \frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}.$$

(2) Let us consider $\frac{5x^2+20x+6}{x^3+2x^2+x}$. Since $x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x+)^2$ we have

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by $x(x + 1)^2$ gives us

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx.$$

Letting $x = 0$ yields $A = 6$ and also $x = -1$ gives $C = 9$. To find B, let $x = 1$. Thus

$$\begin{aligned} 5 + 20 + 6 &= 6 \times 2^2 + B \times 2 + 9 \\ &= 24 + 2B + 9 \\ B &= -1 \end{aligned}$$

Therefore,

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}.$$

7.2.2. Integration using Partial Fractions. The discussion we had in the previous subsection had the aim of enabling us handle rational integrands that require decomposing.

Activity 7.1. *Factorise the denominator of the integrand of the following integral*

$$\int_3^5 \frac{1}{x^2 - x - 2} dx.$$

When evaluating integrals with rational integrands you will need to check if

- it is possible to use simple substitution discussed earlier;
- the denominator can be factorise so that it (integrand) can be decomposed in to partial fractions;
- the degree of the numerator is bigger than that of the denominator. In this case, write the improper fraction as a sum of two components: quotient + $\frac{\text{remainder}}{\text{denominator}}$ then proceed with either (1) or (2);

- you can simply add and subtract a number so that you have the case as in (3). This usually happens when the non constant stem of the denominator is equal to the numerator.

Example 7.4. Evaluate the indefinite integral

$$\int \frac{x^2+12x+12}{x^3-4x}dx.$$

Solution

Consider the integrand $\frac{x^2+12x+12}{x^3-4x}$. The numerator has factors x, $x-2$ and $x+2$. Thus

$$\begin{aligned}\frac{x^2+12x+12}{x^3-4x} &= \frac{x^2+12x+12}{x(x-2)(x+2)}\\ &= \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}\\ &= \frac{A(x-2)(x+2)+Bx(x+2)+Cx(x-2)}{x(x-2)(x+2)}\end{aligned}$$

Letting $x=-2$ yields $4-24+12=8C$ which gives $C=-1$. If we set $x=0$, we get $12=-4A$ so that $A=-3$ and let $x=2$ to find $B=5$. Thus

$$\begin{aligned}\int \frac{x^2+12x+12}{x^3-4x}dx &= -\int \frac{3}{x}dx+5\int \frac{1}{x-2}dx-\int \frac{1}{x+2}dx\\ &= -3\ln(x)+5\ln(x-2)-\ln(x+2)\\ &= \ln\left|\frac{(x-2)^5}{x^3(x+2)}\right|+K\end{aligned}$$

Look at how the following example has been solved

Example 7.5. Evaluate the integral

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx.$$

Solution

The integrand has a numerator with a bigger degree. Thus after long division we find that

$$\begin{aligned}\frac{2x^3-4x^2-15x+5}{x^2-2x-8} &= 2x+\frac{x+5}{x^2-2x-8}\\ &= 2x+\frac{x+5}{(x-4)(x+2)}\end{aligned}$$

Now the proper fraction $\frac{x+5}{(x-4)(x+2)}$ can be written as

$$\frac{x+5}{(x-4)(x+2)} = \frac{A(x+2)+B(x-4)}{(x-4)(x+2)}.$$

Therefore, $x+5 = A(x+2)+B(x-4)$. Let $x=-2$ to get $3=-6B$. Thus $B=-\frac{1}{2}$ and in similar manner we find $A=\frac{3}{2}$. The integral can now be evaluated.

$$\begin{aligned}
\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx &= \int 2x dx + \int \frac{x+5}{x^2-2x-8}dx \\
&= x^2 + \int \frac{3}{2(x-4)}dx - \int \frac{1}{2(x+2)}dx \\
&= {}^2 + \frac{3}{2}\ln(x-4) - \frac{1}{2}\ln(x+2) + K \\
&= x^2 + \frac{1}{2}\ln\left|\frac{(x-4)^3}{x+2}\right| + K
\end{aligned}$$

Integration using partial fractions can also be applied to some problems with integrands that have transcendental functions. You simply need to make a relevant substitution and proceed as in the previous example.

