9. Complete the following assignment prior to Meeting #6

A. Study our notes from Meeting #5

B*. Examine each of the following propositions, determine whether or not it is true, display your choice by circling either "T" or "F," and prove that your choice is correct:

i. $(f \subseteq \{ {}^{-}1, 0, 1 \} \times \{ {}^{-}1, 0, 1 \} \ni f(x) = x^2) \Rightarrow$

$f : \{ {}^{-}1, 0, 1 \} \xrightarrow[onto]{1:1} \{ {}^{-}1, 0, 1 \}$

T F

Sample proof:

Let's examine the elements of f. $f = \{ ({}^{-}1, 1), (0, 0), (1, 1) \}$ we see that although ${}^{-}1 \neq 1$, $f({}^{-}1) = f(1)$. Thus, the proposition is false because f is not an injection. So we're done.

However, note that the following argument would also have been sufficient to prove that the proposition is false:

Let's examine the elements of f. $f = \{ ({}^{-}1, 1), (0, 0), (1, 1) \}$ we see that although ${}^{-}1 \in$ the codomain of f, ${}^{-}1 \notin$ range of f. Thus, the proposition is false because f is not a surjection.

ii. $\{ n^2 : n \in \mathbb{N} \} \sim \mathbb{N}$

T F

Sample proof:

$\exists f \ni f : \mathbb{N} \xrightarrow[onto]{1:1} \{ n^2 : n \in \mathbb{N} \}$ because of the following argument:

Let $f = \{ (1, 1), (2, 4), (3, 9), ... \}$, thus, $f : \mathbb{N} \to \{ n^2 : n \in \mathbb{N} \} \ni f(x) = x^2$. It's clear from examining the elements of f that there is a one-to-one correspondence between $\mathbb{N}$ and $\{ n^2 : n \in \mathbb{N} \}$, but for practice, let's be a bit more formal:

To show that f is an injection, note the following string of deductions:

$f(n_1) = f(n_2) \Rightarrow n_1^2 = n_2^2 \Rightarrow n_1 = n_2$. Keep in mind that since $n_1, n_2 \in \mathbb{N}$, we need not be concerned with either n_1 or n_2 being negative when we computed their square roots in the equation ($n_1^2 = n_2^2$).

To show that f is a surjection, note the following string of deductions:

Suppose y is an arbitrary element of the codomain (i.e., $\{ n^2 : n \in \mathbb{N} \}$). We need to demonstrate that there exist n in the domain (i.e., $\mathbb{N}$) $\ni f(n) = y$. Keeping in mind that y is a perfect square of a natural number, let's select the natural number $\sqrt{y}$ for n. Now, $f(\sqrt{y}) = y \Rightarrow \left(\sqrt{y}\right)^2 = y$ so we've found our domain element that pairs with y.

$\therefore$ we have our bijection that proves $\{ n^2 : n \in \mathbb{N} \} \sim \mathbb{N}$

Q.E.D.

iii. $\mathbb{Z} \sim \mathbb{N}$

T F

Sample proof:

$\exists f \ni f: \mathbb{Z} \xrightarrow[onto]{1:1} \mathbb{N}$ } because of the following argument:

Let f = { ..., ($^-4$, 8), ($^-3$, 6), ($^-2$, 4), ($^-1$, 2) } $\cup$ { (0, 1), (1, 3), (2, 5), (3, 7), ... },

thus, $f: \mathbb{Z} \rightarrow \mathbb{N} \ni (f(x) = {}^-2x \ \forall x < 0 \wedge f(x) = 2x + 1 \ \forall x \geq 0)$. It's clear from examining the elements of f that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$ but for practice, let's be a bit more formal:

To show that f is an injection, note the following string of deductions presented in the two cases:

Case 1 in which $x < 0$ (i.e., $\forall x \in$ { ..., $^-4$, $^-3$, $^-2$, $^-1$ }):

$f(x_1) = f(x_2) \Rightarrow {}^-2x_2 = {}^-2x_1 \Rightarrow x_1 = x_2$.

Case 2 in which $x \geq 0$ (i.e., $\forall x \in$ { 0, 1, 2, 3, ... }):

$f(x_3) = f(x_4) \Rightarrow 2x_3 + 1 = 2x_4 + 1 \Rightarrow x_3 = x_4$.

To show that f is a surjection, note the following string of deductions presented in the two cases:

Case 1 in which $x < 0$ (i.e., $\forall x \in$ { ..., $^-4$, $^-3$, $^-2$, $^-1$ }):

Suppose y_1 is an arbitrary element of the proper subset of the codomain that is relevant to Case 1 (i.e., { even natural numbers }). We need to demonstrate that there exist x_1 in the proper subset of the domain that is relevant to Case 1 (i.e., { negative integers }) $\ni f(x_1) = y_1$. Keeping in mind that y_1 is an even natural number, let's select $^-0.5y_1$ for x_1. So $f(x_1) = f({}^-0.5y_1) = {}^-2({}^-0.5y_1) = y_1$. So we've found our domain element that pairs with y.

