2.1.1 (a) Since

$0.08 + 0.11 + 0.27 + 0.33 + P(X = 4) = 1$

it follows that

$P(X = 4) = 0.21.$

(c) $F(0) = 0.08$

$F(1) = 0.19$

$F(2) = 0.46$

$F(3) = 0.79$

$F(4) = 1.00$

2.1.2

x_i	-4	-1	0	2	3	7
p_i	0.21	0.11	0.07	0.29	0.13	0.19

2.1.3

x_i	1	2	3	4	5	6	8	9	10
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$
$F(x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{8}{36}$	$\frac{10}{36}$	$\frac{14}{36}$	$\frac{16}{36}$	$\frac{17}{36}$	$\frac{19}{36}$

x_i	12	15	16	18	20	24	25	30	36
p_i	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$F(x_i)$	$\frac{23}{36}$	$\frac{25}{36}$	$\frac{26}{36}$	$\frac{28}{36}$	$\frac{30}{36}$	$\frac{32}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

2.1.4 (a)

x_i	0	1	2
p_i	0.5625	0.3750	0.0625

(b)

x_i	0	1	2
$F(x_i)$	0.5625	0.9375	1.000

(c) The value $x = 0$ is the most likely.

Without replacement:

x_i	0	1	2
p_i	0.5588	0.3824	0.0588
$F(x_i)$	0.5588	0.9412	1.000

Again, $x = 0$ is the most likely value.

2.1.5

x_i	-5	-4	-3	-2	-1	0	1	2	3	4	6	8	10	12
p_i	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{3}{36}$
$F(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{9}{36}$	$\frac{12}{36}$	$\frac{14}{36}$	$\frac{19}{36}$	$\frac{20}{36}$	$\frac{24}{36}$	$\frac{27}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	1

2.1.6 (a)

x_i	-6	-4	-2	0	2	4	6
p_i	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

(b)

x_i	-6	-4	-2	0	2	4	6
$F(x_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1

(c) The most likely value is $x = 0$.

2.1.7 (a)

x_i	0	1	2	3	4	6	8	12
p_i	0.061	0.013	0.195	0.067	0.298	0.124	0.102	0.140

(b)

x_i	0	1	2	3	4	6	8	12
$F(x_i)$	0.061	0.074	0.269	0.336	0.634	0.758	0.860	1.000

(c) The most likely value is 4.
P(not shipped) = $P(X \leq 1) = 0.074$

2.1.8

x_i	-1	0	1	3	4	5
p_i	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

2.1.9

x_i	1	2	3	4
p_i	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
$F(x_i)$	$\frac{2}{5}$	$\frac{7}{10}$	$\frac{9}{10}$	1

2.1.10 Since
$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$
it follows that
$P(X = i) = \frac{6}{\pi^2 i^2}$
is a possible set of probability values.

However, since
$\sum_{i=1}^{\infty} \frac{1}{i}$
does not converge, it follows that
$P(X = i) = \frac{c}{i}$
is not a possible set of probability values.

2.1.11 (a) The state space is $\{3, 4, 5, 6\}$.

(b) $P(X = 3) = P(MMM) = \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} = \frac{1}{20}$

$P(X = 4) = P(MMTM) + P(MTMM) + P(TMMM) = \frac{3}{20}$

$P(X = 5) = P(MMTTM) + P(MTMTM) + P(TMMTM)$

$+ P(MTTMM) + P(TMTMM) + P(TTMMM) = \frac{6}{20}$

Finally,
$P(X = 6) = \frac{1}{2}$
since the probabilities sum to one, or since the final appointment made is equally likely to be on a Monday or on a Tuesday.

$P(X \leq 3) = \frac{1}{20}$
$P(X \leq 4) = \frac{4}{20}$
$P(X \leq 5) = \frac{10}{20}$
$P(X \leq 6) = 1$

2.2.1 (a) Continuous

(b) Discrete

(c) Continuous

(d) Continuous

(e) Discrete

(f) This depends on what level of accuracy to which it is measured.
It could be considered to be either discrete or continuous.

2.2.2 (b) $\int_4^6 \frac{1}{x\ln(1.5)}\, dx = \frac{1}{\ln(1.5)} \times [\ln(x)]_4^6$

$= \frac{1}{\ln(1.5)} \times (\ln(6) - \ln(4)) = 1.0$

(c) $P(4.5 \leq X \leq 5.5) = \int_{4.5}^{5.5} \frac{1}{x\ln(1.5)}\, dx$

$= \frac{1}{\ln(1.5)} \times [\ln(x)]_{4.5}^{5.5}$

$= \frac{1}{\ln(1.5)} \times (\ln(5.5) - \ln(4.5)) = 0.495$

(d) $F(x) = \int_4^x \frac{1}{y\ln(1.5)}\, dy$

$= \frac{1}{\ln(1.5)} \times [\ln(y)]_4^x$

$= \frac{1}{\ln(1.5)} \times (\ln(x) - \ln(4))$

for $4 \leq x \leq 6$

2.2.3 (a) Since

$\int_{-2}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx = \frac{7}{16}$

and

$\int_{0}^{3} \left(\frac{3}{8} + cx\right) dx = \frac{9}{8} + \frac{9c}{2}$

it follows that

$\frac{7}{16} + \frac{9}{8} + \frac{9c}{2} = 1$

which gives $c = -\frac{1}{8}$.

(b) $P(-1 \leq X \leq 1) = \int_{-1}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{1} \left(\frac{3}{8} - \frac{x}{8}\right) dx$

$= \frac{69}{128}$

(c) $F(x) = \int_{-2}^{x} \left(\frac{15}{64} + \frac{y}{64}\right) dy$

$= \frac{x^2}{128} + \frac{15x}{64} + \frac{7}{16}$

for $-2 \leq x \leq 0$

$F(x) = \frac{7}{16} + \int_{0}^{x} \left(\frac{3}{8} - \frac{y}{8}\right) dy$

$= -\frac{x^2}{16} + \frac{3x}{8} + \frac{7}{16}$

for $0 \leq x \leq 3$

2.2.4 (b) $P(X \leq 2) = F(2) = \frac{1}{4}$

(c) $P(1 \leq X \leq 3) = F(3) - F(1)$

$= \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$

(d) $f(x) = \frac{dF(x)}{dx} = \frac{x}{8}$

for $0 \leq x \leq 4$

2.2.5 (a) Since $F(\infty) = 1$ it follows that $A = 1$.
Then $F(0) = 0$ gives $1 + B = 0$ so that $B = -1$ and
$F(x) = 1 - e^{-x}$.

(b) $P(2 \leq X \leq 3) = F(3) - F(2)$
$= e^{-2} - e^{-3} = 0.0855$

(c) $f(x) = \frac{dF(x)}{dx} = e^{-x}$

for $x \geq 0$

2.2.6 (a) Since

$\int_{0.125}^{0.5} A\ (0.5 - (x - 0.25)^2)\ dx = 1$

it follows that $A = 5.5054$.

(b) $F(x) = \int_{0.125}^{x} f(y)\ dy$

$= 5.5054 \left(\frac{x}{2} - \frac{(x-0.25)^3}{3} - 0.06315 \right)$

for $0.125 \leq x \leq 0.5$

(c) $F(0.2) = 0.203$

2.2.7 (a) Since

$F(0) = A + B \ln(2) = 0$

and

$F(10) = A + B \ln(32) = 1$

it follows that $A = -0.25$ and $B = \frac{1}{\ln(16)} = 0.361$.

(b) $P(X > 2) = 1 - F(2) = 0.5$

(c) $f(x) = \frac{dF(x)}{dx} = \frac{1.08}{3x+2}$

for $0 \leq x \leq 10$

2.2.8 (a) Since

$\int_0^{10} A\ (e^{10-\theta} - 1)\ d\theta = 1$

it follows that

$A = (e^{10} - 11)^{-1} = 4.54 \times 10^{-5}$.

(b) $F(\theta) = \int_0^{\theta} f(y)\ dy$

$= \frac{e^{10} - \theta - e^{10-\theta}}{e^{10} - 11}$

for $0 \leq \theta \leq 10$

(c) $1 - F(8) = 0.0002$

2.2.9 (a) Since $F(0) = 0$ and $F(50) = 1$
it follows that $A = 1.0007$ and $B = -125.09$.

(b) $P(X \leq 10) = F(10) = 0.964$

(c) $P(X \geq 30) = 1 - F(30) = 1 - 0.998 = 0.002$

(d) $f(r) = \frac{dF(r)}{dr} = \frac{375.3}{(r+5)^4}$

for $0 \leq r \leq 50$

2.2.10 (a) $F(200) = 0.1$

(b) $F(700) - F(400) = 0.65$

2.2.11 (a) Since

$\int_{10}^{11} Ax(130-x^2)\,dx = 1$

it follows that

$A = \frac{4}{819}$.

(b) $F(x) = \int_{10}^{x} \frac{4y(130-y^2)}{819}\,dy$

$= \frac{4}{819}\left(65x^2 - \frac{x^4}{4} - 4000\right)$

for $10 \leq x \leq 11$

(c) $F(10.5) - F(10.25) = 0.623 - 0.340 = 0.283$

2.3.1 $E(X) = (0 \times 0.08) + (1 \times 0.11) + (2 \times 0.27) + (3 \times 0.33) + (4 \times 0.21)$

$= 2.48$

2.3.2 $E(X) = \left(1 \times \frac{1}{36}\right) + \left(2 \times \frac{2}{36}\right) + \left(3 \times \frac{2}{36}\right) + \left(4 \times \frac{3}{36}\right) + \left(5 \times \frac{2}{36}\right) + \left(6 \times \frac{4}{36}\right)$

$+ \left(8 \times \frac{2}{36}\right) + \left(9 \times \frac{1}{36}\right) + \left(10 \times \frac{2}{36}\right) + \left(12 \times \frac{4}{36}\right) + \left(15 \times \frac{2}{36}\right) + \left(16 \times \frac{1}{36}\right)$

$+ \left(18 \times \frac{2}{36}\right) + \left(20 \times \frac{2}{36}\right) + \left(24 \times \frac{2}{36}\right) + \left(25 \times \frac{1}{36}\right) + \left(30 \times \frac{2}{36}\right) + \left(36 \times \frac{1}{36}\right)$

$= 12.25$

2.3.3 With replacement:

$E(X) = (0 \times 0.5625) + (1 \times 0.3750) + (2 \times 0.0625)$

$= 0.5$

Without replacement:

$E(X) = (0 \times 0.5588) + (1 \times 0.3824) + (2 \times 0.0588)$

$= 0.5$

2.3.4 $E(X) = \left(1 \times \frac{2}{5}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{1}{5}\right) + \left(4 \times \frac{1}{10}\right)$

$= 2$

2.3.5

x_i	2	3	4	5	6	7	8	9	10	15
p_i	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$

$E(X) = \left(2 \times \frac{1}{13}\right) + \left(3 \times \frac{1}{13}\right) + \left(4 \times \frac{1}{13}\right) + \left(5 \times \frac{1}{13}\right) + \left(6 \times \frac{1}{13}\right)$
$+ \left(7 \times \frac{1}{13}\right) + \left(8 \times \frac{1}{13}\right) + \left(9 \times \frac{1}{13}\right) + \left(10 \times \frac{1}{13}\right) + \left(15 \times \frac{4}{13}\right)$
$= \$8.77$

If \$9 is paid to play the game, the expected loss would be 23 cents.