Activity 7.2. *Find the indefinite integral*

$$\int \frac{3\cos x}{\sin^2 x + \sin x - 2}dx, \quad \textit{by letting} \quad u = \sin x$$

.

7.3. Integration by Trigonometric Substitution

The objective of this section is to use trigonometric substitution to eliminate a radical in the integrand. We use the identities

$$\begin{aligned}
\cos^2 x &= 1-\sin^2 x \\
\sec^2 x &= 1+\tan^2 x \\
\tan^2 x &= \sec^2 x - 1
\end{aligned}$$

Remark 7.1. (1) For integrals involving $\sqrt{a^2-u^2}$, let $u = a\sin x$, so that $\sqrt{a^2-u^2} = a\cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$;

(2) For integrals involving $\sqrt{a^2+u^2}$, let $u = a\tan x$. Then $\sqrt{a^2+u^2} = a\sec x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$;

(3) For integrals involving $\sqrt{u^2-a^2}$, let $u = a\sec x$. Then $\sqrt{u^2-a^2} = \pm a\tan x$, where $0 \le x \le \frac{\pi}{2}$ or $\frac{\pi}{2} \le x \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.

Example 7.6. Evaluate the indefinite

$$\int \frac{1}{(25-u^2)^{3/2}}du$$

using the substitution $u = 5\sin x$.

Solution

Let $u = 5\sin x$. Then $du/dx = 5\cos x$ and $25 - u^2 = 25 - 25\sin^2 x = 25\cos^2 x$. Therefore,

$$\begin{aligned}
\int \frac{1}{(25-u^2)^{3/2}}du &= \int \frac{5\cos x}{(25-25\sin^2)^{3/2}}dx \\
&= \int \frac{\cos x}{25\cos^3 x}dx \\
&= \frac{1}{25}\int \frac{1}{\cos^2 x}dx \\
&= \frac{1}{25}\int \sec^2 dx \\
&= \frac{1}{25}\tan x + K
\end{aligned}$$

But $u = 5\sin x$ which means $\tan x = \frac{u}{\sqrt{25-u^2}}$ (see triangle in Figure 7.3). Thus

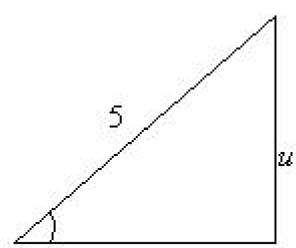


FIGURE 7.1. $u = 5\sin x$, $\tan x = \frac{u}{\sqrt{25-u^2}}$

$$\int \frac{1}{(25-u^2)^{3/2}}du = \frac{1}{25}\frac{u}{\sqrt{25-u^2}} + K.$$

Example 7.7. Evaluate the indefinite

$$\int \frac{x^2}{(1+x^2)^2}dx$$

using the substitution $x = \tan\theta$.

Solution

Let $x = \tan\theta$. Then $dx/d\theta = \sec^2\theta$ and $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$. Therefore,

$$\begin{aligned}
\int \frac{x^2}{(1+x^2)^2}dx &= \int \frac{(\sec^2\theta - 1)\sec^2\theta}{(1+\tan^\theta)^2}d\theta \\
&= \int \frac{(\sec^2\theta - 1)\sec^2\theta}{(\sec^2)^2}d\theta \\
&= \int d\theta - \int \frac{1}{\sec^2\theta}d\theta \\
&= \theta - \int \cos^2\theta d\theta
\end{aligned}$$

But $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ which means

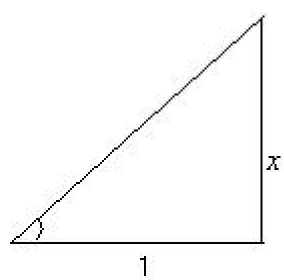


FIGURE 7.2. $x = \tan\theta$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$$\int \cos^2\theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}$$

Now, since $x = \tan\theta$, we have that $\cos\theta = \frac{1}{\sqrt{1+x^2}}$ and $\sin\theta = \frac{x}{\sqrt{1+x^2}}$. Thus

$$\int \frac{x^2}{(1+x^2)^2}dx = \frac{1}{2}\arctan x - \frac{x}{2(1+x^2)} + K.$$

When working with trigonometric substitutions, try to have a sketch right triangle that will connect the old variable and the new variable as we have done in the preceding examples.

