Modeling and Numerical Solution of Thermo-Hydraulic Networks

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1 Configuration of Thermo-Hydraulic Networks

A thermo-hydraulic network has the following characteristics:

- Network is composed of nodes and elements
- Pressure drop is formulated only along elements
- elements can be pipes, pumps, heat exchanges, valves etc.
- Each *element* has exactly one inlet (left) and one outlet (right) in the schematics
- Each *node* has minimum one inlet element (left) and one outlet element (right)
- Nodes connect elements and hold a pressure state (no flow, no pressure drop)
- Mass flux through an element is conserved, i.e. the same at either side.
- Mass flux and pressures are linked in a steady-state relation, i.e. we use an incompressible fluid or (in case of gas) neglect compressibility. Also, mass density of fluid is temperature independent and taken as constant.
- Each *element* conserves energy through enthalpy inflow and outflow (depending on mass flux direction) and optional heat exchange with the environment or other model components. Also, such heat gains/losses can be prescribed as time-series or constant values.
- Each element has a conserved energy state and thus has a finite fluid volume with associated heat capacity. Energy balances of flow elements are transient equations and solved with time integration.

1.1 Hydraulic Network Equations

Hydraulic equations relate pressures p_i at all network nodes i and mass fluxes $\dot{m}_i$ through all elements i.

1.1.1 Nodes

Each node i is characterized by mass fluxes at inlet element $\dot{m}_{in,i,j}$ and at outlet element $\dot{m}_{out,i,j}$ (with j counting the number of inlet and outlet elements). Indeed, mass flux direction is defined negative for inlet nodes and positive for outlet nodes. With this assumption conservation equations may be formulated as:

$$\sum_j \dot{m}_{out,i,j} - \sum_j \dot{m}_{in,i,j} = 0 \tag{1}$$

with

- $\dot{m}$ mass flux into the node ($\dot{m}_{in}$) and out of the node ($\dot{m}_{out}$) [kg/s]

Also, the pressure p_i at some node i is equal to all inlet/outlet pressures of the connected flow elements (no pressure drop/change across a node). This equation is not explicitely formulated but will be used to eliminate variables in the solution process.

1.1.2 Elements

Each element i has one inlet pressure, one outlet pressure and mass flux. Such a flow element can be seen as

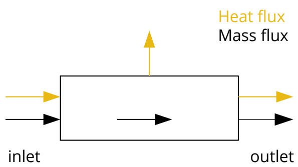

Figure 1: Schematics of an abstract flow element and sign definition of fluxes across model boundary

For each network element the following equations are defined. The system equation for each element (2) defines the relationship between mass flow across the element and the pressures at inlet (in) and outlet (out). The mass flux direction and sign definition is shown in Figure 1. For a typical pipe, the pressure drop would be formulated as $\Delta p_i = p_{in,i} - p_{out,i}$.

$$0 = f_i\left(\dot{m}_i, p_{in,i}, p_{out,i}, T_i\right) \tag{2}$$

with

- p pressure head of the fluid at inlet ($p_{in,i}$) and outlet node ($p_{out,i}$) [Pa]
- $\dot{m}_i$ mass flux through the element [kg/s]
- T_i mean fluid temperature along the flow element

Note that the actual pressure drop equation may be a function of temperature as well. Since a flow element is connected to two nodes, it is hence also in contact with two node temperatures. With respect to numerical stability of the solution procedure, the temperature at the inflow node should be used when evaluating the media properties in the element's system equation, a method called *upwinding*.

1.1.3 Closing/complementory equations

The system itself needs additional constraints to be solved, usually a reference pressure defined at *exactly one node*, for example the node connected to an inlet of a pump. In the solution process it might then possible to eliminate this pressure from the connected flow element's equations. However, a more pragmatic approach from point of view of implementing such a network solver is to define a pressure "boundary condition" as discussed below.

1.2 Temperature/Pressure Controlled Flow Network Elements

Some flow network elements may be controlled, for example pumps and valves. The control equation may use pressures or temperatures of the flow network itself, thus creating a tightly coupled problem. With respect to engineering requirements, often these direct feedback formulations can be softened by introducing typical time scales of reaction. For example, if a signal for pump activation is received (e.g. temperature trigger point passed), it may be possible to introduce a delay of some minutes until the pump has reached the required speed. The time delay must be selected such that the network itself is solved sufficiently correctly.