2.3.6

x_i	1	2	3	4	5	6	7	8	9	10	11	12
p_i	$\frac{6}{72}$	$\frac{7}{72}$	$\frac{8}{72}$	$\frac{9}{72}$	$\frac{10}{72}$	$\frac{11}{72}$	$\frac{6}{72}$	$\frac{5}{72}$	$\frac{4}{72}$	$\frac{3}{72}$	$\frac{2}{72}$	$\frac{1}{72}$

$E(X) = \left(1 \times \frac{6}{72}\right) + \left(2 \times \frac{6}{72}\right) + \left(3 \times \frac{6}{72}\right) + \left(4 \times \frac{6}{72}\right) + \left(5 \times \frac{6}{72}\right) + \left(6 \times \frac{6}{72}\right)$
$+ \left(7 \times \frac{6}{72}\right) + \left(8 \times \frac{6}{72}\right) + \left(9 \times \frac{6}{72}\right) + \left(10 \times \frac{6}{72}\right) + \left(11 \times \frac{6}{72}\right) + \left(12 \times \frac{6}{72}\right)$
$= 5.25$

2.3.7 $P(\text{three sixes are rolled}) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$
$= \frac{1}{216}$
so that
$E(\text{net winnings}) = \left(-\$1 \times \frac{215}{216}\right) + \left(\$499 \times \frac{1}{216}\right)$
$= \$1.31.$

If you can play the game a large number of times then you should play the game as often as you can.

2.3.8 The expected net winnings will be negative.

2.3.9

x_i	0	1	2	3	4	5
p_i	0.1680	0.2816	0.2304	0.1664	0.1024	0.0512

$E(\text{payment}) = (0 \times 0.1680) + (1 \times 0.2816) + (2 \times 0.2304)$
$+ (3 \times 0.1664) + (4 \times 0.1024) + (5 \times 0.0512)$
$= 1.9072$

$E(\text{winnings}) = \$2 - \$1.91 = \$0.09$

The expected winnings increase to 9 cents per game.
Increasing the probability of scoring a three reduces the expected value of the difference in the scores of the two dice.

2.3.10 (a) $E(X) = \int_4^6 x \frac{1}{x \ln(1.5)} \, dx = 4.94$

(b) Solving $F(x) = 0.5$ gives $x = 4.90$.

2.3.11 (a) $E(X) = \int_0^4 x \frac{x}{8} \, dx = 2.67$

(b) Solving $F(x) = 0.5$ gives $x = \sqrt{8} = 2.83$.

2.3.12 $E(X) = \int_{0.125}^{0.5} x \, 5.5054 \, (0.5 - (x - 0.25)^2) \, dx = 0.3095$

Solving $F(x) = 0.5$ gives $x = 0.3081$.

2.3.13 $E(X) = \int_0^{10} \frac{\theta}{e^{10}-11} \, (e^{10-\theta} - 1) \, d\theta = 0.9977$

Solving $F(\theta) = 0.5$ gives $\theta = 0.6927$.

2.3.14 $E(X) = \int_0^{50} \frac{375.3\, r}{(r+5)^4} \, dr = 2.44$

Solving $F(r) = 0.5$ gives $r = 1.30$.

2.3.15 Let $f(x)$ be a probability density function that is symmetric about the point μ, so that $f(\mu+x) = f(\mu-x)$.

Then

$E(X) = \int_{-\infty}^{\infty} \; xf(x) \; dx$

which under the transformation $x = \mu + y$ gives

$E(X) = \int_{-\infty}^{\infty} \; (\mu+y)f(\mu+y) \; dy$

$= \mu \; \int_{-\infty}^{\infty} \; f(\mu+y) \; dy \; + \; \int_{0}^{\infty} \; y \; (f(\mu+y) - f(\mu-y)) \; dy$

$= (\mu \times 1) + 0 = \mu.$

2.3.16 $E(X) = (3 \times \frac{1}{20}) + (4 \times \frac{3}{20}) + (5 \times \frac{6}{20}) + (6 \times \frac{10}{20})$

$= \frac{105}{20} = 5.25$

2.3.17 (a) $E(X) = \int_{10}^{11} \frac{4x^2(130-x^2)}{819} \; dx$

$= 10.418234$

(b) Solving $F(x) = 0.5$ gives the median as 10.385.

2.3.18 (a) Since

$\int_2^3 A(x-1.5)dx = 1$

it follows that

$A\left[x^2 - 1.5x\right]_2^3 = 1$

so that $A = 1$.

(b) Let the median be m.
Then

$\int_2^m (x-1.5)dx = 0.5$

so that

$\left[x^2 - 1.5x\right]_2^m = 0.5$

which gives

$0.5m^2 - 1.5m + 1 = 0.5$.

Therefore,

$m^2 - 3m + 1 = 0$

so that

$m = \frac{3\pm\sqrt{5}}{2}$.

Since $2 \le m \le 3$ it follows that $m = \frac{3+\sqrt{5}}{2} = 2.618$.

2.3.19 $(0 \times 0.38) + (1 \times 0.44) + (2 \times 0.15) + (3 \times 0.03) = 0.83$

2.4.1 (a) $E(X) = \left(-2 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{3}\right) + \left(6 \times \frac{1}{6}\right)$

$= \frac{11}{6}$

(b) $\text{Var}(X) = \left(\frac{1}{3} \times \left(-2 - \frac{11}{6}\right)^2\right) + \left(\frac{1}{6} \times \left(1 - \frac{11}{6}\right)^2\right)$

$+ \left(\frac{1}{3} \times \left(4 - \frac{11}{6}\right)^2\right) + \left(\frac{1}{6} \times \left(6 - \frac{11}{6}\right)^2\right)$

$= \frac{341}{36}$

(c) $E(X^2) = \left(\frac{1}{3} \times (-2)^2\right) + \left(\frac{1}{6} \times 1^2\right) + \left(\frac{1}{3} \times 4^2\right) + \left(\frac{1}{6} \times 6^2\right)$

$= \frac{77}{6}$

$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{77}{6} - \left(\frac{11}{6}\right)^2 = \frac{341}{36}$

2.4.2 $E(X^2) = (0^2 \times 0.08) + (1^2 \times 0.11) + (2^2 \times 0.27)$
$+ (3^2 \times 0.33) + (4^2 \times 0.21) = 7.52$

Then $E(X) = 2.48$ so that
$\text{Var}(X) = 7.52 - (2.48)^2 = 1.37$
and $\sigma = 1.17$.

2.4.3 $E(X^2) = \left(1^2 \times \frac{2}{5}\right) + \left(2^2 \times \frac{3}{10}\right) + \left(3^2 \times \frac{1}{5}\right) + \left(4^2 \times \frac{1}{10}\right)$
$= 5$

Then $E(X) = 2$ so that
$\text{Var}(X) = 5 - 2^2 = 1$
and $\sigma = 1$.

2.4.4 See Problem 2.3.9.

$E(X^2) = (0^2 \times 0.168) + (1^2 \times 0.2816) + (3^2 \times 0.1664)$
$+ (4^2 \times 0.1024) + (5^2 \times 0.0512)$
$= 5.6192$

Then $E(X) = 1.9072$ so that
$\text{Var}(X) = 5.6192 - 1.9072^2 = 1.98$
and $\sigma = 1.41$.

A small variance is generally preferable if the expected winnings are positive.

2.4.5 (a) $E(X^2) = \int_4^6 x^2 \frac{1}{x \ln(1.5)}\, dx = 24.66$

Then $E(X) = 4.94$ so that
$\text{Var}(X) = 24.66 - 4.94^2 = 0.3325$.

(b) $\sigma = \sqrt{0.3325} = 0.5766$

(c) Solving $F(x) = 0.25$ gives $x = 4.43$.
Solving $F(x) = 0.75$ gives $x = 5.42$.

(d) The interquartile range is $5.42 - 4.43 = 0.99$.

2.4.6 (a) $E(X^2) = \int_0^4 x^2 \left(\frac{x}{8}\right) dx = 8$

Then $E(X) = \frac{8}{3}$ so that
$\text{Var}(X) = 8 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}$.

(b) $\sigma = \sqrt{\frac{8}{9}} = 0.94$

(c) Solving $F(x) = 0.25$ gives $x = 2$.
Solving $F(x) = 0.75$ gives $x = \sqrt{12} = 3.46$.

(d) The interquartile range is $3.46 - 2.00 = 1.46$.

2.4.7 (a) $E(X^2) = \int_{0.125}^{0.5} x^2\, 5.5054\, (0.5 - (x - 0.25)^2)\, dx = 0.1073$

Then $E(X) = 0.3095$ so that
$\text{Var}(X) = 0.1073 - 0.3095^2 = 0.0115$.

(b) $\sigma = \sqrt{0.0115} = 0.107$

(c) Solving $F(x) = 0.25$ gives $x = 0.217$.
Solving $F(x) = 0.75$ gives $x = 0.401$.

(d) The interquartile range is $0.401 - 0.217 = 0.184$.