We present three special integral formulas.

Theorem 7.2. (1)

$$\int \sqrt{a^2 - u^2} du = \frac{1}{2}\left(a^2 \arcsin \frac{u}{a} + u\sqrt{a^2 - u^2}\right) + K,$$

(2)

$$\int \sqrt{u^2 - a^2} du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 + a^2}|\right) + K,$$

(3)

$$\int \sqrt{u^2 + a^2} du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}|\right) + K.$$

Example 7.8.

Evaluate the integral

$$\int \sqrt{4 + 9x^2} dx$$

using Theorem 7.2.

Solution

This integral can be re-expressed as

$$\int \sqrt{4 + 9x^2} dx = 3 \int \sqrt{\frac{4}{9} + x^2} dx$$

which is of the form of the third part of Theorem 7.2. Therefore, with $a = \frac{2}{3}$ we have

$$\begin{aligned} \int \sqrt{4 + 9x^2} dx &= 3 \int \sqrt{\frac{4}{9} + x^2} dx \\ &= \frac{3}{2}\left(x\sqrt{x + \frac{4}{9}} + \frac{4}{9} \ln |x + \sqrt{x + \frac{4}{9}}|\right) + K \end{aligned}$$

We close the section by an activity which has one problem involving completion of squares and another that involves an application in finding the area of a circle.

Activity 7.3. (1) *Complete squares in x for the integrand and show that*

$$\int \frac{dx}{(x^2 - 4x)^{\frac{3}{2}}} = -\frac{x - 2}{4\sqrt{x^2 - 4x}} + K.$$

(2) *Find the area of a circle of radius r and centre the origin.*

7.4. Integration by Parts

You will recall the product rule of differentiation for a function $f(x) = u(x)v(x)$. It is given by

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Integration by parts is based on this formula.

Theorem 7.3. *If $u(x)$ and $v(x)$ are functions of x and have continuous derivatives, then*

$$\int uv'dx = uv - \int vu'dx$$

It is worthwhile to note that in integrating by parts, the choices of u and v is important in finding a good solution. You are encouraged to follow the following simples guidelines.

(1) Let v' be the most complicated factor of the integrand that simple to integrate. Then u will be the remaining factor of the integrand.
(2) Let u be the factor of the integrand which can be easily differentiated. Then v' will be the remaining factor of the integrand.

Example 7.9. Evaluate the given integral.

(1) $\int xe^{-2x}dx$,
(2) $\int t\ln(t+1)dt$,
(3) $\int \ln x dx$,
(4) $\int \arccos x dx$,
(5) $\int e^{2x}\sin x$.

Solution

(1) In $\int xe^{-2x}dx$, let $u = x$ and $v' = e^{-2x}$. Then

$$\begin{aligned} u = x \qquad & \text{gives} \quad u' = 1 \\ v' = e^{-2x} \qquad & \text{yields} \quad v = -\frac{1}{2}e^{-2x}. \end{aligned}$$

Now by integrating by parts we get

$$\begin{aligned}\int xe^{-2x}dx &= -\frac{xe^{-2x}}{2}+\frac{1}{2}\int e^{-2x}dx \\ &= -\frac{xe^{-2x}}{2}-\frac{e^{-2x}}{4}+K.\end{aligned}$$

(2) To evaluate $\int t\ln(t+1)dt$, let $u=\ln(t+1)$ so that $u'=\frac{1}{t+1}$ and letting $v'=t$ gives $v=\frac{t^2}{2}$. Therefore,

$$\begin{aligned}\int t\ln(t+1)dt &= \frac{t^2}{2}\ln(t+1)-\frac{1}{2}\int\frac{t^2}{(t+1)}dt \\ &= \frac{t^2}{2}\ln(t+1)-\frac{1}{2}\int\frac{t^2+2t+1-(2t+1)}{t+1}dt \\ &= \frac{t^2}{2}\ln(t+1)-\frac{1}{2}\left[\int(t+1)dt-\int\frac{2t+1+1-1}{t+1}dt\right] \\ &= \frac{t^2}{2}\ln(t+1)-\frac{1}{2}\left[\frac{t^2}{2}+t-2t+\ln(t+1)+K\right] \\ &= \frac{t^2}{2}\ln(t+1)-\frac{t^2}{4}+t-\frac{1}{2}\ln(t+1)+K \\ &= \frac{1}{2}[t^2-1]\ln|t+1|-\frac{t^2}{4}+t+K.\end{aligned}$$

Note that it is necessary to use other techniques you have met earlier to evaluate integral. For instance, we needed to subtract and add a term or a constant as is required.