Suppose we name a control variable $r_i\left(p, T\right)$ and denote by the parameters p and T some dependence of flow network pressures/temperatures. Then the system equation for a pump might read:

$$0 = f_i\left(\dot{m}_i, p_{in,i}, p_{out,i}, r_i\left(p_i, T_i\right)\right)$$

and the resulting network will be fully coupled. If, however, we introduce a time variation on the control variable, we obtain a differential equation:

$$\frac{dr_i}{dt} = \tau\left(r_{set}\left(p_i, T_i\right) - r_i\left(t, p_i, T_i\right)\right)$$

with τ being some time constant parameter. $r_{set}\left(p_i, T_i\right)$ will be the target control signal obtained from the control algorithm equations that use the current network's state. $r_i\left(t, p_i, T_i\right)$ will be a transient quantity that more or less slowly (depending on τ) approaches this set point value. When solving the steady-state hydraulic flow network equations, the control variable r is a constant, which simplifies solution of the equation system significantly.

2 Example Networks and Example Equation Systems

We will illustrate the general solution procedure for all kinds of thermo-hydraulic networks by starting with simple examples.

2.1 Example 1: Pump with Pipes

The first example consists of three nodes and three flow elements (Fig. 2). This is an adiabatic model without energy balances. Already with this simple model we can identify several problems associated with hydraulic networks.

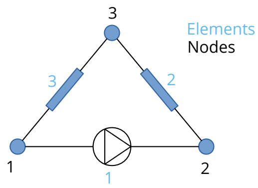

Figure 2: Example 1: Simple flow network with a pump and two pipes

For each of the flow elements we have the system equation. When formulating the equations, we already substitute $p_{in,i}$ and $p_{out,i}$ with the pressure at the connected node.

$$
\begin{aligned}
0 &= f_1\left(\dot{m}_1, p_1, p_2\right) \\
0 &= f_2\left(\dot{m}_2, p_2, p_3\right) \\
0 &= f_3\left(\dot{m}_3, p_3, p_1\right)
\end{aligned}
$$

In addition, we have the nodal equations:

$$
\begin{aligned}
0 &= n_1\left(\dot{m}_1, \dot{m}_3\right) = \dot{m}_3 - \dot{m}_1 \\
0 &= n_2\left(\dot{m}_1, \dot{m}_2\right) = \dot{m}_1 - \dot{m}_2 \\
0 &= n_3\left(\dot{m}_2, \dot{m}_3\right) = \dot{m}_2 - \dot{m}_3
\end{aligned}
$$

We could now formulate the equation system in a generic way as $\boldsymbol{F}\left(\boldsymbol{p}, \dot{\boldsymbol{m}}\right)$ where $\boldsymbol{p}$ and $\dot{\boldsymbol{m}}$ are vectors of the nodal pressures and flows across flow elements. Thus, we have 6 unknowns and the 6 equations above and might assume that this is an easily solvable problem (though non-linear).

However, in this small example we have formulated a linear combination. Try substituting the flow equations: $\dot{m}_3 = \dot{m}_1$ and $\dot{m}_2 = \dot{m}_1$ from the first two equations, an insert these into the last equation. Then we get $\dot{m}_1 = \dot{m}_1$ which obviously doesn't give any new information.

After eliminating the mass fluxes $\dot{m}_3$ and $\dot{m}_2$ this leaves us with 4 unknowns: $p_1, p_2, p_3, \dot{m}$ (we use $\dot{m} = \dot{m}_1$ for simplicity here), but only 3 flow element system equations.

Now the same process of variable elimination can be done with pressures as well. Suppose all flow elements have a rather trivial system equation, that relates the inlet and outlet pressures in a convenient way. Say, we have $p_1 = p_2 + g_1$, $p_2 = p_3 + g_2$ and $p_3 = p_1 + g_3$. We might then eliminate pressures by inserting equations into each other and obtain: $p_1 = p_1 + g_1 + g_2 + g_3$ and consequently $0 = g_1 + g_2 + g_3$. The parameters g are likely functions of the mass flux $\dot{m}$, so solving the equation will give us a solution for $\dot{m}$. However, the corresponding pressures will be arbitrary.

Interestingly, if one were to solve the equation system above using some generic solution method (i.e. algebraic equation system solver library), one *might* even get a result, but that would be ambigous (and depend on some initial guess value of one of the pressures). For example, the result could be $p_1 = 0\,\mathrm{Pa}$, $p_2 = 100\,\mathrm{Pa}$, $p_3 = 70\,\mathrm{Pa}$ or $p_1 = 1000\,\mathrm{Pa}$, $p_2 = 1100\,\mathrm{Pa}$, $p_3 = 1070\,\mathrm{Pa}$ or any other pressure level.