2.4.8 (a) $E(X^2) = \int_0^{10} \frac{\theta^2}{e^{10}-11}(e^{10-\theta} - 1)\, d\theta$
$= 1.9803$

Then $E(X) = 0.9977$ so that
$\text{Var}(X) = 1.9803 - 0.9977^2 = 0.985$.

(b) $\sigma = \sqrt{0.985} = 0.992$

(c) Solving $F(\theta) = 0.25$ gives $\theta = 0.288$.
Solving $F(\theta) = 0.75$ gives $\theta = 1.385$.

(d) The interquartile range is $1.385 - 0.288 = 1.097$.

2.4.9 (a) $E(X^2) = \int_0^{50} \frac{375.3\, r^2}{(r+5)^4}\, dr = 18.80$

Then $E(X) = 2.44$ so that
$\text{Var}(X) = 18.80 - 2.44^2 = 12.8$.

(b) $\sigma = \sqrt{12.8} = 3.58$

(c) Solving $F(r) = 0.25$ gives $r = 0.50$.
Solving $F(r) = 0.75$ gives $r = 2.93$.

(d) The interquartile range is $2.93 - 0.50 = 2.43$.

2.4.10 Adding and subtracting two standard deviations from the mean value gives:
$P(60.4 \leq X \leq 89.6) \geq 0.75$

Adding and subtracting three standard deviations from the mean value gives:
$P(53.1 \leq X \leq 96.9) \geq 0.89$

2.4.11 The interval $(109.55, 112.05)$ is $(\mu - 2.5c, \mu + 2.5c)$
so Chebyshev's inequality gives:
$P(109.55 \leq X \leq 112.05) \geq 1 - \frac{1}{2.5^2} = 0.84$

2.4.12 $E(X^2) = \left(3^2 \times \frac{1}{20}\right) + \left(4^2 \times \frac{3}{20}\right) + \left(5^2 \times \frac{6}{20}\right) + \left(6^2 \times \frac{10}{20}\right)$

$= \frac{567}{20}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= \frac{567}{20} - \left(\frac{105}{20}\right)^2 = \frac{63}{80}$

The standard deviation is $\sqrt{63/80} = 0.887$.

2.4.13 (a) $E(X^2) = \int_{10}^{11} \frac{4x^3(130-x^2)}{819}\, dx$

$= 108.61538$

Therefore,
$\text{Var}(X) = E(X^2) - (E(X))^2 = 108.61538 - 10.418234^2 = 0.0758$
and the standard deviation is $\sqrt{0.0758} = 0.275$.

(b) Solving $F(x) = 0.8$ gives the 80th percentile of the resistance as 10.69,
and solving $F(x) = 0.1$ gives the 10th percentile of the resistance as 10.07.

2.4.14 (a) Since

$1 = \int_2^3 A x^{2.5}\, dx = \frac{A}{3.5} \times (3^{3.5} - 2^{3.5})$

it follows that $A = 0.0987$.

(b) $E(X) = \int_2^3 0.0987\, x^{3.5}\, dx$

$= \frac{0.0987}{4.5} \times (3^{4.5} - 2^{4.5}) = 2.58$

(c) $E(X^2) = \int_2^3 0.0987\, x^{4.5}\, dx$

$= \frac{0.0987}{5.5} \times (3^{5.5} - 2^{5.5}) = 6.741$

Therefore,

$\mathrm{Var}(X) = 6.741 - 2.58^2 = 0.085$

and the standard deviation is $\sqrt{0.085} = 0.29$.

(d) Solving

$0.5 = \int_2^x 0.0987\, y^{2.5}\, dy$

$= \frac{0.0987}{3.5} \times (x^{3.5} - 2^{3.5})$

gives $x = 2.62$.

2.4.15 $E(X) = (-1 \times 0.25) + (1 \times 0.4) + (4 \times 0.35)$

$= \$1.55$

$E(X^2) = ((-1)^2 \times 0.25) + (1^2 \times 0.4) + (4^2 \times 0.35)$

$= 6.25$

Therefore, the variance is

$E(X^2) - (E(X))^2 = 6.25 - 1.55^2 = 3.8475$

and the standard deviation is $\sqrt{3.8475} = \$1.96$.

2.4.16 (a) Since

$1 = \int_3^4 \frac{A}{\sqrt{x}}\, dx = 2A(2 - \sqrt{3})$

it follows that

$A = 1.866$.

(b) $F(x) = \int_3^x \frac{1.866}{\sqrt{y}}\, dy$

$= 3.732 \times (\sqrt{x} - \sqrt{3})$

(c) $E(X) = \int_3^4 x\, \frac{1.866}{\sqrt{x}}\, dx$

$= \frac{2}{3} \times 1.866 \times (4^{1.5} - 3^{1.5}) = 3.488$

(d) $E(X^2) = \int_3^4 x^2\, \frac{1.866}{\sqrt{x}}\, dx$

$= \frac{2}{5} \times 1.866 \times (4^{2.5} - 3^{2.5}) = 12.250$

Therefore,
$\text{Var}(X) = 12.250 - 3.488^2 = 0.0834$
and the standard deviation is $\sqrt{0.0834} = 0.289$.

(e) Solving
$F(x) = 3.732 \times (\sqrt{x} - \sqrt{3}) = 0.5$
gives $x = 3.48$.

(f) Solving
$F(x) = 3.732 \times (\sqrt{x} - \sqrt{3}) = 0.75$
gives $x = 3.74$.

2.4.17 (a) $E(X) = (2 \times 0.11) + (3 \times 0.19) + (4 \times 0.55) + (5 \times 0.15)$
$= 3.74$

(b) $E(X^2) = (2^2 \times 0.11) + (3^2 \times 0.19) + (4^2 \times 0.55) + (5^2 \times 0.15)$
$= 14.70$

Therefore,
$\text{Var}(X) = 14.70 - 3.74^2 = 0.7124$
and the standard deviation is $\sqrt{0.7124} = 0.844$.

2.4.18 (a) $E(X) = \int_{-1}^{1} \frac{x(1-x)}{2}\, dx = -\frac{1}{3}$

(b) $E(X^2) = \int_{-1}^{1} \frac{x^2(1-x)}{2}\, dx = \frac{1}{3}$

Therefore,

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$

and the standard deviation is $\frac{\sqrt{2}}{3} = 0.471$.

(c) Solving

$\int_{-1}^{y} \frac{(1-x)}{2}\, dx = 0.75$

gives $y = 0$.

2.5.1 (a) $P(0.8 \leq X \leq 1, 25 \leq Y \leq 30)$

$= \int_{x=0.8}^{1} \int_{y=25}^{30} \left(\frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000} \right) dx\, dy$

$= 0.092$

(b) $E(Y) = \int_{20}^{35} y \left(\frac{83}{1200} - \frac{(y-25)^2}{10000} \right) dy = 27.36$

$E(Y^2) = \int_{20}^{35} y^2 \left(\frac{83}{1200} - \frac{(y-25)^2}{10000} \right) dy = 766.84$

$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = 766.84 - (27.36)^2 = 18.27$

$\sigma_Y = \sqrt{18.274} = 4.27$

(c) $E(Y|X = 0.55) = \int_{20}^{35} y \left(0.073 - \frac{(y-25)^2}{3922.5} \right) dy = 27.14$

$E(Y^2|X = 0.55) = \int_{20}^{35} y^2 \left(0.073 - \frac{(y-25)^2}{3922.5} \right) dy = 753.74$

$\mathrm{Var}(Y|X = 0.55) = E(Y^2|X = 0.55) - E(Y|X = 0.55)^2$

$= 753.74 - (27.14)^2 = 17.16$

$\sigma_{Y|X=0.55} = \sqrt{17.16} = 4.14$

2.5.2 (a) $p_{1|Y=1} = P(X=1|Y=1) = \frac{p_{11}}{p_{+1}} = \frac{0.12}{0.32} = 0.37500$

$p_{2|Y=1} = P(X=2|Y=1) = \frac{p_{21}}{p_{+1}} = \frac{0.08}{0.32} = 0.25000$

$p_{3|Y=1} = P(X=3|Y=1) = \frac{p_{31}}{p_{+1}} = \frac{0.07}{0.32} = 0.21875$

$p_{4|Y=1} = P(X=4|Y=1) = \frac{p_{41}}{p_{+1}} = \frac{0.05}{0.32} = 0.15625$

$E(X|Y=1) = (1 \times 0.375) + (2 \times 0.25) + (3 \times 0.21875) + (4 \times 0.15625)$
$= 2.15625$

$E(X^2|Y=1) = (1^2 \times 0.375) + (2^2 \times 0.25) + (3^2 \times 0.21875) + (4^2 \times 0.15625)$
$= 5.84375$

$\mathrm{Var}(X|Y=1) = E(X^2|Y=1) - E(X|Y=1)^2$
$= 5.84375 - 2.15625^2 = 1.1943$

$\sigma_{X|Y=1} = \sqrt{1.1943} = 1.093$

(b) $p_{1|X=2} = P(Y=1|X=2) = \frac{p_{21}}{p_{2+}} = \frac{0.08}{0.24} = \frac{8}{24}$

$p_{2|X=2} = P(Y=2|X=2) = \frac{p_{22}}{p_{2+}} = \frac{0.15}{0.24} = \frac{15}{24}$

$p_{3|X=2} = P(Y=3|X=2) = \frac{p_{23}}{p_{2+}} = \frac{0.01}{0.24} = \frac{1}{24}$

$E(Y|X=2) = \left(1 \times \frac{8}{24}\right) + \left(2 \times \frac{15}{24}\right) + \left(3 \times \frac{1}{24}\right)$
$= \frac{41}{24} = 1.7083$

$E(Y^2|X=2) = \left(1^2 \times \frac{8}{24}\right) + \left(2^2 \times \frac{15}{24}\right) + \left(3^2 \times \frac{1}{24}\right)$
$= \frac{77}{24} = 3.2083$

$\mathrm{Var}(Y|X=2) = E(Y^2|X=2) - E(Y|X=2)^2$
$= 3.2083 - 1.7083^2 = 0.290$

$\sigma_{Y|X=2} = \sqrt{0.290} = 0.538$

2.5.3 (a) Since

$\int_{x=-2}^{3}\int_{y=4}^{6} A(x-3)y\ dx\ dy = 1$

it follows that $A = -\frac{1}{125}$.