(3) $\int\ln xdx=\int 1\times\ln xdx$. Let $u=\ln x$ so that $u'=\frac{1}{x}$ and let $v'=1$ yielding $v=x$. The integral can now be evaluated using integration by parts.

$$\begin{aligned}\int\ln xdx &= x\ln x-\int x\times\frac{1}{x}dx \\ &= x\ln x-x+K\end{aligned}$$

(4) $\int\arccos xdx=\int 1\times\arccos xdx$. Now letting $u=\arccos x$ gives $u'=\frac{-1}{\sqrt{1-x^2}}$ and $v'=1$ yields $v=x$. Therefore

$$\begin{aligned}\int\arccos xdx &= x\arccos x-\int x\times\frac{-1}{\sqrt{1-x^2}}dx \\ &= x\arccos x+\int\frac{x}{\sqrt{1-x^2}}dx\end{aligned}$$

The integral component of the term on the right hand side can be evaluated by simple substitution. Letting $w = 1 - x^2$ gives $dw/dx = -2x$ so that $xdx = -dw/2$. Thus

$$\begin{aligned}\int \arccos x dx &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arccos x - \sqrt{1-x^2} + K.\end{aligned}$$

(5) $\int e^{2x} \sin x dx$ is one of the integrals that need integration by parts more than once. Let $u = e^{2x}$ to get $u' = 2e^{2x}$ and $v\prime = \sin x$ to find $v = -\cos x$. This gives

$$\int e^{2x} \sin x dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x dx.$$

Let $u = e^{2x}$ again and $w' = \cos x$ so that $w = \sin x$. Thus

$$\begin{aligned}\int e^{2x} \sin x dx &= -e^{2x} \cos x + 2 \int e^{2x} \cos x dx \\ &= -e^{2x} \cos x + 2 \left[e^{2x} \sin x - 2 \int e^{2x} \sin x dx \right] \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x dx.\end{aligned}$$

Therefore, making $\int e^{2x} \sin x dx$ the subject of the equation we get

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x).$$

7.5. Unit Summary

In this unit you have gained concepts that will assist in evaluating integrals. These techniques include use of partial derivatives, trigonometric substitutions and integration by parts.

In the first case, you will require to look at the integrand and decide if simple substitution is not possible. In that case you decompose the integrand into partial fractions. In using the other two techniques, procedures have been provided so the evaluation of an integral is conveniently executable.

7.6. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

7.7. Exercises

(1) Identify u and v' for evaluating the integral by integration by parts.
   (a) $\int xe^{2x}dx$,
   (b) $\int (\ln x)^2 dx$,
   (c) $\int x\sec^2 x dx$.

(2) Evaluate the integral (by the simplest method because not all need integration by parts).
   (a) $\int x^3 e^{-x}dx$,
   (b) $\int \frac{e^{1/t}}{t^2}dt$,
   (c) $\int \frac{(\ln x)^2}{x}dx$,
   (d) $\int \arctan x dx$.

(3) Evaluate the definite integral

$$\int_0^1 x \arcsin x^2 dx.$$

(4) Match the antiderivative with the correct integral.

(a) $\int \frac{x^2}{\sqrt{16-x^2}}dx$	(a) $4\ln\left\lvert\frac{\sqrt{x^2+16}-4}{x}\right\rvert + \sqrt{x^2+16} + K$
(b) $\int \frac{\sqrt{x^2+16}}{x}dx$	(b) $8\ln\left\lvert\sqrt{x^2-16}+x\right\rvert + \frac{x\sqrt{x^2-16}}{2} + K$
(c) $\int \sqrt{7+6x-x^2}dx$	(c) $8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + K$
(d) $\int \frac{x^2}{\sqrt{x^2-16}}dx$	(d) $8\arcsin\frac{x-3}{4} + \frac{(x-3)\sqrt{7+6x-x^2}}{2} + K$

(5) Evaluate the indefinite integral.
   (a) $\int \frac{x^3+x+1}{x^4+2x^2+1}dx$,
   (b) $\int \frac{1}{t\sqrt{4t^2+9}}dt$,
   (c) $\int u \arcsin u du$.