If we go back to the incomplete equation system problem, we just need to provide the network with additional information by adding another constraint, i.e. by specifying a nodal pressure somewhere. As we've seen above in the elimination of pressures example, *we must not specify a mass flux* as additional constraint, since the mass flux itself is already sufficiently determined.

Specification of a pressure can be done easiest by adding another equation, for example:

$$
p_1 = p_{ref}
$$

With that additional constraint the equation system (the original 6 equations) can be solved together.

2.2 Example 2: Splitter and Mixer

Consider the adjusted example in Figure 3. Here, the nodes 2 and 3 are connected by two parallel pipes with potentially different diameters, length and friction coefficients and thus effictively different $\Delta p\,(\dot{m})$ relations.

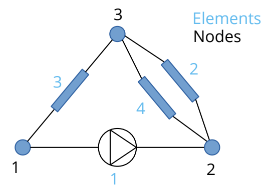

Figure 3: Example 2: Flow network with a mixer and splitter

Following the procedure from example 1, we can formulate again our 4 flow element equations. Also, we have the nodal equations, where node 2 is now a *splitter node* and node 3 becomes a *mixer node* (if we assume mass flux in the direction as indicated by the pump element 1).

$$
\begin{aligned}
0 &= n_1\left(\dot{m}_1, \dot{m}_3\right) = \dot{m}_3 - \dot{m}_1 \\
0 &= n_2\left(\dot{m}_1, \dot{m}_2, \dot{m}_4\right) = \dot{m}_1 - \dot{m}_2 - \dot{m}_4 \\
0 &= n_3\left(\dot{m}_2, \dot{m}_3, \dot{m}_4\right) = \dot{m}_2 + \dot{m}_4 - \dot{m}_3
\end{aligned}
$$

In terms of solving the problem there is nothing different to example 1.

3 Trivial and Non-Existing Solutions

Consider the problem of having a pipe instead of the pump as flow element 1, or when the pump is turned off. In the latter case the pump will then become a simple friction-type flow element just like a heat exchanger or pipe. Then, there is exactly one trivial solution to the problem: zero mass flux. The pressures in the system are all the same, fixed to the given pressure at node 1. This problem is solvable.

For systems with active pumps there can only be a solution if there is at least one flow element in the network that has a positive and *monotonically increasing* $\Delta p\,(\dot{m})$ system equation (i.e. a pipe or other element with flow resistance that increases with flow rate). Otherwise the problem *cannot be solved.*

4 Generic Solution and Equation System Setup

Manual elimination of variables to obtain a reduced set of solution variables is normally not an option. However, we could just formulate the nodal mass balances, insert expressions for the mass fluxes and hereby replace mass fluxes with functions of nodal pressures and simply solve for the pressures. In the example 1 this would lead to the following equations:

$$
\boldsymbol{G}\left(\boldsymbol{p}\right) = 0 = \begin{cases} 0 &= \dot{m}_3\left(p_1, p_3\right) - \dot{m}_1\left(p_1, p_2\right) \quad + p_1 - p_{ref} \\ 0 &= \dot{m}_1\left(p_1, p_2\right) - \dot{m}_2\left(p_2, p_3\right) \\ 0 &= \dot{m}_2\left(p_2, p_3\right) - \dot{m}_3\left(p_1, p_3\right) \end{cases}
$$

Note that we simply added the additional constraint with the pressure at node 1 to the first equation. The equation system for example 1 has six unknowns, three nodal pressures and three mass fluxes, summarized in a vector of solution variables $y = \{p_1, p_2, p_3, \dot{m}_1, \dot{m}_2, \dot{m}_3\}$.