(b) $P(0 \leq X \leq 1, 4 \leq Y \leq 5)$

$= \int_{x=0}^{1}\int_{y=4}^{5} \frac{(3-x)y}{125}\ dx\ dy$

$= \frac{9}{100}$

(c) $f_X(x) = \int_{4}^{6} \frac{(3-x)y}{125}\ dy = \frac{2(3-x)}{25}$

for $-2 \leq x \leq 3$

$f_Y(y) = \int_{-2}^{3} \frac{(3-x)y}{125}\ dx = \frac{y}{10}$

for $4 \leq x \leq 6$

(d) The random variables X and Y are independent since
$f_X(x) \times f_Y(y) = f(x,y)$
and the ranges of the random variables are not related.

(e) Since the random variables are independent it follows that
$f_{X|Y=5}(x)$ is equal to $f_X(x)$.

2.5.4 (a)

X\Y	0	1	2	3	p_{i+}
0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{2}{16}$
1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	0	$\frac{6}{16}$
2	0	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{6}{16}$
3	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$
p_{+j}	$\frac{2}{16}$	$\frac{6}{16}$	$\frac{6}{16}$	$\frac{2}{16}$	1

(b) See the table above.

(c) The random variables X and Y are not independent.
For example, notice that
$p_{0+} \times p_{+0} = \frac{2}{16} \times \frac{2}{16} = \frac{1}{4} \neq p_{00} = \frac{1}{16}$.

(d) $E(X) = \left(0 \times \frac{2}{16}\right) + \left(1 \times \frac{6}{16}\right) + \left(2 \times \frac{6}{16}\right) + \left(3 \times \frac{2}{16}\right) = \frac{3}{2}$

$$E(X^2) = \left(0^2 \times \frac{2}{16}\right) + \left(1^2 \times \frac{6}{16}\right) + \left(2^2 \times \frac{6}{16}\right) + \left(3^2 \times \frac{2}{16}\right) = 3$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

The random variable Y has the same mean and variance as X.

(e) $E(XY) = \left(1 \times 1 \times \frac{3}{16}\right) + \left(1 \times 2 \times \frac{2}{16}\right) + \left(2 \times 1 \times \frac{2}{16}\right)$

$+ \left(2 \times 2 \times \frac{3}{16}\right) + \left(2 \times 3 \times \frac{1}{16}\right) + \left(3 \times 2 \times \frac{1}{16}\right) + \left(3 \times 3 \times \frac{1}{16}\right)$

$= \frac{44}{16}$

$\mathrm{Cov}(X, Y) = E(XY) - (E(X) \times E(Y))$

$= \frac{44}{16} - \left(\frac{3}{2} \times \frac{3}{2}\right) = \frac{1}{2}$

(f) $P(X = 0|Y = 1) = \frac{p_{01}}{p_{+1}} = \frac{\left(\frac{1}{16}\right)}{\left(\frac{6}{16}\right)} = \frac{1}{6}$

$P(X = 1|Y = 1) = \frac{p_{11}}{p_{+1}} = \frac{\left(\frac{3}{16}\right)}{\left(\frac{6}{16}\right)} = \frac{1}{2}$

$P(X = 2|Y = 1) = \frac{p_{21}}{p_{+1}} = \frac{\left(\frac{2}{16}\right)}{\left(\frac{6}{16}\right)} = \frac{1}{3}$

$P(X = 3|Y = 1) = \frac{p_{31}}{p_{+1}} = \frac{0}{\left(\frac{6}{16}\right)} = 0$

$E(X|Y = 1) = \left(0 \times \frac{1}{6}\right) + \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{3}\right) + (3 \times 0) = \frac{7}{6}$

$E(X^2|Y = 1) = \left(0^2 \times \frac{1}{6}\right) + \left(1^2 \times \frac{1}{2}\right) + \left(2^2 \times \frac{1}{3}\right) + (3^2 \times 0) = \frac{11}{6}$

$\mathrm{Var}(X|Y = 1) = E(X^2|Y = 1) - E(X|Y = 1)^2$

$= \frac{11}{6} - \left(\frac{7}{6}\right)^2 = \frac{17}{36}$

2.5.5 (a) Since

$\int_{x=1}^{2}\int_{y=0}^{3} A(e^{x+y} + e^{2x-y})\,dx\,dy = 1$

it follows that $A = 0.00896$.

(b) $P(1.5 \leq X \leq 2, 1 \leq Y \leq 2)$

$= \int_{x=1.5}^{2}\int_{y=1}^{2} 0.00896\,(e^{x+y} + e^{2x-y})\,dx\,dy$

$= 0.158$

(c) $f_X(x) = \int_0^3 0.00896\,(e^{x+y} + e^{2x-y})\,dy$

$= 0.00896\,(e^{x+3} - e^{2x-3} - e^x + e^{2x})$

for $1 \leq x \leq 2$

$f_Y(y) = \int_1^2 0.00896\,(e^{x+y} + e^{2x-y})\,dx$

$= 0.00896\,(e^{2+y} + 0.5e^{4-y} - e^{1+y} - 0.5e^{2-y})$

for $0 \leq y \leq 3$

(d) No, since $f_X(x) \times f_Y(y) \neq f(x,y)$.

(e) $f_{X|Y=0}(x) = \frac{f(x,0)}{f_Y(0)} = \frac{e^x+e^{2x}}{28.28}$

2.5.6 (a)

X\Y	0	1	2	p_{i+}
0	$\frac{25}{102}$	$\frac{26}{102}$	$\frac{6}{102}$	$\frac{57}{102}$
1	$\frac{26}{102}$	$\frac{13}{102}$	0	$\frac{39}{102}$
2	$\frac{6}{102}$	0	0	$\frac{6}{102}$
p_{+j}	$\frac{57}{102}$	$\frac{39}{102}$	$\frac{6}{102}$	1

(b) See the table above.

(c) No, the random variables X and Y are not independent.
For example,
$p_{22} \neq p_{2+} \times p_{+2}$.

(d) $E(X) = \left(0 \times \frac{57}{102}\right) + \left(1 \times \frac{39}{102}\right) + \left(2 \times \frac{6}{102}\right) = \frac{1}{2}$

$E(X^2) = \left(0^2 \times \frac{57}{102}\right) + \left(1^2 \times \frac{39}{102}\right) + \left(2^2 \times \frac{6}{102}\right) = \frac{21}{34}$

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{21}{34} - \left(\frac{1}{2}\right)^2 = \frac{25}{68}$

The random variable Y has the same mean and variance as X.

(e) $E(XY) = 1 \times 1 \times p_{11} = \frac{13}{102}$

$\mathrm{Cov}(X, Y) = E(XY) - (E(X) \times E(Y))$

$= \frac{13}{102} - \left(\frac{1}{2} \times \frac{1}{2}\right) = -\frac{25}{204}$

(f) $\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = -\frac{1}{3}$

(g) $P(Y = 0|X = 0) = \frac{p_{00}}{p_{0+}} = \frac{25}{57}$

$P(Y = 1|X = 0) = \frac{p_{01}}{p_{0+}} = \frac{26}{57}$

$P(Y = 2|X = 0) = \frac{p_{02}}{p_{0+}} = \frac{6}{57}$

$P(Y = 0|X = 1) = \frac{p_{10}}{p_{1+}} = \frac{2}{3}$

$P(Y = 1|X = 1) = \frac{p_{11}}{p_{1+}} = \frac{1}{3}$

$P(Y = 2|X = 1) = \frac{p_{12}}{p_{1+}} = 0$

2.5.7 (a)

X\Y	0	1	2	p_{i+}
0	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{9}{16}$
1	$\frac{4}{16}$	$\frac{2}{16}$	0	$\frac{6}{16}$
2	$\frac{1}{16}$	0	0	$\frac{1}{16}$
p_{+j}	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$	1

(b) See the table above.

(c) No, the random variables X and Y are not independent.
For example,
$p_{22} \neq p_{2+} \times p_{+2}$.

(d) $E(X) = \left(0 \times \frac{9}{16}\right) + \left(1 \times \frac{6}{16}\right) + \left(2 \times \frac{1}{16}\right) = \frac{1}{2}$

$E(X^2) = \left(0^2 \times \frac{9}{16}\right) + \left(1^2 \times \frac{6}{16}\right) + \left(2^2 \times \frac{1}{16}\right) = \frac{5}{8}$

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{5}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} = 0.325$

The random variable Y has the same mean and variance as X.

(e) $E(XY) = 1 \times 1 \times p_{11} = \frac{1}{8}$

$\text{Cov}(X, Y) = E(XY) - (E(X) \times E(Y))$

$= \frac{1}{8} - \left(\frac{1}{2} \times \frac{1}{2}\right) = -\frac{1}{8}$

(f) $\text{Corr}(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = -\frac{1}{3}$

(g) $P(Y = 0|X = 0) = \frac{p_{00}}{p_{0+}} = \frac{4}{9}$

$P(Y = 1|X = 0) = \frac{p_{01}}{p_{0+}} = \frac{4}{9}$

$P(Y = 2|X = 0) = \frac{p_{02}}{p_{0+}} = \frac{1}{9}$

$P(Y = 0|X = 1) = \frac{p_{10}}{p_{1+}} = \frac{2}{3}$

$P(Y = 1|X = 1) = \frac{p_{11}}{p_{1+}} = \frac{1}{3}$

$P(Y = 2|X = 1) = \frac{p_{12}}{p_{1+}} = 0$

2.5.8 (a) Since

$\int_{x=0}^{5}\int_{y=0}^{5} A\ (20 - x - 2y)\ dx\ dy = 1$

it follows that $A = 0.0032$

(b) $P(1 \leq X \leq 2, 2 \leq Y \leq 3)$

$= \int_{x=1}^{2}\int_{y=2}^{3} 0.0032\ (20 - x - 2y)\ dx\ dy$

$= 0.0432$

(c) $f_X(x) = \int_{y=0}^{5} 0.0032\ (20 - x - 2y)\ dy = 0.016\ (15 - x)$

for $0 \leq x \leq 5$

$f_Y(y) = \int_{x=0}^{5} 0.0032\ (20 - x - 2y)\ dx = 0.008\ (35 - 4y)$

for $0 \leq y \leq 5$

(d) No, the random variables X and Y are not independent since $f(x, y) \neq f_X(x) f_Y(y)$.