(6) Evaluate the definite integral using partial fractions.

(a)

$$\int_1^2 \frac{x+1}{x(x^2+1)}dx,$$

(b)

$$\int_0^1 \frac{3}{2x^2+5x+2}dx.$$

(7) A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected people at time t. The common epidermic model assumes that the disease spreads at a rate proportional to the product of the total number of infected and the number not yet infected. Thus

$$\frac{dx}{dt} = k(x+1)(n-x)$$

and we obtain

$$\int \frac{1}{(x+1)(n-x)}dx = \int dt.$$

Solve for x as a function of t.

UNIT 8

NUMERICAL INTEGRATION

8.1. Introduction

Now you have the impression that almost any function may be integrated after sufficient thought. This is not the case because some functions do not possess antiderivatives that are elementary. For instance,

$$\int \frac{\cos x}{x} dx, \quad \int e^{x^2} dx, \quad \int \sqrt{1-x^3} dx$$

cannot be evaluated in terms of other elementary functions. Their definite integrals can only be approximated by the technique we shall discuss in this unit.
The unit has the following topics:

- Trapezoidal Rule,
- Simpson's Rule for n even.

8.1.1. Learning objectives. By the end of this unit you should be able to:

- use the Trapezoidal Rule to approximate a definite integral,
- find an approximation of a definite integral Simpson's method,
- Find errors in the trapezoidal and Simpson's rules.

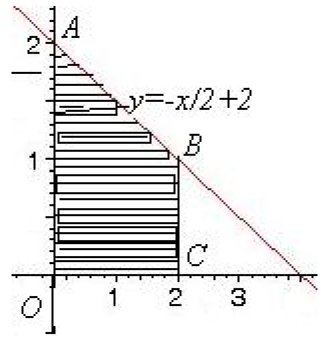
8.1.2. Prerequisite knowledge. You will require some algebraic concepts.

8.1.3. Time. You should be able to complete this unit in $1\frac{1}{2}$ hours.

8.2. The Trapezoidal Rule

We motivate the discussion by starting with an activity.

Activity 8.1. *Find the area of the trapezium bounded by* $y = -1/2x + 2$, $y = 0$, $x = 0$ *and* $x = 2$ *(shown in the figure below) using (a) the formula, (b) calculus.*



Suppose we require to evaluate the integral

$$\int_b^c f(x)dx.$$

where $f(x)$ is some given function defined on $[b,\ c]$. We divide the interval $[b,\ c]$ into n equal subintervals

$$b = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-2} < x_{n-1} < x_n = c$$

Let

$$h = x_i - x_{i-1} = \frac{c-b}{n}, \quad i = 1,\ 2,\ 3,\ \ldots,\ n.$$

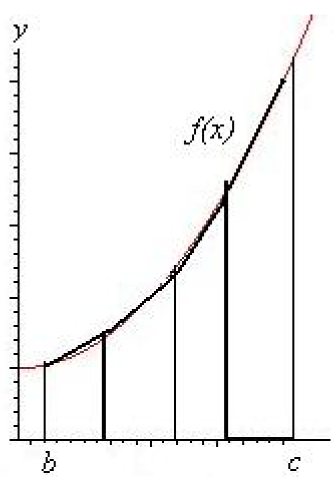


FIGURE 8.1. Trapezoids

Denote $f(x_i)$ by f_i to shorten the notation. Complete trapezoids by joining points $(x_{i-1},\ f_{i-1})$ and $(x_i,\ f_i)$. The results is n trapezoids. An illustration is show in Figure 8.1. Then the area under the curve, which by definition is just the integral, is approximately the sum of areas of the trapezoids formed.