Case 2 in which $x \geq 0$ (i.e., $\forall x \in$ { 0, 1, 2, 3, ... }):

Suppose y_2 is an arbitrary element of the proper subset of the codomain that is relevant to Case 1 (i.e., { odd natural numbers }). We need to demonstrate that there exist x_2 in the proper subset of the domain that is relevant to Case 1 (i.e., { non-negative integers }) $\ni f(x_2) = y_2$. Keeping in mind that y_1 is an odd

natural number, let's select for $0.5(y_2 - 1)$ for x_2. So $f(x_2) = f(0.5(y_2 - 1)) = 2(0.5(y_2 - 1) + 1 = y_2 - 1 + 1 = y_2$
So we've found our domain element that pairs with y.

$\therefore$ we have our bijection that proves $\mathbb{Z} \sim \mathbb{N}$ *Q.E.D.*

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iv. $[0, 1] \sim [2, 3]$

T F

Sample proof:

Note that $0 \le x \le 1 \Rightarrow 2 \le x + 2 \le 3$. So consider $f \ni (f : [0, 1] \to [2, 3]$ with $f(x) = x + 2$.

To show that f is an injection:

$$f(x_1) = f(x_2) \Rightarrow x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$$

To show that f is a surjection:

Suppose $y \in [2, 3]$, then $y - 2 \in [0, 1]$. And we choose $y - 2$ for our x to associate with y and thus demonstrate that $\forall y$ in the codomain of f, $\exists x \in$ in the domain of $f \ni f(x) = y$:

$$f(y - 2) = y - 2 + 2 = y$$

Q.E.D.

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v. $[\,^{-}1,\ 0\,] \sim [\,0, 0.25\,]$

T F

Sample proof:

Note that $^{-}1 \le x \le 0 \Rightarrow 0 \le \left(\frac{x+1}{4}\right) \le 0.25$. So consider $f \ni (\, f : [\,^{-}1,\ 0\,] \rightarrow [\,0, 0.25\,]$ with $f(x) = \left(\frac{x+1}{4}\right)$.

To show that f is an injection:

$$f(x_1) = f(x_2) \Rightarrow \left(\frac{x_1+1}{4}\right) = \left(\frac{x_2+1}{4}\right) \Rightarrow x_1 = x_2$$

To show that f is a surjection:

Suppose $y \in [\,0,\ 0.25\,]$, then $4y - 1 \in [\,0, 0.25\,]$. And we choose $4y - 1$ for our x to associate with y and thus demonstrate that $\forall\, y$ in the codomain of f, $\exists\, x \in$ in the domain of $f \ni f(x) = y$:

$$f(4y - 1) = \frac{4y-1+1}{4} = y$$

Q.E.D.

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vi. $V = \mathbb{Z} \Rightarrow \{\,^{-}n : n \in \omega\,\}^{c} = \mathbb{N}$

T F

Sample proof:

$\{\,^{-}n : n \in \omega\,\} = \{\,\ldots, ^{-}4,\ ^{-}3.\ ^{-}2,\ ^{-}1,\ 0\,\}$. So for $V = \mathbb{Z}$, the complement of

$\{\,\ldots, ^{-}4,\ ^{-}3.\ ^{-}2,\ ^{-}1,\ 0\,\}$ is $\{\,1, 2, 3, 4, \ldots\,\} = \mathbb{Z} - \omega = \mathbb{N}$

Q.E.D.

C*. The following proposition is, of course, true:

$$\mathbb{Z} \sim \mathbb{R} \;\veebar\; \mathbb{Z} \nsim \mathbb{R}$$

Which one of the following propositions do you think is true? Circle either "$\mathbb{Z} \sim \mathbb{R}$" or "$\mathbb{Z} \nsim \mathbb{R}$" to indicate your choice: $\mathbb{Z} \sim \mathbb{R}$ $\mathbb{Z} \nsim \mathbb{R}$

Write a paragraph that explain the rationale for your choice.

Sample explanation:

It's true that both sets are infinite, but I think in a weird way $|\mathbb{Z}|$ is somehow less than $|\mathbb{R}|$. $\mathbb{Z}$ is a discrete set in which no integers exists between consecutive integers (e.g., between 40 and 41). On the other hand, between any two different real numbers (e.g., between 4 and 4.1), there exists infinitely more real numbers.

D. Compare your responses to the homework prompts to those Jim posted in *Canvas* on the usual page.