(e) $E(X) = \int_0^5 x\ 0.016\ (15 - x)\ dx = \frac{7}{3}$

$E(X^2) = \int_0^5 x^2\ 0.016\ (15 - x)\ dx = \frac{15}{2}$

$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{15}{2} - \left(\frac{7}{3}\right)^2 = \frac{37}{18}$

(f) $E(Y) = \int_0^5 \ y \, 0.008 \, (35 - 4y) \ dy = \frac{13}{6}$

$E(Y^2) = \int_0^5 \ y^2 \, 0.008 \, (35 - 4y) \ dy = \frac{20}{3}$

$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{20}{3} - \left(\frac{13}{6}\right)^2 = \frac{71}{36}$

(g) $f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{17-2y}{60}$

for $0 \leq y \leq 5$

(h) $E(XY) = \int_{x=0}^5 \int_{y=0}^5 \ 0.0032 \, xy \, (20 - x - 2y) \ dx \ dy = 5$

$\mathrm{Cov}(X,Y) = E(XY) - (E(X) \times (EY))$

$= 5 - \left(\frac{7}{3} \times \frac{13}{6}\right) = -\frac{1}{18}$

(i) $\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = -0.0276$

2.5.9 (a) $P(\text{same score}) = P(X = 1, Y = 1) + P(X = 2, Y = 2)$
$+ P(X = 3, Y = 3) + P(X = 4, Y = 4)$
$= 0.80$

(b) $P(X < Y) = P(X = 1, Y = 2) + P(X = 1, Y = 3) + P(X = 1, Y = 4)$
$+ P(X = 2, Y = 3) + P(X = 2, Y = 4) + P(X = 3, Y = 4)$
$= 0.07$

(c)

x_i	1	2	3	4
p_{i+}	0.12	0.20	0.30	0.38

$E(X) = (1 \times 0.12) + (2 \times 0.20) + (3 \times 0.30) + (4 \times 0.38) = 2.94$

$E(X^2) = (1^2 \times 0.12) + (2^2 \times 0.20) + (3^2 \times 0.30) + (4^2 \times 0.38) = 9.70$

$\text{Var}(X) = E(X^2) - E(X)^2 = 9.70 - (2.94)^2 = 1.0564$

(d)

y_j	1	2	3	4
p_{+j}	0.14	0.21	0.30	0.35

$E(Y) = (1 \times 0.14) + (2 \times 0.21) + (3 \times 0.30) + (4 \times 0.35) = 2.86$

$E(Y^2) = (1^2 \times 0.14) + (2^2 \times 0.21) + (3^2 \times 0.30) + (4^2 \times 0.35) = 9.28$

$\text{Var}(Y) = E(Y^2) - E(Y)^2 = 9.28 - (2.86)^2 = 1.1004$

(e) The scores are not independent.
For example, $p_{11} \neq p_{1+} \times p_{+1}$.
The scores would not be expected to be independent since they apply to the two inspectors' assessments of the same building. If they were independent it would suggest that one of the inspectors is randomly assigning a safety score without paying any attention to the actual state of the building.

(f) $P(Y = 1|X = 3) = \frac{p_{31}}{p_{3+}} = \frac{1}{30}$

$P(Y = 2|X = 3) = \frac{p_{32}}{p_{3+}} = \frac{3}{30}$

$P(Y = 3|X = 3) = \frac{p_{33}}{p_{3+}} = \frac{24}{30}$

$P(Y = 4|X = 3) = \frac{p_{34}}{p_{3+}} = \frac{2}{30}$

(g) $E(XY) = \sum_{i=1}^{4} \sum_{j=1}^{4} \; i \; j \; p_{ij} = 9.29$

$\mathrm{Cov}(X, Y) = E(XY) - (E(X) \times E(Y))$
$= 9.29 - (2.94 \times 2.86) = 0.8816$

(h) $\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X \ \mathrm{Var}Y}} = \frac{0.8816}{\sqrt{1.0564 \times 1.1004}} = 0.82$

A high positive correlation indicates that the inspectors are consistent.
The closer the correlation is to one the more consistent the inspectors are.

2.5.10 (a) $\int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} \; \frac{3xyz^2}{32} \; dx \; dy \; dz = 1$

(b) $\int_{x=0}^{1} \int_{y=0.5}^{1.5} \int_{z=1}^{2} \; \frac{3xyz^2}{32} \; dx \; dy \; dz = \frac{7}{64}$

(c) $f_X(x) = \int_{z=0}^{2} \int_{y=0}^{2} \; \frac{3xyz^2}{32} \; dy \; dz = \frac{x}{2}$

for $0 \leq 2 \leq x$

2.6.1 (a) $E(3X+7) = 3E(X) + 7 = 13$

$\mathrm{Var}(3X+7) = 3^2\mathrm{Var}(X) = 36$

(b) $E(5X-9) = 5E(X) - 9 = 1$

$\mathrm{Var}(5X-9) = 5^2\mathrm{Var}(X) = 100$

(c) $E(2X+6Y) = 2E(X) + 6E(Y) = -14$

$\mathrm{Var}(2X+6Y) = 2^2\mathrm{Var}(X) + 6^2\mathrm{Var}(Y) = 88$

(d) $E(4X-3Y) = 4E(X) - 3E(Y) = 17$

$\mathrm{Var}(4X-3Y) = 4^2\mathrm{Var}(X) + 3^2\mathrm{Var}(Y) = 82$

(e) $E(5X-9Z+8) = 5E(X) - 9E(Z) + 8 = -54$

$\mathrm{Var}(5X-9Z+8) = 5^2\mathrm{Var}(X) + 9^2\mathrm{Var}(Z) = 667$

(f) $E(-3Y-Z-5) = -3E(Y) - E(Z) - 5 = -4$

$\mathrm{Var}(-3Y-Z-5) = (-3)^2\mathrm{Var}(Y) + (-1)^2\mathrm{Var}(Z) = 25$

(g) $E(X+2Y+3Z) = E(X) + 2E(Y) + 3E(Z) = 20$

$\mathrm{Var}(X+2Y+3Z) = \mathrm{Var}(X) + 2^2\mathrm{Var}(Y) + 3^2\mathrm{Var}(Z) = 75$

(h) $E(6X+2Y-Z+16) = 6E(X) + 2E(Y) - E(Z) + 16 = 14$

$\mathrm{Var}(6X+2Y-Z+16) = 6^2\mathrm{Var}(X) + 2^2\mathrm{Var}(Y) + (-1)^2\mathrm{Var}(Z) = 159$

2.6.2 $E(aX+b) = \int (ax+b)\ f(x)\ dx$

$= a\int x\ f(x)\ dx\ +\ b\int\ f(x)\ dx$

$= aE(X) + b$

$\mathrm{Var}(aX+b) = E((aX+b-E(aX+b))^2)$

$= E((aX-aE(X))^2)$

$= a^2E((X-E(X))^2)$

$= a^2\mathrm{Var}(X)$

2.6.3 $E(Y) = 3E(X_1) = 3\mu$

$\text{Var}(Y) = 3^2\text{Var}(X_1) = 9\sigma^2$

$E(Z) = E(X_1) + E(X_2) + E(X_3) = 3\mu$

$\text{Var}(Z) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = 3\sigma^2$

The random variables Y and Z have the same mean
but Z has a smaller variance than Y.

2.6.4 length $= A_1 + A_2 + B$

$E(\text{length}) = E(A_1) + E(A_2) + E(B) = 37 + 37 + 24 = 98$

$\text{Var}(\text{length}) = \text{Var}(A_1) + \text{Var}(A_2) + \text{Var}(B) = 0.7^2 + 0.7^2 + 0.3^2 = 1.07$

2.6.5 Let the random variable X_i be the winnings from the i^{th} game.
Then

$E(X_i) = \left(10 \times \frac{1}{8}\right) + \left((-1) \times \frac{7}{8}\right) = \frac{3}{8}$

and

$E(X_i^2) = \left(10^2 \times \frac{1}{8}\right) + \left((-1)^2 \times \frac{7}{8}\right) = \frac{107}{8}$

so that

$\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{847}{64}$.

The total winnings from 50 (independent) games is

$Y = X_1 + \ldots + X_{50}$

and

$E(Y) = E(X_1) + \ldots + E(X_{50}) = 50 \times \frac{3}{8} = \frac{75}{4} = \$18.75$

with

$\text{Var}(Y) = \text{Var}(X_1) + \ldots + \text{Var}(X_{50}) = 50 \times \frac{847}{64} = 661.72$

so that $\sigma_Y = \sqrt{661.72} = \$25.72$.

2.6.6 (a) $E(\text{average weight}) = 1.12$ kg

$\text{Var}(\text{average weight}) = \frac{0.03^2}{25} = 3.6 \times 10^{-5}$

The standard deviation is $\frac{0.03}{\sqrt{25}} = 0.0012$ kg.

(b) It is required that $\frac{0.03}{\sqrt{n}} \leq 0.005$ which is satisfied for $n \geq 36$.

2.6.7 Let the random variable X_i be equal to 1 if an ace is drawn on the i^{th} drawing (which happens with a probability of $\frac{1}{13}$) and equal to 0 if an ace is not drawn on the i^{th} drawing (which happens with a probability of $\frac{12}{13}$).

Then the total number of aces drawn is $Y = X_1 + \ldots + X_{10}$.

Notice that $E(X_i) = \frac{1}{13}$ so that regardless of whether the drawing is performed with or without replacement it follows that

$E(Y) = E(X_1) + \ldots + E(X_{10}) = \frac{10}{13}$.

Also, notice that $E(X_i^2) = \frac{1}{13}$ so that

$\text{Var}(X_i) = \frac{1}{13} - \left(\frac{1}{13}\right)^2 = \frac{12}{169}$.

If the drawings are made *with replacement* then the random variables X_i are independent so that

$\text{Var}(Y) = \text{Var}(X_1) + \ldots + \text{Var}(X_{10}) = \frac{120}{169}$.