Now, the area of the first trapezium is

$$\frac{f_0 + f_1}{2}h$$

area of the second trapezium is

$$\frac{f_1 + f_2}{2}h$$

area of nth trapezoid

$$\frac{f_{n-1} + f_n}{2}h.$$

and so on. We obtain

$$\int_b^c f(x)dx \approx \frac{f_0 + f_1}{2}h + \frac{f_1 + f_2}{2}h + \frac{f_2 + f_3}{2}h + \ldots + \frac{f_{n-1} + f_n}{2}h$$

or equivalently

$$\int_b^c f(x)dx \approx \frac{h}{2}\left[f_0 + f_n + 2\sum_{i=1}^{n-1} f(i)\right] \tag{8.1}$$

This is known as the *Trapezoidal Rule*.

We begin with a define integral that can be easily without employing this method in order for you to appreciate the usability of the method.

Example 8.1. Approximate

$$\int_0^2 x^3 dx$$

using the Trapezoidal with $n = 4$ and then $n = 8$.

Solution

Let us begin with $n = 4$. This gives us $h = (0 + 2)/4 = 1/2$ and the following table

x_i	0	0.5	1	1.5	2
$f(x_i)$	0	0.125	1	3.375	8

Thus

$$\begin{aligned}\int_0^2 x^3 dx &\approx \frac{1/2}{2}\left[0 + 8 + 2(0.125 + 1 + 3.375)\right] \\ &\approx 4.25\end{aligned}$$

With $n = 8$, $h = 2/8 = 1/4$ we have the following table

x_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x_i)$	0	0.015625	0.125	0.421875	1	1.953125	3.375	5.359375	8

Therefore,

$$\begin{aligned}\int_0^2 x^3 dx &\approx \frac{1/4}{2}\left[0+8+2(0.015625+0.125+0.421875+1+1.953125+3.375+5.359375)\right]\\ &\approx 4.0625\end{aligned}$$

If you evaluate the integral using elementary techniques, you find that the exact value of the integral is 4. You also note that as the number of stripes increases, the approximation improves.

Example 8.2. Approximate the definite integral using the Trapezoidal Rule with $n = 4$.

(1)

$$\int_0^2 \frac{1}{\sqrt{1+x^3}},$$

(2)

$$\int_0^{\pi/2} \sqrt{1+\cos^2 x}dx.$$

Solution

(1) With $n = 4$, $h = (0+2)/4 = \frac{1}{2}$ and the following table is generated

x_i	0	0.5	1	1.5	2
$f(x_i)$	1	0.9428	0.7071	0.4781	0.3333

Thus

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{1+x^3}}dx &\approx \frac{1/2}{2}\left[1+0.3333+2(0.9428+0.7071+0.4781)\right]\\ &\approx 1.3973\end{aligned}$$

(2) In this case the interval is $[0,\ \pi/2]$. Thus with $n = 4$,

$$h = \frac{(0+\frac{\pi}{2})}{4} = \frac{\pi}{8}$$

and the following table is generated

x_i	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$f(x_i)$	1	1.0743	1.2716	1.5453	1.8621

Thus

$$\int_0^{\pi/2} \sqrt{1+\cos x^2}dx \approx \frac{\pi}{16}[1+1.8621+2(1.0743+1.2716+1.5453)]$$
$$\approx 1.8578$$

8.3. Simpson's Rule

While the trapezoidal is easy to use, it is not particularly accurate. A more accurate method is *Simpson's Rule*. In this we approximate the curve by a number of parabolic arcs. We will need this simple theorem.

Theorem 8.1. *Let $p(x) = Ax^2 + Bx + C$. Show that*

$$\int_b^c p(x)dx = \frac{c-b}{6}\left[p(b) + 4p\left(\frac{b+c}{2}\right) + p(c)\right]$$

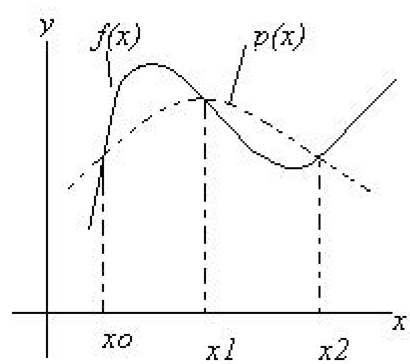


FIGURE 8.2. $\int_{x_0}^{x_2} p(x)dx \approx \int_{x_0}^{x_2} f(x)dx$

Now any parabola is determined by three points. Let us partition $[b,\ c]$ into n equal intervals, each of width $\Delta x = \frac{c-b}{n}$. In this method, however, we require that n be even and that intervals be grouped in pairs:

$$b = \underbrace{x_0 < x_1 < x_2}_{[x_0,\ x_2]} \underbrace{< x_3 < x_4}_{[x_2,\ x_4]} < \ldots < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2},\ x_n]}$$