The equation system is complemented by the flow element system equations, yielding 6 equations:

$$
\boldsymbol{G}\left(\boldsymbol{y}\right) = 0 = \begin{cases} 0 &= \dot{m}_3\left(p_1, p_3\right) - \dot{m}_1\left(p_1, p_2\right) \quad + p_1 - p_{ref} \\ 0 &= \dot{m}_1\left(p_1, p_2\right) - \dot{m}_2\left(p_2, p_3\right) \\ 0 &= \dot{m}_2\left(p_2, p_3\right) - \dot{m}_3\left(p_1, p_3\right) \\ 0 &= f_1\left(\dot{m}_1, p_1, p_2\right) \\ 0 &= f_2\left(\dot{m}_2, p_2, p_3\right) \\ 0 &= f_3\left(\dot{m}_3, p_3, p_1\right) \end{cases}
$$

In general, we need to differentiate between node counter *node* and element counter i. Each element is connected to inlet node $node(in,i)$ and outlet node $node(out,i)$. With this assumptions, the system may be formulated as follows

$$\boldsymbol{G}(\boldsymbol{y}) = 0 = \begin{cases} & \vdots \\ 0 &= \sum_{inlet} \dot{m}_i \left(p_{node}, p_{node(out,i)}\right) - \sum_{outlet} \dot{m}_i \left(p_{node(in,i)}, p_{node}\right) \quad + p_{node} - p_{ref} \\ & \vdots \\ 0 &= f_i \left(\dot{m}_i, p_{node(in,i)}, p_{node(out,i)}\right) \\ & \vdots \end{cases} \tag{3}$$

with

- $\dot{m}_i$ mass flux through element i [kg/s]
- p_{node} nodal pressure [Pa]
- $p_{node(in,i)}$ nodal pressure at inlet of element i [Pa]
- $p_{node(out,i)}$ nodal pressure at outlet of element i [Pa]

This non-linear equation system can be solved efficiently with a Newton-Raphson method. This requires the evaluation of the system function $G(y)$. Since nodal pressures and mass fluxes are given, these system functions for the individual flow elements f_i can be easily evaluated.

4.1 Newton Method Convergence and Requirements on Flow Element System Functions

The Newton method may fail to converge if the system functions include discontinuities or clipping. While this cannot be avoided in all cases, care has to be taken in formulating the system functions.

5 Thermal network equations

Thermal equations calculate specific enthalpies h_i and temperatures T_i at all network nodes i as well as heat losses $\dot{Q}_i$ through all elements i. Note, that conservation equations are transient. For this purpose, we additionally take into consideration internal energy states of the fluid inside a fluid element as enthalpy H_j and temperature T_j. An element may either be associated with a single volume or a selection of discretization volumes j, whereby an energy balance is solved for each of the discretization volumes.

5.1 Nodes

In general, each node is inlet and outlet node for minimim one element in,i and out,i. Conservation equation covers corresponding enthalpy fluxes:

$$\sum_i \dot{H}_{in,i} - \sum_i \dot{H}_{out,i} = 0$$

with

- $\dot{H}_{in,i}$ enthalpy flux into element i [W]
- $\dot{H}_{out,i}$ enthalpy flux out of element i [W]

In this context, enthalpy fluxes are defined in the direction of mass fluxes (negative for inlet nodes and positive for outlet nodes).

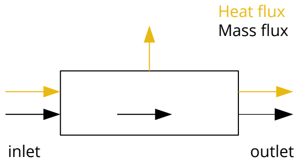

Figure 4: Schematics of an abstract flow element and sign definition of fluxes across model boundary. Note: the labels 'inlet' and 'outlet' are fixed to the installation direction of the element and are independent of the actual flow direction. Hence, for negative mass flows the fluid leaves the element at 'inlet'.

We first formulate all fluxes with positive direction into the node ($\dot{m}_i \geq 0$):

$$\begin{aligned}\dot{H}_{in,i}|_{\dot{m}_i\geq 0} &= \dot{m}_i h_{node} &&= \dot{m}_i c_p T_{node}\\ \dot{H}_{out,i}|_{\dot{m}_i\geq 0} &= \dot{m}_i h_i &&= \dot{m}_i c_p T_i.\end{aligned}$$

with

- $\dot{m}_i$ mass flux through element i [kg/s]
- h_i fluid specific enthalpy of element i [J/kg]
- h_{node} nodal specific enthalpy = specific enthalpy at inlet of element i [J/kg]
- T_i fluid temperature of element i [K]
- T_{node} nodal temperature = temperature at inlet of element i [K]
- c_p specific heat capacity (at constant pressure) [J/kgK]

Note, that we use upwinding to ensure numerical stability of the solution procedure. In other words, the specific enthalpy and temperature of the outflowing fluid is set equal to element's fluid specifc enthalpy h_i and temperature T_i, a method called *upwinding*. Consequently, for all inflows (from node into elements) the fluid temperature and specific enthalpy are set equal to nodal values $h_{in,i}$ and $T_{in,i}$.

