7.

B*. Examine each one of the following propositions to determine whether or not its true; indicate your choice by circling either "T" or "F" then prove that the choice is correct:

i. $f : \mathbb{R} \to \mathbb{R} \ni f(x) = x^2 - 1 \Rightarrow f : \mathbb{R} \xrightarrow{1:1} \mathbb{R}$

T F

---

Sample proof:

Given that $f : \mathbb{R} \to \mathbb{R} \ni f(x) = x^2 - 1$, we need to demonstrate that $f$ is not an injection. Our definition of injection (#013-C in our glossary) is as follows:

$$\text{Given } A, B \in \{ \text{ sets } \}, \left( f : A \xrightarrow{1:1} B \Leftrightarrow f : A \to B \ni ( (x_1, y_1), (x_2, y_1) \in f \Rightarrow x_1 = x_2 ) \right)$$

Consider two *different* domain elements ${}^{-}3$ and 3:

$$(( f({}^{-}3) = ({}^{-}3)^2 - 1 = 8 ) \wedge ( f(3) = 3^2 - 1 = 8 )) \Rightarrow ({}^{-}3, 8), (3, 8) \in f$$

$\therefore f$ is not an injection. *Q.E.D.* ( *quod erat demonstrandum* )

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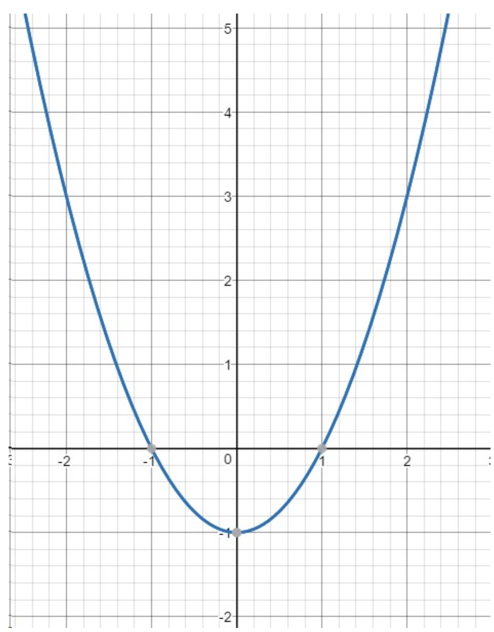
An aside: Check out the graph of $f$.

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ii. $f : \mathbb{R} \to \mathbb{R} \ni f(x) = x^2 - 1 \Rightarrow f : \mathbb{R} \xrightarrow{onto} \mathbb{R}$

T F

---

Sample proof:

Given that $f : \mathbb{R} \to \mathbb{R} \ni f(x) = x^2 - 1$, we need to demonstrate that $f$ is not a surjection. Our definition of surjection (#013-D in our glossary) is as follows:

Given $A, B \in \{ \text{sets} \}$, ( $f : A \xrightarrow{onto} B \Leftrightarrow f : A \to B \ni$ the *range* of $f = B$)

So we need to display an element from the codomain that is not also an element of the range. Consider the codomain element $^-3$:

Suppose $\exists\, x \in \mathbb{R} \ni f(x) = {}^-3$, then we have the following string of deductions:

$$f(x) = {}^-3 \Rightarrow x^2 - 1 = {}^-3 \Rightarrow x^2 = {}^-2 \Rightarrow x \in \{ {}^-\sqrt{2}\, i, \sqrt{2}\, i, \} \Rightarrow x \notin \mathbb{R}$$

Thus, $\nexists\, x \in \mathbb{R} \ni f(x) = {}^-3$

$\therefore f$ is not a surjection. *Q.E.D.*

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iii. $g : \mathbb{R} \to \mathbb{R} \ni g(x) = \sqrt[3]{x} \Rightarrow g : \mathbb{R} \xrightarrow{1:1} \mathbb{R}$

T F

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Sample proof:

Given that $g : \mathbb{R} \to \mathbb{R} \ni g(x) = \sqrt[3]{x}$, we need to demonstrate that $f$ is an injection. Our definition of injection (#013-C in our glossary) is as follows:

Given $A, B \in$ { sets }, ( $f : A \xrightarrow{1:1} B \Leftrightarrow f : A \to B \ni ( (x_1, y_1), (x_2, y_1) \in f \Rightarrow x_1 = x_2 )$)