The area under $f(x)$ on $[x_{i-2},\ x_i]$ is approximated by a second degree polynomial, $p(x) = Ax^2 + Bx + C$. This polynomial must pass through $(x_{i-2},\ y_{i-2})$, $(x_{i-1},\ y_{i-1})$

and $(x_i,\ y_i)$. Note that $x_2 - x_0 = 2h$ and $(x_0 + x_2)/2 = x_1$. By Theorem 8.1

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx &= \frac{x_2 - x_0}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\
&= \frac{2h}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\
&= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \qquad (8.2)
\end{aligned}$$

Now, we approximate the area under the curve by the sum of the areas under the parabolic arcs show in Figure 8.2. Using the result in Equation 8.2 we have

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &\approx \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \ldots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n] \\
\int_{x_0}^{x_2} f(x)dx &\approx \frac{h}{3}[(f_0 + f_n) + 2(f_2 + f_4 + \ldots + f_{n-2}) + 4(f_1 + f_3 + \ldots + f_{n-3} + f_{n-1})]
\end{aligned}$$

This is called *Simpson's Rule*

We now present two examples one of which will be a repeat from the previous section.

Example 8.3. Use the Simpson's Rule to approximate

$$\int_0^2 x^3 dx$$

Take $n = 4$ and then $n = 8$ and compare the results with those obtained using the Trapezoidal Rule.

Solution

We begin with $n = 4$. Then $h = 1/2$ and the other values are given in the following table.

x_i	0	0.5	1	1.5	2
$f(x_i)$	0	0.125	1	3.375	8

Now, the rule tells us that

$$\begin{aligned}
\int_0^2 x^3 dx &\approx \frac{1/2}{3}[(f_0 + f_4) + 2f_2 + 4(f_1 + f_3)] \\
&\approx \frac{1}{6}[(0 + 8) + 2(1) + 4(0.125 + 3.375)] \\
&\approx 4.00.
\end{aligned}$$

But this is the exact value of this definite integral and, therefore, it is not necessary to use $n = 8$.

Example 8.4. Use Simpson's Rue with $n = 4$ to evaluate

$$\int_0^{\pi/4} \tan x^2 dx$$

Solution

In this case $h = \frac{(\pi/4)}{4} = \pi/16$. This gives the following values of $f(x)$:

x_i	0	$\pi/16$	$\pi/8$	$3\pi/16$	$\pi/4$
$(x_i)^2$	0	$(\pi/16)^2$	$(\pi/8)^2$	$(3\pi/16)^2$	$(\pi/4)^2$
f_i	0	0.03857	0.15545	0.36161	0.70916

Therefore, by Simpson's Rule we have

$$\begin{aligned} \int_0^{\pi/4} \tan x^2 dx &\approx \frac{\pi/16}{3}[(f_0 + f_4) + 2f_2 + 4(f_1 + f_3)] \\ &\approx \frac{\pi}{48}[(0 + 0.70916) + 2(0.15545) + 4(0.03857 + 0.36161)] \\ &\approx 0.17153 \end{aligned}$$

Remark 8.1. (1) Just like in the case of the Trapezoidal Rule, the approximation improves with the number of partitions.

(2) You will also note from Examples 8.1 and 8.3 that Simpson's Rule is more accurate than the Trapezoidal Rule.

8.4. Error Analysis

When using approximation techniques, we often are interested in the degree of accuracy of the technique. The following theorem provides formulae for estimating errors in using the two techniques discussed above.

Theorem 8.2. (1) *Let $f(x)$ have a continuous second derivative on $[b,\ c]$. Then the error, E, in approximating $\int_b^c f(x)dx$ by Trapezoidal Rule is given by*

$$E \leq \frac{(c-b)^3}{12n^2}[max|f''(x)|],\ b \leq x \leq c. \tag{8.3}$$

(2) *If $f(x)$ is a function such that $f^{(4)}(x)$ is defined on $[b,\ c]$, then the error, E, in approximating $\int_b^c f(x)dx$ by Simpson's Rule is given by*

$$E \leq \frac{(c-b)^5}{180n^4}[max|f^{(4)}(x)|],\ b \leq x \leq c. \tag{8.4}$$

You will note that for Example 8.3 the error is 0 because $f^{(4)}(x) = 0$. Now look at the following examples.