1. Fluxes with negative direction are formulated analogue:

$$\begin{aligned}\dot{H}_{in,i}|_{\dot{m}_i<0} &= \dot{m}_i h_i &&= \dot{m}_i c_p T_i\\ \dot{H}_{out,i}|_{\dot{m}_i<0} &= \dot{m}_i h_{node} &&= \dot{m}_i c_p T_{node}\end{aligned}$$

- h_{node} nodal specific enthalpy = specific enthalpy at outlet of element i [J/kg]
- T_{node} nodal temperature = temperature at outlet of element i [K]

As a result, nodal temperature and enthalpy values at the inlet and outlet of all connected elements are evaluated using the following equations:

$$h_{node} = \frac{\sum_i \dot{m}_i c_p T_i|_{\dot{m}_i\geq 0} - \sum_i \dot{m}_i c_p T_i|_{\dot{m}_i<0}}{-\sum_i \dot{m}_i|_{\dot{m}_i<0} + \sum_i \dot{m}_i|_{\dot{m}_i\geq 0}} \tag{4}$$

$$T_{node} = \frac{h_{node}}{c_P}. \tag{5}$$

5.2 Elements

For each network element the following transient equation is fulfilled:

$$\rho V c_P \frac{\partial T_i}{\partial t} = \dot{H}_{in,i} - \dot{H}_{out,i} - \dot{Q}_{loss,i} \tag{6}$$

with

- T temperature of the fluid volume [K]
- c_p specific heat capacity (at constant pressure) [J/kgK]

- ρ fluid density
- $\dot{H}$ enthalpy flux at inlet node ($\dot{H}_{in}$) and outlet node ($\dot{H}_{out}$) of the element [W]
- $\dot{Q}_{loss}$ heat loss of the fluid along the element [W], either given or computed based on fluid elements' temperature (e.g. heat exchange with a construction or zone or ...)
- V pipe internal volume [m³]

Note: Mass and energy storage capacity of the element itself (i.e. pipe walls/body of heat exchangers/coils) is ignored. This might be quite an error source and may be necessary to consider when heating/cooling dynamics of the network system itself are of importance.

In the special case of an adiabatic element (adiabatic with respect to heat exchange towards surroundings) the equation reduces to

$$\rho V c_p \frac{\partial T_i}{\partial t} = \dot{H}_{in,i} - \dot{H}_{out,i} \tag{7}$$

Equations (6) and 7 are the energy balances across the flow element, which typically involves the energy quantities/enthalpies transported with the mass flow across the inlet and outlet. For positive mass fluxes ($\dot{m}_i \geq 0$) we use the following equation:

$$\begin{aligned} \dot{H}_{in,i} &= \dot{m}_i c_p T_{node(in,i)} \\ \dot{H}_{out,i} &= \dot{m}_i c_p T_i. \end{aligned} \tag{8}$$

with

- T temperature of element volume [K]
- $T_{node(in,i)}$ nodal temperature at elements inlet [K]

For negative mass fluxes ($\dot{m}_i < 0$) we obtain:

$$\begin{aligned} \dot{H}_{in,i} &= \dot{m}_i c_p T_i \\ \dot{H}_{out,i} &= \dot{m}_i c_p T_{node(out,i)}. \end{aligned} \tag{9}$$

with

- $T_{node(out,i)}$ nodal temperature at elements outlet [K]

In the case of a discretized element we formulate the same balance equation for each partial volume k:

$$\rho V_k c_P \frac{\partial T_k}{\partial t} = \dot{m}_i c_P \left(T_{k-1} - T_k\right) - \dot{Q}_{loss,k} \tag{10}$$

with the boundary conditions

$$\begin{aligned} T_0 = T_{node(in,i)} &\quad \text{for} \quad \dot{m}_i \geq 0 \\ T_{n-1} = T_{node(out,i)} &\quad \text{for} \quad \dot{m}_i < 0 \end{aligned}$$

with

- T_k temperature of discretization element [K]
- $\dot{Q}_{loss,k}$ heat loss of the fluid along the discretization element [W]
- V_k volume of discretization element [m³]
- n number of discretization elements

The special case of an adiabatic element leads to the modified equation

$$\rho V_k c_P \frac{\partial T_k}{\partial t} = \dot{m}_i c_P \left(T_{k-1} - T_k\right) \tag{11}$$