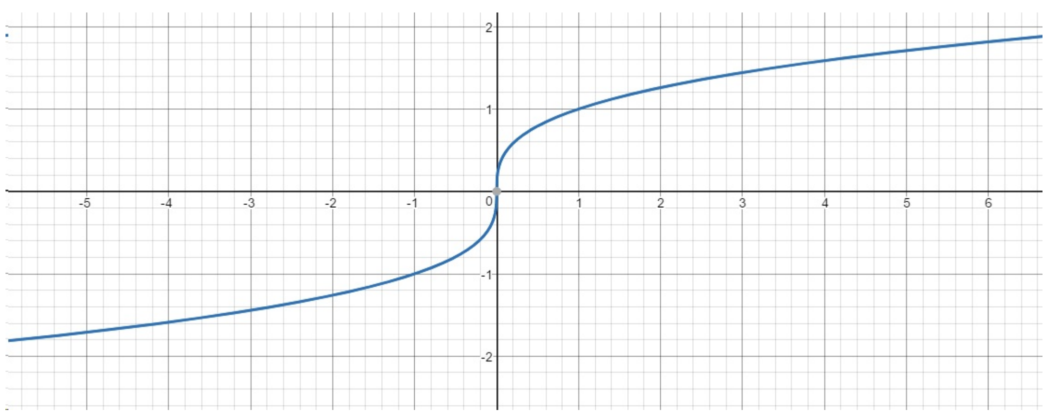
So we need to demonstrate that $( g(x_1) = y_1 \wedge g(x_2) = y_1) \Rightarrow x_1 = x_2$. Here we go:

$$( g(x_1) = y_1 \wedge g(x_2) = y_1) \Rightarrow ( \sqrt[3]{x_1} = y_1 \wedge \sqrt[3]{x_2} = y_1 ) \Rightarrow ( x_1 = y_1^3 \wedge x_2 = y_1^3 ) \Rightarrow x_1 = x_2$$ ☺

$\therefore$ $g$ is an injection. *Q.E.D.*

---

An aside: Check out the graph of $g$.

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iv. $g : \mathbb{R} \to \mathbb{R} \ni g(x) = \sqrt[3]{x} \Rightarrow g : \mathbb{R} \xrightarrow{onto} \mathbb{R}$

T F

---

A sample proof:

Given $g : \mathbb{R} \to \mathbb{R} \ni g(x) = \sqrt[3]{x}$, we need to demonstrate that $g$ is surjection. Our definition of surjection (#013-D in our glossary) is as follows:

> Given $A, B \in$ { sets }, ( $f : A \xrightarrow{onto} B \Leftrightarrow f : A \to B \ni$ the *range* of $f = B$)

$g$ is a surjection because the codomain is the range. Here is a proof of that fact:

> Suppose $y \in \mathbb{R}$ (i.e., an arbitrary element in the codomain), we need to demonstrate that $\exists\, x \in \mathbb{R}$ (i.e., an element in the domain ) $\ni g(x) = y$. Consider $x = y^3$. Note that since $y \in \mathbb{R}$, we know that $y^3 \in \mathbb{R}$. Also $g(y^3) = \sqrt[3]{y^3} = y$. That bit of deduction makes me smile. *Q.E.D.*

Although revealing the scratch work behind my choice of $y^3$ for $x$ in the proof above is not a necessary part of the proof itself, I want to show it to you anyway:

> $g(x) = y \Rightarrow \sqrt[3]{x^3} = y \Rightarrow x = y^3$

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v. $h:\mathbb{R}\to\mathbb{R} \ni h(x)=\sqrt{x} \Rightarrow h:\mathbb{R}\xrightarrow{1:1}\mathbb{R}$

T F

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Sample proof:

This is just weird. In the first place the hypothesis of the proposition ( i.e., $h:\mathbb{R}\to\mathbb{R} \ni h(x)=\sqrt{x}$ ) is false because since negative real numbers are in the domain but the principal square root of a negative number is not a real number. But let's check out a couple of rows from our truth table from Line 002-H of our glossary:

| $p$ | $q$ | $\overline{p}$ | $p \vee q$ | $p \wedge q$ | $p \veebar q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $\overline{q} \Rightarrow \overline{p}$ | $p \Leftrightarrow q$ |
|---|---|---|---|---|---|---|---|---|---|
| T | T | F | T | T | F | T | T | T | T |
| T | F | F | T | F | T | F | T | F | F |
| F | T | T | T | F | T | T | F | T | F |
| F | F | T | F | F | F | T | T | T | T |

So the proposition ( $h:\mathbb{R}\to\mathbb{R} \ni h(x)=\sqrt{x} \Rightarrow h:\mathbb{R}\xrightarrow{1:1}\mathbb{R}$ ) is vacuously true. *Q.E. D.*

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vi. $(s \subseteq (\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q} \ni s = \{ ((x, y),\ x \cdot y) : x, y \in \mathbb{Q} \}) \Rightarrow s : \mathbb{Q} \times \mathbb{Q} \xrightarrow{1:1} \mathbb{Q}$

T F

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Just a note: In a subsequent stage of Math 4200, we will refer to functions such as $s$ in this proposition as a "binary operation on a set." In this case, the set is $\mathbb{Q}$ and the binary operation is ordinary multiplication. Okay, on to the business of proving this proposition false.

A sample proof:

> It is true that $s : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, but it is not an injection. Note that $(3, 2), (12,\ 0.5) \in \mathbb{Q} \times \mathbb{Q}$. Furthermore, $3 \cdot 2 = 6 \wedge 12 \cdot 0.5 = 6$. Thus, we have $s((3, 2)) = s((12,\ 0.5))$ although $(3, 3) \neq (12,\ 0.5)$.

$\therefore$ $s$ is not an injection. *Q.E.D.*

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