C*. Examine each of the following propositions to determine whether or not it is true; indicate your determination in the usual way and then prove that the determination is correct (Please post the resulting PDF using the appropriate Canvas Assignment link):

i. $p(A^{c}) = 1 - p(A)$

T F

Sample proof:

Call to mind our definitions of *probability measure* from Glossary entry 030 and *complement of an event* from Glossary entry 029-F:

Given Ω is a sample space $\wedge E = \{$ events of $\Omega \}$,
$(p \in \{$ probability measures on $\Omega \} \Leftrightarrow$
$p : E \to [0, 1] \ni (p(\Omega) = 1 \wedge (A_1 \subseteq E \wedge A_2 \subseteq E \wedge A_1 \cap A_2 = \varnothing) \Rightarrow$
$p(A_1 \cup A_2)) = p(A_1) + p(A_2))$

$A^{c} = \{ X : X \subseteq \Omega \wedge X \cap A = \varnothing \}$

Since $(p(\Omega) = 1 \wedge A^{c} \cup A = \Omega \wedge p(A^{c} \cup A)) = p(A^{c}) + p(A))$, we have
$p(A^{c} \cup A)) = p(A^{c}) + p(A) \Rightarrow 1 = p(A^{c}) + p(A) \Rightarrow 1 - p(A) = p(A^{c})$

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ii. $p(\varnothing) = 0$

T F

Sample proof:

Keep in mind that since $\varnothing$ is a subset of any set, $\varnothing \subseteq \Omega$ and is thus an event. So $p(\varnothing) \in [0, 1]$. From the definition of complement of an event, we have $A^{c} \cap A = \varnothing \Rightarrow$
$| A^{c} \cap A | = 0 \Rightarrow p(\varnothing) = 0$.

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iii. $A \subseteq B \Rightarrow p(A) \leq p(B)$

T F ?

Sample proof with a caveat:

I only tried to design a proof for the case in which $\Omega \in$ { finite sets } $\cup$ { countable sets}. I'm not sure that it is true for $\Omega \in$ { uncountably infinite sets }. And here is my attempt:

$$(p(A) = | A | \div | \Omega | \wedge | B | \div | \Omega | \wedge | A |, | B | \in \omega) \wedge (A \subseteq B \Rightarrow | A | \leq | B |))$$
$$\Rightarrow (| A | \div | \Omega | \leq | B | \div | \Omega |) \Rightarrow p(A) \leq p(B)$$

Note: I think I proved that the proposition in question is true for a finite or countable sample space; however, I didn't expatiate my reasons for the above string of deductions very well. So I'm walking away from this attempt with an uncomfortable feeling.

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iv. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

T F

Sample proof:

We'll employ our definition for probability measure:

Given Ω is a sample space $\wedge E =$ { events of Ω },
($p \in$ { probability measures on Ω } $\Leftrightarrow$
$p : E \rightarrow [0, 1] \ni (p(\Omega) = 1 \wedge (A_1 \subseteq E \wedge A_2 \subseteq E \wedge A_1 \cap A_2 = \varnothing) \Rightarrow$
$p(A_1 \cup A_2)) = p(A_1) + p(A_2))$

I rewrote the proposition in question so that the symbols matched the symbols in the definition just to make the connection between the proposition and the definition easier to see. I supplanted "A" with "A_1" and "B" with "A_2" .

So we need to prove that $p(A_1 \cup A_2) = p(A_1) + p(A_2) - p(A_1 \cap A_2)$:

The definition stipulates that $(A_1 \subseteq A \wedge A_2 \subseteq A \wedge A_1 \cap A_2 = \varnothing) \Rightarrow$
$p(A_1 \cup A_2)) = p(A_1) + p(A_2))$. So for the case that $A_1 \cap A_2 = \varnothing$, we're done since $p(A_1 \cap A_2) = 0$.

But since the proposition in question doesn't stimulate whether or not $p(A_1 \cap A_2) =$ 0, we must deal with a second case (i.e., $p(A_1 \cap A_2) \neq 0$). So for this second case the computation of $p(A) + p(B)$ is greater than $p(A_1 \cup A_2)$ because $| A_1 | + | A_2 | >$ $| A_1 \cup A_2 |$. This is true because elements in $A_1 \cap A_2$ are double-counted. Thus, $p(A_1 \cup A_2) = p(A_1) + p(A_2) - p(A_1 \cap A_2)$ (i.e., the probability that the event A_1 or event A_2 randomly occurs is the probability of **either** A_1 **or** A_2 randomly occurring.

Sidebar note: It would be okay to revisit Lines 002-D and 002-H from our Glossary.