However, if the drawings are made *without replacement* then the random variables X_i are not independent.

2.6.8 $F_X(x) = P(X \leq x) = x^2$ for $0 \leq x \leq 1$

(a) $F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = F_X(y^{1/3}) = y^{2/3}$

and so

$f_Y(y) = \frac{2}{3}y^{-1/3}$

for $0 \leq y \leq 1$

$E(y) = \int_0^1 \, y \; f_Y(y) \; dy = 0.4$

(b) $F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2) = y^4$

and so

$f_Y(y) = 4y^3$

for $0 \leq y \leq 1$

$E(y) = \int_0^1 \, y \; f_Y(y) \; dy = 0.8$

(c) $F_Y(y) = P(Y \leq y) = P(\frac{1}{1+X} \leq y) = P(X \geq \frac{1}{y} - 1)$

$= 1 - F_X(\frac{1}{y} - 1) = \frac{2}{y} - \frac{1}{y^2}$

and so

$f_Y(y) = -\frac{2}{y^2} + \frac{2}{y^3}$

for $\frac{1}{2} \leq y \leq 1$

$E(y) = \int_{0.5}^1 \, y \; f_Y(y) \; dy = 0.614$

(d) $F_Y(y) = P(Y \leq y) = P(2^X \leq y) = P\left(X \leq \frac{\ln(y)}{\ln(2)}\right)$

$= F_X\left(\frac{\ln(y)}{\ln(2)}\right) = \left(\frac{\ln(y)}{\ln(2)}\right)^2$

and so

$f_Y(y) = \frac{2 \, \ln(y)}{y \, (\ln(2))^2}$

for $1 \leq y \leq 2$

$E(y) = \int_1^2 \, y \; f_Y(y) \; dy = 1.61$

2.6.9 (a) Since

$\int_0^2 A(1-(r-1)^2)\ dr = 1$

it follows that $A = \frac{3}{4}$.

This gives

$F_R(r) = \frac{3r^2}{4} - \frac{r^3}{4}$

for $0 \leq r \leq 2$.

(b) $V = \frac{4}{3}\pi r^3$

Since

$F_V(v) = P(V \leq v) = P\left(\frac{4}{3}\pi r^3 \ \leq \ v\right) = F_R\left(\left(\frac{3v}{4\pi}\right)^{1/3}\right)$

it follows that

$f_V(v) = \frac{1}{2}(\frac{3}{4\pi})^{2/3}v^{-1/3} \ - \ \frac{3}{16\pi}$

for $0 \leq v \leq \frac{32\pi}{3}$.

(c) $E(V) = \int_0^{\frac{32\pi}{3}} v\ f_V(v)\ dv = \frac{32\pi}{15}$

2.6.10 (a) Since

$\int_0^L Ax(L-x)\, dx = 1$

it follows that $A = \frac{6}{L^3}$.

Therefore,

$F_X(x) = \frac{x^2(3L-2x)}{L^3}$

for $0 \leq x \leq L$.

(b) The random variable corresponding to the difference between the lengths of the two pieces of rod is
$W = |L - 2X|$.

Therefore,

$F_W(w) = P\left(\frac{L}{2} - \frac{w}{2} \leq X \leq \frac{L}{2} + \frac{w}{2}\right) = F_X\left(\frac{L}{2} + \frac{w}{2}\right) - F_X\left(\frac{L}{2} - \frac{w}{2}\right)$

$= \frac{w(3L^2-w^2)}{2L^3}$

and

$f_W(w) = \frac{3(L^2-w^2)}{2L^3}$

for $0 \leq w \leq L$.

(c) $E(W) = \int_0^L \; w \; f_W(w)\; dw = \frac{3}{8}L$

2.6.11 (a) The return has an expectation of \$100, a standard deviation of \$20, and a variance of 400.

(b) The return has an expectation of \$100, a standard deviation of \$30, and a variance of 900.

(c) The return from fund A has an expectation of \$50, a standard deviation of \$10, and a variance of 100.

The return from fund B has an expectation of \$50, a standard deviation of \$15, and a variance of 225.

Therefore, the total return has an expectation of \$100 and a variance of 325, so that the standard deviation is \$18.03.

(d) The return from fund A has an expectation of $\$0.1x$,
a standard deviation of $\$0.02x$,
and a variance of $0.0004x^2$.

The return from fund B has an expectation of $\$0.1(1000 - x)$,
a standard deviation of $\$0.03(1000 - x)$,
and a variance of $0.0009(1000 - x)^2$.

Therefore, the total return has an expectation of \$100
and a variance of $0.0004x^2 + 0.0009(1000 - x)^2$.

This variance is minimized by taking $x = \$692$,
and the minimum value of the variance is 276.9
which corresponds to a standard deviation of \$16.64.

This problem illustrates that the variability of the return on an investment can be reduced by *diversifying* the investment, so that it is spread over several funds.

2.6.12 The expected value of the total resistance is

$5 \times E(X) = 5 \times 10.418234 = 52.09.$

The variance of the total resistance is

$5 \times \text{Var}(X) = 5 \times 0.0758 = 0.379$

so that the standard deviation is $\sqrt{0.379} = 0.616.$

2.6.13 (a) The mean is

$$E(X) = \left(\tfrac{1}{3} \times E(X_1)\right) + \left(\tfrac{1}{3} \times E(X_2)\right) + \left(\tfrac{1}{3} \times E(X_3)\right)$$

$$= \left(\tfrac{1}{3} \times 59\right) + \left(\tfrac{1}{3} \times 67\right) + \left(\tfrac{1}{3} \times 72\right) = 66$$

The variance is

$$\mathrm{Var}(X) = \left(\left(\tfrac{1}{3}\right)^2 \times \mathrm{Var}(X_1)\right) + \left(\left(\tfrac{1}{3}\right)^2 \times \mathrm{Var}(X_2)\right) + \left(\left(\tfrac{1}{3}\right)^2 \times \mathrm{Var}(X_3)\right)$$

$$= \left(\left(\tfrac{1}{3}\right)^2 \times 10^2\right) + \left(\left(\tfrac{1}{3}\right)^2 \times 13^2\right) + \left(\left(\tfrac{1}{3}\right)^2 \times 4^2\right) = \tfrac{95}{3}$$

so that the standard deviation is $\sqrt{95/3} = 5.63$.

(b) The mean is

$$E(X) = (0.4 \times E(X_1)) + (0.4 \times E(X_2)) + (0.2 \times E(X_3))$$

$$= (0.4 \times 59) + (0.4 \times 67) + (0.2 \times 72) = 64.8.$$

The variance is

$$\mathrm{Var}(X) = (0.4^2 \times \mathrm{Var}(X_1)) + (0.4^2 \times \mathrm{Var}(X_2)) + (0.2^2 \times \mathrm{Var}(X_3))$$

$$= (0.4^2 \times 10^2) + (0.4^2 \times 13^2) + (0.2^2 \times 4^2) = 43.68$$

so that the standard deviation is $\sqrt{43.68} = 6.61$.

2.6.14 $1000 = E(Y) = a + bE(X) = a + (b \times 77)$

$10^2 = \mathrm{Var}(Y) = b^2\mathrm{Var}(X) = b^2 \times 9^2$

Solving these equations gives $a = 914.44$ and $b = 1.11$,
or $a = 1085.56$ and $b = -1.11$.

2.6.15 (a) The mean is $\mu = 65.90$.
The standard deviation is $\frac{\sigma}{\sqrt{5}} = \frac{0.32}{\sqrt{5}} = 0.143$.

(b) The mean is $8\mu = 8 \times 65.90 = 527.2$.
The standard deviation is $\sqrt{8}\sigma = \sqrt{8} \times 0.32 = 0.905$.

2.6.16 (a) $E(A) = \frac{E(X_1)+E(X_2)}{2} = \frac{W+W}{2} = W$

$\text{Var}(A) = \frac{\text{Var}(X_1)+\text{Var}(X_2)}{4} = \frac{3^2+4^2}{4} = \frac{25}{4}$

The standard deviation is $\frac{5}{2} = 2.5$.

(b) $\text{Var}(B) = \delta^2\text{Var}(X_1) + (1-\delta)^2\text{Var}(X_2) = 9\delta^2 + 16(1-\delta)^2$

This is minimized when $\delta = \frac{16}{25}$ and the minimum value is $\frac{144}{25}$

so that the minimum standard deviation is $\frac{12}{5} = 2.4$.

2.6.17 When a die is rolled once the expectation is 3.5 and the standard deviation is 1.71 (see Games of Chance in section 2.4).

Therefore, the sum of eighty die rolls has an expectation of $80 \times 3.5 = 280$ and a standard deviation of $\sqrt{80} \times 1.71 = 15.3$.

2.6.18 (a) The expectation is $4 \times 33.2 = 132.8$ seconds.
The standard deviation is $\sqrt{4} \times 1.4 = 2.8$ seconds.

(b) $E(A_1 + A_2 + A_3 + A_4 - B_1 - B_2 - B_3 - B_4)$
$= E(A_1) + E(A_2) + E(A_3) + E(A_4) - E(B_1) - E(B_2) - E(B_3) - E(B_4)$
$= (4 \times 33.2) - (4 \times 33.0) = 0.8$

$\text{Var}(A_1 + A_2 + A_3 + A_4 - B_1 - B_2 - B_3 - B_4)$
$= \text{Var}(A_1) + \text{Var}(A_2) + \text{Var}(A_3) + \text{Var}(A_4)$
$+ \text{Var}(B_1) + \text{Var}(B_2) + \text{Var}(B_3) + \text{Var}(B_4)$
$= (4 \times 1.4^2) + (4 \times 1.3^2) = 14.6$

The standard deviation is $\sqrt{14.6} = 3.82$.

(c) $E\left(A_1 - \frac{A_2+A_3+A_4}{3}\right)$

$= E(A_1) - \frac{E(A_2)}{3} - \frac{E(A_3)}{3} - \frac{E(A_4)}{3} = 0$

$\text{Var}\left(A_1 - \frac{A_2+A_3+A_4}{3}\right)$

$= \text{Var}(A_1) + \frac{\text{Var}(A_2)}{9} + \frac{\text{Var}(A_3)}{9} + \frac{\text{Var}(A_4)}{9}$

$= \frac{4}{3} \times 1.4^2 = 2.613$

The standard deviation is $\sqrt{2.613} = 1.62$.