Example 8.5. Find the maximum possible error in approximating

$$\int_0^1 \frac{1}{1+x}dx,\ n = 4$$

using (a) Trapezoidal Rule and (b) using Simpson's Rule.

Solution

(a) Let the error be E. Now, $c - b = 1$, $n = 4$ and $f''(x) = \frac{2}{(1+x)^3}$. The maximum value of $f''(x)$ on $[0,\ 1]$ is $f''(0) = 2$. Thus

$$E \leq \frac{(1)^3}{12 \times 4^2}[2] = \frac{1}{96} = 0.01042.$$

(b) In this case $f^{(4)}(x) = \frac{24}{(1+x)^5}$ and its maximum value on $[0,\ 1]$ is 24. Therefor, the maximum error in using Simpson's Rule is

$$E \leq \frac{(1)^5}{180 \times 4^4}[24] = \frac{1}{1920} = 0.00052.$$

Again you see how good Simpson's Rule is.

Example 8.6. Find n so that the error in the approximation of

$$\int_0^2 (x+1)^{2/3}dx$$

is less than 0.00001 using the Trapezoidal Rule.

Solution

The length of the interval is $c - b = 1$ and

$$f''(x) = -\frac{2}{9(x+1)^{4/3}}.$$

This has a maximum absolute value when $x = 0$. Therefore,

$$E \leq \frac{(1)^3}{12 \times n^2}[\frac{2}{9}] = \frac{2}{108n^2}$$

This value must be less than $\frac{1}{100000}$. Thus

$$\begin{aligned} \frac{2}{108n^2} &\leq \frac{1}{100000}, \\ 54n^2 &\geq 100000 \end{aligned}$$

$$n \geq 43.033$$

So we require about 44 stripes to achieve this accuracy.

8.5. Unit Summary

In this unit you have seen two techniques used for approximating definite integrals. It has been show by way of examples that the Simpson's method is more accurate than the Trapezoidal Rule. However, the latter method, as has been seen, is easier to use.

You have also learnt how to errors in using the Trapezoidal or the Simpson's Rule.

8.6. References

(1) Anton, H.; Bivens, I and Davis, S. (2005), Calculus. John Wiley and Sons, New Jersey.

(2) Larson,R. E.; Hostetler, R. P.; Edwards, B. H. and Heyd, D. E. (1998). Calculus of a Single Variable. Houghton Mifflin Company,Boston.

8.7. Exercises

(1) Use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Round the answer to four decimal places and compare the results with the exact value of the definite integral.

  (a) $\int_4^9 \sqrt{x}dx,\ n = 8,$

  (b) $\int_0^1 (\frac{x^2}{2} + 1)dx,\ n = 4,$

(2) Approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$.

  (a) $\int_2^4 e^{x^3} dx,$

  (b) $\int_0^{\sqrt{\pi/2}} \cos x^2 dx,$

  (c) $\int_3^7 (\ln x)^3 dx,$

(3) Evaluate the definite integral

$$\int_0^1 x \arcsin x^2 dx.$$

(4) Use Trapezoidal Rule with $n = 10$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x} \cos x$, $y = 0$, $x = 0$ and $x = \pi/2$.

(5) Find the maximum error,with $n = 8$, in approximating

$$\sqrt{3}\int_0^{\pi/2} \sqrt{1 - \frac{2}{3}\sin^2\theta d\theta}$$

using the (a) Trapezoidal Rule (b) Simpson's Rule.

(6) The value of π is usually approximated by the equation

$$\pi = \int_0^1 \frac{4}{1+x^2}dx$$

Use Simpson's Rule with $n = 14$ to make the approximation.

(7) Find n such that the error in the approximation of

$$\int_1^2 \frac{\sin x}{x}dx$$

is less than 0.0001 using (a) Trapezoidal Rule (b) Simpson's Rule.