5.3 Solution of the thermal network equations

Solution of the transient equations is done using numerical integration. Classical transient methods solve problems of the following type:

$$\dot{\boldsymbol{y}} = \boldsymbol{f}\left(\mathbf{y}, t\right). \tag{12}$$

The vector $\mathbf{y}$ selects the internal states of all hydraulic elements, the function $\mathbf{f}$ the sum of heat and enthalpy fluxes inside all elements. For a single volume the vector entries are formulated as follows:

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} \vdots \\ y_i \\ \vdots \end{pmatrix} \qquad \text{with } y_i = \rho V_i c_p T_i, \\ \boldsymbol{f}\left(\mathbf{y}, t\right) &= \begin{pmatrix} \vdots \\ f_i(y_i, t) \\ \vdots \end{pmatrix} \quad . \end{aligned} \tag{13}$$

A discretized element (i.e.pipe) generates n vector entries.

$$\begin{aligned} \mathbf{y} &= \begin{pmatrix} \vdots \\ \mathbf{y}_i \\ \vdots \end{pmatrix} \qquad \text{with } \mathbf{y}_i = \rho c_P \begin{pmatrix} V_0 T_0 \\ \vdots \\ V_k T_k \\ \vdots \\ V_{n-1} T_{n-1} \end{pmatrix}, \\ \boldsymbol{f}\left(\mathbf{y}, t\right) &= \begin{pmatrix} \vdots \\ \mathbf{f}_i(\mathbf{y}_i, t) \\ \vdots \end{pmatrix} \quad . \end{aligned} \tag{14}$$

6 Thermo-Hydraulic Network Solution Algorithm

The numerical integration algorithm requires the implementation of the system function $\boldsymbol{f}\left(\mathbf{y}, t\right)$. Within the function the following algorithm is used:

1. The integrator suggests a solution in vector $\mathbf{y}$.
2. The internal specific enthalpies and temperatures T_i are calculated in all flow elements using 15. For single equations, the element temperature T_i is computed from the respective energy density y_i. For elements with discretized fluid volumes, each fluid volume's temperature is computed from the respective y component.
$$T_i = \frac{y_i}{\rho V c_P} \tag{15}$$
3. Solve hydraulic network equation system and compute mass fluxes $\dot{m}_i$ and pressures p_i in the network with 3 using a Newton-Raphson iteration.
4. Update of thermal network node states due to equations 4 and 5. Afterwards, all nodal inlet and outlet temperatures are available. We locally store $\dot{m} = \dot{m}_i$, $T_{in} = T_{node(in,i)}$, $T_{out} = T_{node(out,i)}$ in each flow element.
5. Calculate current individual heat loss $\dot{Q}_{loss} \widehat{=} \dot{Q}_{loss,i}$ along the element or its discretization volumes. This heat loss calculation may depend on mass flow rates, for example when heat transfer coefficients from fluid to surface depend on Reynolds/Prandtl-numbers.
Note: results of this calculation must not be used to formulate additional resistances (valves) or other terms that impact the flow element system equations!

6. Now the time derivatives of all flow element energy balances are computed and the vector $\dot{\mathbf{y}}$ is composed using equations 16 and 17:

$$\dot{y}_i = \begin{cases} \dot{m}\left(c_p T_{in} - c_p T\right) - \dot{Q}_{loss} & \text{for } \dot{m} \geq 0 \\ \dot{m}\left(c_p T - c_P T_{out}\right) - \dot{Q}_{loss} & \text{for } \dot{m} < 0 \end{cases} \tag{16}$$

$$\dot{\mathbf{y}}_i = \begin{cases} \begin{pmatrix} \dot{m}c_p\left(T_{in} - T_0\right) - \dot{Q}_{loss,0} \\ \vdots \\ \dot{m}c_p - \dot{Q}_{loss,k} \\ \vdots \\ \dot{m}c_p\left(T_{n-2} - T_{n-1}\right) - \dot{Q}_{loss,n-1} \end{pmatrix} & \text{for } \dot{m} \geq 0 \\ \begin{pmatrix} \dot{m}c_p\left(T_0 - T_1\right) - \dot{Q}_{loss,0} \\ \vdots \\ \dot{m}c_p\left(T_k - T_{k+1}\right) - \dot{Q}_{loss,k} \\ \vdots \\ \dot{m}c_p\left(T_{n-1} - T_{out}\right) - \dot{Q}_{loss,n-1} \end{pmatrix} & \text{for } \dot{m} < 0 \end{cases} \tag{17}$$