2.6.19 Let X be the temperature in Fahrenheit and let Y be the temperature in Centigrade.

$E(Y) = E\left(\frac{5(X-32)}{9}\right) = \left(\frac{5(E(X)-32)}{9}\right) = \left(\frac{5(110-32)}{9}\right) = 43.33$

$\text{Var}(Y) = \text{Var}\left(\frac{5(X-32)}{9}\right) = \left(\frac{5^2\text{Var}(X)}{9^2}\right) = \left(\frac{5^2 \times 2.2^2}{9^2}\right) = 1.49$

The standard deviation is $\sqrt{1.49} = 1.22$.

2.6.20 $\text{Var}(0.5X_\alpha + 0.3X_\beta + 0.2X_\gamma)$

$= 0.5^2\text{Var}(X_\alpha) + 0.3^2\text{Var}(X_\beta) + 0.2^2\text{Var}(X_\gamma)$

$= (0.5^2 \times 1.2^2) + (0.3^2 \times 2.4^2) + (0.2^2 \times 3.1^2) = 1.26$

The standard deviation is $\sqrt{1.26} = 1.12$.

2.6.21 The inequality $\frac{56}{\sqrt{n}} \leq 10$ is satisfied for $n \geq 32$.

2.6.22 (a) $E(X_1 + X_2) = E(X_1) + E(X_2) = 7.74$

$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 0.0648$

The standard deviation is $\sqrt{0.0648} = 0.255$.

(b) $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 11.61$

$\text{Var}(X_1 + X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = 0.0972$

The standard deviation is $\sqrt{0.0972} = 0.312$.

(c) $E\left(\frac{X_1+X_2+X_3+X_4}{4}\right)$

$= \frac{E(X_1)+E(X_2)+E(X_3)+E(X_4)}{4}$

$= 3.87$

$\text{Var}\left(\frac{X_1+X_2+X_3+X_4}{4}\right)$

$= \frac{\text{Var}(X_1)+\text{Var}(X_2)+\text{Var}(X_3)+\text{Var}(X_4)}{16}$

$= 0.0081$

The standard deviation is $\sqrt{0.0081} = 0.09$.

(d) $E\left(X_3 - \frac{X_1+X_2}{2}\right) = E(X_3) - \frac{E(X_1)+E(X_2)}{2} = 0$

$\text{Var}\left(X_3 - \frac{X_1+X_2}{2}\right) = \text{Var}(X_3) + \frac{\text{Var}(X_1)+\text{Var}(X_2)}{4} = 0.0486$

The standard deviation is $\sqrt{0.0486} = 0.220$.

2.6.23 (a) $\sqrt{14000^2 + 14000^2} = \$19,799$

(b) $14000/\sqrt{2} = \$9,899$

2.6.24 Expectation $= \$30,000 + \$45,000 = \$75,000$
Standard deviation $= \sqrt{4000^2 + 3000^2} = \$5,000$

2.9.1 (a)

x_i	2	3	4	5	6
p_i	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(b) $E(X) = \left(2 \times \frac{1}{15}\right) + \left(3 \times \frac{2}{15}\right) + \left(4 \times \frac{3}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(6 \times \frac{5}{15}\right) = \frac{14}{3}$

2.9.2 (a)

x_i	0	1	2	3	4	5	6
$F(x_i)$	0.21	0.60	0.78	0.94	0.97	0.99	1.00

(b) $E(X) = (0 \times 0.21) + (1 \times 0.39) + (2 \times 0.18) + (3 \times 0.16)$
$+ (4 \times 0.03) + (5 \times 0.02) + (6 \times 0.01)$
$= 1.51$

(c) $E(X^2) = (0^2 \times 0.21) + (1^2 \times 0.39) + (2^2 \times 0.18) + (3^2 \times 0.16)$
$+ (4^2 \times 0.03) + (5^2 \times 0.02) + (6^2 \times 0.01)$
$= 3.89$

$\mathrm{Var}(X) = 3.89 - (1.51)^2 = 1.61$

(d) The expectation is $1.51 \times 60 = 90.6$
and the variance is $1.61 \times 60 = 96.6$.

2.9.3 (a)

x_i	2	3	4	5
p_i	$\frac{2}{30}$	$\frac{13}{30}$	$\frac{13}{30}$	$\frac{2}{30}$

(b) $E(X) = \left(2 \times \frac{2}{30}\right) + \left(3 \times \frac{13}{30}\right) + \left(4 \times \frac{13}{30}\right) + \left(5 \times \frac{2}{30}\right) = \frac{7}{2}$

$E(X^2) = \left(2^2 \times \frac{2}{30}\right) + \left(3^2 \times \frac{13}{30}\right) + \left(4^2 \times \frac{13}{30}\right) + \left(5^2 \times \frac{2}{30}\right) = \frac{383}{30}$

$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{383}{30} - \left(\frac{7}{2}\right)^2 = \frac{31}{60}$

(c)

x_i	2	3	4	5
p_i	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

$E(X) = \left(2 \times \frac{2}{10}\right) + \left(3 \times \frac{3}{10}\right) + \left(4 \times \frac{3}{10}\right) + \left(5 \times \frac{2}{10}\right) = \frac{7}{2}$

$E(X^2) = \left(2^2 \times \frac{2}{10}\right) + \left(3^2 \times \frac{3}{10}\right) + \left(4^2 \times \frac{3}{10}\right) + \left(5^2 \times \frac{2}{10}\right) = \frac{133}{10}$

$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{133}{10} - \left(\frac{7}{2}\right)^2 = \frac{21}{20}$

2.9.4 Let X_i be the value of the i^{th} card dealt.

Then

$E(X_i) = \left(2 \times \frac{1}{13}\right) + \left(3 \times \frac{1}{13}\right) + \left(4 \times \frac{1}{13}\right) + \left(5 \times \frac{1}{13}\right) + \left(6 \times \frac{1}{13}\right)$

$+ \left(7 \times \frac{1}{13}\right) + \left(8 \times \frac{1}{13}\right) + \left(9 \times \frac{1}{13}\right) + \left(10 \times \frac{1}{13}\right) + \left(15 \times \frac{4}{13}\right) = \frac{114}{13}$

The total score of the hand is

$Y = X_1 + \ldots + X_{13}$

which has an expectation

$E(Y) = E(X_1) + \ldots + E(X_{13}) = 13 \times \frac{114}{13} = 114.$

2.9.5 (a) Since

$\int_1^{11} A \left(\frac{3}{2}\right)^x dx = 1$

it follows that $A = \frac{\ln(1.5)}{1.5^{11}-1.5} = \frac{1}{209.6}$.

(b) $F(x) = \int_1^x \frac{1}{209.6} \left(\frac{3}{2}\right)^y dy$

$= 0.01177 \left(\frac{3}{2}\right)^x - 0.01765$

for $1 \leq x \leq 11$

(c) Solving $F(x) = 0.5$ gives $x = 9.332$.

(d) Solving $F(x) = 0.25$ gives $x = 7.706$.
Solving $F(x) = 0.75$ gives $x = 10.305$.
The interquartile range is $10.305 - 7.706 = 2.599$.

2.9.6 (a) $f_X(x) = \int_1^2 4x(2-y)\, dy = 2x \quad \text{for} \quad 0 \leq x \leq 1$

(b) $f_Y(y) = \int_0^1 4x(2-y)\, dx = 2(2-y) \quad \text{for} \quad 1 \leq y \leq 2$

Since $f(x,y) = f_X(x) \times f_Y(y)$ the random variables are independent.

(c) $\text{Cov}(X,Y) = 0$ because the random variables are independent.

(d) $f_{X|Y=1.5}(x) = f_X(x)$ because the random variables are independent.

2.9.7 (a) Since

$\int_5^{10} A\left(x+\frac{2}{x}\right)dx = 1$

it follows that $A = 0.02572$.

(b) $F(x) = \int_5^x 0.02572\left(y+\frac{2}{y}\right)dy$

$= 0.0129x^2 + 0.0514\ln(x) - 0.404$

for $5 \leq x \leq 10$

(c) $E(X) = \int_5^{10} 0.02572\, x\left(x+\frac{2}{x}\right)dx = 7.759$

(d) $E(X^2) = \int_5^{10} 0.02572\, x^2\left(x+\frac{2}{x}\right)dx = 62.21$

$\text{Var}(X) = E(X^2) - E(X)^2 = 62.21 - 7.759^2 = 2.01$

(e) Solving $F(x) = 0.5$ gives $x = 7.88$.

(f) Solving $F(x) = 0.25$ gives $x = 6.58$.
Solving $F(x) = 0.75$ gives $x = 9.00$.
The interquartile range is $9.00 - 6.58 = 2.42$.

(g) The expectation is $E(X) = 7.759$.

The variance is $\frac{\text{Var}(X)}{10} = 0.0201$.

2.9.8 $\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n + b)$
$= \text{Var}(a_1X_1) + \ldots + \text{Var}(a_nX_n) + \text{Var}(b)$
$= a_1^2\text{Var}(X_1) + \ldots + a_n^2\text{Var}(X_n) + 0$

2.9.9 $Y = \frac{5}{3}X - 25$

2.9.10 Notice that $E(Y) = aE(X) + b$ and $\text{Var}(Y) = a^2\text{Var}(X)$.

Also,

$\text{Cov}(X,Y) = E(\ (X - E(X))\ (Y - E(Y))\ )$

$= E(\ (X - E(X))\ a(X - E(X))\ )$

$= a\text{Var}(X)$.

Therefore,

$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{a\text{Var}(X)}{\sqrt{\text{Var}(X)a^2\text{Var}(X)}} = \frac{a}{|a|}$

which is 1 if $a > 0$ and is -1 if $a < 0$.

2.9.11 The expected amount of a claim is

$E(X) = \int_0^{1800} x\ \frac{x(1800-x)}{972{,}000{,}000}\ dx = \$900.$

Consequently, the expected profit from each customer is

$\$100 - \$5 - (0.1 \times \$900) = \$5.$

The expected profit from 10,000 customers is therefore $10,000 \times \$5 = \$50,000$.

The profits may or may not be independent depending on the type of insurance and the pool of customers.

If large natural disasters affect the pool of customers all at once then the claims would not be independent.

2.9.12 (a) The expectation is $5 \times 320 = 1600$ seconds.

The variance is $5 \times 63^2 = 19845$

and the standard deviation is $\sqrt{19845} = 140.9$ seconds.

(b) The expectation is 320 seconds.

The variance is $\frac{63^2}{10} = 396.9$

and the standard deviation is $\sqrt{396.9} = 19.92$ seconds.

2.9.13 (a) The state space is the positive integers from 1 to n, with each outcome having a probability value of $\frac{1}{n}$.

(b) $E(X) = \left(\frac{1}{n} \times 1\right) + \left(\frac{1}{n} \times 2\right) + \ldots + \left(\frac{1}{n} \times n\right) = \frac{n+1}{2}$

(c) $E(X^2) = \left(\frac{1}{n} \times 1^2\right) + \left(\frac{1}{n} \times 2^2\right) + \ldots + \left(\frac{1}{n} \times n^2\right) = \frac{(n+1)(2n+1)}{6}$

Therefore,
$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$.

2.9.14 (a) Let X_T be the amount of time that Tom spends on the bus and let X_N be the amount of time that Nancy spends on the bus. Therefore, the sum of the times is $X = X_T + X_N$ and
$E(X) = E(X_T) + E(X_N) = 87 + 87 = 174$ minutes.

If Tom and Nancy ride on different buses then the random variables X_T and X_N are independent so that
$\mathrm{Var}(X) = \mathrm{Var}(X_T) + \mathrm{Var}(X_N) = 3^2 + 3^2 = 18$
and the standard deviation is $\sqrt{18} = 4.24$ minutes.

(b) If Tom and Nancy ride together on the same bus then $X_T = X_N$ so that $X = 2 \times X_T$, twice the time of the ride.
In this case
$E(X) = 2 \times E(X_T) = 2 \times 87 = 174$ minutes
and
$\mathrm{Var}(X) = 2^2 \times \mathrm{Var}(X_1) = 2^2 \times 3^2 = 36$
so that the standard deviation is $\sqrt{36} = 6$ minutes.

2.9.15 (a) Two heads gives a total score of 20.
One head and one tail gives a total score of 30.
Two tails gives a total score of 40.
Therefore, the state space is $\{20, 30, 40\}$.

(b) $P(X = 20) = \frac{1}{4}$

$P(X = 30) = \frac{1}{2}$

$P(X = 40) = \frac{1}{4}$

(c) $P(X \leq 20) = \frac{1}{4}$

$P(X \leq 30) = \frac{3}{4}$

$P(X \leq 40) = 1$

(d) $E(X) = \left(20 \times \frac{1}{4}\right) + \left(30 \times \frac{1}{2}\right) + \left(40 \times \frac{1}{4}\right) = 30$

(e) $E(X^2) = \left(20^2 \times \frac{1}{4}\right) + \left(30^2 \times \frac{1}{2}\right) + \left(40^2 \times \frac{1}{4}\right) = 950$

$\text{Var}(X) = 950 - 30^2 = 50$

The standard deviation is $\sqrt{50} = 7.07$.

2.9.16 (a) Since

$\int_5^6 Ax\; dx = \frac{A}{2} \times (6^2 - 5^2) = 1$

it follows that $A = \frac{2}{11}$.

(b) $F(x) = \int_5^x \frac{2y}{11}\; dy = \frac{x^2 - 25}{11}$

(c) $E(X) = \int_5^6 \frac{2x^2}{11}\; dx = \frac{2\times(6^3-5^3)}{33} = \frac{182}{33} = 5.52$

(d) $E(X^2) = \int_5^6 \frac{2x^3}{11}\; dx = \frac{6^4-5^4}{22} = \frac{671}{22} = 30.5$

$\text{Var}(X) = 30.5 - \left(\frac{182}{33}\right)^2 = 0.083$

The standard deviation is $\sqrt{0.083} = 0.29$.

2.9.17 (a) The expectation is $3 \times 438 = 1314$.
The standard deviation is $\sqrt{3} \times 4 = 6.93$.

(b) The expectation is 438.
The standard deviation is $\frac{4}{\sqrt{8}} = 1.41$.

2.9.18 (a) If a 1 is obtained from the die the net winnings are $(3 \times \$1) - \$5 = -\$2$
If a 2 is obtained from the die the net winnings are $\$2 - \$5 = -\$3$
If a 3 is obtained from the die the net winnings are $(3 \times \$3) - \$5 = \$4$
If a 4 is obtained from the die the net winnings are $\$4 - \$5 = -\$1$
If a 5 is obtained from the die the net winnings are $(3 \times \$5) - \$5 = \$10$
If a 6 is obtained from the die the net winnings are $\$6 - \$5 = \$1$

Each of these values has a probability of $\frac{1}{6}$.

(b) $E(X) = \left(-3 \times \frac{1}{6}\right) + \left(-2 \times \frac{1}{6}\right) + \left(-1 \times \frac{1}{6}\right)$

$+ \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(10 \times \frac{1}{6}\right) = \frac{3}{2}$

$E(X^2) = \left((-3)^2 \times \frac{1}{6}\right) + \left((-2)^2 \times \frac{1}{6}\right) + \left((-1)^2 \times \frac{1}{6}\right)$

$+ \left(1^2 \times \frac{1}{6}\right) + \left(4^2 \times \frac{1}{6}\right) + \left(10^2 \times \frac{1}{6}\right) = \frac{131}{6}$

The variance is

$\frac{131}{6} - \left(\frac{3}{2}\right)^2 = \frac{235}{12}$

and the standard deviation is $\sqrt{\frac{235}{12}} = \$4.43$.

(c) The expectation is $10 \times \frac{3}{2} = \$15$.
The standard deviation is $\sqrt{10} \times 4.43 = \$13.99$.

2.9.19 (a) False

(b) True

(c) True

(d) True

(e) True

(f) False

2.9.20 $E(\text{total time}) = 5 \times \mu = 5 \times 22 = 110$ minutes

The standard deviation of the total time is $\sqrt{5}\sigma = \sqrt{5} \times 1.8 = 4.02$ minutes.

$E(\text{average time}) = \mu = 22$ minutes

The standard deviation of the average time is $\frac{\sigma}{\sqrt{5}} = \frac{1.8}{\sqrt{5}} = 0.80$ minutes.

2.9.21 (a) $E(X) = (0 \times 0.12) + (1 \times 0.43) + (2 \times 0.28) + (3 \times 0.17) = 1.50$

(b) $E(X^2) = (0^2 \times 0.12) + (1^2 \times 0.43) + (2^2 \times 0.28) + (3^2 \times 0.17) = 3.08$
The variance is $3.08 - 1.50^2 = 0.83$
and the standard deviation is $\sqrt{0.83} = 0.911$.

(c) $E(X_1 + X_2) = E(X_1) + E(X_2) = 1.50 + 1.50 = 3.00$
$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 0.83 + 0.83 = 1.66$
The standard deviation is $\sqrt{1.66} = 1.288$.

2.9.22 (a) $E(X) = (-22 \times 0.3) + (3 \times 0.2) + (19 \times 0.1) + (23 \times 0.4) = 5.1$

(b) $E(X^2) = ((-22)^2 \times 0.3) + (3^2 \times 0.2) + (19^2 \times 0.1) + (23^2 \times 0.4) = 394.7$
$\text{Var}(X) = 394.7 - 5.1^2 = 368.69$
The standard deviation is $\sqrt{368.69} = 19.2$.

2.9.23 (a) Since

$\int_2^4 f(x)\ dx = \int_2^4 \frac{A}{x^2}\ dx = \frac{A}{4} = 1$

it follows that $A = 4$.

(b) Since

$\frac{1}{4} = \int_2^y f(x)\ dx = \int_2^y \frac{4}{x^2}\ dx = \left(2 - \frac{4}{y}\right)$

it follows that $y = \frac{16}{7} = 2.29$.

2.9.24 (a) $100 = E(Y) = c + dE(X) = c + (d \times 250)$
$1 = \text{Var}(Y) = d^2\text{Var}(X) = d^2 \times 16$

Solving these equations gives $d = \frac{1}{4}$ and $c = \frac{75}{2}$
or $d = -\frac{1}{4}$ and $c = \frac{325}{2}$.

(b) The mean is $10 \times 250 = 1000$.
The standard deviation is $\sqrt{10} \times 4 = 12.65$.

2.9.25 Since

$E(c_1X_1 + c_2X_2) = c_1E(X_1) + c_2E(X_2) = (c_1 + c_2) \times 100 = 100$

it is necessary that $c_1 + c_2 = 1$.

Also,

$\text{Var}(c_1X_1 + c_2X_2) = c_1^2\text{Var}(X_1) + c_2^2\text{Var}(X_2) = (c_1^2 \times 144) + (c_2^2 \times 169) = 100.$

Solving these two equations gives $c_1 = 0.807$ and $c_2 = 0.193$

or $c_1 = 0.273$ and $c_2 = 0.727$.

2.9.26 (a) The mean is $3\mu_A = 3 \times 134.9 = 404.7$.

The standard deviation is $\sqrt{3}\ \sigma_A = \sqrt{3} \times 0.7 = 1.21$.

(b) The mean is $2\mu_A + 2\mu_B = (2 \times 134.9) + (2 \times 138.2) = 546.2$.

The standard deviation is $\sqrt{0.7^2 + 0.7^2 + 1.1^2 + 1.1^2} = 1.84$.

(c) The mean is $\frac{4\mu_A+3\mu_B}{7} = \frac{(4\times 134.9)+(3\times 138.2)}{7} = 136.3$.

The standard deviation is $\frac{\sqrt{0.7^2+0.7^2+0.7^2+0.7^2+1.1^2+1.1^2+1.1^2}}{7} = 0.34$.

2.9.27 B

2.9.28 Expectation $= \$150,000 - \$175,000 = -\$25,000$

Standard deviation $= \sqrt{30000^2 + 40000^2} = \$50,000$